PARTIAL DIFFERENTIAL EQUATIONS

Math 124A – Fall 2010

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These lecture notes arose from the course "Partial Differential Equations" – Math 124A taught by the author in the Department of Mathematics at UCSB in the fall quarters of 2009 and 2010. The selection of topics and the order in which they are introduced is based on [Str]. Most of the problems appearing in this text are also borrowed from Strauss. A list of other references that were consulted while teaching this course appears in the bibliography at the end.

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`http://www.math.ucsb.edu/~grigoryan/124A.pdf`

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1 Introduction

Recall that an ordinary differential equation (ODE) contains an independent variable x and a dependent variable u, which is the unknown in the equation. The defining property of an ODE is that derivatives of the unknown function $u' = \frac{du}{dx}$ enter the equation. Thus, an equation that relates the independent variable x, the dependent variable u and derivatives of u is called an *ordinary differential equation*. Some examples of ODEs are:

$$\begin{aligned} u'(x) &= u \\ u'' + 2xu &= e^x \\ u'' + x(u')^2 + \sin u &= \ln x \end{aligned}$$

In general, and ODE can be written as $F(x, u, u', u'', \dots) = 0$.

In contrast to ODEs, a partial differential equation (PDE) contains partial derivatives of the dependent variable, which is an unknown function in more than one variable $x, y, \dots$. Denoting the partial derivative of $\frac{\partial u}{\partial x} = u_x$, and $\frac{\partial u}{\partial y} = u_y$, we can write the general first order PDE for $u(x, y)$ as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \tag{1.1}$$

Although one can study PDEs with as many independent variables as one wishes, we will be primarily concerned with PDEs in two independent variables. A *solution* to the PDE (1.1) is a function $u(x, y)$ which satisfies (1.1) for all values of the variables x and y. Some examples of PDEs (of physical significance) are:

$$u_x + u_y = 0 \qquad \text{transport equation} \tag{1.2}$$
$$u_t + uu_x = 0 \qquad \text{inviscid Burger's equation} \tag{1.3}$$
$$u_{xx} + u_{yy} = 0 \qquad \text{Laplace's equation} \tag{1.4}$$
$$u_{tt} - u_{xx} = 0 \qquad \text{wave equation} \tag{1.5}$$
$$u_t - u_{xx} = 0 \qquad \text{heat equation} \tag{1.6}$$
$$u_t + uu_x + u_{xxx} = 0 \qquad \text{KdV equation} \tag{1.7}$$
$$iu_t - u_{xx} = 0 \qquad \text{Shrödinger's equation} \tag{1.8}$$

It is generally nontrivial to find the solution of a PDE, but once the solution is found, it is easy to verify whether the function is indeed a solution. For example to see that $u(t, x) = e^{t-x}$ solves the wave equation (1.5), simply substitute this function into the equation:

$$(e^{t-x})_{tt} - (e^{t-x})_{xx} = e^{t-x} - e^{t-x} = 0.$$

1.1 Classification of PDEs

There are a number of properties by which PDEs can be separated into families of similar equations. The two main properties are *order* and *linearity*.

Order. The order of a partial differential equation is the order of the highest derivative entering the equation. In examples above (1.2), (1.3) are of first order; (1.4), (1.5), (1.6) and (1.8) are of second order; (1.7) is of third order.

Linearity. Linearity means that all instances of the unknown and its derivatives enter the equation linearly. To define this property, rewrite the equation as

$$\mathcal{L}u = 0, \tag{1.9}$$

where $\mathcal{L}$ is an operator, which assigns u a new function $\mathcal{L}u$. For example $\mathcal{L} = \frac{\partial^2}{\partial x^2} + 1$, then $\mathcal{L}u = u_{xx} + u$. The operator $\mathcal{L}$ is called linear if

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \quad \text{and} \quad \mathcal{L}(cu) = c\mathcal{L}u \tag{1.10}$$

for any functions u, v and constant c. The equation (1.9) is called linear, if $\mathcal{L}$ is a linear operator. In our examples above (1.2), (1.4), (1.5), (1.6), (1.8) are linear, while (1.3) and (1.7) are nonlinear (*i.e.* not linear). To see this, let us check, e.g. (1.6) for linearity:

$$\mathcal{L}(u+v) = (u+v)_t - (u+v)_{xx} = u_t + v_t - u_{xx} - v_{xx} = (u_t - u_{xx}) + (v_t - v_{xx}) = \mathcal{L}u + \mathcal{L}v,$$

and

$$\mathcal{L}(cu) = (cu)_t - (cu)_{xx} = cu_t - cu_{xx} = c(u_t - u_{xx}) = c\mathcal{L}u.$$

So, indeed, (1.6) is a linear equation, since it is given by a linear operator. To understand how linearity can fail, let us see what goes wrong for equation (1.3):

$$\mathcal{L}(u+v) = (u+v)_t + (u+v)(u+v)_x = u_t + v_t + (u+v)(u_x+v_x) = (u_t+uu_x)+(v_t+vv_x)+uv_x+vu_x \neq \mathcal{L}u+\mathcal{L}v.$$

You can check that the second condition of linearity fails as well. This happens precisely due to the nonlinearity of the uu_x term, which is quadratic in "u and its derivatives".

Notice that for a linear equation, if u is a solution, then so is cu, and if v is another solution, then $u + v$ is also a solution. In general any linear combination of solutions

$$c_1u_1(x, y) + c_2u_2(x, y) + \cdots + c_nu_n(x, y) = \sum_{i=1}^{n} c_iu_i(x, y)$$

will also solve the equation.

The linear equation (1.9) is called *homogeneous* linear PDE, while the equation

$$\mathcal{L}u = g(x, y) \tag{1.11}$$

is called *inhomogeneous* linear equation. Notice that if u^h is a solution to the homogeneous equation (1.9), and u^p is a particular solution to the inhomogeneous equation (1.11), then $u^h + u^p$ is also a solution to the inhomogeneous equation (1.11). Indeed

$$\mathcal{L}(u^h + u^p) = \mathcal{L}u^h + \mathcal{L}u^p = 0 + g = g.$$

Thus, in order to find the general solution of the inhomogeneous equation (1.11), it is enough to find the general solution of the homogeneous equation (1.9), and add to this a particular solution of the inhomogeneous equation (check that the difference of any two solutions of the inhomogeneous equation is a solution of the homogeneous equation). In this sense, there is a similarity between ODEs and PDEs, since this principle relies only on the linearity of the operator $\mathcal{L}$.

1.2 Examples

Example 1.1. $u_x = 0$

Remember that we are looking for a function $u(x, y)$, and the equation says that the partial derivative of u with respect to x is 0, so u does not depend on x. Hence $u(x, y) = f(y)$, where $f(y)$ is an arbitrary function of y. Alternatively, we could simply integrate both sides of the equation with respect to x. More on this in the following examples. □

Example 1.2. $u_{xx} + u = 0$

Similar to the previous example, we see that only the partial derivative with respect to one of the variables enters the equation. In such cases we can treat the equation as an ODE in the variable in which partial derivatives enter the equation, keeping in mind that the constants of integration may depend on the other variables. Rewrite the equation as

$$u_{xx} = -u,$$

which, as an ODE, has the general solution

$$u = c_1 \cos x + c_2 \sin x.$$

Since the constants may depend on the other variable y, the general solution of the PDE will be

$$u(x, y) = f(y)\cos x + g(y)\sin x,$$

where f and g are arbitrary functions. To check that this is indeed a solution, simply substitute the expression back into the equation. $\square$

Example 1.3. $u_{xy} = 0$
We can think of this equation as an ODE for u_x in the y variable, since $(u_x)_y = 0$. Then similar to the first example, we can integrate in y to obtain

$$u_x = f(x).$$

This is an ODE for u in the x variable, which one can solve by integrating with respect to x, arriving at at the solution

$$u(x, y) = F(x) + G(y).$$

$\square$

1.3 Conclusion

Notice that where the solution of an ODE contains arbitrary constants, the solution to a PDE contains arbitrary functions. In the same spirit, while an ODE of order m has m linearly independent solutions, a PDE has infinitely many (there are arbitrary functions in the solution!). These are consequences of the fact that a function of two variables contains immensely more (a whole dimension worth) of information than a function of only one variable.

Problem Set 1

1. (#1.1.2 in [Str]) Which of the following operators are linear?

   (a) $\mathcal{L}u = u_x + xu_y$

   (b) $\mathcal{L}u = u_x + uu_y$

   (c) $\mathcal{L}u = u_x + u_y^2$

   (d) $\mathcal{L}u = u_x + u_y + 1$

   (e) $\mathcal{L}u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$

2. (#1.1.3 in [Str]) For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

   (a) $u_t - u_{xx} + 1 = 0$

   (b) $u_t - u_{xx} + xu = 0$

   (c) $u_t - u_{xxt} + uu_x = 0$

   (d) $u_{tt} - u_{xx} + x^2 = 0$

   (e) $iu_t - u_{xx} + u/x = 0$

   (f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$

   (g) $u_x + e^y u_y = 0$

   (h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

3. Show that $\cos(x - ct)$ is a solution of $u_t + cu_x = 0$.

4. (#1.1.10 in [Str]) Show that the solutions of the differential equation $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis of it.

5. (#1.1.11 in [Str]) Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

6. (#1.1.12 in [Str]) Verify by direct substitution that

$$u_n(x, y) = \sin nx \sinh ny$$

   is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

7. Find the general solution of

$$u_{xy} + u_x = 0.$$

   (Hint: first treat it as an ODE for u_x).

2 First-order linear equations

Last time we saw how some simple PDEs can be reduced to ODEs, and subsequently solved using ODE methods. For example, the equation

$$u_x = 0 \tag{2.1}$$

has "constant in x" as its general solution, and hence u depends only on y, thus $u(x,y) = f(y)$ is the general solution, with f an arbitrary function of a single variable. Today we will see that any linear first order PDE can be reduced to an ordinary differential equation, which will then allow as to tackle it with already familiar methods from ODEs.

Let us start with a simple example. Consider the following constant coefficient PDE

$$au_x + bu_y = 0. \tag{2.2}$$

Here a and b are constants, such that $a^2+b^2 \neq 0$, i.e. at least one of the coefficients is nonzero (otherwise this would not be a differential equation). Using the inner (scalar or dot) product in $\mathbb{R}^2$, we can rewrite the left hand side of (2.2) as

$$(a,b)\cdot(u_x,u_y) = 0, \qquad \text{or} \qquad (a,b)\cdot\nabla u = 0.$$

Denoting the vector $(a,b) = \mathbf{v}$, we see that the left hand side of the above equation is exactly $D_{\mathbf{v}}u(x,y)$, the directional derivative of u in the direction of the vector $\mathbf{v}$. Thus the solution to (2.2) must be constant in the direction of the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

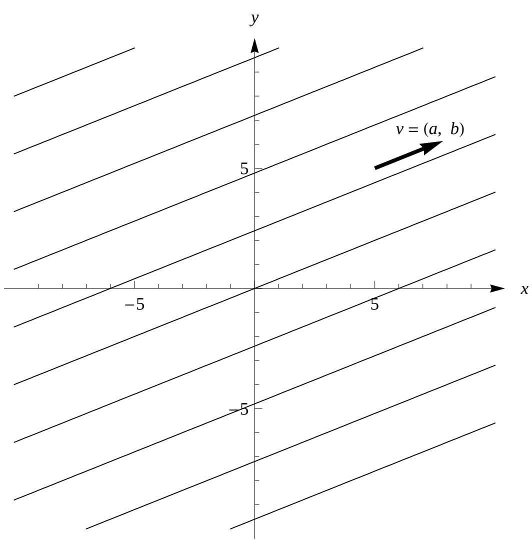

Figure 2.1: Characteristic lines $bx - ay = c$.

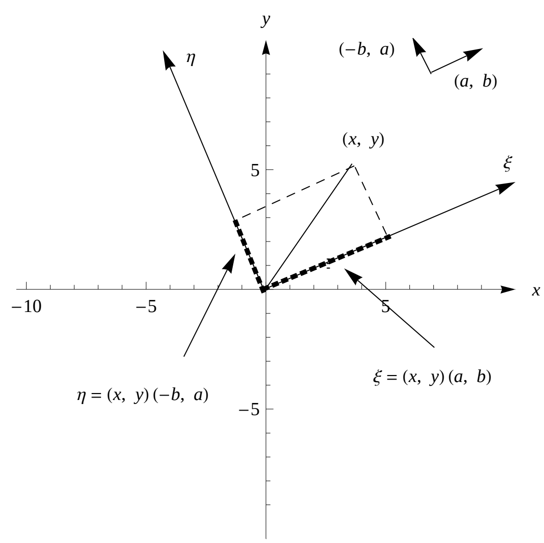

Figure 2.2: Change of coordinates.

The lines parallel to the vector $\mathbf{v}$ have the equation

$$bx - ay = c, \tag{2.3}$$

since the vector $(b,-a)$ is orthogonal to $\mathbf{v}$, and as such is a normal vector to the lines parallel to $\mathbf{v}$. In equation (2.3) c is an arbitrary constant, which uniquely determines the particular line in this family of parallel lines, called *characteristic lines* for the equation (2.2).

As we saw above, $u(x,y)$ is constant in the direction of $\mathbf{v}$, hence also along the lines (2.3). The line containing the point (x,y) is determined by $c = bx - ay$, thus u will depend only on $bx - ay$, that is

$$u(x,y) = f(bx - ay), \tag{2.4}$$

where f is an arbitrary function. One can then check that this is the correct solution by plugging it into the equation. Indeed,

$$a\partial_x f(bx-ay)+b\partial_y f(bx-ay)=abf'(bx-ay)-baf'(bx-ay)=0.$$

The geometric viewpoint that we used to arrive at the solution is akin to solving equation (2.1) simply by recognizing that a function with a vanishing derivative must be constant. However one can approach equation (2.2) from another perspective, by trying to reduce it to an ODE.

2.1 The method of characteristics

To have an ODE, we need to eliminate one of the partial derivatives in the equation. But we know that the directional derivative vanishes in the direction of the vector (a,b). Let us then make a change of the coordinate system to one that has its "x-axis" parallel to this vector, as in Figure 2. In this coordinate system

$$(\xi,\eta)=((x,y)\cdot(a,b),(x,y)\cdot(b,-a))=(ax+by,bx-ay).$$

So the change of coordinates is

$$\begin{cases}\xi=ax+by,\\ \eta=bx-ay.\end{cases}\tag{2.5}$$

To rewrite the equation (2.2) in this coordinates, notice that

$$\begin{aligned}u_x&=u_\xi\frac{\partial\xi}{\partial x}+u_\eta\frac{\partial\eta}{\partial x}=au_\xi+bu_\eta,\\ u_y&=u_\xi\frac{\partial\xi}{\partial y}+u_\eta\frac{\partial\eta}{\partial y}=bu_\xi-au_\eta.\end{aligned}$$

Thus,

$$0=au_x+bu_y=a(au_\xi+bu_\eta)+b(bu_\xi-au_\eta)=(a^2+b^2)u_\xi.$$

Now, since $a^2+b^2\neq 0$, then, as we anticipated,

$$u_\xi=0,$$

which is an ODE. We can solve this last equation just as we did in the case of equation (2.1), arriving at the solution

$$u(\xi,\eta)=f(\eta).$$

Changing back to the original coordinates gives $u(x,y)=f(bx-ay)$. This is the same solution that we derived with the geometric deduction. This method of reducing the PDE to an ODE is called *the method of characteristics*, and the coordinates (ξ,η) given by formulas (2.5) are called *characteristic coordinates*.

Example 2.1. Find the solution of the equation $3u_x-5u_y=0$ satisfying the condition $u(0,y)=\sin y$.

From the above discussion we know that u will depend only on $\eta=-5x-3y$, so $u(x,y)=f(-5x-3y)$. The solution also has to satisfy the additional condition (called initial condition), which we verify by plugging in $x=0$ into the general solution.

$$\sin y=u(0,y)=f(-3y).$$

So $f(z)=\sin(-\frac{z}{3})$, and hence $u(x,y)=\sin\left(\frac{5x+3y}{3}\right)$, which one can verify by substituting into the equation and the initial condition. □

2.2 General constant coefficient equations

We can easily adapt the method of characteristics to general constant coefficient linear first-order equations

$$au_x + bu_y + cu = g(x, y). \tag{2.6}$$

Recall that to find the general solution of this equation it is enough to find the general solution of the homogeneous equation

$$au_x + bu_y + cu = 0, \tag{2.7}$$

and add to this a particular solution of the inhomogeneous equation (2.6). Notice that in the characteristic coordinates (2.5), equation (2.7) will take the form

$$(a^2 + b^2)u_\xi + cu = 0, \qquad \text{or} \qquad u_\xi + \frac{c}{a^2 + b^2}u = 0,$$

which can be treated as an ODE in ξ. The solution to this ODE has the form

$$u_h(\xi, \eta) = e^{-\frac{c}{a^2+b^2}\xi} f(\eta),$$

with f again being an arbitrary single-variable function. Changing the coordinates back to the original (x, y), we will obtain the general solution to the homogeneous equation

$$u_h(x, y) = e^{-\frac{c(ax+by)}{a^2+b^2}} f(bx - ay).$$

You should verify that this indeed solves equation (2.7).

To find a particular solution of (2.6), we can use the characteristic coordinates to reduce it to the inhomogeneous ODE

$$(a^2 + b^2)u_\xi + cu = g(\xi, \eta), \qquad \text{or} \qquad u_\xi + \frac{c}{a^2 + b^2}u = \frac{g(\xi, \eta)}{a^2 + b^2}.$$

Having found the solution to the homogeneous ODE, we can find the solution to this inhomogeneous equation by e.g. variation of parameters. So the particular solution will be

$$u_p = e^{-\frac{c}{a^2+b^2}\xi} \int \frac{g(\xi, \eta)}{a^2 + b^2} e^{\frac{c}{a^2+b^2}\xi} \, d\xi.$$

The general solution of (2.6) is then

$$u(\xi, \eta) = u_h + u_p = e^{-\frac{c}{a^2+b^2}\xi} \left(f(\eta) + \int \frac{g(\xi, \eta)}{a^2 + b^2} e^{\frac{c}{a^2+b^2}\xi} \, d\xi \right).$$

To find the solution in terms of (x, y), one needs to first carry out the integration in ξ in the above formula, then replace ξ and η by their expressions in terms of x and y.

Example 2.2. Find the general solution of $-2u_x + 4u_y + 5u = e^{x+3y}$.

The characteristic change of coordinates for this equation is given by

$$\begin{cases} \xi = -2x + 4y, \\ \eta = 4x + 2y. \end{cases}$$

From these we can also find the expressions of x and y in terms of (ξ, η). In particular notice that $x + 3y = \frac{\xi+\eta}{2}$. In the characteristic coordinates the equation reduces to

$$20u_\xi + 5u = e^{\frac{\xi+\eta}{2}}.$$

The general solution of the homogeneous equation associated with the above equation is

$$u_h = e^{-\frac{1}{4}\xi} f(\eta),$$

and the particular solution will be

$$u_p = e^{-\frac{1}{4}\xi} \int \frac{e^{\frac{\xi+\eta}{2}}}{20} e^{\frac{1}{4}\xi}\, d\xi = e^{-\frac{1}{4}\xi} \cdot \frac{1}{15} e^{\frac{\eta}{2}} e^{\frac{3}{4}\xi} = e^{-\frac{1}{4}\xi} \cdot \frac{1}{15} e^{\frac{1}{4}(3\xi+2\eta)}.$$

Adding these two will give the general solution to the inhomogeneous equation

$$u(\xi, \eta) = e^{-\frac{1}{4}\xi} \left(f(\eta) + \frac{1}{15} e^{\frac{1}{4}(3\xi+2\eta)} \right).$$

Finally, substituting the expressions for ξ and η in terms of (x, y), we will obtain the solution

$$u(x, y) = e^{-\frac{1}{4}(2x-4y)} \left(f(4x + 2y) + \frac{1}{15} e^{\frac{1}{4}(2x+16y)} \right).$$

You should check that this indeed solves the differential equation. □

2.3 Variable coefficient equations

The method of characteristics can be generalized to variable coefficient first-order linear PDEs as well, albeit the change of variables may no longer be orthogonal. The general variable coefficient linear first-order equations is

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \tag{2.8}$$

Let us first consider the following simple particular case

$$u_x + yu_y = 0. \tag{2.9}$$

Using our geometric intuition from the constant coefficient equations, we see that the directional derivative of u in the direction of the vector $\mathbf{v} = (1, y)$ is constant. Notice that the vector $\mathbf{v}$ itself is no longer constant, and varies with y. The curves that have $\mathbf{v}$ as their tangent vector have slope $\frac{y}{1}$, and thus satisfy

$$\frac{dy}{dx} = \frac{y}{1}.$$

We can solve this equation as an ODE, and obtain the general solution

$$y = Ce^x, \qquad \text{or} \qquad e^{-x}y = C. \tag{2.10}$$

As in the case of the constant coefficients, the solution to the equation (2.9) will be constant along these curves, called *characteristic curves*. This family of non-intersecting curves fills the entire coordinate plane, and the curve containing the point (x, y) is uniquely determined by $C = e^{-x}y$, which implies that the general solution to (2.9) is

$$u(x, y) = f(C) = f(e^{-x}y).$$

As always, we can check this by substitution.

$$u_x + yu_y = -f'(e^{-x}y)e^{-x}y + yf'(e^{-x}y)e^{-x} = 0.$$

Let us now try to generalize the method of characteristics to the equation

$$a(x, y)u_x + b(x, y)u_y = 0. \tag{2.11}$$

The idea is again to introduce new coordinates (ξ, η), which will reduce (2.11) to an ODE. Suppose

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y) \end{cases} \tag{2.12}$$

gives such a change of variables. To rewrite the equation in this coordinates, we compute

$$u_x = u_\xi \xi_x + u_\eta \eta_x,$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y,$$

and substitute these into equation (2.11) to get

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta = 0.$$

Since we would like this to give us an ODE, say in ξ, we require the coefficient of u_η to be zero,

$$a\eta_x + b\eta_y = 0.$$

Without loss of generality, we may assume that $a \neq 0$ (locally). Notice that for curves $y(x)$ that have the slope $\frac{dy}{dx} = \frac{b}{a}$ we have

$$\frac{d}{dx}\eta(x, y(x)) = \eta_x + \eta_y \frac{dy}{dx} = \eta_x + \frac{b}{a}\eta_y = 0.$$

So the characteristic curves, just as before, are given by

$$\frac{dy}{dx} = \frac{b}{a}. \tag{2.13}$$

The general solution to this ODE will be $\eta(x, y) = C$, with $\eta_y \neq 0$ (otherwise $\eta_x = 0$ as well, and this will not be a solution). This is how we find the new variable η, for which our PDE reduces to an ODE. We choose $\xi(x, y) = x$ as the other variable. For this change of coordinates the Jacobian determinant is

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \eta_y \neq 0.$$

Thus, (2.12) constitutes a non-degenerate change of coordinates. In the new variables equation (2.11) reduces to

$$a(\xi, \eta)u_\xi = 0, \quad \text{hence} \quad u_\xi = 0,$$

which has the solution

$$u = f(\eta).$$

The general variable coefficient equation (2.8) in these coordinates will reduce to

$$a(\xi, \eta)u_\xi + c(\xi, \eta)u = d(\xi, \eta),$$

which is called the *canonical form* of equation (2.8). This equation, as in previous cases, can be solved by standard ODE methods.

Example 2.3. Find the general solution of the equation

$$xu_x - yu_y + y^2 u = y^2, \qquad x, y \neq 0.$$

Condition (2.13) in this case is $\frac{dy}{dx} = -\frac{y}{x}$. This is a separable ODE, which can be solved to obtain the general solution $xy = C$. Thus, our change of coordinates will be

$$\begin{cases} \xi = x, \\ \eta = xy. \end{cases}$$

In these coordinates the equation takes the form

$$\xi u_\xi + \frac{\eta^2}{\xi^2} u = \frac{\eta^2}{\xi^2}, \quad \text{or} \quad u_\xi + \frac{\eta^2}{\xi^3} u = \frac{\eta^2}{\xi^3}.$$

Using the integrating factor

$$e^{\int \frac{\eta^2}{\xi^3}\, d\xi} = e^{-\frac{\eta^2}{2\xi^2}},$$

the above equation can be written as

$$\left(e^{-\frac{\eta^2}{2\xi^2}} u\right)_\xi = e^{-\frac{\eta^2}{2\xi^2}} \frac{\eta^2}{\xi^3}.$$

Integrating both sides in ξ, we arrive at

$$e^{-\frac{\eta^2}{2\xi^2}} u = \int e^{-\frac{\eta^2}{2\xi^2}} \frac{\eta^2}{\xi^3}\, d\xi = e^{-\frac{\eta^2}{2\xi^2}} + f(\eta).$$

Thus, the general solution will be given by

$$u(\xi, \eta) = e^{\frac{\eta^2}{2\xi^2}} \left(f(\eta) + e^{-\frac{\eta^2}{2\xi^2}} \right) = e^{\frac{\eta^2}{2\xi^2}} f(\eta) + 1.$$

Finally, substituting the expressions of ξ and η in terms of (x, y) into the solution, we obtain

$$u(x, y) = f(xy) e^{\frac{y^2}{2}} + 1.$$

One should again check by substitution that this is indeed a solution to the PDE. $\square$

2.4 Conclusion

The method of characteristics is a powerful method that allows one to reduce *any* first-order linear PDE to an ODE, which can be subsequently solved using ODE techniques. We will see in later lectures that a subclass of second order PDEs – second order hyperbolic equations can be also treated with a similar characteristic method.

3 Method of characteristics revisited

3.1 Transport equation

A particular example of a first order constant coefficient linear equation is the transport, or advection equation $u_t + cu_x = 0$, which describes motions with constant speed. One way to derive the transport equation is to consider the dynamics of the concentration of a pollutant in a stream of water flowing through a thin tube at a constant speed c.

Let $u(t, x)$ denote the concentration of the pollutant in gr/cm (unit mass per unit length) at time t. The amount of pollutant in the interval $[a, b]$ at time t is then

$$\int_a^b u(x,t)\,dx.$$

Due to conservation of mass, the above quantity must be equal to the amount of the pollutant after some time h. After the time h, the pollutant would have flown to the interval $[a + ch, b + ch]$, thus the conservation of mass gives

$$\int_a^b u(x,t)\,dx = \int_{a+ch}^{b+ch} u(x,t+h)\,dx.$$

To derive the dynamics of the concentration $u(x, t)$, differentiate the above identity with respect to b to get

$$u(b,t) = u(b+ch, t+h).$$

Notice that this equation asserts that the concentration at the point b at time t is equal to the concentration at the point $b + ch$ at time $t + h$, which is to be expected, due to the fact that the water containing the pollutant particles flows with a constant speed. Since b is arbitrary in the last equation, we replace it with x. Now differentiate both sides of the equation with respect to h, and set h equal to zero to obtain the following differential equation for $u(x, t)$.

$$0 = cu_x(x,t) + u_t(x,t),$$

or

$$u_t + cu_x = 0. \tag{3.1}$$

Since equation (3.1) is a first order linear PDE with constant coefficients, we can solve it by the method of characteristics. First, we rewrite the equation as

$$(1, c) \cdot \nabla u = 0,$$

which implies that the slope of the characteristic lines is given by

$$\frac{dx}{dt} = \frac{c}{1}.$$

Integrating this equation, one arrives at the equation for the characteristic lines

$$x = ct + x(0), \tag{3.2}$$

where $x(0)$ is the coordinate of the point at which the characteristic line intersects the x-axis. The solution to the PDE (3.1) can then be written as

$$u(t,x) = f(x - ct) \tag{3.3}$$

for any arbitrary single-variable function f.

Let us now consider a particular initial condition for $u(t,x)$

$$u(0,x) = \begin{cases} x & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

According to (3.3), $u(0,x) = f(x)$, which determines the function f. Having found the function from the initial condition, we can now evaluate the solution $u(t,x)$ of the transport equation from (3.3). Indeed

$$u(t,x) = f(x-ct) = \begin{cases} x-ct & 0 < x-ct < 1 \\ 0 & \text{otherwise} \end{cases}$$

Noticing that the inequalities $0 < x - ct < 1$ imply that x is in-between ct and $ct+1$, we can rewrite the above solution as

$$u(t,x) = \begin{cases} x-ct & ct < x < ct+1, \\ 0 & \text{otherwise,} \end{cases}$$

which is exactly the initial function $u(0,x)$, given by (3.4), moved to the right along the x-axis by ct units. Thus, the *initial data* $u(0,x)$ travels from left to right with constant speed c.

We can alternatively understand the dynamics by looking at the characteristic lines in the xt coordinate plane. From (3.2), we can rewrite the characteristics as

$$t = \frac{1}{c}(x - x(0)).$$

Along these characteristics the solution remains constant, and one can obtain the value of the solution at any point (t,x) by tracing it back to the x-axis:

$$u(t,x) = u(t-t, x-ct) = u(0, x(0)).$$

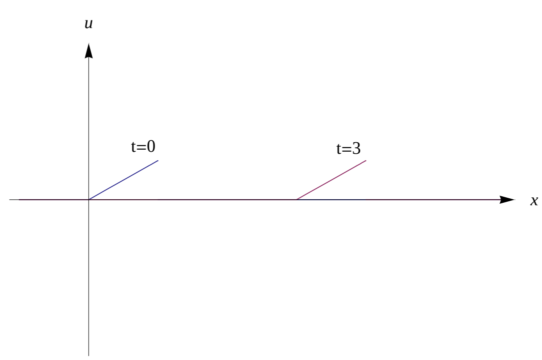

Figure 3.1: $u(t,x)$ at two different times t.

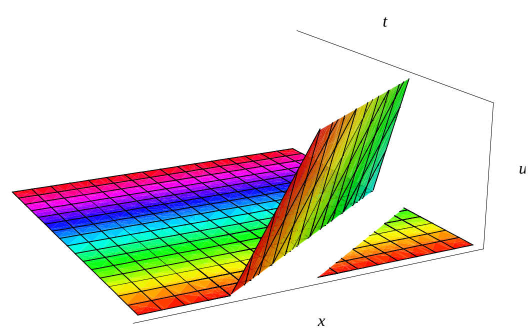

Figure 3.2: The graph of $u(t,x)$ colored with respect to time t.

Figure 1 gives the graphs of $u(t,x)$ at two different times, while Figure 3.1 gives the three dimensional graph of $u(t,x)$ as a function of two variables.

3.2 Quasilinear equations

We next look at a simple nonlinear equation, for which the method of characteristics can be applied as well. The general first order quasilinear equation has the following form

$$a(x,y,u)u_x + b(x,y,u)u_y = g(x,y,u).$$

We can see that the highest order derivatives, in this case the first order derivatives, enter the equation linearly, although the coefficients depend on the unknown u. A very particular example of first order quasilinear equations is the inviscid Burger's equation

$$u_t + uu_x = 0. \tag{3.5}$$

As before, we can rewrite this equation in the form of a dot product, which is a vanishing condition for a certain directional derivative

$$(1, u) \cdot (u_t, u_x) = 0, \qquad \text{or} \qquad (1, u) \cdot \nabla u = 0.$$

This shows that the tangent vector of the characteristic curves, $\mathbf{v} = (1, u)$, depends on the unknown function u.

3.3 Rarefaction

Let as now look at a particular initial condition, and try to construct solutions along the characteristic curves. Suppose

$$u(0, x) = \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases} \tag{3.6}$$

The slope of the characteristic curves satisfies

$$\frac{dx}{dt} = u(t, x(t)) = u(0, x(0)).$$

Here we used the fact that the directional derivative of the solution vanishes in the direction of the tangent vector of the characteristic curves. This implies that the solution remains constant along the characteristics, i.e. $u(t, x(t))$ remains constant for all values of t. We can find the equation of the characteristics by integrating the above equation, which gives

$$x(t) = u(0, x(0))t + x(0). \tag{3.7}$$

Using the initial condition (3.6), this equation will become

$$x(t) = \begin{cases} t + x(0) & \text{if } x(0) < 0, \\ 2t + x(0) & \text{if } x(0) > 0. \end{cases}$$

Thus, the characteristics have different slopes depending on whether they intersect the x axis at a positive, or negative intercept $x(0)$. We can express the characteristic lines to give t as a function of x, so that the initial condition is defined along the horizontal x axis.

$$t = \begin{cases} x - x(0) & \text{if } x(0) < 0, \\ \frac{1}{2}(x - x(0)) & \text{if } x(0) > 0. \end{cases} \tag{3.8}$$

Some of the characteristic lines corresponding to the initial condition (3.6) are sketched in Figure 3 below. The solid lines are the two families of characteristics with different slopes.

Notice that in this case the waves originating at $x(0) > 0$ move to the right faster than the waves originating at points $x(0) < 0$. Thus an increasing gap is formed between the faster moving wave front and the slower one. One can also see from Figure 3, that there are no characteristic lines from either of the two families given by (3.8) passing through the origin, since there is a jump discontinuity at $x = 0$ in the initial condition (3.6). In fact, in this case we can imagine that there are infinitely many characteristics originating from the origin with slopes ranging between $\frac{1}{2}$ and 1 (the dotted lines in Figure 3). The proper way to see this is to notice that in the case of $x(0) = 0$, (3.7) implies that

$$u = \frac{x}{t}, \qquad \text{if} \qquad t < x < 2t.$$

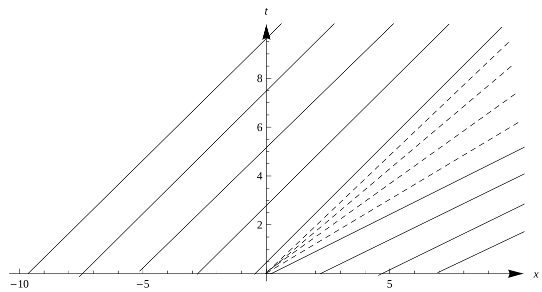

Figure 3.3: Characteristic lines illustrating rarefaction.

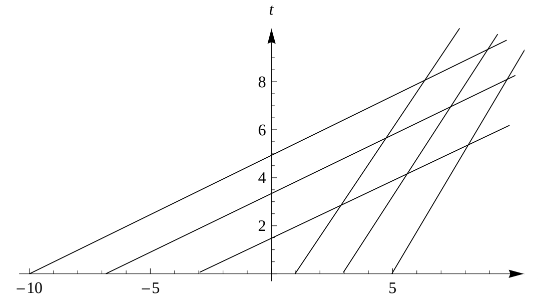

Figure 3.4: Characteristic lines illustrating shock wave formation.

This type of waves, which arise from decompression, or *rarefaction* of the medium due to the increasing gap formed between the wave fronts traveling at different speeds are called *rarefaction waves*. Putting all the pieces together, we can write the solution to equation (3.5) satisfying initial condition (3.6) as follows.

$$u(t,x) = \begin{cases} 1 & \text{if } x < t, \\ \frac{x}{t} & \text{if } t < x < 2t, \\ 2 & \text{if } x > 2t. \end{cases}$$

3.4 Shock waves

A completely opposite phenomenon to rarefaction is seen when one has a faster wave moving from left to right catching up to a slower wave. To illustrate this, let us consider the following initial condition for the Burger's inviscid equation

$$u(0,x) = \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{3.9}$$

Then the characteristic lines (3.7) will take the form

$$x(t) = \begin{cases} 2t + x(0) & \text{if } x(0) < 0, \\ t + x(0) & \text{if } x(0) > 0. \end{cases}$$

Or expressing t in terms of x, we can write the equations as

$$t = \begin{cases} \frac{1}{2}(x - x(0)) & \text{if } x(0) < 0, \\ x - x(0) & \text{if } x(0) > 0. \end{cases} \tag{3.10}$$

Thus, the characteristics origination from $x(0) < 0$ have smaller slope (corresponding to faster speed), than the characteristics originating from $x(0) > 0$. In this case the characteristics from the two families will intersect eventually, as seen in Figure 4. At the intersection points the solution u becomes multi-valued, since the point can be traced back along either of the characteristics to an initial value of either 1, or 2, given by the initial condition (3.9). This phenomenon is known as *shock waves*, since the faster moving wave catches up to the slower moving wave to form a multivalued (multicrest) wave.

3.5 Conclusion

We saw that the method of characteristics can be generalized to quasilinear equations as well. Using the method of characteristics for the inviscid Burger's equations, we discovered that in the case of nonlinear equations one may encounter characteristics that diverge from each other to give rise to an unexpected solution in the widening region in-between, as well as intersecting characteristics, leading to multivalued solutions. These are nonlinear phenomena, and do not arise in the study of linear PDEs.

Problem Set 2

1. (#1.2.1 in [Str]) Solve the first order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

2. (#1.2.2 in [Str]) Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)

3. (#1.2.3 in [Str]) Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.

4. (#1.2.6 in [Str]) Solve the equation $\sqrt{1 - x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.

5. (#1.2.7 in [Str])

   (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.

   (b) In which region of the xy plane is the solution uniquely determined?

6. (#1.2.10 in [Str]) Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.

7. Show that the characteristics of $u_t + 5uu_x = 0$ with $u(x, 0) = f(x)$ are straight lines.

In this lecture we will derive the wave and heat equations from physical principles. These are second order constant coefficient linear PDEs, which we will study in detail for the rest of the quarter.

4.1 Vibrating string

Consider a thin string of length l, which undergoes relatively small transverse vibrations (think of a string of a musical instrument, say a violin string). Let ρ be the linear density of the string, measured in units of mass per unit of length. We will assume that the string is made of homogeneous material and its density is constant along the entire length of the string. The displacement of the string from its equilibrium state at time t and position x will be denoted by $u(t,x)$. We will position the string on the x-axis with its left endpoint coinciding with the origin of the xu coordinate system.

Considering the motion of a small portion of the string sitting atop the interval $[a,b]$, which has mass $\rho(b-a)$, and acceleration u_{tt}, we can write Newton's second law of motion (balance of forces) as follows

$$\rho(b-a)u_{tt} = \text{Total force.} \tag{4.1}$$

Having a thin string with negligible mass, we can ignore the effect of gravity on the string, as well as air resistance, and other external forces. The only force acting on the string is then the tension force. Assuming that the string is perfectly flexible, the tension force will have the direction of the tangent vector along the string. At a fixed time t the position of the string is given by the parametric equations

$$\begin{cases} x = x, \\ u = u(x,t), \end{cases}$$

where x plays the role of a parameter. The tangent vector is then $(1, u_x)$, and the unit tangent vector will be $\left(\frac{1}{\sqrt{1+u_x^2}}, \frac{u_x}{\sqrt{1+u_x^2}}\right)$. Thus, we can write the tension force as

$$\mathbf{T}(x,t) = T(x,t)\left(\frac{1}{\sqrt{1+u_x^2}}, \frac{u_x}{\sqrt{1+u_x^2}}\right) = \frac{1}{\sqrt{1+u_x^2}}(T, Tu_x), \tag{4.2}$$

where $T(t,x)$ is the magnitude of the tension. Since we consider only small vibrations, it is safe to assume that u_x is small, and the following approximation via the Taylor's expansion can be used

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + o(u_x^4) \approx 1.$$

Substituting this approximation into (4.2), we arrive at the following form of the tension force

$$\mathbf{T} = (T, Tu_x).$$

With our previous assumption that the motion is transverse, i.e. there is no longitudinal displacement (along the x-axis), we arrive at the following identities for the balance of forces (4.1) in the x, respectively u directions

$$\begin{aligned} 0 &= T(b,t) - T(a,t) \\ \rho(b-a)u_{tt} &= T(b,t)u_x(b,t) - T(a,t)u_x(a,t). \end{aligned}$$

The first equation above merely states that in the x direction the tensions from the two edges of the small portion of the string balance each other out (no longitudinal motion). From this we can also see that the tension force is independent of the position on the string. Then the second equation can be rewritten as

$$\rho u_{tt} = T\frac{u_x(b,t) - u_x(a,t)}{b-a}.$$

Passing to the limit $b \to a$, we arrive at the wave equation

$$\rho u_{tt} = T u_{xx}, \qquad \text{or} \qquad u_{tt} - c^2 u_{xx} = 0,$$

where we used the notation $c^2 = T/\rho$. As we will see later, c will be the speed of wave propagation, similar to the constant appearing in the transport equation.

One can generalize the wave equation to incorporate effects of other forces. Some examples follow.

a) With air resistance: force is proportional to the speed u_t

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \qquad r > 0.$$

b) With transverse elastic force: force is proportional to the displacement u

$$u_{tt} - c^2 u_{xx} + k u = 0, \qquad k > 0.$$

c) With an externally applied force

$$u_{tt} - c^2 u_{xx} = f(x, t) \qquad \text{(inhomogeneous)}.$$

4.2 Vibrating drumhead

Similar to the vibrating string, one can consider a vibrating drumhead (elastic membrane), and look at the dynamic of the displacement $u(x, y, t)$, which now depends on the two spatial variables (x, y) denoting the point on the 2-dimensional space of the equilibrium state. Taking a small region on the drumhead, the tension force will again be directed tangentially to the surface, in this case along the normal vector to the boundary of the region. Its vertical component will be $T\frac{\partial u}{\partial \mathbf{n}}$, where $\frac{\partial u}{\partial \mathbf{n}}$ denotes the derivative of u in the normal direction to the boundary of the region. The vertical component of the cumulative tension force will then be

$$T_{\text{vert}} = \int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}}\, ds = \int_{\partial D} T \nabla u \cdot \mathbf{n}\, ds,$$

where ∂D denotes the boundary of the region D, and the integral is taken with respect to the length element along the boundary of D. The second law of motion will then take the form

$$\int_{\partial D} T \nabla u \cdot \mathbf{n}\, ds = \iint_D \rho u_{tt}\, dxdy.$$

Using Green's theorem, we can convert the line integral on the left hand side to a two dimensional integral, and arrive at the following identity

$$\iint_D \nabla \cdot (T \nabla u)\, dxdy = \iint_D \rho u_{tt}\, dxdy.$$

Since the region D was taken arbitrarily, the integrands on both sides must be the same, resulting in

$$\rho u_{tt} = T(u_{xx} + u_{yy}), \qquad \text{or} \qquad u_{tt} - c^2(u_{xx} + u_{yy}) = 0.$$

This is the wave equation in two spatial dimensions. Three dimensional vibrations can be treated in much the same way, leading to the three dimensional wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0.$$

Often one makes use of the operator notation

$$\Delta = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \dots$$

to write the wave equation in any dimension as

$$u_{tt} - c^2 \Delta u = 0.$$

The Δ operator is called Laplace's operator or the *Laplacian*.

4.3 Heat flow

Let $u(x,t)$ denote the temperature at time t at the point x in some thin insulated rod. Again, think of the rod as positioned along the x-axis in the xu coordinate system, where the vertical axis will measure the temperature. The heat, or thermal energy of a small portion of the rod situated at the interval $[a,b]$ is given by

$$H(x,t) = \int_a^b c\rho u \, dx,$$

where ρ is the linear density of the rod (mass per unit of length), and c denotes the *specific heat capacity* of the material of the rod. The instantaneous change of the heat with respect to time will be the time derivative of the above expression

$$\frac{dH}{dt} = \int_a^b c\rho u_t \, dx.$$

Since the heat cannot be lost or gained in the absence of an external heat source, the change in the heat must be balanced by the heat flux across the cross-section of the cylindrical piece around the interval $[a,b]$ (we assume that the lateral boundary of the rod is perfectly insulated). According to Fourier's law, the heat flux across the boundary will be proportional to the derivative of the temperature in the direction of the outward normal to the boundary, in this case the x-derivative.

$$\text{Heat flux} = \kappa u_x,$$

where κ denotes the thermal conductivity. Using this expression for the heat flux, and noting that the change in the internal heat of the portion of the rod is equal to the combined flux through the two ends of this portion, we have

$$\int_a^b c\rho u_t \, dx = \kappa(u_x(b,t)) - u_x(a,t)).$$

Differentiating this identity with respect to b, we arrive at the heat equation

$$c\rho u_t = \kappa u_{xx}, \qquad \text{or} \qquad u_t - k u_{xx} = 0,$$

where we denoted $k = \frac{\kappa}{c\rho}$.

As in the case of the wave equation, one can consider higher dimensional heat flows (heat flow in a two dimensional plate, or a three dimensional solid) to arrive at the general heat equation

$$u_t - k\Delta u = 0.$$

We also note that diffusion phenomena lead to an equation which has the same form as the heat equation (cf. Strauss for the actual derivation, where instead of Fourier's law of heat conduction one uses Fick's law of diffusion).

4.4 Stationary waves and heat distribution

If one looks at vibrations or heat flows where the physical state does not change with time, then $u_t = u_{tt} = 0$, and both the wave and the heat equations reduce to

$$\Delta u = 0. \tag{4.3}$$

This equation is called the *Laplace equation*. Notice that in the one dimensional case (4.3) reduces to

$$u_{xx} = 0,$$

which has the general solution $u(x,t) = c_1 x + c_2$ (remember that u is independent of t). The solutions to the Laplace equation are called *harmonic functions*, and we will see later in the course that one encounters nontrivial harmonic function in higher dimensions.

4.5 Boundary conditions

We saw in previous lectures that PDEs generally have infinitely many solutions. One then imposes some auxiliary conditions to single out relevant solutions. Since the equations that we study come from physical considerations, the auxiliary conditions must be physically relevant as well. These conditions come in two different varieties, initial conditions and boundary value conditions.

The initial condition, familiar from the theory of ODEs, specifies the physical state at some particular time t_0. For example for the heat equation one would specify the initial temperature, which in general will be different at different points of the rod,

$$u(x, 0) = \phi(x).$$

For the vibrating string, one needs to specify both the initial position of (each point of) the string, and the initial velocity, since the equation is of second order in time,

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x).$$

In the physical examples that we considered at the beginning of this lecture, it is clear that there is a domain on which the solutions must live. For example in the case of the vibrating string of length l, we only look at the amplitude $u(t, x)$, where $0 \leq x \leq l$. Similarly, for the heat conduction in an insulated rod. In higher dimensions the domain is bounded by curves (2d), surfaces (3d) or higher dimensional geometric shapes. The boundary of this domain is where the system interacts with the external factors, and one needs to impose boundary conditions to account for these interactions.

There are three important kinds of boundary conditions:

(D) the value (on the boundary) of u is specified (*Dirichlet condition*)

(N) the value of the normal derivative $\partial u/\partial n$ is specified (*Neumann condition*)

(R) the value of $\partial u/\partial n + au$ is specified (*Robin condition*, a is a function of $x, y, z, \dots$ and t)

These conditions are usually written as equations, for example the Dirichlet condition

$$u(\mathbf{x}, t)|_{\partial D} = f(\mathbf{x}, t),$$

or the Robin condition

$$\frac{\partial u}{\mathbf{n}} + au|_{\partial D} = h(\mathbf{x}, t).$$

The condition is called *homogeneous*, if the right hand side vanishes for all values of the variables.

In one dimensional case, the domain D is an interval $(0 < x < l)$, hence the boundary consists of just two points $x = 0$, and $x = l$. The boundary conditions then take the simple form

(D) $u(0, t) = g(t)$ and $u(l, t) = h(t)$

(N) $u_x(0, t) = g(t)$ and $u_x(l, t) = h(t)$

(R) $u(0, t) + a(t)u_x(0, t) = g(t)$ and $u(l, t) + b(t)u_x(l, t) = h(t)$

4.6 Examples of physical boundary conditions

In the case of a vibrating string, one can impose the condition that the endpoints of the string remain fixed (the case for strings of musical instruments). This gives the Dirichlet conditions $u(0, t) = u(l, t) = 0$.

If one end is allowed to move freely in the transverse direction, then the lack of any tension force at this endpoint can be expressed as the Neumann condition $u_x(l, 0) = 0$.

A Robin condition will correspond to the case when one endpoint is allowed to move transversely, but the motion is restricted by a force proportional to the amplitude u (think of a coiled spring attached to the endpoint).

In the case of heat conduction in a rod, the perfect insulation of the rod surface leads to the Neumann condition $\partial u/\partial n = 0$ (no exchange of heat with the ambient space).

If the endpoints of the rod are kept in a thermal balance with the ambient temperature $g(t)$, then one has the Dirichlet condition $u = g(t)$ on the boundary.

If one allows thermal exchange between the rod and the ambient space obeying Newton's law of cooling, then he boundary condition takes the form of a Robin condition $u_x(l,t) = -a(u(l,t) - g(t))$.

4.7 Conclusion

We derived the wave and heat equations from physical principles, identifying the unknown function with the amplitude of a vibrating string in the first case and the temperature in a rod in the second case. Understanding the physical significance of these PDEs will help us better grasp the qualitative behavior of their solutions, which will be derived by purely mathematical techniques in the subsequent lectures. The physicality of the initial and boundary conditions will also help us immediately rule out solutions that do not conform to the physical laws behind the appropriate problems.

5 Classification of second order linear PDEs

Last time we derived the wave and heat equations from physical principles. We also saw that Laplace's equation describes the steady physical state of the wave and heat conduction phenomena. Today we will consider the general second order linear PDE and will reduce it to one of three distinct types of equations that have the wave, heat and Laplace's equations as their canonical forms. Knowing the type of the equation allows one to use relevant methods for studying it, which are quite different depending on the type of the equation. One should compare this to the conic sections, which arise as different types of second order algebraic equations (quadrics). Since the hyperbola, given by the equation $x^2 - y^2 = 1$, has very different properties from the parabola $x^2 - y = 0$, it is expected that the same holds true for the wave and heat equations as well. For conic sections, one uses change of variables to reduce the general second order equation to a simpler form, which are then classified according to the form of the reduced equation. We will see that a similar procedure works for second order PDEs as well.

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x, y, but do not depend on the unknown function u. The classification of second order equations depends on the form of the leading part of the equations consisting of the second order terms. So, for simplicity of notation, we combine the lower order terms and rewrite the above equation in the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0. \tag{5.1}$$

As we will see, the type of the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y), \tag{5.2}$$

which is called the *discriminant* for (5.1). The classification of second order linear PDEs is given by the following.

Definition 5.1. At the point (x_0, y_0) the second order linear PDE (5.1) is called

i) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

ii) *elliptic*, if $\Delta(x_0, y_0) < 0$

Notice that in general a second order equation may be of one type at a specific point, and of another type at some other point. In order to illustrate the significance of the discriminant $\Delta = B^2 - 4AC$, we next describe how one reduces equation (5.1) to a canonical form. Similar to the second order algebraic equations, we use change of coordinates to reduce the equation to a simpler form. Define the new variables as

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \end{cases} \qquad \text{with} \qquad J = \det\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0. \tag{5.3}$$

We then use the chain rule to compute the terms of the equation (5.1) in these new variables.

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y. \end{aligned}$$

To express the second order derivatives in terms of the (ξ, η) variables, differentiate the above expressions for the first derivatives using the chain rule again.

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + \text{l.o.t}, \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + u_{\eta\eta}\eta_x\eta_y + \text{l.o.t}, \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + \text{l.o.t}. \end{aligned}$$

Here *l.o.t.* stands for the low order terms, which contain only one derivative of the unknown u. Using these expressions for the second order derivatives of u, we can rewrite equation (5.1) in these variables as

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + I^*(\xi, \eta, u, u_\xi, u_\eta) = 0, \tag{5.4}$$

where the new coefficients of the higher order terms A^*, B^* and C^* are expressed via the original coefficients and the change of variables formulas as follows.

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \tag{5.5}$$

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y, \tag{5.6}$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \tag{5.7}$$

One can form the discriminant for the equation in the new variables via the new coefficients in the obvious way,

$$\Delta^* = (B^*)^2 - 4A^*C^*.$$

We need to guarantee that the reduced equation will have the same type as the original equation. Otherwise, the classification given by Definition 5.1 is meaningless, since in that case the same physical phenomenon will be described by equations of different types, depending on the particular coordinate system in which one chooses to view them. The following statement provides such a guarantee.

Theorem 5.2. *The discriminant of the equation in the new variables can be expressed in terms of the discriminant of the original equation* (5.1) *as follows*

$$\Delta^* = J^2\Delta,$$

where J is the Jacobian determinant of the change of variables given by (5.3).

As a simple corollary, the type of the equation is invariant under nondegenerate coordinate transformations, since the signs of Δ and Δ^* coincide.

This theorem can be proved by a straightforward, although somewhat messy, calculation to express Δ^* in terms of the coefficients of the original equation (5.1). The bottom line of the theorem is that we can perform any nondegenerate change of variables to reduce the equation, while the type remains unchanged. Let us now try to construct such transformations, which will make one, or possibly two of the coefficients of the leading second order terms of equation (5.4) vanish, thus reducing the equation to a simpler form.

For simplicity, we assume that the coefficients A, B and C are constant. The material of this lecture can be extended to the variable coefficient case with minor changes, but we will not study variable coefficient second order PDEs in this class.

Notice that the expressions for A^* and C^* in (5.5), respectively (5.7) have the same form, with the only difference being in that the first equation contains the variable ξ, while the second one has η. Due to this, we can try to chose a transformation, which will make both A^* and C^* vanish. This is equivalent to the following equation

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0. \tag{5.8}$$

We use ζ (*zeta*) in place of both ξ and η. The solutions to this equation are called *characteristic curves* for the second order PDE (5.1) (compare this to the characteristic curves for first order PDEs, where the idea was again to reduce the equation to a simpler form, in which only one of the first order derivatives appears). We divide both sides of the above equation by ζ_y^2 to get

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \tag{5.9}$$

Without loss of generality we can assume that $A \neq 0$. Indeed, if $A = 0$, but $C \neq 0$, one can proceed in a similar way, by considering the ratio ζ_y/ζ_x instead of ζ_x/ζ_y. Otherwise, if both $A = 0$, and $C = 0$, then the equation is already in the reduced form, and there is nothing to do. Now recall that we are trying to find change of variables formulas, which are given as curves $\zeta(x,y) = \text{const}$ (fix the new variable, e.g. $\xi(x,y) = \xi_0$). Along such curves we have

$$d\zeta = \zeta_x dx + \zeta_y dy = 0.$$

Hence, the slope of the characteristic curve is given by

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}.$$

Substituting this into equation (5.9), we arrive at the following equation for the slope of the characteristic curve

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0.$$

Since the above is a quadratic equation, it has 2, 1, or 0 real solutions, depending on the sign of the discriminant, $B^2 - 4AC$, and the solutions are given by the quadratic formulas

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{5.10}$$

5.1 Hyperbolic equations

If the discriminant $\Delta > 0$, then the quadratic formulas (5.10) give two distinct families of characteristic curves, which will define the change of variables (5.3). To derive these change of variables formulas, integrate (5.10) to get

$$y = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}x + c, \qquad \text{or} \qquad \frac{B \pm \sqrt{B^2 - 4AC}}{2A}x - y = c.$$

These equations give the following change of variables

$$\begin{cases} \xi = \frac{B + \sqrt{B^2 - 4AC}}{2A}x - y \\ \eta = \frac{B - \sqrt{B^2 - 4AC}}{2A}x - y \end{cases} \tag{5.11}$$

In these new variables $A^* = C^* = 0$, while for B^* we have from (5.6)

$$B^* = 2A\left(\frac{B^2 - (B^2 - 4AC)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C = 4C - \frac{B^2}{A} = -\frac{\Delta}{A} \neq 0.$$

One can then divide equation (5.4) by B^*, to arrive at the reduced equation

$$u_{\xi\eta} + \cdots = 0. \tag{5.12}$$

This is called the first canonical form for hyperbolic equations. Under the orthogonal transformations

$$\begin{cases} x' = \xi + \eta \\ y' = \xi - \eta \end{cases}$$

equation (5.12) becomes

$$u_{x'x'} - u_{y'y'} + \cdots = 0,$$

which is the second canonical form for hyperbolic equations. Notice that the last equation has exactly the same form in its leading terms as the wave equation (with $c = 1$).

5.2 Parabolic equations

In the case of parabolic equations $\Delta = B^2 - 4AC = 0$, and the quadratic formulas (5.10) give only one family of characteristic curves. This means that there is no change of variables that makes both A^* and C^* vanish. However we can make one of this vanish, for example A^*, by choosing ξ to be the unique solution of equation (5.10). We can then chose η arbitrarily, as long as the change of coordinates formulas (5.3) define a nondegenerate transformation. Notice that in such a case, according to Theorem 5.2, $\Delta^* = (B^*)^2 - 4A^*C^* = 0$. But then we have $B^* = \pm 2\sqrt{A^*C^*} = 0$.

We solve equation (5.10) by integration to get ξ, and set $\eta = x$ to arrive at the following change of variables formulas

$$\begin{cases} \xi = \frac{B}{2A}x - y \\ \eta = x \end{cases} \tag{5.13}$$

The Jacobian determinant of this transformation is

$$J = \begin{vmatrix} B/(2A) & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0.$$

Thus, the transformation (5.13) is indeed nondegenerate, and reduces equation (5.1) to the following form (after division by C^*)

$$u_{\eta\eta} + \cdots = 0,$$

which is the canonical form for parabolic PDEs. Notice that this equation has the same leading terms as the heat equation $u_{xx} - u_t = 0$.

5.3 Elliptic equations

In the case of elliptic equations $\Delta = B^2 - 4AC < 0$, and the quadratic formulas (5.10) give two complex conjugate solutions. We can formally solve for ξ similar to the hyperbolic case, and arrive at the formula

$$\xi = \left(\frac{B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A}i\right)x - y.$$

We define new variables (α, β) by taking respectively the real and imaginary parts of ξ.

$$\begin{cases} \alpha = \frac{B}{2A}x - y \\ \beta = \frac{\sqrt{B^2-4AC}}{2A}x \end{cases} \tag{5.14}$$

In these variables equation (5.1) has the form

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + I^{**}(\alpha, \beta, u, u_\alpha, u_\beta) = 0, \tag{5.15}$$

in which the coefficients will be given by formulas similar to (5.5)-(5.7) with ξ replaced by α, and η replaced by β. Computing these new coefficients we get

$$\begin{aligned} A^{**} &= A\left(\frac{B}{2A}\right)^2 - \frac{B^2}{2A} + C = \frac{4AC - B^2}{4A}, \\ B^{**} &= 2A\frac{B}{2A}\frac{\sqrt{4AC - B^2}}{2A} - B\frac{\sqrt{4AC - B^2}}{2A} = 0, \\ C^{**} &= A\frac{4AC - B^2}{2A^2} = \frac{4AC - B^2}{4A}. \end{aligned}$$

As we can see, $A^{**} = C^{**}$, and $B^{**} = 0$. This is a direct consequence of the fact that $\xi = \alpha + \beta i$ and $\eta = \alpha - \beta i$ solve equation (5.8). One can then divide both sides of equation (5.4) by $A^{**} = C^{**} \neq 0$, to arrive at the reduced equation

$$u_{\alpha\alpha} + u_{\beta\beta} + \cdots = 0,$$

which is the canonical form for elliptic PDEs. Notice that this equation has the same leading terms as the Laplace's equation.

Example 5.1. Determine the regions in the xy plane where the following equation is hyperbolic, parabolic, or elliptic.

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0.$$

The coefficients of the leading terms in this equation are

$$A = 1, B = 0, C = y.$$

The discriminant is then $\Delta = B^2 - 4AC = -4y$. Hence the equation is hyperbolic when $y < 0$, parabolic when $y = 0$, and elliptic when $y > 0$. □

5.4 Conclusion

The second order linear PDEs can be classified into three types, which are invariant under changes of variables. The types are determined by the sign of the discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. We saw that hyperbolic equations have two distinct families of (real) characteristic curves, parabolic equations have a single family of characteristic curves, and the elliptic equations have none. All the three types of equations can be reduced to canonical forms. Hyperbolic equations reduce to a form coinciding with the wave equation in the leading terms, the parabolic equations reduce to a form modeled by the heat equation, and the Laplace's equation models the canonical form of elliptic equations. Thus, the wave, heat and Laplace's equations serve as canonical models for all second order constant coefficient PDEs. We will spend the rest of the quarter studying the solutions to the wave, heat and Laplace's equations.

Problem Set 3

1. (#1.3.5 in [Str]) Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

2. (#1.5.1 in [Str]) Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$
$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution *unique, or not*? Does the answer depend on L?

3. (#1.5.2 in [Str]) Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.

   (a) Is the solution *unique*? Explain.

   (b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

4. (#1.6.1 in [Str]) What is the type of each of the following equations?

   (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.

   (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

5. (#1.6.2 in [Str]) Find the regions in the xy plane where the equation

$$(1 + x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

6. (#1.6.4 in [Str]) What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

7. Find the general solution of the equation $u_{xx} - 2u_{xy} - 3u_{yy} = 0$.

In this lecture we will solve the wave equation on the entire real line $x \in \mathbb{R}$. This corresponds to a string of infinite length. Although physically unrealistic, as we will see later, when considering the dynamics of a portion of the string away from the endpoints, the boundary conditions have no effect for some (nonzero) finite time. Due to this, one can neglect the endpoints, and make the assumption that the string is infinite.

The wave equation, which describes the dynamics of the amplitude $u(x,t)$ of the point at position x on the string at time t, has the following form

$$u_{tt} = c^2 u_{xx},$$

or

$$u_{tt} - c^2 u_{xx} = 0. \tag{6.1}$$

As we saw in the last lecture, the wave equation has the second canonical form for hyperbolic equations. One can then rewrite this equation in the first canonical form, which is

$$u_{\xi\eta} = 0. \tag{6.2}$$

This is achived by passing to the characteristic variables

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases} \tag{6.3}$$

To see that (6.2) is equivalent to (6.1), let us compute the partial derivatives of u with respect to x and t in the new variables using the chain rule.

$$\begin{aligned} u_t &= cu_\xi - cu_\eta, \\ u_x &= u_\xi + u_\eta. \end{aligned}$$

We can differentiate the above first order partial derivatives with respect to t, respectively x using the chain rule again, to get

$$\begin{aligned} u_{tt} &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, \\ u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}. \end{aligned}$$

Substituting this expressions into the left hand side of equation (6.1), we see that

$$u_{tt} - c^2 u_{xx} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} - c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = -4c^2 u_{\xi\eta} = 0,$$

which is equivalent to (6.2).

Equation (6.2) can be treated as a pair two successive ODEs. Integrating first with respect to the variable η, and then with respect to ξ, we arrive at the solution

$$u(\xi, \eta) = f(\xi) + g(\eta).$$

Recalling the definition of the characteristic variables (6.3), we can switch back to the original variables (x,t), and obtain the general solution

$$u(x,t) = f(x + ct) + g(x - ct). \tag{6.4}$$

Another way to solve equation (6.1) is to realize that the second order linear operator of the wave equation factors into two first order operators

$$\mathcal{L} = \partial_t^2 - c^2 \partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x).$$

Hence, the wave equation can be thought of as a pair of transport (advection) equations

$$(\partial_t - c\partial_x)v = 0, \tag{6.5}$$
$$(\partial_t + c\partial_x)u = v. \tag{6.6}$$

It is no coincidence, of course, that

$$x + ct = \text{constant}, \tag{6.7}$$

and

$$x - ct = \text{constant}, \tag{6.8}$$

are the characteristic lines for the transport equations (6.5), and (6.6) respectively, hence our choice of the characteristic coordinates (6.3). We also see that for each point in the xt plane there are two distinct characteristic lines, each belonging to one of the two families (6.7) and (6.8), that pass through the point. This is illustrated in Figure 6.1 below.

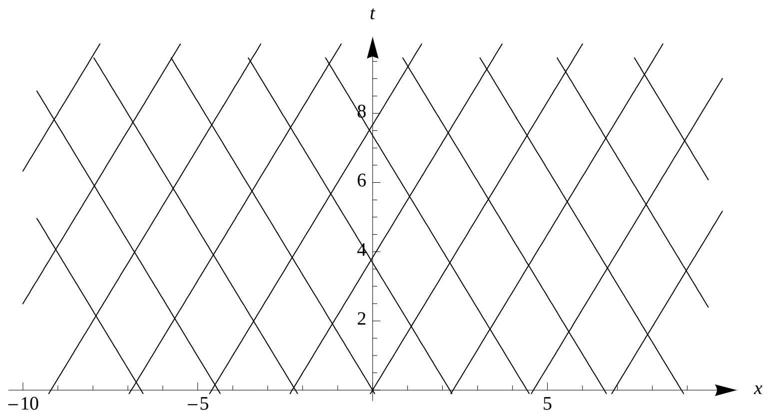

Figure 6.1: Characteristic lines for the wave equation with $c = 0.6$.

6.1 Initial value problem

Along with the wave equation (6.1), we next consider some initial conditions, to single out a particular physical solution from the general solution (6.4). The equation is of second order in time t, so initial values must be specified both for the initial dispalcement $u(x, 0)$, and the initial velocity $u_t(x, 0)$. We study the following *initial value problem*, or IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for} -\infty < x < +\infty, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{6.9}$$

where ϕ and ψ are arbitrary functions of single variable, and together are called the *initial data* of the IVP. The solution to this problem is easily found from the general solution (6.4). All we need to do is find f and g from the initial conditions of the IVP (6.9). To check the first initial condition, set $t = 0$ in (6.4),

$$u(x, 0) = \phi(x) = f(x) + g(x). \tag{6.10}$$

To check the second initial condition, we differentiate (6.4) with respect to t, and set $t = 0$

$$u_t(x, 0) = \psi(x) = cf'(x) - cg'(x). \tag{6.11}$$

Equations (6.10) and (6.11) can be treated as a system of two equations for f and g. To solve this system, we first integrate both sides of (6.11) from 0 to x to get rid of the derivatives on f and g (alternatively we could differentiate (6.10) instead), and rewrite the equations as

$$\begin{aligned} f(x) + g(x) &= \phi(x), \\ f(x) - g(x) &= \frac{1}{c}\int_0^x \psi(s)\,ds + f(0) - g(0). \end{aligned}$$

We can solve this system by adding the equations to eliminate g, snd subtracting them to eliminate f. This leads to the solution

$$
\begin{aligned}
f(x) &= \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds + \frac{1}{2}[f(0)-g(0)],\\
g(x) &= \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds - \frac{1}{2}[f(0)-g(0)].
\end{aligned}
$$

Finally, substituting this expressions for f and g back into the solution (6.4), we obtain the solution of the IVP (6.9)

$$
u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)\,ds.
$$

Combining the integrals using additivity, and the fact that flipping the integration limits changes the sign of the integral, we arrive at the following form for the solution

$$
u(x,t) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \tag{6.12}
$$

This is d'Alambert's solution for the IVP (6.9). It implies that as long as ϕ is twice continuously differentiable ($\phi \in C^2$), and ψ is continuously differentiable ($\psi \in C^1$), (6.12) gives a solution to the IVP. We will also consider examples with discontinuous initial data, which after plugging into (6.12) produce *weak solutions*. This notion will be made precise in later lectures.

Example 6.1. Solve the initial value problem (6.9) with the initial data

$$
\phi(x) \equiv 0, \qquad \psi(x) = \sin x.
$$

Substituting ϕ and ψ into d'Alambert's formula, we obtain the solution

$$
u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\,ds = \frac{1}{2c}(-\cos(x+ct)+\cos(x-ct)).
$$

Using the trigonometric identities for the cosine of a sum and difference of two angles, we can simplify the above to get

$$
u(x,t) = \frac{1}{c}\sin x \sin(ct).
$$

You should verify that this indeed solves the wave equation and satisfies the given initial conditions. □

6.2 The Box wave

When solving the transport equation, we saw that the initial values simply travel to the right, when the speed is positive (they propagate along the characteristics (6.8)); or to the left (they propagate along the characteristics (6.7)), when the speed is negative. Since the wave equation is made up of two of these type of equations, we expect similar behavior for the solutions of the IVP (6.9) as well. To see this, let us consider the following example with simplified initial data.

Example 6.2. Find the solution of IVP (6.9), with the initial data

$$
\begin{aligned}
\phi(x) &= \begin{cases} h, & |x| \le a,\\ 0, & |x| > a,\end{cases}\\
\psi(x) &\equiv 0.
\end{aligned} \tag{6.13}
$$

This data corresponds to an initial disturbance of the string centered at $x = 0$ of height h, and zero initial velocity. Notice that $\phi(x)$ is not continuous, let alone twice differentiable, though one can still substitute it into d'Alambert's solution (6.12) and get a function u, which will be a weak solution of the wave equation.

Since $\psi(x) \equiv 0$, only the first term in (6.12) is nonzero. We compute this term using the particular $\phi(x)$ in (6.13). First notice that

$$\phi(x+ct) = \begin{cases} h, & |x+ct| \le a, \\ 0, & |x+ct| > a, \end{cases} \tag{6.14}$$

and

$$\phi(x-ct) = \begin{cases} h, & |x-ct| \le a, \\ 0, & |x-ct| > a. \end{cases} \tag{6.15}$$

Hence, the solution

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$$

is piecewise defined in 4 different regions in the xt half-plane (we consider only positive time $t \ge 0$), which correspond to pairings of the intervals in the expressions (6.14) and (6.15). These regions are

$$\begin{aligned} \mathrm{I}: &\quad \{|x+ct| \le a, |x-ct| \le a\}, && u(x,t) = h \\ \mathrm{II}: &\quad \{|x+ct| \le a, |x-ct| > a\}, && u(x,t) = \frac{h}{2} \\ \mathrm{III}: &\quad \{|x+ct| > a, |x-ct| \le a\}, && u(x,t) = \frac{h}{2} \\ \mathrm{IV}: &\quad \{|x+ct| > a, |x-ct| > a\}, && u(x,t) = 0 \end{aligned} \tag{6.16}$$

The regions are depicted in Figure 6.2. Notice that $|x+ct| \le a$ is equivalent to

$$-a \le x + ct \le a, \qquad \text{or} \qquad -\frac{1}{c}(x+a) \le t \le -\frac{1}{c}(x-a).$$

Similarly for the other inequalities.

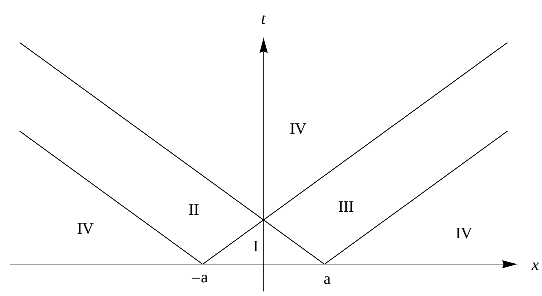

Figure 6.2: Regions where u has different values.

Using the values for the solution in (6.16), we can draw the graph of u at different times, some of which are depicted in Figures 6.3-6.6.

We see that the initial box-like disturbance centered at $x = 0$ splits into two disturbances of half the size, which travel in opposite directions with speed c.

The graphs hint that the initial disturbance will not be felt at a point x on the string (for $|x| > a$) before the time $t = \frac{1}{c}||x| - a|$. You will shortly see that this is a general property for the wave equation. In this particular case a box-like disturbance appears at the time $t = \frac{1}{c}||x| - a|$, and lasts exactly $t = \frac{2a}{c}$ units of time, after which it completely moves along. In general, the initial velocity may slow down the speed, and subsequently make the disturbance "last" longer, however, the speed cannot exceed c. $\square$

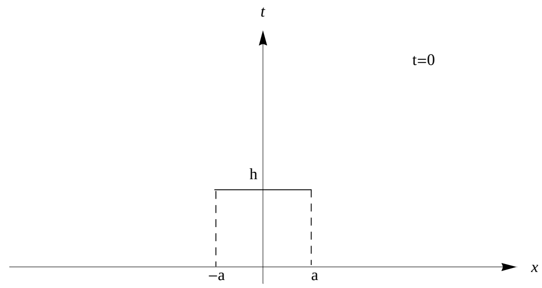

Figure 6.3: The solution at $t = 0$.

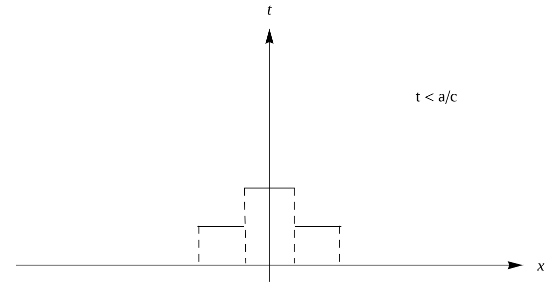

Figure 6.4: The solution for $0 < t < a/c$.

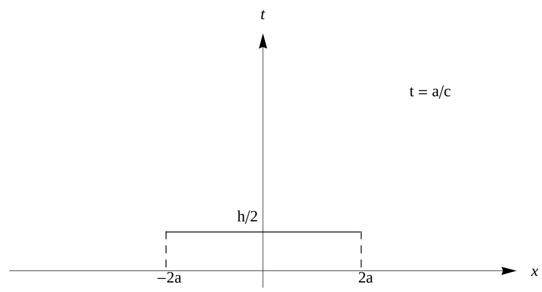

Figure 6.5: The solution at $t = a/c$.

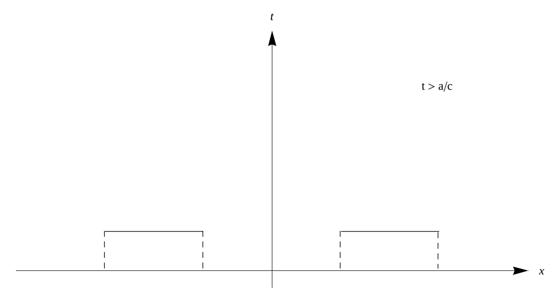

Figure 6.6: The solution for $t > a/c$.

6.3 Causality

The value of the solution to the IVP (6.9) at a point (x_0, t_0) can be found from d'Alambert's formula (6.12)

$$u(x_0, t_0) = \frac{1}{2}[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)\, ds. \tag{6.17}$$

We can see that this value depends on the values of ϕ at only two points on the x axis, $x_0 + ct_0$, and $x_0 - ct_0$, and the values of ψ only on the interval $[x_0 - ct_0, x_0 + ct_0]$. For this reason the interval $[x_0 - ct_0, x_0 + ct_0]$ is called *interval of dependence* for the point (x_0, t_0). Sometimes the entire triangular region with vertices at $x_0 - ct_0$ and $x_0 + ct_0$ on the x axis and the vertex (x_0, t_0) is called the *domain of dependence*, or the *past history* of the point (x_0, t_0), as depicted in Figure 6.7. The sides of this triangle are segments of characteristic lines passing through the point (x_0, t_0). Thus, we see that the initial data travels along the characteristics to give the values at later times. In the previous example of the box wave, we saw that the jump discontinuities travel along the characteristic lines as well.

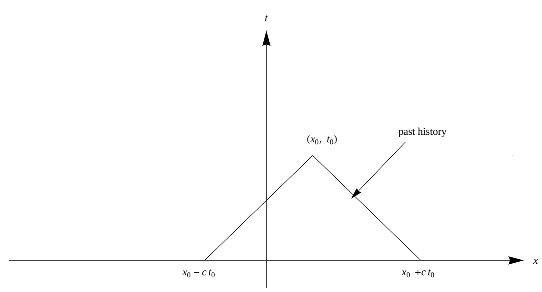

Figure 6.7: Domain of dependence for (x_0, t_0).

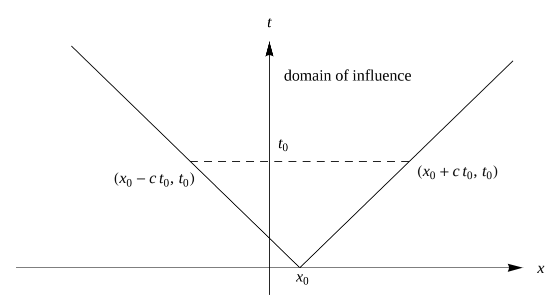

Figure 6.8: Domain of influence of x_0.

An inverse notion to the domain of dependence is the notion of *domain of influence* of the point x_0 on the x axis. This is the region in the xt plane consisting of all the points, whose domain of dependence contains the point x_0. The region has an upside-down triangular shape, with the sides being the characteristic lines emanating from the point x_0, as shown in Figure 6.8. This also means that the value of the initial data at the point x_0 effects the values of the solution u at all the points in the domain of influence. Notice that at a fixed time t_0, only the points satisfying $x_0 - ct_0 \leq x \leq x_0 + ct_0$ are influenced by the point x_0 on the x axis.

Example 6.3 (The hammer blow)**.** Analyze the solution to the IVP (6.9) with the following initial data

$$\begin{aligned} &\phi(x) \equiv 0, \\ &\psi(x) = \begin{cases} h, & |x| \leq a, \\ 0, & |x| > a. \end{cases} \end{aligned} \tag{6.18}$$

From d'Alambert's formula (6.12), we obtain the following solution

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds. \tag{6.19}$$

Similar to the previous example of the box wave, there are several regions in the xt plane, in which u has different values. Indeed, since the initial velocity $\psi(x)$ is nonzero only in the interval $[-a, a]$, the integral in (6.19) must be computed differently according to how the intervals $[-a, a]$ and $[x - ct, x + ct]$ intersect. This corresponds to the cases when ψ is zero on the entire integration interval in (6.19), on a part of it, or is nonzero on the entire integration interval. These different cases are:

$$\begin{aligned} \text{I}: &\quad \{x - ct < x + ct < -a < a\}, & u(x,t) &= 0 \\ \text{II}: &\quad \{x - ct < -a < x + ct < a\}, & u(t,x) &= \frac{1}{2c}\int_{-a}^{x+ct} h\, ds = h\frac{x + ct + a}{2c} \\ \text{III}: &\quad \{x - ct < -a < a < x + ct\}, & u(t,x) &= \frac{1}{2c}\int_{-a}^{a} h\, ds = h\frac{a}{c} \\ \text{IV}: &\quad \{-a < x - ct < x + ct < a\}, & u(t,x) &= \frac{1}{2c}\int_{x-ct}^{x+ct} h\, ds = ht \\ \text{V}: &\quad \{-a < x - ct < a < x + ct\}, & u(t,x) &= \frac{1}{2c}\int_{x-ct}^{a} h\, ds = h\frac{a - (x - ct)}{2c} \\ \text{VI}: &\quad \{-a < a < x - ct < x + ct\}, & u(x,t) &= 0 \end{aligned} \tag{6.20}$$

The regions are depicted in Figure 6.9 below. Notice that it is relatively simple to figure out the value of the solution at a point (x_0, t_0) by simply tracing back the point to the x axis along the characteristic lines, and determining how the interval of dependence intersects the segment $[-a, a]$.

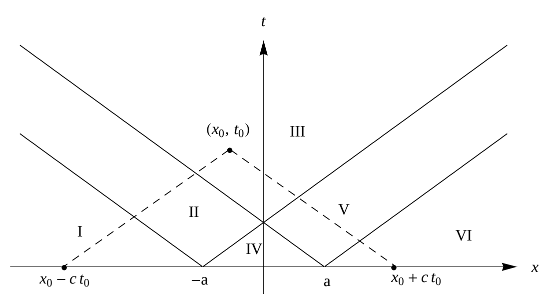

Figure 6.9: Regions where u has different values.

□

6.4 Conclusion

The wave equation, being a constant coefficient second order PDE of hyperbolic type, posseses two families of characteristic lines, which correspond to constant values of respective characteristic variables. Using these variables the equation can be treated as a pair of successive ODEs, integrating which leads to the general solution. This general solution was used to arrive at d'Alambert's solution (6.12) for the IVP on the whole line. Unfortunatelly this simple derivative relies on having two families of characteristics and does not work for the heat and Laplace equations.

Exploring a few examples of initial data, we established causality between the initial data and the values of the solution at later times. In particular, we saw that the initial values travel with speeds bounded by the wave speed c, and that the discountinuities of the initial data travel along the characteristic lines.

7.1 Energy for the wave equation

Let us consider an infinite string with constant linear density ρ and tension magnitude T. The wave equation describing the vibrations of the string is then

$$\rho u_{tt} = T u_{xx}, \qquad -\infty < x < \infty. \tag{7.1}$$

Since this equation describes the mechanical motion of a vibrating string, we can compute the kinetic energy associated with the motion of the string. Recall that the kinetic energy is $\frac{1}{2}mv^2$. In this case the string is infinite, and the speed differs for different points on the string. However, we can still compute the energy of small pieces of the string, add them together, and pass to a limit in which the lengths of the pieces go to zero. This will result in the following integral

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 \, dx.$$

We will assume that the initial data vanishes outside of a large interval $|x| \leq R$, so that the above integral is convergent due to the finite speed of propagation. We would like to see if the kinetic energy KE is conserved in time. For this, we differentiate the above integral with respect to time to see whether it is zero, as is expected for a constant function, or is different from zero.

$$\frac{d}{dt}KE = \frac{1}{2}\rho \int_{-\infty}^{\infty} 2u_t u_{tt} \, dx = \int_{-\infty}^{\infty} \rho u_t u_{tt} \, dx.$$

Using the wave equation (7.1), we can replace the ρu_{tt} by $T u_{xx}$, obtaining

$$\frac{d}{dt}KE = T\int_{-\infty}^{\infty} u_t u_{xx} \, dx.$$

The last quantity does not seem to be zero in general, thus the next best thing we can hope for, is to convert the last integral into a full derivative in time. In that case the difference of the kinetic energy and some other quantity will be conserved. To see this, we perform an integration by parts in the last integral

$$\frac{d}{dt}KE = T u_t u_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_{xt} u_x \, dx.$$

Due to the finite speed of propagation, the endpoint terms vanish. The last integral is a full derivative, thus we have

$$\frac{d}{dt}KE = -\int_{-\infty}^{\infty} T u_{xt} u_x \, dx = -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} T u_x^2 \, dx\right).$$

Defining

$$PE = \frac{1}{2}T\int_{-\infty}^{\infty} u_x^2 \, dx,$$

we see that

$$\frac{d}{dt}KE = -\frac{d}{dt}PE, \qquad \text{or} \qquad \frac{d}{dt}(KE + PE) = 0.$$

The quantity $E = KE + PE$ is then conserved, which is the total energy of the string undergoing vibrations. Notice that PE plays the role of the potential energy of a stretched string, and the conservation of energy implies conversion of the kinetic energy into the potential energy and back without a loss.

Another way to see that the energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) \, dx \tag{7.2}$$

is conserved, is to multiply equation (7.1) by u_t and integrate with respect to x over the real line.

$$0 = \int_{-\infty}^{\infty} \rho u_{tt} u_t \, dx - \int_{-\infty}^{\infty} T u_{xx} u_t \, dx.$$

The first integral above is a full derivative in time. Integrating by parts in the second term, and realizing that the subsequent integral is a full derivative as well, while the boundary terms vanish, we obtain the identity

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2 \, dx\right) = 0,$$

which is exactly the conservation of total energy.

The conservation of energy provides a straightforward way of showing that the solution to an IVP associated with the linear equation is unique. We demonstrate this for the wave equation next, while a similar procedure will be applied to establish uniqueness of solutions for the heat IVP later on.

Example 7.1. Show that the initial value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{for } -\infty < x < +\infty, \\ u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x), \end{cases} \tag{7.3}$$

has a unique solution.

Arguing from the inverse, let as assume that the IVP (7.3) has two distinct solutions, u and v. But then their difference $w = u - v$ will solve the homogeneous wave equation, and will have the initial data

$$\begin{aligned} w(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) \equiv 0, \\ w_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) \equiv 0. \end{aligned}$$

Hence the energy associated with the solution w at time $t = 0$ is

$$E[w](0) = \frac{1}{2}\int_{-\infty}^{\infty} [(w_t(x,0))^2 + c^2 (w_x(x,0))^2] \, dx = 0$$

This differs from the energy defined above by a constant factor of $1/\rho$ (recall that $T/\rho = c^2$), and is thus still a conserved quantity. It will subsequently be zero at any later time as well. Thus,

$$E[w](t) = \frac{1}{2}\int_{-\infty}^{\infty} [(w_t(x,t))^2 + c^2 (w_x(x,t))^2] \, dx = 0, \qquad \forall t.$$

But since the integrand in the expression of the energy is nonnegative, the only way the integral can be zero, is if the integrand is uniformly zero. That is,

$$\nabla w(t,x) = (w_t(x,t), w_x(x,t)) = 0, \qquad \forall x, t.$$

This implies that w is constant for all values of x and t, but since $w(x,0) \equiv 0$, the constant value must be zero. Thus,

$$u(x,t) - v(x,t) = w(x,t) \equiv 0,$$

which is in contradiction with our initial assumption of distinctness of u and v. This implies that the solution to the IVP (7.3) is unique. □

The procedure used in the last example, called the *energy method*, is quite general, and works for other linear evolution equations possessing a conserved (or decaying) positive definite energy. The heat equation, considered next, is one such case.

7.2 Energy for the heat equation

We next consider the (inhomogeneous) heat equation with some auxiliary conditions, and use the energy method to show that the solution satisfying those conditions must be unique. Consider the following mixed initial-boundary value problem, which is called the *Dirichlet problem for the heat equation*

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for} \quad 0 \le x \le l, \quad t > 0 \\ u(x,0) = \phi(x), \\ u(0,t) = g(t), \quad u(l,t) = h(t), \end{cases} \tag{7.4}$$

for given functions f, ϕ, g, h.

Example 7.2. Show that there is at most one solution to the Dirichlet problem (7.4).

Just as in the case of the wave equation, we argue from the inverse by assuming that there are two functions, u, and v, that both solve the inhomogeneous heat equation and satisfy the initial and Dirichlet boundary conditions of (7.4). Then their difference, $w = u - v$, satisfies the homogeneous heat equation with zero initial-boundary conditions, i.e.

$$\begin{cases} w_t - kw_{xx} = 0 & \text{for} \quad 0 \le x \le l, \quad t > 0 \\ w(x,0) = 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \end{cases} \tag{7.5}$$

Now define the following "energy"

$$E[w](t) = \frac{1}{2}\int_0^l [w(x,t)]^2\,dx, \tag{7.6}$$

which is always positive, and decreasing, if w solves the heat equation. Indeed, differentiating the energy with respect to time, and using the heat equation we get

$$\frac{d}{dt}E = \int_0^l ww_t\,dx = k\int_0^l ww_{xx}\,dx.$$

Integrating by parts in the last integral gives

$$\frac{d}{dt}E = kww_x\Big|_0^l - \int_0^l w_x^2\,dx \le 0,$$

since the boundary terms vanish due to the boundary conditions in (7.5), and the integrand in the last term is nonnegative.

Due to the initial condition in (7.5), the energy at time $t = 0$ is zero. But then using the fact that the energy is a nonnegative decreasing quantity, we get

$$0 \le E[w](t) \le E[w](0) = 0.$$

Hence,

$$\frac{1}{2}\int_0^l [w(x,t)]^2\,dx = 0, \quad \text{for all } t \ge 0,$$

which implies that the nonnegative continuous integrand must be identically zero over the integration interval, i.e $w \equiv 0$, for all $x \in [0,l], t > 0$. Hence also

$$u_1 \equiv u_2,$$

which finishes the proof of uniqueness. □

The energy (7.6) arises if one multiplies the heat equation by w and integrates in x over the interval $[0,l]$. Then realizing that the first term will be the time derivative of the energy, and performing the same integration by parts on the second term as above, we can reprove that this energy is decreasing.

Notice that all of the above arguments hold for the case of the infinite interval $-\infty < x < \infty$ as well. In this case one ignores the effect of the infinitely far endpoints and considers an IVP.

7.3 Conclusion

Using the energy motivated by the vibrating string model behind the wave equation, we derived a conserved quantity, which corresponds to the total energy of motion for the infinite string. This positive definite quantity was then used to prove uniqueness of solution to the wave IVP via the energy method, which essentially asserts that zero initial total energy precludes any (nonzero) dynamics. A similar approach was used to prove uniqueness for the heat IBVP, concluding that zero initial heat implies steady zero temperatures at later times.

Problem Set 4

1. (#2.1.1 in [Str]) Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

2. (#2.1.5 in [Str]) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c}\{\text{length of } (x-ct, x+ct) \cap (-a,a)\}.$$

Then $u(x, a/2c) = (1/2c)\{\text{length of } (x - a/2, x + a/2) \cap (-a.a)\}$. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

3. (#2.1.7 in [Str]) If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

4. (#2.1.9 in [Str]) Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (*Hint:* Factor the operator as the textbook does for the wave equation.)

5. (#2.2.3 in [Str]) Show that the wave equation has the following invariance properties.

   (a) Any translate $u(x-y,t)$, where y is fixed, is also a solution.

   (b) Any derivative, say u_x, of a solution is also a solution.

   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.

6. (#2.2.4 in [Str]) If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h)$$

for all x, t, h and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

7. (#2.2.5 in [Str]) For the damped string, $u_{tt} - c^2 u_{xx} + r u_t = 0, r > 0$, show that the energy decreases.

We would like to solve the heat (diffusion) equation

$$u_t - ku_{xx} = 0, \tag{8.1}$$

and obtain a solution formula depending on the given initial data, similar to the case of the wave equation. However the methods that we used to arrive at d'Alambert's solution for the wave IVP do not yield much for the heat equation. To see this, recall that the heat equation is of parabolic type, and hence, it has only one family of characteristic lines. If we rewrite the equation in the form

$$ku_{xx} + \cdots = 0,$$

where the dots stand for the lower order terms, then you can see that the coefficients of the leading order terms are

$$A = k, \quad B = C = 0.$$

The slope of the characteristic lines are then

$$\frac{dt}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{B}{2A} = 0.$$

Consequently, the single family of characteristic lines will be given by

$$t = c.$$

These characteristic lines are not very helpful, since they are parallel to the x axis. Thus, one cannot trace points in the xt plane along the characteristics to the x axis, along which the initial data is defined. Notice that there is also no way to factor the heat equation into first order equations, either, so the methods used for the wave equation do not shed any light on the solutions of the heat equation.

Instead, we will study the properties of the heat equation, and use the gained knowledge to devise a way of reducing the heat equation to an ODE, as we have done for every PDE we have solved so far.

8.1 The maximum principle

The first properties that we need to make sure of, are the uniqueness and stability for the solution of the problem with certain auxiliary conditions. This would guarantee that the problem is wellpossed, and the chosen auxiliary conditions do not break the physicality of the problem. We begin by establishing the following property, that will be later used to prove uniqueness and stability.

Maximum Principle. If $u(x,t)$ satisfies the heat equation (8.1) in the rectangle $R = \{0 \leq x \leq l, 0 \leq t \leq T\}$ in space-time, then the maximum value of $u(x,t)$ over the rectangle is assumed either initially ($t = 0$), or on the lateral sides ($x = 0$, or $x = l$).

Mathematically, the maximum principle asserts that the maximum of $u(x,t)$ over the three sides must be equal to the maximum of the $u(x,t)$ over the entire rectangle. If we denote the set of points comprising the three sides by $\Gamma = \{(x,t) \in R \mid t = 0 \quad \text{or} \quad x = 0 \quad \text{or} \quad x = l\}$, then the maximum principle can be written as

$$\max_{(x,t)\in\Gamma}\{u(x,t)\} = \max_{(x,t)\in R}\{u(x,t)\}. \tag{8.2}$$

If you think of the heat conduction phenomena in a thin rod, then the maximum principle makes physical sense, since the initial temperature, as well as the temperature at the endpoints will dissipate through conduction of heat, and at no point the temperature can rise above the highest initial or endpoint temperature. In fact, a stronger version of the maximum principle holds, which asserts that the maximum over the rectangle R can not be attained at a point not belonging to Γ, unless $u \equiv$ constant, i.e. for nonconstant solutions the following strict inequality holds

$$\max_{(x,t)\in R\setminus\Gamma}\{u(x,t)\} < \max_{(x,t)\in R}\{u(x,t)\},$$

where $R \setminus \Gamma$ is the set of all points of R that are not in Γ (difference of sets). This makes physical sense as well, since the heat from the point of highest initial or boundary temperature will necessarily transfer to points of lower temperature, thus decreasing the highest temperature of the rod.

We finally note, that the maximum principle also implies a *minimum principle*, since one can apply it to the function $-u(x,t)$, which also solves the heat equation, and make use of the fact that

$$\min\{u(x,t)\} = -\max\{-u(x,t)\}.$$

Thus, the minima points of the function $u(x,t)$ will exactly coincide with the maxima points of $-u(x,t)$, of which, by the maximum principle, there must necessarily be in Γ.

Proof of the maximum principle. If the maximum of the function $u(x,t)$ over the rectangle R is assumed at an internal point (x_0,t_0), then the gradient of u must vanish at that point, i.e. $u_t(x_0,t_0) = u_x(x_0,t_0) = 0$. If in addition we had the strict inequality $u_{xx}(x_0,t_0) < 0$, then one would get a contradiction by plugging the point (x_0,t_0) into the heat equation. Indeed, we would have

$$u_t(x_0,t_0) - ku_{xx}(x_0,t_0) = -ku_{xx}(x_0,t_0) > 0.$$

This contradicts the heat equation (8.1), which must hold for all values of x and t. Thus, the contradiction would imply that the maximum point (x_0,t_0) cannot be an internal point. However, from the second derivative test we have the weaker inequality $u_{xx}(x_0,t_0) \leq 0$ (the point would not be a maximum if $u_{xx}(x_0,t_0) > 0$), which is not enough for this argument to go through.

The way out, is to recycle the above argument with a slight modification to the function u. Define a new function

$$v(x,t) = u(x,t) + \epsilon x^2, \tag{8.3}$$

where $\epsilon > 0$ is a constant that can be taken as small as one wants. Now let M be the maximum value of u over the three sides, which we denoted by Γ above. That is

$$M = \max_{(x,t)\in\Gamma}\{u(x,t)\}.$$

To prove the maximum principle, we need to establish (8.2). The maximum over Γ is always less than or equal to the maximum over R, since $\Gamma \subset R$. So we only need to show the opposite inequality, which would follow from showing that

$$u(x,t) \leq M, \qquad \text{for all the points } (x,t) \in R. \tag{8.4}$$

Notice that from the definition of v, we have that at the points of Γ, $v(x,t) \leq M + \epsilon l^2$, since the maximum value of ϵx^2 on Γ is ϵl^2. Then, instead of proving inequality (8.4), we will prove that

$$v(x,t) \leq M + \epsilon l^2, \qquad \text{for all the points } (x,t) \in R, \tag{8.5}$$

which implies (8.4). Indeed, from the definition of v in (8.3), we have that in the rectangle R

$$u(x,t) \leq v(x,t) - \epsilon x^2 \leq M + \epsilon(l^2 - x^2),$$

where we used (8.5) to bound $v(x,t)$. Now, since the point (x,t) is taken from the rectangle R, we have that $0 \leq x \leq l$, and the difference $l^2 - x^2$ is bounded. But then the right hand side of the above inequality can be made as close to M as possible by taking ϵ small enough, which implies the bound (8.4).

If we formally apply the heat operator to the function v, and use the definition (8.3), we will get

$$v_t - kv_{xx} = u_t - k(u_{xx} + 2\epsilon) = (u_t - ku_{xx}) - 2k\epsilon < 0,$$

since both $k, \epsilon > 0$, and u satisfies the heat equation (8.1) on R. Thus, v satisfies the *heat inequality* in R

$$v_t - kv_{xx} < 0. \tag{8.6}$$

We now recycle the above argument, which barely failed for u, applying it to v instead. Suppose $v(x,t)$ attains its maximum value at an internal point (x_0,t_0). Then necessarily $v_t(x_0,t_0) = 0$, and $v_{xx}(x_0,t_0) \leq 0$. Hence, at this point we have

$$v_t(x_0,t_0) - kv_{xx}(x_0,t_0) = -kv_{xx}(x_0,t_0) \geq 0,$$

which contradicts the heat inequality (8.6). Thus, v cannot have an internal maximum point in R.

Similarly, suppose that $v(x,t)$ attains its maximum value at a point (x_0,t_0) on the fourth side of the rectangle R, i.e. when $t_0 = T$. Then we still have that $v_x(x_0,t_0) = 0$, and $v_{xx}(x_0,t_0) \leq 0$, but $v_t(x_0,t_0)$ does not have to be zero, since $t_0 = T$ is an endpoint in the t direction. However, from the definition of the derivative, and our assumption that (x_0,t_0) is a point of maximum, we have

$$v_t(x_0,t_0) = \lim_{\delta\to 0+} \frac{v(x_0,t_0) - v(x_0,t_0-\delta)}{\delta} \geq 0.$$

So at this point we still have

$$v_t(x_0,t_0) - kv_{xx}(x_0,t_0) \geq 0,$$

which again contradicts the heat inequality (8.6).

Now, since the continuous function $v(x,t)$ must attain its maximum value somewhere in the closed rectangle R, this must happen on one of the remaining three sides, which comprise the set Γ. Hence,

$$v(x,t) \leq \max_{(x,t)\in R}\{v(x,t)\} = \max_{(x,t)\in\Gamma}\{v(x,t)\} \leq M + \epsilon l^2,$$

which finishes the proof of (8.5). □

8.2 Uniqueness

Consider the Dirichlet problem for the heat equation,

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for} \quad 0 \leq x \leq l, \quad t > 0 \\ u(x,0) = \phi(x), \\ u(0,t) = g(t), \quad u(l,t) = h(t), \end{cases} \tag{8.7}$$

for given functions f, ϕ, g, h. We will use the maximum principle to show uniqueness and stability for the solutions of this problem (recall that last time we used the energy method to prove uniqueness for this problem).

Uniqueness of solutions. There is at most one solution to the Dirichlet problem (8.7).

Indeed, arguing from the inverse, suppose that there are two functions, u, and v, that both solve the inhomogeneous heat equation and satisfy the initial and Dirichlet boundary conditions of (8.7). Then their difference, $w = u - v$, satisfies the homogeneous heat equation with zero initial-boundary conditions, i.e.

$$\begin{cases} w_t - kw_{xx} = 0 & \text{for} \quad 0 \leq x \leq l, \quad t > 0 \\ w(x,0) = 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \end{cases} \tag{8.8}$$

But from the maximum principle, we know that w assumes its maximum and minimum values on one of the three sides $t = 0$, $x = 0$, and $x = l$. And since $w = 0$ on all of these three sides from the initial and boundary conditions in (8.8), we have that for $x \in [0,l], t > 0$

$$0 \leq w \leq 0 \quad \Rightarrow \quad w(x,t) \equiv 0.$$

Hence,

$$u - v = w \equiv 0, \qquad \text{or} \qquad u \equiv v,$$

and the solution must indeed be unique.

Notice again that all of the above arguments hold for the case of the infinite interval $-\infty < x < \infty$ as well. In this case one ignores the effect of the infinitely far endpoints and considers an IVP. And the maximum principle simply asserts that the maximum of the solutions must be attained initially. We will use this in the next lecture when deriving the solution for the IVP for the heat equation on the entire real line $x \in \mathbb{R}$.

8.3 Stability

Stability of solutions with respect to the auxiliary conditions is the third ingredient of well-posedness, after existence and uniqueness. It asserts that *close* auxiliary conditions lead to *close* solutions. One may have different ways of measuring the closeness of the solutions, and the initial and boundary data. Consider two solutions, u_1, u_2, of the heat equation (8.1) for $x \in [0, l], t > 0$, which satisfy the following initial-boundary conditions

$$\begin{cases} u_1(x,0) = \phi_1(x), \\ u_1(0,t) = g_1(t), \quad u_1(l,t) = h_1(t), \end{cases} \qquad \begin{cases} u_2(x,0) = \phi_2(x), \\ u_2(0,t) = g_2(t), \quad u_2(l,t) = h_2(t). \end{cases} \tag{8.9}$$

Stability of solutions means that *closeness* of ϕ_1 to ϕ_2, g_1 to g_2 and h_1 to h_2 implies the closeness of u_1 to u_2. Notice that the difference $w = u_1 - u_2$ solves the heat equation as well, and satisfies the following initial-boundary conditions

$$\begin{cases} w_1(x,0) = \phi_1(x) - \phi_2(x), \\ w(0,t) = g_1(t) - g_2(t), \quad w(l,t) = h_1(t) - h_2(t). \end{cases}$$

But then the maximum and minimum principles imply

$$-\max_{(x,t)\in\Gamma}\{|w(x,t)|\} \leq \max_{\substack{0\leq x\leq l\\ t\geq 0}}\{w(x,t)\} \leq \max_{(x,t)\in\Gamma}\{|w(x,t)|\},$$

and hence, the absolute value of the difference $u_1 - u_2$ will be bounded by

$$\begin{aligned}\max_{\substack{0\leq x\leq l\\ t\geq 0}}\{|u_1(x,t) - u_2(x,t)|\} = \max_{\substack{0\leq x\leq l\\ t\geq 0}}\{|w(x,t)|\} &\leq \max_{(x,t)\in\Gamma}\{|w(x,t)|\} \\ &= \max_{\substack{0\leq x\leq l\\ t\geq 0}}\{|\phi_1(x) - \phi_2(x)|, |g_1(t) - g_2(t)|, |h_1(t) - h_2(t)|\}.\end{aligned}$$

Thus, the smallness of the maximum of the differences $|\phi_1 - \phi_2|$, $|g_1 - g_2|$ and $|h_1 - h_2|$ implies the smallness of the maximum of the difference of solutions $|u_1 - u_2|$. In this case the stability is said to be in the *uniform* sense, i.e. smallness is understood to hold uniformly in the (x, t) variables.

An alternate way of showing the stability is provided by the energy method. Suppose u_1 and u_2 solve the heat equation with initial data ϕ_1 and ϕ_2 respectively, and zero boundary conditions. This would be the case for the problem over the entire real line $x \in \mathbb{R}$, or if $g_1 = g_2 = h_1 = h_2 = 0$ in (8.9). In this case the energy method for the difference $w = u_1 - u_2$ implies that $E[w](t) \leq E[w](0)$ for all $t \geq 0$, or

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\, dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2\, dx, \qquad \text{for all} \quad t \geq 0.$$

Thus the closeness of ϕ_1 to ϕ_2 in the sense of the square integral of the difference implies the closeness of the respective solutions in the same sense. This is called stability in the *square integral* (L^2) sense.

8.4 Conclusion

As expected, the method of characteristics is inefficient for solving the heat equation. We then need to find an alternative method of reducing the equation to an ODE. But before embarking on this path, we first study the properties of the heat equation, which will serve as beacons in the later reduction to an ODE. Today we established the maximum principle for the heat equation, which immediately implied the uniqueness and stability for the solution. Next time we will look at the invariance properties of the equation and derive the solution using these properties.

9 Heat equation: solution

Equipped with the uniqueness property for the solutions of the heat equation with appropriate auxiliary conditions, we will next present a way of deriving the solution to the heat equation

$$u_t - ku_{xx} = 0. \tag{9.1}$$

Considering the equation on the entire real line $x \in \mathbb{R}$ simplifies the problem by eliminating the effect of the boundaries, we will first concentrate on this case, which corresponds to the dynamics of the temperature in a rod of infinite length. We want to solve the IVP

$$\begin{cases} u_t - ku_{xx} = 0 \quad (-\infty < x < \infty, 0 < t < \infty), \\ u(x,0) = \phi(x). \end{cases} \tag{9.2}$$

Since the solution to the above IVP is not easy to derive directly, unlike the case of the wave IVP, we will first derive a particular solution for a special simple initial data, and try to produce solutions satisfying all other initial conditions by exploiting the invariance properties of the heat equation.

9.1 Invariance properties of the heat equation

The heat equation (9.1) is invariant under the following transformations

(a) Spatial translations: If $u(x,t)$ is a solution of (9.1), then so is the function $u(x-y,t)$ for any fixed y.

(b) Differentiation: If u is a solution of (9.1), then so are u_x, u_t, u_{xx} and so on.

(c) Linear combinations: If $u_1, u_2, \ldots, u_n$ are solutions of (9.1), then so is $u = c_1u_1 + c_2u_2 + \ldots c_nu_n$ for any constants $c_1, c_2, \ldots, c_n$.

(d) Integration: If $S(x,t)$ is a solution of (9.1), then so is the integral

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as the improper integral converges (we will ignore the issue of the convergence for the time being).

(e) Dilation (scaling): If $u(x,t)$ is a solution of (9.1), then so is the dilated function $v(x,t) = u(\sqrt{a}x, at)$ for any constant $a > 0$ (compare this to the scaling property of the wave equation, which is invariant under the dilation $u(x,t) \mapsto u(ax,at)$ for all $a \in \mathbb{R}$).

Properties (a), (b) and (c) are trivial (check by substitution), while property (d) is the limiting case of property (c). Indeed, if we use the notation $u^y(x,t) = S(x-y,t)$, and $c^y = g(y)\Delta y$, then u^y is also a solution by property (a), and we have the formal limit

$$\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy = \lim_{\Delta y \to 0} \sum_y c^y u^y.$$

To make this precise, we need to consider a finite interval of integration, which is partitioned by points $\{y_i\}_{i=1}^n$ into subintervals of length Δy, and use the definition of the integral as the limit of the corresponding Riemann sum to write

$$\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy = \lim_{b\to\infty} \int_{-b}^{b} S(x-y,t)g(y)\,dy = \lim_{b\to\infty}\lim_{n\to\infty} \sum_{i=1}^{n} S(x-y_i)g(y_i)\Delta y,$$

where $-b = y_1 < y_2 < \ldots y_n = b$ is a partition of the interval $[-b,b]$.

Finally, property (e) can be checked by direct substitution as well. Notice that we cannot formally reverse the time by dilating with the factor $a = -1$, as was the case for the wave equation, since the $\sqrt{a}$ factor in front of the x argument would make the dilated function complex, which is not allowed in the theory of real PDEs (what is the meaning of complex valued temperature?!). We will later see that the heat equation is indeed time irreversible.

9.2 Solving a particular IVP

As a special initial data we take the following function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{9.3}$$

which is called the Heaviside step function. We consider the IVP

$$\begin{cases} Q_t - kQ_{xx} = 0 & (-\infty < x < \infty, 0 < t < \infty), \\ Q(x,0) = H(x). \end{cases} \tag{9.4}$$

We will solve this IVP in successive steps.

Step 1: Reduction to an ODE. Notice that the Heaviside function (9.3) is invariant under the dilation $x \mapsto \sqrt{a}x$, i.e. $H(\sqrt{a}x) = H(x)$. From the dilation property of the heat equation, we know that $Q(\sqrt{a}x, at)$ also solves the heat equation. But $Q(\sqrt{a}x, 0) = H(\sqrt{a}x) = H(x)$, thus $Q(\sqrt{a}x, at)$ and $Q(x,t)$ both solve the IVP (9.4). The uniqueness of solutions then implies that $Q(\sqrt{a}x, at) = Q(x,t)$ for all $x \in \mathbb{R}, t > 0$, so Q is invariant under the dilation $(x,t) \mapsto (\sqrt{a}x, at)$ as well.

Due to this invariance, Q can depend only on the ratio $\frac{x}{\sqrt{t}}$, that is $Q(x,t) = q(\frac{x}{\sqrt{t}})$. To see this, define the function q in the following way $q(z) = Q(z,1)$. But then for fixed (x,t), we have

$$Q(x,t) = Q\left(\frac{1}{\sqrt{t}}x, \frac{1}{t}t\right) = Q\left(\frac{x}{\sqrt{t}}, 1\right) = q\left(\frac{x}{\sqrt{t}}\right).$$

Thus Q is completely determined by the function of one variable q.

For convenience of future calculations we pass to the function $g(z) = q(\sqrt{4k}z)$, so that

$$Q(x,t) = q\left(\frac{x}{\sqrt{t}}\right) = g\left(\frac{x}{\sqrt{4kt}}\right) = g(p),$$

where we used the notation $p = x/\sqrt{4kt}$. We next compute the derivatives of Q in terms of g, and substitute them into the heat equation in order to obtain an ODE for g. Using the chain rule, one gets

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{4k}{2}\frac{x}{(\sqrt{4kt})^3}g'(p) = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p), \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p), \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial t} = \frac{1}{4kt}g''(p). \end{aligned}$$

The heat equation then implies

$$0 = Q_t - kQ_{xx} = \frac{1}{4t}[-2pg'(p) - g''(p)],$$

which gives the following equation for g

$$g'' + 2pg' = 0. \tag{9.5}$$

Step 2: Solving the ODE. Using the integrating factor $\exp(\int 2p\,dp) = e^{p^2}$, the ODE (9.5) reduces to

$$[e^{p^2}g'(p)]' = 0.$$

Thus, we have

$$e^{p^2} g'(p) = c_1.$$

Solving for $g'(p)$, and integrating, we obtain

$$g(p) = c_1 \int e^{-p^2}\, dp + c_2.$$

Step 3: Checking the initial condition. Recalling that $Q(x,t) = g(p)$, where $p = x/\sqrt{4kt}$, we obtain the following explicit formula for Q

$$Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp + c_2. \tag{9.6}$$

Notice that we chose a particular antiderivative, which we are free to do due to the presence of the arbitrary constants. Also note that the above formula is only valid for $t > 0$, so to check the initial condition, we need to take the limit $t \to 0+$. Recalling the initial condition from (9.4), we have that,

$$\begin{aligned} &\text{if} \quad x > 0, \qquad 1 = \lim_{t\to 0+} Q(x,t) = c_1 \int_0^{+\infty} e^{-p^2}\, dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2, \\ &\text{if} \quad x < 0, \qquad 0 = \lim_{t\to 0+} Q(x,t) = c_1 \int_0^{-\infty} e^{-p^2}\, dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2, \end{aligned}$$

where we used the fact that $\int_0^\infty e^{-p^2}\, dp = \sqrt{\pi}/2$ to compute the improper integrals. The above identities give

$$c_1 \frac{\sqrt{\pi}}{2} + c_2 = 1, \qquad -c_1 \frac{\sqrt{\pi}}{2} + c_2 = 0.$$

Solving for c_1 and c_2, we get $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. Substituting these into (9.6) gives the unique solution of the IVP (9.4),

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp, \qquad \text{for} \quad t > 0. \tag{9.7}$$

9.3 Solving the general IVP

Returning to the general IVP (9.2), we would like to derive a solution formula, which will express the solution to the IVP in terms of the initial data (similar to d'Alambert's solution for the wave equation).

We first define the function

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t), \tag{9.8}$$

where $Q(x,t)$ is the solution to the particular IVP (9.4), and is given by (9.7). Then, by the invariance properties of the heat equation, $S(x,t)$ also solves the heat equation, and so does

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy, \qquad \text{for} \quad t > 0. \tag{9.9}$$

We claim that this u is the unique solution of the IVP (9.2). To verify this claim one only needs to check the initial condition of (9.2). Notice that using $S = Q_x$, we can rewrite u as follows

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\, dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\, dy$$

Integrating by parts in the last integral, we get

$$u(x,t) = -Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy.$$

We assume that the boundary terms vanish, which can be guaranteed for example by assuming that $\phi(y)$ vanishes for large $|y|$ (this is not strictly necessary, since $S(x-y,t)$ decays rapidly as $|y-x|$ becomes large, as we will shortly see). Now plugging in $t=0$, and using that Q has the Heaviside function (9.3) as its initial data, we have

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy = \int_{-\infty}^{x} \phi'(y)\,dy = \phi(y)\Big|_{y=-\infty}^{y=x} = \phi(x).$$

So $u(x,t)$ indeed satisfies the initial condition of (9.2).

We can compute $S(x,t)$ from (9.8), which will give

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}, \tag{9.10}$$

Using this expression of $S(x,t)$, we can now rewrite the solution given by (9.9) in the following explicit form

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy, \qquad \text{for} \quad t>0. \tag{9.11}$$

The function $S(x,t)$ is known as the *heat kernel*, *fundamental solution*, *source function*, *Green's function*, or *propagator* of the heat equation. Notice that it gives a way of *propagating* the initial data ϕ to later times, giving the solution at any time $t>0$.

It is clear that formula (9.11) does not make sense for $t=0$, although one can compute the limit of $u(t,x)$ as $t \to 0+$ in that formula, which will give an alternate way of checking the initial condition of (9.2).

9.4 Conclusion

We derived the solution to the heat equation by first looking at a particular initial data, which was invariant under dilation. This guaranteed that the solution corresponding to this initial data is also dilation invariant, which reduced the heat equation to an ODE. After solving this ODE, and obtaining the solution, we saw that the solution to the general heat IVP can be written in an integral form using this particular solution. Next time we will explore the solution given by formula (9.11), and will study its qualitative behavior.

Problem Set 5

1. (#2.3.1 in [Str]) Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \leq x \leq 1, 0 \leq t \leq T\}$.

2. (#2.3.5 in [Str]) The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

   (a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$.

   (b) Where precisely does the proof of the maximum principle break down for this equation?

3. (#2.3.6 in [Str]) Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for $0 \leq t < \infty, 0 \leq x \leq l$.

4. (#2.3.8 in [Str]) Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$, and $u_x(l,t) + a_l u(l,t) = 0$. If $a_0 > 0$, and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

5. (#2.4.1 in [Str]) Solve the diffusion equation with the initial condition

   $$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

   Write your answer in terms of $\mathcal{E}\mathrm{rf}(x)$.

6. (#2.4.6 in [Str]) Compute $\int_0^\infty e^{-x^2}\,dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)

7. (#2.4.9 in [Str]) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x,t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.

Last time we considered the IVP for the heat equation on the whole line

$$\begin{cases} u_t - ku_{xx} = 0 & (-\infty < x < \infty, 0 < t < \infty), \\ u(x,0) = \phi(x), \end{cases} \tag{10.1}$$

and derived the solution formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \qquad \text{for} \quad t > 0, \tag{10.2}$$

where $S(x,t)$ is the heat kernel,

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}. \tag{10.3}$$

Substituting this expression into (10.2), we can rewrite the solution as

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy, \qquad \text{for} \quad t > 0. \tag{10.4}$$

Recall that to derive the solution formula we first considered the heat IVP with the following particular initial data

$$Q(x,0) = H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{10.5}$$

Then using dilation invariance of the Heaviside step function $H(x)$, and the uniqueness of solutions to the heat IVP on the whole line, we deduced that Q depends only on the ratio $x/\sqrt{t}$, which lead to a reduction of the heat equation to an ODE. Solving the ODE and checking the initial condition (10.5), we arrived at the following explicit solution

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp, \qquad \text{for} \quad t > 0. \tag{10.6}$$

The heat kernel $S(x,t)$ was then defined as the spatial derivative of this particular solution $Q(x,t)$, i.e.

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t), \tag{10.7}$$

and hence it also solves the heat equation by the differentiation property.

The key to understanding the solution formula (10.2) is to understand the behavior of the heat kernel $S(x,t)$. To this end some technical machinery is needed, which we develop next.

10.1 Dirac delta function

Notice that, due to the discontinuity in the initial data of Q, the derivative $Q_x(x,t)$, which we used in the definition of the function S in (10.7), is not defined in the traditional sense when $t = 0$. So how can one make sense of this derivative, and what is the initial data for $S(x,t)$?

It is not difficult to see that the problem is at the point $x = 0$. Indeed, using that $Q(x,0) = H(x)$ is constant for any $x \neq 0$, we will have $S(x,0) = 0$ for all x different from zero. However, $H(x)$ has a jump discontinuity at $x = 0$, as is seen in Figure 10.1, and one can imagine that at this point the rate of growth of H is infinite. Then the "derivative"

$$\delta(x) = H'(x) \tag{10.8}$$

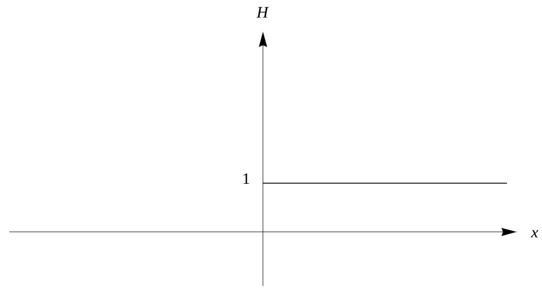

Figure 10.1: The graph of the Heaviside step function.

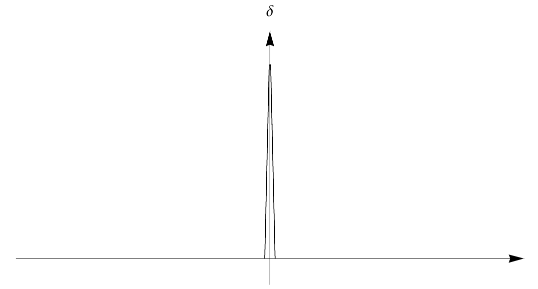

Figure 10.2: The sketch of the Dirac δ function.

is zero everywhere, except at $x = 0$, where it has a spike of zero width and infinite height. Refer to Figure 10.2 below for an intuitive sketch of the graph of δ. Of course, δ is not a function in the traditional sense, but is rather a *generalized function*, or *distribution*. Unlike regular functions, which are characterized by their finite values at every point in their domains, distributions are characterized by how they act on regular functions.

To make this rigorous, we define the set of *test functions* $\mathcal{D} = C_c^\infty$, the elements of which are smooth functions with compact support. So $\phi \in \mathcal{D}$, if and only if ϕ has continuous derivatives of any order $k \in \mathbb{N}$, and the closure of the support of ϕ,

$$\operatorname{supp}(\phi) = \{x \in \mathbb{R} \mid \phi(x) \neq 0\},$$

is compact. Recall that compact sets in $\mathbb{R}$ are those that are closed and bounded. In particular for any test function ϕ there is a rectangle $[-R, R]$, outside of which ϕ vanishes. Notice that derivatives of test functions are also test functions, as are sums, scalar multiples and products of test functions.

Distributions are continuous linear *functionals* on $\mathcal{D}$, that is, they are continuous linear maps from $\mathcal{D}$ to the real numbers $\mathbb{R}$. Notice that for any regular function f, we can define the functional

$$f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx, \tag{10.9}$$

which makes f into a distribution, since to every $\phi \in \mathcal{D}$ it assigns the number $\int_{-\infty}^{\infty} f(x)\phi(x)\, dx$. This integral will converge under very weak conditions on f ($f \in L^1_{loc}$), due to the compact support of ϕ. In particular, f can certainly have jump discontinuities. Notice that we committed an abuse of notation to identify the distribution associated with f by the same letter f. The particular notion in which we use the function will be clear from the context.

One can also define the *distributional derivative* of f to be the distribution, which acts on the test functions as follows

$$f'[\phi] = -\int_{-\infty}^{\infty} f(x)\phi'(x)\, dx.$$

Notice that integration by parts and the compact support of test functions makes this definition consistent with the regular derivative for differentiable functions (check that the distribution formed as in (10.9) by the derivative of f coincides with the distributional derivative of f).

We can also apply the notion of the distributional derivative to the Heaviside step function $H(x)$, and think of the definition (10.8) in the sense of distributional derivatives. Let us now compute how δ, called the *Dirac delta function*, acts on test functions. By the definition of the distributional derivative,

$$\delta[\phi] = -\int_{-\infty}^{\infty} H(x)\phi'(x)\, dx.$$

Recalling the definition of $H(x)$ in (10.5), we have that

$$\delta[\phi] = -\int_0^{\infty} \phi'(x)\, dx = -\phi(x)\Big|_0^{\infty} = \phi(0). \tag{10.10}$$

Thus, the Dirac delta function maps test functions to their values at $x = 0$. We can make a translation in the x variable, and define $\delta(x - y) = H'(x - y)$, i.e. $\delta(x - y)$ is the distributional derivative of the distribution formed by the function $H(x - y)$. Then it is not difficult to see that $\delta(x - y)[\phi] = \phi(y)$. That is, $\delta(x-y)$ maps test functions to their values at y. We will make the abuse of notation mentioned above, and write this as

$$\int_{-\infty}^{\infty} \delta(x - y)\phi(x)\, dx = \phi(y).$$

We also note that $\delta(x - y) = \delta(y - x)$, since δ is even, if we think of it as a regular function with a centered spike (one can prove this from the definition of δ as a distribution).

Using these new notions, we can make sense of the initial data for $S(x, y)$. Indeed,

$$S(x, 0) = \delta(x). \tag{10.11}$$

Since the initial data is a distribution, one then thinks of the equation to be in the sense of distributions as well, that is, treat the derivatives appearing in the equation as distributional derivatives. This requires the generalization of the idea of a distribution to two dimensions. We call this type of solutions *weak solutions* (recall the solutions of the wave equation with discontinuous data). Thus $S(x, t)$ is a week solution of the heat equation, if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x, t)[\phi_t(x, t) - k\phi_{xx}(x, t)]\, dxdt = 0,$$

for any test function ϕ of two variables. This means that the distribution $(\partial_t - k\partial_x^2)S$, with the derivatives taken in the distributional sense, is the zero distribution. Notice that the weak solution $S(x, t)$ arising from the initial data (10.11) has the form (10.3), which is an infinitely differentiable function of x and t. This is in stark contrast to the case of the wave equation, where, as we have seen in the examples, the discontinuity of the initial data is preserved in time.

Having the δ function in our arsenal of tools, we can now give an alternate proof that (10.2) satisfies the initial conditions of (10.1). Directly plugging in $t = 0$ into (10.2), which we are now allowed to do by treating it as a distribution, and using (10.11), we get

$$u(x, 0) = \int_{-\infty}^{\infty} \delta(x - y)\phi(y)\, dy = \phi(x).$$

10.2 Interpretation of the solution

Let us look at the solution (10.4) in detail, and try to understand how the heat kernel $S(x, t)$ propagates the initial data $\phi(x)$. Notice that $S(x, t)$, given by (10.3), is a well-defined function of (x, t) for any $t > 0$. Moreover, $S(x, t)$ is positive, is even in the x variable, and for a fixed t has a bell-shaped graph. In general, the function

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is called *Gaussian function* or *Gaussian*. In the probability theory, it gives the density of the normal distribution with mean μ and standard deviation σ. The graph of the Gaussian is a bell-curve with its peak of height $1/\sqrt{2\pi\sigma^2}$ at $x = \mu$ and the width of the bell at mid-height roughly equal to 2σ. Thus, for some fixed time t the height of $S(x, t)$ at its peak $x = 0$ is $\frac{1}{\sqrt{4\pi kt}}$, which decays as t grows.

Notice that as $t \to 0+$, the height of the peak becomes arbitrarily large, and the width of the bell-curve, $\sqrt{2kt}$ goes to zero. This, of course, is expected, since $S(x, t)$ has the initial data (10.11). One can think of $S(x, t)$ as the temperature distribution at time t that arises from the initial distribution given by the Dirac delta function. With passing time the highest temperature at $x = 0$ gets gradually transferred to the other points of the rod. It also makes sense, that points closer to $x = 0$ will have higher temperature than those farther away. Graphs of $S(x, t)$ for three different times are sketched in Figure 10.3 below.

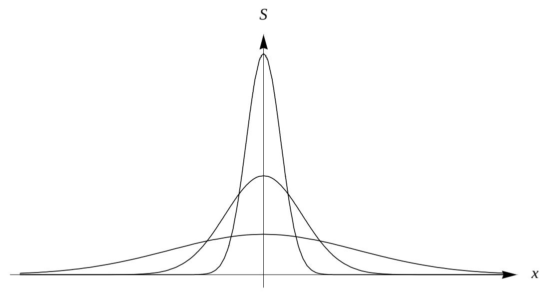

Figure 10.3: The graphs of the heat kernel at different times.

From the initial condition (10.11), we see that initially the temperature at every point $x \neq 0$ is zero, but $S(x,t) > 0$ for any x and $t > 0$. This means that heat is instantaneously transferred to all points of the rod (closer points get more heat), so the speed of heat conduction is infinite. Compare this to the finite speed of propagation for the wave equation. One can also compute the area below the graph of $S(x,t)$ at any time $t > 0$ to get

$$\int_{-\infty}^{\infty} S(x,t)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp = 1,$$

where we used the change of variables $p = x/\sqrt{4kt}$. At $t = 0$, we have

$$\int_{-\infty}^{\infty} S(x,0)\,dx = \int_{-\infty}^{\infty} \delta(x)\,dx = 1,$$

where we think of the last integral as the δ distribution applied to the constant function 1 (more precisely, a test function that is equal to 1 in some open interval around $x = 0$). This shows that the area below the graph of $S(x,t)$ is preserved in time and is equal to 1, so for any fixed time $t \geq 0$, $S(x,t)$ can be thought of as a probability density function. At time $t = 0$ its the probability density that assigns probability 1 to the point $x = 0$, as was seen in (10.10), and for times $t > 0$ it is a normal distribution with mean $x = 0$ and standard deviation $\sigma = \sqrt{2kt}$ that grows with time. As we mentioned earlier, $S(x,t)$ is smooth, in spite of having a discontinuous initial data. We will see in the next lecture that this is true for any solution of the heat IVP (10.1) with general initial data.

We now look at the solution (10.4) with general data $\phi(x)$. First, notice that the integrand in (10.2),

$$S(x-y,t)\phi(y),$$

measures the effect of $\phi(y)$ (the initial temperature at the point y) felt at the point x at some later time t. The source function $S(x-y,t)$, which has its peak precisely at y, *weights* the contribution of $\phi(y)$ according to the distance of y from x and the elapsed time t.

Since the value of $u(x,t)$ (temperature at the point x at time t) is the total sum of contributions from the initial temperature at all points y, we have the formal sum

$$u(x,t) \approx \sum_{y} S(x-y,t)\phi(y),$$

which in the limit gives formula (10.2). So, the heat kernel $S(x,t)$ gives a way of propagating the initial data ϕ to later times. Of course the contribution from a point y_1 closer to x has a bigger weight $S(x-y_1,t)$, than the contribution from a point y_2 farther away, which gets weighted by $S(x-y_2,t)$.

The function $S(x,t)$ appears in various other physical situations. For example in the random (Brownian) motion of a particle in one dimension. If the probability of finding the particle at position x initially is given by the density function $\phi(x)$, then the density defining the probability of finding the particle at position x at time t is given by the same formula (10.2).

Example 10.1. Solve the heat equation with the initial condition $u(x,0) = e^x$.

Using the solution formula (10.4), we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^y \, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{[-x^2+2xy-y^2+4kty]/4kt} \, dy$$

We can complete the squares in the numerator of the exponent, writing it as

$$\begin{aligned} \frac{-x^2+2xy-y^2+4kty}{4kt} &= \frac{-x^2+2(x+2kt)y-y^2}{4kt} \\ &= \frac{-(y-2kt-x)^2+4ktx+4k^2t^2}{4kt} = -\left(\frac{y-2kt-x}{\sqrt{4kt}}\right)^2 + x + kt. \end{aligned}$$

We then have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{x+kt} e^{[(y-2kt-x)/\sqrt{4kt}]^2} \, dy = \frac{1}{\sqrt{\pi}} e^{x+kt} \int_{-\infty}^{\infty} e^{-p^2} \, dp = e^{kt+x}.$$

Notice that $u(x,t)$ grows with time, which may seem to be in contradiction with the maximum principle. However, thinking in terms of heat conduction, we see that the initial temperature $u(x,0) = e^x$ is itself infinitely large at the far right end of the rod $x = +\infty$. So the temperature does not grow out of nowhere, but rather gets transferred from right to left with the "speed" k. Thus the initial exponential distribution of the temperature "travels" from right to left with the speed k as t grows. Compare this to the example in Strauss, where the initial temperature $u(x,0) = e^{-x}$ "travels" from left to right, since the initial temperature peaks at the far left end $x = -\infty$. □

In the above example we were able to compute the solution explicitly, however, the integral in (10.4) may be impossible to evaluate completely in terms of elementary functions for general initial data $\phi(x)$. Due to this, the answers for particular problems are usually written in terms of the error function in statistics,

$$\mathcal{E}\text{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} \, dp.$$

Notice that $\mathcal{E}\text{rf}(0) = 0$, and $\lim_{x\to\infty} \mathcal{E}\text{rf}(x) = 1$. Using this function, we can rewrite the function $Q(x,t)$ given by (10.6), which solves the heat IVP with Heaviside initial data, as follows

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathcal{E}\text{rf}\left(\frac{x}{\sqrt{4kt}}\right).$$

10.3 Conclusion

Using the notions of distribution and distributional derivative, we can make sense of the heat kernel $S(x,t)$ that has the Dirac δ function as its initial data. Comparing the expression of the heat kernel (10.3) with the density function of the normal (Gaussian) distribution, we saw that the solution formula (10.2) essentially weights the initial data by the bell-shaped curve $S(x,t)$, thus giving the contribution from the initial heat at different points towards the temperature at point x at time t.

11 Comparison of wave and heat equations

In the last several lectures we solved the initial value problems associated with the wave and heat equations on the whole line $x \in \mathbb{R}$. We would like to summarize the properties of the obtained solutions, and compare the propagation of waves to conduction of heat.

Recall that the solution to the wave IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{11.1}$$

is given by d'Alambert's formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds. \tag{11.2}$$

Most of the properties of this solution can be deduced from the solution formula, which can be understood fairly well, if one thinks in terms of the characteristic coordinates. This is how we arrived at the properties of finite speed of propagation, propagation of discontinuities of the data along the characteristics, and others.

On the other hand, the solution to the heat IVP on the whole line

$$\begin{cases} u_t - k u_{xx} = 0, \\ u(x,0) = \phi(x), \end{cases} \tag{11.3}$$

is given by the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\, dy. \tag{11.4}$$

We saw some of the properties of the solutions to the heat IVP, for example the smoothing property, in the case of the fundamental solution or the heat kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}, \tag{11.5}$$

which had the Dirac delta function as its initial data. The solution u given by (11.4) can be written in terms of the heat kernel, and we use this to prove the properties for solutions to the general IVP (11.3). In terms of the heat kernel the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy = \int_{-\infty}^{\infty} S(z,t)\phi(x-z)\, dz,$$

where we made the change of variables $z = x - y$ to arrive at the last integral. Making a further change of variables $p = z/\sqrt{kt}$, the above can be written as

$$u(x,t) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-p^2/4}\phi(x - p\sqrt{kt})\, dp. \tag{11.6}$$

This last form of the solution will be handy when proving the smoothing property of the heat equation, the precise statement of which is contained in the following.

Theorem 11.1. *Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Then* (11.4) *defines an infinitely differentiable function $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$, which satisfies the heat equation, and* $\lim_{t\to 0+} u(x,t) = \phi(x)$, $\forall x \in \mathbb{R}$.

The proof is rather straightforward, and amounts to pushing the derivatives of $u(x,t)$ onto the heat kernel inside the integral. All one needs to guarantee for this procedure to go through is the uniform convergence of the resulting improper integrals. Let us first take a look at the solution itself given by (11.4). Notice that using the form in (11.6), we have

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left| e^{-p^2/4}\phi(x - p\sqrt{kt}) \right| dp \leq \frac{1}{\sqrt{4\pi}}(\max|\phi|) \int_{-\infty}^{\infty} e^{-p^2/4}\, dp = \max|\phi|,$$

which shows that u, given by the improper integral, is well-defined, since ϕ is bounded. One can also see the maximum principle in the above inequality. We will use similar logic to show that the improper integrals appearing in the derivatives of u converge uniformly in x and t.

Notice that formally

$$\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\, dy. \tag{11.7}$$

To make this rigorous, one must prove the uniform convergence of the integral. For this, we use expression (11.5) for the heat kernel to write

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\, dy &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[e^{-(x-y)^2/4kt}\right]\phi(y)\, dy \\ &= -\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{x-y}{2kt} e^{-(x-y)^2/4kt}\phi(y)\, dy.\end{aligned}$$

Making the change of variables $p = (x-y)/\sqrt{kt}$ in the above integral, we get

$$\int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\, dy = \frac{1}{4\sqrt{\pi kt}} \int_{-\infty}^{\infty} pe^{-p^2/4}\phi(x - p\sqrt{kt})\, dp \leq \frac{c}{\sqrt{t}}(\max|\phi|) \int_{-\infty}^{\infty} |p|e^{-p^2/4}\, dp,$$

where $c = 1/(4\sqrt{\pi k})$ is a constant. The last integral is finite, so the integral in the formal derivative (11.7) converges uniformly and absolutely for all $x \in \mathbb{R}$ and $t > \epsilon > 0$, where ϵ can be taken arbitrarily small. So the derivative $u_x = \partial u/\partial x$ exists and is given by (11.7).

The above argument works for the t derivative, and all the higher order derivatives as well, since for the n^{th} order derivatives one will end up with the integral $\int_{-\infty}^{\infty} |p|^n e^{-p^2/4}\, dp$, which is finite for all $n \in \mathbb{N}$. This proves the infinite differentiability of the solution, even though the initial data is only continuous.

We have already seen that u given by (11.4) solves the heat equation, due to the invariance properties. It then only remains to prove that $\lim_{t\to 0+} u(x,t) = \phi(x)$, $\forall x$. Recall that our previous proofs of this used the derivative of $\phi(x)$, or the language of distributions to employ the Dirac δ, where we assumed that ϕ is a test function, i.e. infinitely differentiable with compact support. To prove that $u(x,t)$ satisfies the initial condition in (11.3) in the case of continuous initial data ϕ as well, one can either use a *density argument*, in which $\phi(x)$ is uniformly approximated by smooth functions, and make a use of our earlier proofs, or provide a direct proof. The basic idea behind the direct proof is given next. We need to show that the difference $u(x,t) - \phi(x)$ becomes arbitrarily small when $t \to 0+$. First notice that

$$u(x,t) - \phi(x) = \int_{-\infty}^{\infty} S(x-y,t)[\phi(y) - \phi(x)]\, dy = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp, \tag{11.8}$$

where we used the same change of variables as before, $p = (x-y)/\sqrt{kt}$. To see that the last integral becomes arbitrarily small as t goes to zero, notice that if $p\sqrt{kt}$ is small, then $|\phi(x - p\sqrt{kt}) - \phi(x)|$ is small due to the continuity of ϕ, and the rest of the integral is finite. Otherwise, when $p\sqrt{kt}$ is large, then p is large, and the exponential in the integral becomes arbitrarily small, while the ϕ term is bounded. Thus, one estimates the above integral by breaking it into the following two integrals

$$\begin{aligned}&\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp \\ &\quad = \frac{1}{\sqrt{4\pi}} \int_{|p|<\delta/\sqrt{kt}} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp + \frac{1}{\sqrt{4\pi}} \int_{|p|\geq\delta/\sqrt{kt}} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp.\end{aligned}$$

For some small δ, the first integral is small due to the continuity of ϕ, while for arbitrarily small t the second integral is the tail of a converging improper integral, and is hence small. You should try to fill in the rigorous details. This completes the proof of Theorem 11.1.

It turns out, that the result in the above theorem can be proved even if the assumption of continuity of ϕ is relaxed to piecewise continuity. One then has the following.

Theorem 11.2. *Let $\phi(x)$ be a bounded piecewise-continuous function for $-\infty < x < \infty$. Then* (11.4) *defines an infinitely differentiable function $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$, which satisfies the heat equation, and*

$$\lim_{t\to 0+} u(x,t) = \frac{1}{2}[\phi(x+) + \phi(x-)], \qquad \text{for all } x \in \mathbb{R}, \tag{11.9}$$

where $\phi(x+)$ and $\phi(x-)$ stand for the right hand side and left hand side limits of ϕ at x.

Of course the fact that ϕ has jump discontinuities will not effect the convergence of the improper integrals encountered in the proof of Theorem 11.1. To see why (11.9) holds, notice that the integral in the right hand side of (11.6) can be broken into integrals over positive and negative half-lines

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{0} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp + \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp.$$

Then, since $p < 0$ in the first integral, $\phi(x - p\sqrt{kt})$ goes to $\phi(x+)$ as $t \to 0+$, while it goes to $\phi(x-)$ in the second integral, due to p being positive. So one can make the obvious changes in the proof of the previous theorem to show (11.9). This curious fact is one of the reasons why some people prefer to define the value of the Heaviside step function at $x = 0$ to be $H(0) = \frac{1}{2}$. Then one has

$$\lim_{t\to 0+} Q(x,t) = H(x) \qquad \text{for all } x \in \mathbb{R} \quad (\text{including } x = 0!),$$

where $Q(x,t)$ was the solution arising from the initial data given by $H(x)$.

11.1 Comparison of wave to heat

We now summarize and compare the fundamental properties of the wave and heat equations in the table below. Brief discussion of each of the properties will follow.

	Property	**Wave** ($u_{tt} - c^2 u_{xx} = 0$)	**Heat** ($u_t - k u_{xx} = 0$)
(i)	Speed of propagation	Finite (speed $\leq c$)	Infinite
(ii)	Singularities for $t > 0$	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$	Yes	Yes (for bounded solutions)
(iv)	Well-posed for $t < 0$	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behaviour as $t \to \infty$	Does not decay	Decays to zero (if ϕ is integrable)
(vii)	Information	Transported	Lost gradually

Let us now recall why each of the properties listed in the table holds or does not for each equation.

(i) Finite speed of propagation for the wave equation is immediately seen from d'Alambert's formula (11.2).

The infinite speed of propagation for the heat equation was seen in the example of the heat kernel, which is strictly positive for all $x \in \mathbb{R}$ for $t > 0$, but has Dirac δ function as its initial data, and hence is zero for all $x \neq 0$ initially.

(ii) We saw in the box-wave (initial displacement in the form of a box, no initial velocity) and the "hammer blow" (no initial displacement, initial box-shaped velocity) that singularities are preserved and are transported along the characteristics. The same is seen from (11.2).

For the heat equation we saw in the last section that the solution (11.4) is infinitely differentiable even for piecewise continuous initial data (this is true for even weaker conditions on ϕ).

(iii) Well-posedness for the wave IVP is seen immediately from d'Alambert's formula.

In the case of the heat equation, we proved uniqueness and stability using either the maximum principle, or alternatively, the energy method. Existence follows from our construction of the explicit solution (11.4).

(iv) For the wave equation, this follows from the invariance under time reversion. Indeed, if $u(x,t)$ is a solution, then so is $u(x,-t)$, which has data $(\phi(x), -\psi(x))$.

If we reverse the time in the heat equation, we get $u_t + ku_{xx} = 0, t > 0$. One can solve this equation in much the same way as the heat equation, and due to the symmetry in t, will get the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{(x-y)^2/4kt} \phi(y)\, dy,$$

which diverges for all $x \in \mathbb{R}$ (unless ϕ decays to zero faster than e^{-p^2}). So the heat equation is not well-posed *backward in time*. This makes physical sense as well, since the processes described by the heat equation, namely diffusion, heat flow and random motion are irreversible processes.

(v) The fact that there is no maximum principle for the wave equation is apparent from the "hammer blow" example, where the solution was everywhere zero initially, but due to the nonzero initial velocity, had nonzero displacement for any time $t > 0$. For the heat equation, the maximum principle was proved rigorously in a previous lecture.

(vi) We saw that the energy is conserved for the wave equation, so the solutions do not decay. We also saw this in the box-wave example, in which the initial box-shaped data split into two box-shaped waves of half the height that traveled in opposite directions without changing the shape.

For the heat equation, the decay is seen from formula (11.4), since $S(x-y,t) \to 0$ as $t \to \infty$, and the integral will be bounded if ϕ is integrable. Notice that in the example we considered in the last lecture, with $\phi(x) = e^x$, the solution did not decay, but rather "traveled" from right to left. This was due to ϕ being non-integrable.

(vii) The fact that information is transported by the solutions of the wave equation is seen from the fact that the initial data is propagated along the characteristics. So the information will travel along the characteristics as well.

In the case of the heat equation, the information is gradually lost, which can be seen from the graph of a typical solution (think of the heat kernel). The heat from the higher temperatures gets dissipated and after a while it is not clear what the original temperatures were.

11.2 Conclusion

Although the wave and heat equations are both second order linear constant coefficient PDEs, their respective solutions posses very different properties. By now we have learned how to solve the initial value problems on the whole line for both of these equations, and understood these solutions in terms of the physics behind the corresponding problems. We also saw that the properties of the solutions of the respective equations correspond to our intuition for each of the physical phenomena described by the equations.

Problem Set 6

1. (#2.4.4 in [Str]) Solve the heat equation if $\phi(x) = e^{-x}$ for $x > 0$, and $\phi(x) = 0$ for $x < 0$.

2. (#2.4.11 (a) in [Str]) Consider the heat equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)

3. (#2.4.15 in [Str]) Prove the uniqueness of the heat problem with Neumann boundary conditions:

$$\begin{cases} u_t - ku_{xx} = f(x,t) \quad \text{for } 0 < x < l, t > 0, \\ u(x,0) = \phi(x), \\ u_x(0,t) = g(t), \quad u_x(l,t) = h(t), \end{cases}$$

by the energy method.

4. (#2.4.16 in [Str]) Solve the initial value problem for the diffusion equation with constant dissipation:

$$\begin{cases} u_t - ku_{xx} + bu = 0 \quad \text{for } -\infty < x < \infty, \\ u(x,0) = \phi(x), \end{cases}$$

where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x, t) = e^{-bt}v(x, t)$.)

5. (#2.5.4 in [Str]) Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u(x,s)\, ds.$$

   (a) Show that $v(x, t)$ solves the diffusion equation!

   (b) Show that $\lim_{t\to 0} v(x, t) = u(x, 0)$.

   (*Hint:* (a) Write the formula as $v(x, t) = \int_{-\infty}^{\infty} H(s, t)u(x, s)\, ds$, where $H(x, t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x, t)$, assuming that you can freely differentiate inside the integral. (b) Use the fact that $H(s, t)$ is essentially the diffusion (heat) kernel with the spatial variable s. You can use the fact that the diffusion kernel has the Dirac delta function as its initial data.)

6. Solve the heat equation with the initial data $\phi(x) = \delta(x - 2) + 3\delta(x)$.

7. Show that the distributional derivative of the Dirac delta function acts on test functions as

$$\delta'[f] = -f'(0).$$

In previous lectures we completely solved the initial value problem for the heat equation on the whole line, i.e. in the absence of boundaries. Next, we turn to problems with physically relevant boundary conditions. Let us first add a boundary consisting of a single endpoint, and consider the heat equation on the half-line $D = (0, \infty)$. The following initial/boundary value problem, or IBVP, contains a Dirichlet boundary condition at the endpoint $x = 0$.

$$\begin{cases} v_t - kv_{xx} = 0, \quad 0 < x < \infty, 0 < t < \infty, \\ v(x,0) = \phi(x), \quad x > 0, \\ v(0,t) = 0, \quad t > 0. \end{cases} \tag{12.1}$$

If the solution to the above mixed initial/boundary value problem exists, then we know that it must be unique from an application of the maximum principle. In terms of the heat conduction, one can think of v in (12.1) as the temperature in an infinite rod, one end of which is kept at a constant zero temperature. The initial temperature of the rod is then given by $\phi(x)$.

Our goal is to solve the IBVP (12.1), and derive a solution formula, much like what we did for the heat IVP on the whole line. But instead of constructing the solution from scratch, it makes sense to try to reduce this problem to the IVP on the whole line, for which we already have a solution formula. This is achieved by extending the initial data $\phi(x)$ to the whole line. We have a choice of how exactly to extend the data to the negative half-line, and one should try to do this in such a fashion that the boundary condition of (12.1) is automatically satisfied by the solution to the IVP on the whole line that arises from the extended data. This is the case, if one chooses the *odd extension* of $\phi(x)$, which we describe next.

By the definition a function $\psi(x)$ is odd, if $\psi(-x) = -\psi(x)$. But then plugging in $x = 0$ into this definition, one gets $\psi(0) = 0$ for any odd function. Recall also that the solution $u(x,t)$ to the heat IVP with odd initial data is itself odd in the x variable. This follows from the fact that the sum $[u(x,t) + u(-x,t)]$ solves the heat equation and has zero initial data, hence, it is the identically zero function by the uniqueness of solutions. Then, by our above observation for odd functions, we would have that $u(0,t) = 0$ for any $t > 0$, which is exactly the boundary condition of (12.1).

This shows that if one extends $\phi(x)$ to an odd function on the whole line, then the solution with the extended initial data automatically satisfies the boundary condition of (12.1). Let us then define

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ -\phi(-x) & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{12.2}$$

It is clear that ϕ_{odd} is an odd function, since we defined it for negative x by reflecting the $\phi(x)$ with respect to the vertical axis, and then with respect to the horizontal axis. This procedure produces a function whose graph is symmetric with respect to the origin, and thus it is odd. One can also verify this directly from the definition of odd functions. Now, let $u(x,t)$ be the solution of the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{odd}}(x). \end{cases} \tag{12.3}$$

From previous lectures we know that the solution to (12.3) is given by the formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)\,dy, \qquad t > 0. \tag{12.4}$$

Restricting the x variable to only the positive half-line produces the function

$$v(x,t) = u(x,t)\big|_{x \geq 0}. \tag{12.5}$$

We claim that this $v(x,t)$ is the unique solution of IBVP (12.1). Indeed, $v(x,t)$ solves the heat equation on the positive half-line, since so does $u(x,t)$. Furthermore,

$$v(x,0) = u(x,0)\Big|_{x>0} = \phi_{\text{odd}}(x)\Big|_{x>0} = \phi(x),$$

and $v(0,t) = u(0,t) = 0$, since $u(x,t)$ is an odd function of x. So $v(x,t)$ satisfies the initial and boundary conditions of (12.1).

Returning to formula (12.4), we substitute the expressions for ϕ_{odd} from (12.2) and write

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\phi_{\text{odd}}(y)\,dy + \int_{-\infty}^0 S(x-y,t)\phi_{\text{odd}}(y)\,dy \\ &= \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_{-\infty}^0 S(x-y,t)\phi(-y)\,dy. \end{aligned}$$

Making the change of variables $y \mapsto -y$ in the second integral on the right, and flipping the integration limits gives

$$u(x,t) = \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_0^\infty S(x+y,t)\phi(y)\,dy.$$

Using (12.5) and the above expression for $u(x,t)$, as well as the expression of the heat kernel $S(x,t)$, we can write the solution formula for the IBVP (12.1) as follows

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right] \phi(y)\,dy. \tag{12.6}$$

The method used to arrive at this solution formula is called the *method of odd extensions* or the *reflection method*. We can make a physical sense of formula (12.6) by interpreting the integrand as the contribution from the point y minus the heat loss from this point due to the constant zero temperature at the endpoint.

Example 12.1. Solve the IBVP (12.1) with the initial data $\phi(x) = e^x$.

Using the solution formula (12.6), we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} e^y - e^{-(x+y)^2/4kt} e^y \right] dy. \tag{12.7}$$

Combining the exponential factors of the first product under the integral, we will get an exponential with the following exponent

$$\frac{-[y^2 - 2(x+2kt)y + x^2]}{4kt} = -\left(\frac{y-(x+2kt)}{\sqrt{4kt}}\right)^2 + kt - x = -p^2 + kt + x,$$

where we made the obvious notation

$$p = \frac{y-x-2kt}{\sqrt{4kt}}.$$

Similarly, the exponent of the combined exponential from the second product under integral (12.7) is

$$\frac{-[y^2 + 2(x-2kt)y + x^2]}{4kt} = -\left(\frac{y+x-2kt}{\sqrt{4kt}}\right)^2 + kt - x = -q^2 + kt - x,$$

with

$$q = \frac{y+x-2kt}{\sqrt{4kt}}.$$

Braking integral (12.7) into a difference of two integrals, and making the changes of variables $y \mapsto p$, and $y \mapsto q$ in the respective integrals, we will get

$$v(x,t) = e^{kt+x}\frac{1}{\sqrt{\pi}}\int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\,dp - e^{kt-x}\frac{1}{\sqrt{\pi}}\int_{\frac{x-2kt}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,dq. \tag{12.8}$$

Notice that

$$\frac{1}{\sqrt{\pi}}\int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\,dp = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_{\frac{-x-2kt}{\sqrt{4kt}}}^{0} e^{-p^2}\,dp = \frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right),$$

and similarly for the second integral. Putting this back into (12.8), we will arrive at the solution

$$v(x,t) = e^{kt+x}\left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right] - e^{kt-x}\left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right].$$

□

12.1 Neumann boundary condition

Let us now turn to the Neumann problem on the half-line,

$$\begin{cases} w_t - kw_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), & x > 0 \\ w_x(0,t) = 0, & t > 0. \end{cases} \tag{12.9}$$

To find the solution of (12.9), we employ a similar idea used in the case of the Dirichlet problem. That is, we seek to reduce the IBVP to an IVP on the whole line by extending the initial data $\phi(x)$ to the negative half-axis in such a fashion that the boundary condition is automatically satisfied.

Notice that if $\psi(s)$ is an even function, i.e. $\psi(-x) = \psi(x)$, then its derivative function will be odd. Indeed, differentiating in the definition of the even function, we get $-\psi'(-x) = \psi'(x)$, which is the same as $\psi'(-x) = -\psi'(x)$. Hence, for an arbitrary even function $\psi(x)$, $\psi'(0) = 0$. It is now clear that extending the initial data so that the resulting function is even will produce solutions to the IVP on the whole line that automatically satisfy the Neumann condition of (12.9).

We define the even extension of $\phi(x)$,

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \tag{12.10}$$

and consider the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x). \end{cases} \tag{12.11}$$

It is clear that the solution $u(x,t)$ of the IVP (12.11) will be even in x, since the difference $[u(-x,t) - u(x,t)]$ solves the heat equation and has zero initial data. We then use the solution formula for the IVP on the whole line to write

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\,dy, \qquad t > 0, \tag{12.12}$$

and take

$$w(x,t) = u(x,t)\big|_{x\geq 0},$$

similar to the case of the Dirichlet problem. One can show that this $w(x,t)$ solves the IBVP (12.9), and use the expression for the heat kernel, as well as the definition (12.10), to write the solution formula as follows

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right] \phi(y)\, dy. \tag{12.13}$$

Notice that the formulas (12.6) and (12.13) differ only by the sign between the two exponential terms inside the integral.

In terms of heat conduction, the Neumann condition in (12.9) means that there is no heat exchange between the rod and the environment (recall that the heat flux is proportional to the spatial derivative of the temperature). The physical interpretation of formula (12.13) is that the integrand is the contribution of $\phi(y)$ plus an additional contribution, which comes from the lack of heat transfer to the points of the rod with negative coordinates.

Example 12.2. Solve the IBVP (12.9) with the initial data $\phi(x) \equiv 1$.

Using the formula (12.13), we can write the solution as

$$\begin{aligned} w(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2}\, dq, \end{aligned}$$

where we made the changes of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \text{and} \qquad q = \frac{y+x}{\sqrt{4kt}}.$$

Using the same idea as in the previous example, we can write the solution in terms of the $\mathcal{E}\mathrm{rf}$ function as follows

$$w(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] + \left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] \equiv 1.$$

So the solution is identically 1, which is clear if one thinks in terms of heat conduction. Indeed, problem (12.9) describes the temperature dynamics with identically 1 initial temperature, and no heat loss at the endpoint. Obviously there is no heat transfer between points of equal temperature, so the temperatures remain steady along the entire rod. $\square$.

12.2 Conclusion

We derived the solution to the heat equation on the half-line by reducing the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In the case of zero Dirichlet boundary condition the odd extension of the initial data automatically guarantees that the solution will satisfy the boundary condition. While for the case of zero Neumann boundary condition the appropriate choice is the even extension. This reflection method relies on the fact that the solution to the heat equation on the whole line with odd initial data is odd, while the solution with even initial data is even.

Similar to the last lecture on the heat equation on the half-line, we will use the reflection method to solve the boundary value problems associated with the wave equation on the half-line $0 < x < \infty$. Let us start with the Dirichlet boundary condition first, and consider the initial boundary value problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), & x > 0, \\ v(0,t) = 0, \quad t > 0. \end{cases} \tag{13.1}$$

For the vibrating string, the boundary condition of (13.1) means that the end of the string at $x = 0$ is held fixed. We reduce the Dirichlet problem (13.1) to the whole line by the reflection method. The idea is again to extend the initial data, in this case ϕ, ψ, to the whole line, so that the boundary condition is automatically satisfied for the solutions of the IVP on the whole line with the extended initial data. Since the boundary condition is in the Dirichlet form, one must take the odd extensions

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) & \text{for } x < 0. \end{cases} \qquad \psi_{\text{odd}}(x) = \begin{cases} \psi(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\psi(-x) & \text{for } x < 0. \end{cases} \tag{13.2}$$

Consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{odd}}(x), u_t(x,0) = \psi_{\text{odd}}(x). \end{cases} \tag{13.3}$$

Since the initial data of the above IVP are odd, we know from a homework problem that the solution of the IVP, $u(x,t)$, will also be odd in the x variable, and hence $u(0,t) = 0$ for all $t > 0$. Then defining the restriction of $u(x,t)$ to the positive half-line $x \geq 0$,

$$v(x,t) = u(x,t)\big|_{x \geq 0}, \tag{13.4}$$

we automatically have that $v(0,t) = u(0,t) = 0$. So the boundary condition of the Dirichlet problem (13.1) is satisfied for v. Obviously the initial conditions are satisfied as well, since the restrictions of $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$ to the positive half-line are $\phi(x)$ and $\psi(x)$ respectively. Finally, $v(x,t)$ solves the wave equation for $x > 0$, since $u(x,t)$ satisfies the wave equation for all $x \in \mathbb{R}$, and in particular for $x > 0$. Thus, $v(x,t)$ defined by (13.4) is a solution of the Dirichlet problem (13.1). It is clear that the solution must be unique, since the odd extension of the solution will solve IVP (13.3), and therefore must be unique.

Using d'Alambert's formula for the solution of (13.3), and taking the restriction (13.4), we have that for $x \geq 0$,

$$v(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)\, ds. \tag{13.5}$$

Notice that if $x \geq 0$ and $t > 0$, then $x+ct > 0$, and $\phi_{\text{odd}}(x+ct) = \phi(x+ct)$. If in addition $x-ct > 0$, then $\phi_{\text{odd}}(x-ct) = \phi(x-ct)$, and over the interval $s \in [x-ct, x+ct]$, $\psi_{odd}(s) = \psi(s)$. Thus, for $x > ct$, we have

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds, \tag{13.6}$$

which is exactly d'Alambert's formula.

For $0 < x < ct$, the argument $x - ct < 0$, and using (13.2) we can rewrite the solution (13.5) as

$$v(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\left[\int_{x-ct}^{0} -\psi(-s)\, ds + \int_{0}^{x+ct} \psi(s)\, ds\right].$$

Making the change of variables $s \mapsto -s$ in the first integral on the right, we get

$$v(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(s)\,ds + \int_{0}^{x+ct} \psi(s)\,ds\right]$$
$$= \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds.$$

One could also use the fact that the integral of the odd function $\psi_{\text{odd}}(s)$ over the symmetric interval $[x-ct, ct-x]$ is zero, thus $\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)\,ds = \int_{ct-x}^{x+ct} \psi(s)\,ds$.

The two different cases giving different expressions are illustrated in Figures 13.1 and 13.2 below. Notice how one of the characteristics from a point with $x_0 < ct_0$ gets reflected from the "wall" at $x = 0$ in Figure 13.2.

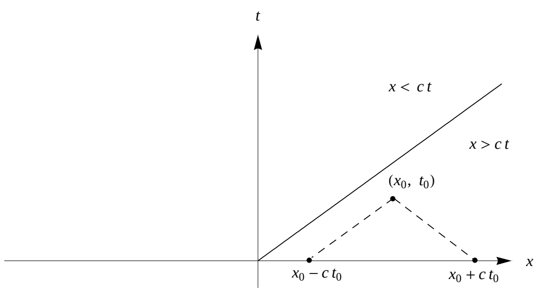

Figure 13.1: The case with $x_0 > ct_0$.

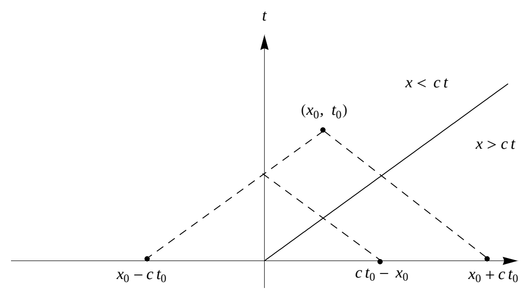

Figure 13.2: The case with $x_0 < ct_0$.

Combining the two expressions for $v(t,x)$ over the two regions, we arrive at the solution

$$v(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, & \text{for } x > ct \\ \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds, & \text{for } 0 < x < ct. \end{cases} \tag{13.7}$$

The minus sign in front of $\phi(ct-x)$ in the second expression above, as well as the reduction of the integral of ψ to the smaller interval are due to the cancellation stemming from the *reflected wave*. The next example illustrates this phenomenon.

Example 13.1. Solve the Dirichlet problem (13.1) with the following initial data

$$\phi(x) = \begin{cases} h & \text{for } a < x < 2a, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \psi(x) \equiv 0.$$

The initial data defines a box-like displacement over the interval $(a, 2a)$ and zero initial velocity for the string. It is clear that for some small time the wave will propagate just like in the case of the IVP on the whole line, i.e. without "seeing" the boundary. This is due to the finite speed of propagation property of the wave equation, according to which it takes some time, specifically a/c time for the initial displacement to reach the boundary. Thus, we expect that the box-like wave will break into two box-waves, each with half the height of the initial box-like displacement, which travel with speed c in opposite directions. The box-wave traveling in the right direction will never hit the boundary at $x = 0$, and will continue traveling unaltered for all time. However, the left box-wave hits the wall (the fixed end of the vibrating string), and gets reflected in an odd fashion, that is the displacement gets the minus sign in the second expression of (13.7).

To find the values of the solution at any point (x_0, t_0), we draw the backward characteristics from that point and trace the point back to the x axis, where the initial data is defined. Since the initial data is

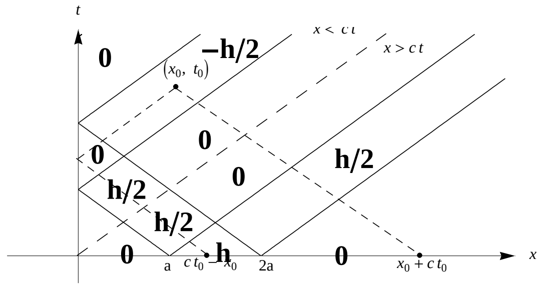

Figure 13.3: The values of $v(x,t)$ carried forward in time by the characteristics.

nonzero only over the interval $(a, 2a)$, only those characteristics that hit the x axis between the points a and $2a$ will carry nonzero values forward in time. Notice also that if a characteristic hits the interval $(a, 2a)$ after being reflected from the wall $x = 0$, then the value it gets must be taken with a minus sign.

The different values determined by this method are illustrated in Figure 13.3 above. Notice how the left box-wave with height $h/2$ flips after hitting the boundary, and travels in the opposite direction with negative height $-h/2$.

One can then use the values from Figure 13.3 to draw the profile of the string at any time t. □

13.1 Neumann boundary condition

For the Neumann problem on the half-line,

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), & x > 0, \\ w_x(0,t) = 0, \quad t > 0, \end{cases} \tag{13.8}$$

we use the reflection method with even extensions to reduce the problem to an IVP on the whole line. Define the even extensions of the initial data

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \qquad \psi_{\text{even}} = \begin{cases} \psi(x) & \text{for } x \geq 0, \\ \psi(-x) & \text{for } x \leq 0. \end{cases} \tag{13.9}$$

and consider the following IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x), \quad u_t(x,0) = \psi_{\text{even}}(x). \end{cases} \tag{13.10}$$

Clearly, the solution $u(x,t)$ to the IVP (13.10) will be even in x, and since the derivative of an even function is odd, $u_x(x,t)$ will be odd in x, and hence $u_x(0,t) = 0$ for all $t > 0$. Similar to the case of the Dirichlet problem, the restriction

$$w(x,t) = u(x,t)\big|_{x \geq 0}$$

will be the unique solution of the Neumann problem (13.8).

Using d'Alambert's formula for the solution $u(x,t)$ of (13.10), and taking the restriction to $x \geq 0$, we get

$$w(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(s)\, ds. \tag{13.11}$$

One again needs to consider the two cases $x > ct$ and $0 < x < ct$ separately. Notice that with the even extensions we will get additions, rather than cancellations. Using the definitions (13.9), the solution (13.11) can be written as

$$w(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds, & \text{for } x > ct \\ \frac{1}{2}[\phi(x+ct) + \phi(ct-x)] & \\ \qquad + \frac{1}{2c}\left[\int_0^{ct-x} \psi(s)\, ds + \int_0^{x+ct} \psi(s)\, ds\right], & \text{for } 0 < x < ct. \end{cases}$$

The Neumann boundary condition corresponds to a vibrating string with a free end at $x = 0$, since the string tension, which is proportional to the derivative $v_x(x, t)$, vanishes at $x = 0$. In this case the reflected wave adds to the original wave, rather than canceling it.

Example 13.2. Solve the Neumann problem (13.8) with the following initial data

$$\phi(x) = \begin{cases} h & \text{for } a < x < 2a, \\ 0 & \text{otherwise,} \end{cases} \qquad \psi(x) \equiv 0.$$

The initial data is exactly the same as in the previous example for the Dirichlet problem. The only difference from the Dirichlet case is that the free end reflects the wave with a plus sign. The values carried forward in time by the characteristics are determined in the same way as before. Figure 13.4 illustrates this method. Notice that the reflected wave has the same (positive) height $h/2$ as the wave right before the reflection.

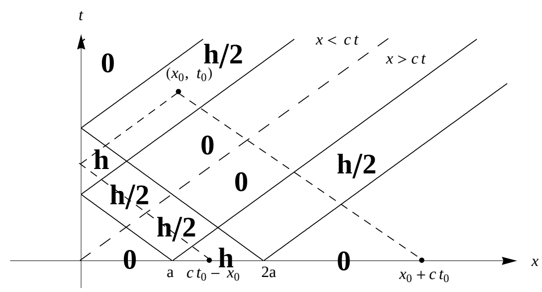

Figure 13.4: The values of $v(x, t)$ carried forward in time by the characteristics.

One can again use the values from Figure 13.4 to draw the profile of the string at any time t. □

13.2 Conclusion

We derived the solution to the wave equation on the half-line in much the same way as was done for the heat equation. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In this case the characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall $x = 0$ carry the values of the initial data with a minus sign in the case of the Dirichlet boundary condition, and with a plus sign in the case of the Neumann boundary condition. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.

14 Waves on the finite interval

In the last lecture we used the reflection method to solve the boundary value problem for the wave equation on the half-line. We would like to apply the same method to the boundary value problems on the finite interval, which correspond to the physically realistic case of a finite string. Consider the Dirichlet wave problem on the finite line

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < l, 0 < t < \infty, \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \quad x > 0, \\ v(0,t) = v(l,t) = 0, \quad t > 0. \end{cases} \tag{14.1}$$

The homogeneous Dirichlet conditions correspond to the vibrating string having fixed ends, as is the case for musical instruments. Using our intuition from the half-line problems, where the wave reflects from the fixed end, we can imagine that in the case of the finite interval the wave bounces back and forth infinitely many times between the endpoints. In spite of this, we can still use the reflection method to find the value of the solution to problem (14.1) at any point (x,t).

Recall that the idea of the reflection method is to extend the initial data to the whole line in such a way, that the boundary conditions are automatically satisfied. For the homogeneous Dirichlet data the appropriate choice is the odd extension. In this case, we need to extend the initial data ϕ, ψ, which are defined only on the interval $0 < x < l$, in such a way that the resulting extensions are odd with respect to both $x = 0$, and $x = l$. That is, the extensions must satisfy

$$f(-x) = -f(x) \qquad \text{and} \qquad f(l-x) = -f(l+x). \tag{14.2}$$

Notice that for such a function $f(0) = -f(0)$ from the first condition, and $f(l) = -f(l)$ from the second condition, hence, $f(0) = f(l) = 0$. Subsequently, the solution to the IVP with such data will be odd with respect to both $x = 0$ and $x = l$, and the boundary conditions will be automatically satisfied. Notice that the conditions (14.2) imply that functions that are odd with respect to both $x = 0$ and $x = l$ satisfy $f(2l + x) = -f(-x) = f(x)$, which means that such functions must be $2l-$periodic. Using this we can define the extensions of the initial data ϕ, ψ as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < l, \\ -\phi(-x) & \text{for } -l < x < 0, \\ \text{extended to be } 2l-\text{periodic}, \end{cases} \qquad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{for } 0 < x < l, \\ -\psi(-x) & \text{for } -l < x < 0, \\ \text{extended to be } 2l-\text{periodic}. \end{cases} \tag{14.3}$$

Now, consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{ext}}(x), u_t(x,0) = \psi_{\text{ext}}(x). \end{cases} \tag{14.4}$$

For the solution of this IVP we automatically have $u(0,t) = u(l,t) = 0$, and the restriction

$$v(x,t) = u(x,t)\big|_{0 \le x \le l},$$

will solve the boundary value problem (14.1). By d'Alambert's formula, the solution will be given as

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds \tag{14.5}$$

for $0 < x < l$. Although formula (14.5) contains all the information about our solution, we would like to have an expression in terms of the original initial data, so that the values of the solution can be directly computed using the given functions $\phi(x)$ and $\psi(x)$. For this, we need to "bring" the points $x - ct$ and $x + ct$ into the interval $(0, l)$ using the periodicity and oddity of the extended data. To illustrate how this is done, let us fix a point (x,t) and try to find the value of the solution at this point by tracing it

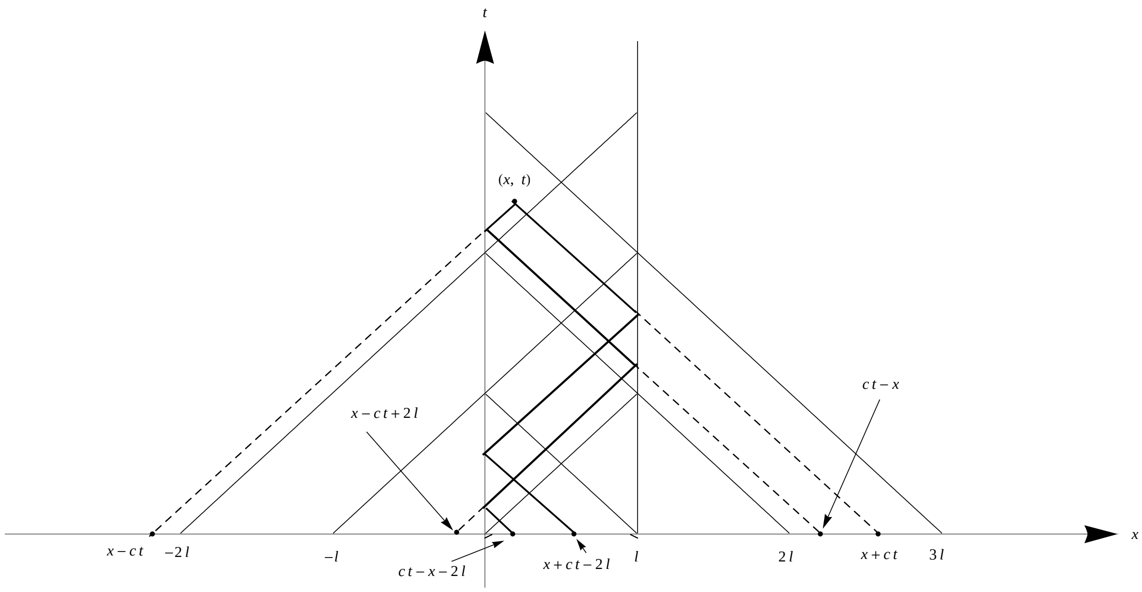

Figure 14.1: The backwards characteristics from the point (x, t).

back in time along the characteristics to the initial data. The sketch of the backwards characteristics from this point appears in Figure 14.1 above.

In general, the points $x+ct$ and $x-ct$ will end up either in the interval $(0, l)$ or $(-l, 0)$ after finitely many translations by the period $2l$. If the point ends up in $(0, l)$ (even number of reflections), then the value of the initial data picked up by the reflected characteristic will be taken with a positive sign. If, however, the point ends up in the interval $(-l, 0)$ (odd number of reflections), then we need to reflect this point with respect to $x = 0$, and the corresponding value of the initial data will be taken with a negative sign.

From Figure 14.1 we see that $x + ct$ goes into the interval $(0, l)$ (2 reflections) after translating it to the left by one period $2l$, but the point $x - ct$ goes into the interval $(-l, 0)$ (3 reflections) after a right translation by $2l$, so we need to reflect the resulting point $x - ct + 2l$ to arrive at the point $ct - x - 2l$ in the interval $(0, l)$. The solution will then be

$$u(x, t) = \frac{1}{2}[\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds.$$

For the integral term, we can break it into two integrals as follows

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds = \frac{1}{2c}\int_{x-ct}^{ct-x} \psi_{\text{ext}}(s)\, ds + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi_{\text{ext}}(s)\, ds.$$

Notice that from the oddity of ψ_{ext}, the integral over the interval $[x - ct, ct - x]$ will be zero, while by periodicity, we can bring the interval $[ct - x, x + ct]$ into the interval $(0, l)$ by subtracting one period $2l$. Thus, the solution can be written as

$$u(x, t) = \frac{1}{2}[\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c}\int_{ct-x-2l}^{x+ct-2l} \psi(s)\, ds. \tag{14.6}$$

Clearly, the derivation of the above expression for the solution depends on the chosen point, which in turn determines how many reflections the backward characteristics undergo before arriving at the x axis. Hence, the solution will be given by different expressions, depending on the region from which the point is taken. These different regions are depicted in Figure 14.2, where the labels (m, n) show how many times each of the two backward characteristics gets reflected before reaching the x axis. Expression (14.6) will be valid for all the points in the region $(3, 2)$.

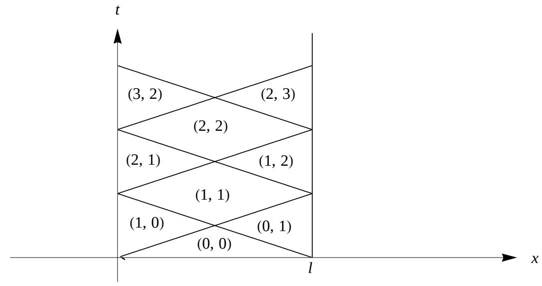

Figure 14.2: Regions of $(x, t) \in (0, l) \times (0, \infty)$ with the different number of reflections.

The method used to arrive at the expression (14.6) can be used to find the value of the solution at any point (x, t), although it is quite impractical to derive the expression for each of the regions depicted in Figure 14.2. Furthermore, it does not generalize to higher dimensions, nor does it apply to the heat equation (no characteristics to trace back). Later we will study another method, which allows for a more general way of approaching boundary value problems on finite intervals.

Example 14.1. Consider the Dirichlet wave problem on the finite interval

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad \text{for } 0 < x < 1, \\ u(x, 0) = x(1 - x), u_t(x, 0) = x^2, \\ u(0, t) = u(1, t) = 0. \end{cases}$$

Find the value of the solution at the point $(\frac{3}{4}, \frac{5}{2})$.

Notice that in this problem $c = 1$, and $l = 1$, so the period of the extended data will be $2l = 2$. The sketch of the backward characteristics from the point $(x, t) = (\frac{3}{4}, \frac{5}{2})$ appears in the figure below.

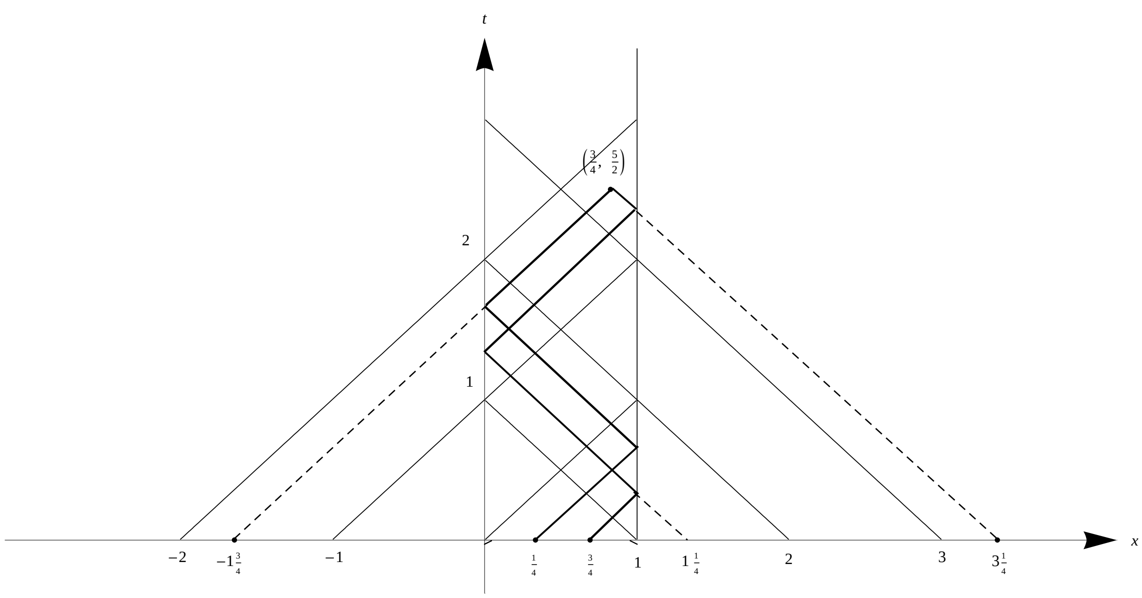

Figure 14.3: The backwards characteristics from the point $(\frac{3}{4}, \frac{5}{2})$.

The characteristics intersect the x axis at the points

$$x - t = \tfrac{3}{4} - \tfrac{5}{2} = -1\tfrac{3}{4} \qquad \text{and} \qquad x + t = \tfrac{3}{4} + \tfrac{5}{2} = 3\tfrac{1}{4}.$$

The point $-1\frac{3}{4}$ goes to the point $\frac{1}{4}$ after a right translation by one period, while the point $3\frac{1}{4}$ goes to the point $1\frac{1}{4}$ after a left translation by one period. After a reflection with respect to $x = 1$, this point will end up at $\frac{3}{4}$, thus, the value of the initial data must be taken with a negative sign at this point. Also, the integral over the interval $[\frac{3}{4}, 1\frac{1}{4}]$ of ψ_{ext} will be zero due to its oddity with respect to $x = 1$. The value of the solution is then

$$u(\tfrac{3}{4}, \tfrac{5}{2}) = \frac{1}{2}[-\phi(\tfrac{3}{4}) + \phi(\tfrac{1}{4})] + \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} \psi(s)\, ds = \frac{1}{2}\left[-\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4}\right] + \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} x^2\, dx$$

$$= \left.\frac{x^3}{6}\right|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{6}\left(\frac{27}{64} - \frac{1}{64}\right) = \frac{13}{192}.$$

□

14.1 The parallelogram rule

Recall from a homework problem, that for the vertices of a characteristic parallelogram A, B, C and D as for example in Figure 14.4, the values of the solution of the wave equation are related as follows

$$u(A) + u(C) = u(B) + u(D).$$

Hence, we can find the value at the vertex A from the values at the three other vertices.

$$u(A) = u(B) + u(D) - u(C).$$

Notice that the values at the vertices B and C in Figure 14.4 can be found from the expression of the solution for the region $(0, 0)$, while the value at D comes from the boundary data. Thus we reduced finding the value at a point in the region $(1, 0)$ to finding values in the region $(0, 0)$. One can always follow this procedure to evaluate the solution in the regions $(m + 1, n)$ and $(m, n + 1)$ via the values in the region (m, n), provided the boundary condition is in the Dirichlet form.

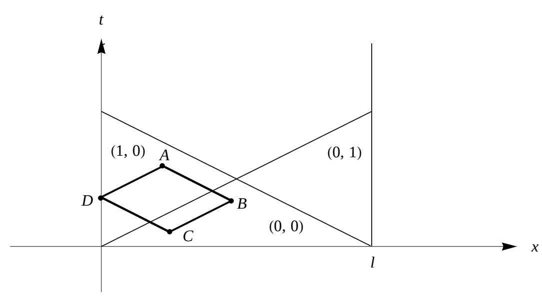

Figure 14.4: The parallelogram rule.

14.2 Conclusion

We applied the reflection method to derive expressions for the solution to the Dirichlet wave problem on the finite interval. However, the method yields infinitely many expressions for different regions in $(x, t) \in (0, l) \times (0, \infty)$, depending on the number of times the backward characteristics from a point get reflected before reaching the x axis, where the initial data is defined. This makes the method impractical in applications, and is not generalizable to higher dimensions and other PDEs. An alternative method (separation of variables) of solving boundary value problems on the finite interval will be described later in the course.

Problem Set 7

1. (#3.1.1 in [Str]) Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.

2. (#3.1.2 in [Str]) Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.

3. Solve the following Neumann problem for the heat equation on the half-line

$$\begin{cases} u_t - ku_{xx} = 0 & \text{in } 0 < x < \infty, t > 0, \\ u(x,0) = \delta(x-2), \\ u_x(0,t) = 0, \end{cases}$$

   with δ being the Dirac delta function. Explain your solution in terms of heat propagation.

4. (#3.2.5 in [Str]) Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) \equiv 1$, $u_t(x,0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.

5. Consider the Dirichlet problem for the "hammer blow" on the half line

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 & \text{in } 0 < x < \infty, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \\ u(0,t) = 0, \end{cases} \qquad \text{with} \qquad \begin{array}{l} \phi(x) \equiv 0, \\ \psi(x) = \begin{cases} h, & a < x < 2a, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

   Sketch the regions in the xt plane corresponding to the different expressions for the solution. Draw the profile of the string at several times $a/2c$, a/c, $3a/2c$, $5a/2c$ to understand how the wave gets reflected from the boundary.

6. (#3.2.9 in [Str])

   (a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1$, $u(x,0) = x^2(1-x)$, $u_t(x,0) = (1-x)^2$, $u(0,t) = u(1,t) = 0$.

   (b) Find $u(\frac{1}{4}, \frac{7}{2})$.

7. (#3.2.10 in [Str]) Solve the initial boundary value problem on the finite interval (Neumann condition at the left endpoint, Dirichlet condition at the right endpoint)

$$\begin{cases} u_{tt} = 9u_{xx} & \text{in } 0 < x < \pi/2, \\ u(x,0) = \cos x, u_t(x,0) = 0, \\ u_x(0,t) = 0, u(\pi/2,t) = 0. \end{cases}$$

15 Heat with a source

So far we considered homogeneous wave and heat equations and the associated initial value problems on the whole line, as well as the boundary value problems on the half-line and the finite line (for wave only). The next step is to extend our study to the inhomogeneous problems, where an external heat source, in the case of heat conduction in a rod, or an external force, in the case of vibrations of a string, are also accounted for. We first consider the inhomogeneous heat equation on the whole line,

$$\begin{cases} u_t - ku_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{15.1}$$

where $f(x,t)$ and $\phi(x)$ are arbitrary given functions. The right hand side of the equation, $f(x,t)$ is called the *source* term, and measures the physical effect of an external heat source. It has units of heat flux (left hand side of the equation has the units of u_t, i.e. change in temperature per unit time), thus it gives the instantaneous temperature change due to an external heat source.

From the superposition principle, we know that the solution of the inhomogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the inhomogeneous equation. We can thus break problem (15.1) into the following two problems

$$\begin{cases} u^h_t - ku^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \end{cases} \tag{15.2}$$

and

$$\begin{cases} u^p_t - ku^p_{xx} = f(x,t), \\ u^p(x,0) = 0. \end{cases} \tag{15.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (15.1).

Notice that we solve for the general solution of the homogeneous equation with arbitrary initial data in (15.2), while in the second problem (15.3) we solve for a particular solution of the inhomogeneous equation, namely the solution with zero initial data. This reduction of the original problem to two simpler problems (homogeneous, and inhomogeneous with zero data) using the superposition principle is a standard practice in the theory of linear PDEs.

We have solved problem (15.2) before, and arrived at the solution

$$u^h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \tag{15.4}$$

where $S(x,t)$ is the heat kernel. Notice that the physical meaning of expression (15.4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out, too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (15.3) is given by

$$u^p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds. \tag{15.5}$$

Notice that the time integration is only over the interval $[0,t]$, since the heat contribution at later times can not effect the temperature at time t. Combining (15.4) and (15.5) we obtain the following solution to the IVP (15.1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds, \tag{15.6}$$

or, substituting the expression of the heat kernel,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4k(t-s)}}{\sqrt{4\pi k(t-s)}} f(y,s)\, dy\, ds.$$

One can draw parallels between formula (15.6) and the solution to the inhomogeneous ODE analogous to the heat equation. Indeed, consider the IVP for the following ODE.

$$\begin{cases} \frac{d}{dt}u(t) - Au(t) = f(t), \\ u(0) = \phi, \end{cases} \tag{15.7}$$

where A is a constant (more generally, for vector valued u, the equation will be a system of ODEs for the components of u, and A will be a matrix with constant entries). Using an integrating factor e^{-At}, the ODE in (15.7) yields

$$\frac{d}{dt}\left(e^{-At}u\right) = e^{-At}\frac{du}{dt} - Ae^{-At}u = e^{-At}(u' - Au) = e^{-At}f(t).$$

But then

$$e^{-At}u = \int_0^t e^{-As} f(s)\, ds + e^{-A\cdot 0}u(0),$$

and multiplying both sides by e^{At} gives

$$u(t) = e^{At}\phi + \int_0^t e^{A(t-s)} f(s)\, ds. \tag{15.8}$$

The operator $\mathscr{S}(t)\phi = e^{At}\phi$, called the *propagator* operator, maps the initial value ϕ to the solution of the homogeneous equation at later times. In terms of this operator, we can rewrite solution (15.8) as

$$u(t) = \mathscr{S}(t)\phi + \int_0^t \mathscr{S}(t-s) f(s)\, ds. \tag{15.9}$$

In the case of the heat equation, the heat propagator operator is

$$\mathscr{S}(t)\phi = \int_{-\infty}^{\infty} S(x-y,t)\phi(t)\, dy,$$

which again maps the initial data ϕ to the solution of the homogeneous equation at later times. Using the heat propagator, we can rewrite formula (15.6) in exactly the same form as (15.9).

We now show that (15.6) indeed solves the problem (15.1) by a direct substitution. Since we have solved the homogeneous equation before, it suffices to show that u^p given by (15.5) solves problem (15.3). Differentiating (15.5) with respect to t gives

$$\partial_t u^p = \int_{-\infty}^{\infty} S(x-y,0) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s) f(y,s)\, dy\, ds.$$

Recall that the heat kernel solves the heat equation and has the Dirac delta function as its initial data, i.e. $S_t = kS_{xx}$, and $S(x-y,0) = \delta(x-y)$. Hence,

$$\begin{aligned} \partial_t u^p &= \int_{-\infty}^{\infty} \delta(x-y) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s) f(y,s)\, dy\, ds \\ &= f(x,t)\, dy + k\frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\, dy\, ds = f(x,t) + ku^p_{xx}, \end{aligned}$$

which shows that u^p solves the inhomogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u^p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\, dy\, ds = 0.$$

Thus, u^p given by (15.5) indeed solves problem (15.3), which finishes the proof that (15.6) solves the original IVP (15.1).

Example 15.1. Find the solution of the inhomogeneous heat equation with the source $f(x,t) = \delta(x-2)\delta(t-1)$ and zero initial data.

Using formula (15.6), and substituting the expression for $f(x,t)$, and $\phi(x) = 0$, we get

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s)\delta(y-2)\delta(s-1)\, dy\, ds = \int_0^t S(x-2, t-s)\delta(s-1)\, ds.$$

For the last integral, notice that if $t < 1$, then $\delta(s-1) = 0$ for all $s \in [0,t]$, and if $t > 1$, then the delta function will act on the heat kernel by assigning its value at $s = 1$. Hence,

$$u(x,t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ S(x-2, t-1) & \text{for } t > 1. \end{cases}$$

This, of course, coincides with our intuition of heat conduction, since the external heat source in this case gives an instantaneous temperature boost to the point $x = 1$ at time $t = 1$. Henceforth, the temperature in the rod will remain zero till the time $t = 1$, and afterward the heat will transfer exactly as in the case of the homogeneous heat equation with data given at time $t = 1$ as $u(x,1) = \delta(x-2)$. □

15.1 Source on the half-line

We will use the reflection method to solve the inhomogeneous heat equation on the half-line. Consider the Dirichlet heat problem

$$\begin{cases} v_t - kv_{xx} = f(x,t), & \text{for } 0 < x < \infty, \\ v(x,0) = \phi(x), \\ v(0,t) = h(t). \end{cases} \tag{15.10}$$

Notice that in the above problem not only the equation is inhomogeneous, but the boundary data is given by an arbitrary function $h(t)$. In this case the Dirichlet condition is called inhomogeneous. We can reduce the above problem to one with zero initial data by the following subtraction method. Defining the new quantity

$$V(x,t) = v(x,t) - h(t), \tag{15.11}$$

we have that

$$\begin{aligned} &V_t - kV_{xx} = v_t - h'(t) - kv_{xx} = f(x,t) - h'(t), \\ &V(x,0) = v(x,0) - h(0) = \phi(x) - h(0), \\ &V(0,t) = v(0,t) - h(t) = h(t) - h(t) = 0. \end{aligned}$$

Thus, $v(x,t)$ solves problem (15.10) if and only if $V(x,t)$ solves the Dirichlet problem

$$\begin{cases} V_t - kV_{xx} = f(x,t) - h'(t), & \text{for } 0 < x < \infty, \\ V(x,0) = \phi(x) - h(0), \\ V(0,t) = 0. \end{cases} \tag{15.12}$$

With this procedure, we essentially combined the heat source given as the boundary data at the endpoint $x = 0$ with the external heat source $f(x,t)$. Notice that $h(t)$ has units of temperature, so its derivative will have units of heat flux, which matches the units of $f(x,t)$. We will denote the combined source in the last problem by $F(x,t) = f(x,t) - h'(t)$, and the initial data by $\Phi(x) = \phi(x) - h(0)$. Since the Dirichlet boundary condition for V is homogeneous, we can extend $F(x,t)$ and $\Phi(x,t)$ to the whole line in an odd fashion, and use the reflection method to solve (15.12). The extensions are

$$\Phi_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) + h(0) & \text{for } x < 0, \end{cases} \qquad F_{\text{odd}}(x,t) = \begin{cases} f(x,t) - h'(t) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -f(-x,t) + h'(t) & \text{for } x < 0. \end{cases}$$

Clearly, the solution to the problem

$$\begin{cases} U_t - kU_{xx} = F_{\text{odd}}(x,t), & \text{for } -\infty < x < \infty, \\ U(x,0) = \Phi_{\text{odd}}(x), \end{cases}$$

is odd, since $U(x,t) + U(-x,t)$ will solve the homogeneous heat equation with zero initial data. Then $U(0,t) = 0$, and the restriction to $x \geq 0$ will solve the Dirichlet problem (15.12) on the half-line. Thus, for $x > 0$,

$$V(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi_{\text{odd}}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)F_{\text{odd}}(y,s)\,dy\,ds.$$

Proceeding exactly as in the case of the (homogeneous) heat equation on the half-line, we will get

$$\begin{aligned} V(x,t) = &\int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

Finally, using that $v(x,t) = V(x,t) + h(t)$, we have

$$\begin{aligned} v(x,t) = h(t) + &\int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

15.2 Conclusion

Using our intuition of heat conduction as an averaging process with the weight given by the heat kernel, we guessed formula (15.6) for the solution of the inhomogeneous heat equation, treating the inhomogeneity as an external heat source. Employing the propagator operator, this formula coincided exactly with the solution formula for the analogous inhomogeneous ODE, which further hinted at the correctness of the formula. However, to obtain a rigorous proof that formula (15.6) indeed gives the unique solution, we verified that the function given by the formula satisfies the equation and the initial condition by a direct substitution. One can then use this formula along with the reflection method to also find the solution for the inhomogeneous heat equation on the half-line.

Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{16.1}$$

where $f(x,t)$, $\phi(x)$ and $\psi(x)$ are arbitrary given functions. Similar to the inhomogeneous heat equation, the right hand side of the equation, $f(x,t)$, is called the *source* term. In the case of the string vibrations this term measures the external force (per unit mass) applied on the string, and the equation again arises from Newton's second law, in which one now also has a nonzero external force.

As was done for the inhomogeneous heat equation, we can use the superposition principle to break problem (16.1) into two simpler ones:

$$\begin{cases} u^h_{tt} - c^2 u^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \quad u^h_t(x,t) = \psi(x), \end{cases} \tag{16.2}$$

and

$$\begin{cases} u^p_{tt} - c^2 u^p_{xx} = f(x,t), \\ u^p(x,0) = 0, \quad u^p_t(x,t) = 0. \end{cases} \tag{16.3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (16.1). u^h solves the homogeneous equation, so it is given by d'Alambert's formula. Thus, we only need to solve the inhomogeneous equation with zero data, i.e. problem (16.3). We will show that the solution to the original IVP (16.1) is

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{16.4}$$

The first two terms in the above formula come from d'Alambert's formula for the homogeneous solution u^h, so to prove formula (16.4), it suffices to show that the solution to the IVP (16.3) is

$$u^p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{16.5}$$

For simplicity, we will seize specifying the superscript and write $u = u^p$ (this corresponds to the assumption $\phi(x) \equiv \psi(x) \equiv 0$, which is the only remaining case to solve).

Recall that we have already solved inhomogeneous hyperbolic equations by the method of characteristics, which we will apply to the inhomogeneous wave equation as well. The change of variables into the characteristic variables and back are given by the following formulas

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct, \end{cases} \qquad \begin{cases} t = \frac{\xi-\eta}{2c}, \\ x = \frac{\xi+\eta}{2}. \end{cases} \tag{16.6}$$

To write the equation in the characteristic variables, we compute u_{tt} and u_{xx} in terms of (ξ,η) using the chain rule.

$$\begin{aligned} u_t &= cu_\xi - cu_\eta, & u_x &= u_\xi + u_\eta, \\ u_{tt} &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}, & u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \end{aligned}$$

so

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}. \tag{16.7}$$

Notice that $u(x,t) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$, and we made an abuse of notation above to identify u with the function $U(\xi,\eta) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$. In the same way, we will identify f with the function $F(\xi,\eta) = f(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$, and will implicitly understand that the functions in terms of (ξ,η) depend on $(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$.

Using (16.7), we can rewrite the inhomogeneous wave equation in terms of the characteristic variables as

$$u_{\xi\eta} = -\frac{1}{4c^2} f(\xi, \eta). \tag{16.8}$$

To solve this equation, we need to successively integrate in terms of η and then ξ. Recall that in previous examples of inhomogeneous hyperbolic equations we performed these integrations explicitly, then changed the variables back to (x, t), and determined the integration constants from the initial conditions. In our present case, however, we would like to obtain a formula for the general function f, so explicit integration is not an option. Thus, to determine the constants of integration, we need to rewrite the initial conditions in terms of the characteristic variables.

Notice that from (16.6), $t = 0$ is equivalent to $(\xi - \eta)/2c = 0$, or $\xi = \eta$. The initial conditions of (16.3) then imply

$$\begin{aligned}
&u(\xi, \xi) = 0,\\
&cu_\xi(\xi, \xi) - cu_\eta(\xi, \xi) = 0,\\
&u_\xi(\xi, \xi) + u_\eta(\xi, \xi) = 0,
\end{aligned}$$

where the last identity is equivalent to the identity $u_x(x, 0) = 0$, which can be obtained by differentiating the first initial condition of (16.3). From the last two conditions above, it is clear that $u_\xi(\xi, \xi) = u_\eta(\xi, \xi) = 0$, so the initial conditions in terms of the characteristic variables are

$$u(\xi, \xi) = u_\xi(\xi, \xi) = u_\eta(\xi, \xi) = 0. \tag{16.9}$$

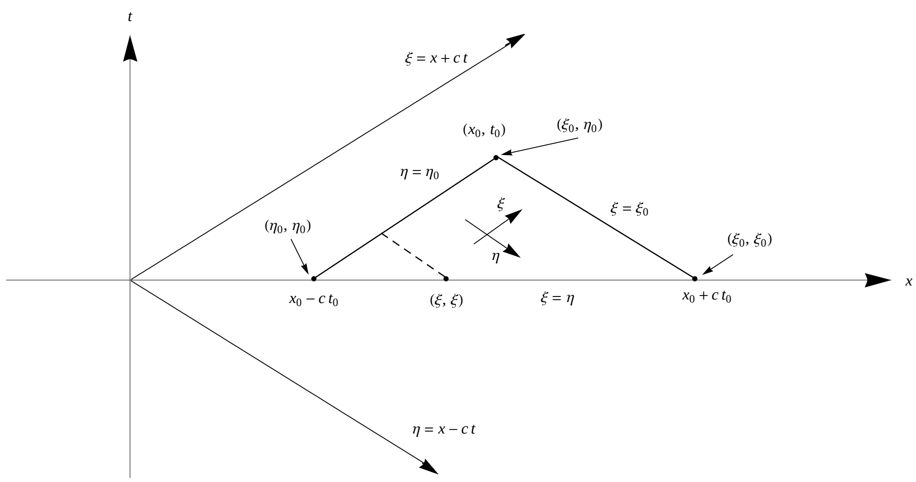

Figure 16.1: The triangle of dependence of the point (x_0, t_0).

Now fix a point (x_0, t_0) for which we will show formula (16.5). This point has the coordinates (ξ_0, η_0) in the characteristic variables. To find the value of the solution at this point, we first integrate equation (16.8) in terms of η from ξ to η_0

$$\int_\xi^{\eta_0} u_{\xi\eta}\, d\eta = -\frac{1}{4c^2} \int_\xi^{\eta_0} f(\xi, \eta)\, d\eta.$$

But

$$\int_\xi^{\eta_0} u_{\xi\eta}\, d\eta = u_\xi(\xi, \eta_0) - u_\xi(\xi, \xi) = u_\xi(\xi, \eta_0)$$

due to (16.9) (this is precisely the reason for the choice of the lower limit), so we have

$$u_\xi(\xi, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi} f(\xi, \eta)\, d\eta.$$

Integrating this identity with respect to ξ from η_0 to ξ_0 gives

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi, \eta_0)\, d\xi = \frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi} f(\xi, \eta)\, d\eta\, d\xi.$$

Similar to the previous integral,

$$\int_{\eta_0}^{\xi_0} u_\xi(\xi, \eta_0)\, d\xi = u(\xi_0, \eta_0) - u_\xi(\eta_0, \eta_0) = u(\xi_0, \eta_0)$$

due to (16.9). We then have

$$u(\xi_0, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi} f(\xi, \eta)\, d\eta\, d\xi = \frac{1}{4c^2} \iint_{\triangle} f(\xi, \eta)\, d\xi\, d\eta, \tag{16.10}$$

where the double integral is taken over the triangle of dependence of the point (x_0, t_0), as depicted in Figure 16.1. Using the change of variables (16.6), and computing the Jacobian,

$$J = \frac{\partial(\xi, \eta)}{\partial(x, t)} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c,$$

we can transform the double integral in (16.10) to a double integral in terms of the (x, t) variables to get

$$u(x_0, t_0) = \frac{1}{4c^2} \iint_{\triangle} f(x, t)|J|\, dx\, dt = \frac{1}{2c} \iint_{\triangle} f(x, t)\, dx\, dt.$$

Finally, rewriting the last double integral as an iterated integral, we will arrive at formula (16.5). This finishes the proof that (16.4) is the unique solution of the IVP (16.1). One can alternatively show that formula (16.4) gives the solution by directly substituting it into (16.1), which is left as a homework problem.

Example 16.1. Solve the inhomogeneous wave IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^x, \\ u(x, 0) = u_t(x, 0) = 0. \end{cases}$$

Using formula (16.4) with $\phi = \psi = 0$, we get

$$\begin{aligned} u(x, t) &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^y\, dy\, ds = \frac{1}{2c} \int_0^t [e^{x+c(t-s)} - e^{x-c(t-s)}]\, ds \\ &= \frac{e^x}{2c} \left(-\frac{1}{c} e^{c(t-s)} \Big|_0^t - \frac{1}{c} e^{-c(t-s)} \Big|_0^t \right) = \frac{e^x}{2c^2} (e^{ct} + e^{-ct} - 2). \end{aligned}$$

□

16.1 Source on the half-line

Consider the following inhomogeneous Dirichlet wave problem on the half-line

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x, t), \quad \text{for } 0 < x < \infty, t > 0, \\ v(x, 0) = \phi(x), \quad v_t(x, t) = \psi(x), \\ v(0, t) = h(t). \end{cases} \tag{16.11}$$

One can employ the subtraction method that we used for the heat equation to reduce the problem to one with zero Dirichlet data, and then use the reflection method to derive a solution formula for the reduced problem. An alternative simple way, however, is to derive the solution from scratch as follows. Since we know how to find the solution for zero Dirichlet data (use the standard reflection method), we treat the complementary case, that is, assume that the boundary data is nonzero, while $f(x,t) \equiv \phi(x) \equiv \psi(x) \equiv 0$.

From the method of characteristics, we know that the solution can be written as

$$v(x,t) = j(x+ct) + g(x-ct). \tag{16.12}$$

The zero initial conditions then give

$$\begin{aligned} v(x,0) &= j(x) + g(x) = 0, \\ v_t(x,0) &= cj'(x) - cg'(x) = 0, \end{aligned}$$

for $x > 0$. Differentiating the first identity, and dividing the second identity by c, we arrive at the following system for j' and g'

$$\begin{cases} j'(x) + g'(x) = 0, \\ j'(x) - g'(x) = 0, \end{cases} \qquad \Rightarrow \qquad j'(x) = g'(x) = 0.$$

This means that for $s > 0$,

$$j(s) = -g(s) = a$$

for some constant a. On the other hand, the boundary condition for $v(x,t)$ implies

$$v(0,t) = j(ct) + g(-ct) = h(t).$$

But since $ct > 0$, we have $j(ct) = a$, and

$$g(-ct) = h(t) - a, \qquad \text{or} \qquad g(s) = h(-s/c) - a$$

for $s < 0$. Returning to (16.12), notice that the argument of the j term is always positive, so

$$v(x,t) = \begin{cases} a - a & \text{for } x > ct, \\ a + h\left(t - \frac{x}{c}\right) - a & \text{for } x < ct. \end{cases} = \begin{cases} 0 & \text{for } x > ct, \\ h\left(t - \frac{x}{c}\right) & \text{for } x < ct. \end{cases}$$

Thus, for $x > ct$ the solution of (16.11) will be given by (16.4), while for $x < ct$ we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy + h\left(t - \frac{x}{c}\right) + \frac{1}{2c}\iint_D f(y,s)\,dy\,ds,$$

where D is the domain of dependence of the point (x,t).

16.2 Conclusion

The superposition principle was again used to write the solution to the IVP for the inhomogeneous wave equation as a sum of the general homogeneous solution, and the inhomogeneous solution with zero initial data. The inhomogeneous solution was obtained by the method of characteristics through a successive integration in terms of the characteristic variables. One can also derive the solution formula for the inhomogeneous wave equation by simply integrating the equation over the domain of dependence, and using Green's theorem to compute the integral of the left hand side. Yet another way is to approach the solution of the inhomogeneous equation by studying the propagator operator of the wave equation, similar to what we did for the heat equation. These methods are discussed in the appendix.

17 Waves with a source: the operator method

In the previous lecture we used the method of characteristics to solve the initial value problem for the inhomogeneous wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{17.1}$$

and obtained the formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds. \tag{17.2}$$

Another way to derive the above solution formula is to integrate both sides of the inhomogeneous wave equation over the triangle of dependence and use Green's theorem.

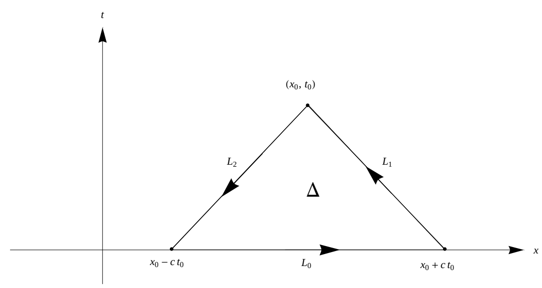

Figure 17.1: The triangle of dependence of the point (x_0, t_0).

Fix a point (x_0, t_0), and integrate both sides of the equation in (17.1) over the triangle of dependence for this point.

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\, dx\, dt = \iint_\triangle f(x,t)\, dx\, dt. \tag{17.3}$$

Recall that by Green's theorem

$$\iint_D (Q_x - P_t)\, dx\, dt = \oint_{\partial D} P\, dx + Q\, dt,$$

where ∂D is the boundary of the region D with counterclockwise orientation. We thus have

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\, dx\, dt = \iint_\triangle (-c^2 u_x)_x - (-u_t)_t\, dx\, dt = \oint_{\partial\triangle} -u_t\, dx - c^2 u_x\, dt.$$

The boundary of the triangle of dependence consists of three sides, $\partial\triangle = L_0 + L_1 + L_2$, as can be seen in Figure 17.1, so

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\, dx\, dt = \int_{L_0 + L_1 + L_2} -u_t\, dx - c^2 u_x\, dt,$$

and we have the following relations on each of the sides

$$\begin{aligned} L_0 &: \quad dt = 0 \\ L_1 &: \quad dx = -cdt \\ L_2 &: \quad dx = cdt \end{aligned}$$

Using these, we get

$$\int_{L_0} -c^2 u_x\,dt - u_t\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$
$$\int_{L_1} -c^2 u_x\,dt - u_t\,dx = c\int_{L_1} du = c[u(x_0,t_0) - u(0,x_0+ct_0)] = cu(x_0,t_0) - c\phi(x_0+ct_0),$$
$$\int_{L_2} -c^2 u_x\,dt - u_t\,dx = -c\int_{L_2} du = -c[u(0,x_0-ct_0) - u(x_0,t_0)] = cu(x_0,t_0) - c\phi(x_0-ct_0).$$

Putting all the sides together gives

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = 2cu(x_0,t_0) - c[\phi(x_0+ct_0) + \phi(x_0-ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$

and using (17.3), we obtain

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx + \frac{1}{2c}\iint_\triangle f(x,t)\,dx\,dt,$$

which is equivalent to formula (17.2).

17.1 The operator method

For the inhomogeneous heat equation we interpreted the solution formula in terms of the heat propagator, which also showed the parallels between the heat equation and the analogous ODE. We would like to obtain such a description for the solution formula (17.2) as well. For this, consider the ODE analog of the wave equation with the associated initial conditions

$$\begin{cases} \frac{d^2u}{dt^2} + A^2 u = f(t), \\ u(0) = \phi, \quad u'(0) = \psi, \end{cases} \tag{17.4}$$

where A is a constant (a matrix, if we allow u to be vector valued). To find the solution of the inhomogeneous ODE, we need to first solve the homogeneous equation, and then use variation of parameters to find a particular solution of the inhomogeneous equation. The solution of the homogeneous equation is

$$u^h(t) = c_1\cos(At) + c_2\sin(At),$$

and the initial conditions imply that $c_1 = \phi$, and $c_2 = A^{-1}\psi$. To obtain a particular solution of the inhomogeneous equation we assume that c_1 and c_2 depend on t,

$$u^p(t) = c_1(t)\cos(At) + c_2(t)\sin(At),$$

and substitute u^p into the equation to solve for $c_1(t)$ and $c_2(t)$. This procedure leads to

$$c_1(t) = -\int_0^t A^{-1}\sin(As)f(s)\,dt, \qquad c_2(t) = \int_0^t A^{-1}\cos(As)f(s)\,ds.$$

Putting everything together, the solution to (17.4) will be

$$u(t) = \cos(At)\phi + A^{-1}\sin(At)\psi + \int_0^t A^{-1}\sin(A(t-s))f(s)\,ds.$$

If we now define the propagator

$$\mathscr{S}(t)\psi = A^{-1}\sin(At)\psi,$$

then the solution to (17.4) can be written as

$$u(t) = \mathscr{S}'(t)\phi + \mathscr{S}(t)\psi + \int_0^t \mathscr{S}(t-s)f(s)\,ds. \tag{17.5}$$

For the wave equation, similarly denoting the operator acting on ψ from d'Alambert's formula by

$$\mathscr{S}(t)\psi = \int_{x-ct}^{x+ct} \psi(y)\,dy,$$

we can rewrite formula (17.2) in exactly the same form as (17.4). The moral of this story is that having solved the homogeneous equation and found the propagator, we have effectively derived the solution of the inhomogeneous equation as well. The rigorous connection between the solution of the homogeneous equation and that of the inhomogeneous wave equation is contained in the following statement.

Duhamel's Principle. Consider the following 1-parameter family of wave IVPs

$$\begin{cases} u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s) = 0, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \end{cases} \tag{17.6}$$

then the function

$$v(x,t) = \int_0^t u(x,t;s)\,ds$$

solves the inhomogeneous wave equation with vanishing data, i.e.

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = f(x,t), \\ v(x,0) = 0, \quad v_t(x,0) = 0. \end{cases}$$

Note that the initial conditions of the s-IVP (17.6) are given at time $t = s$, and the initial velocity is $\psi(x;s) = f(x,s)$. Duhamel's principle has the physical description of replacing the external force by its effect on the velocity. From Newton's second law, the force is responsible for acceleration, or change in velocity per unit time. So if we can account for the effect of the external force on the instantaneous velocity, then the the solution of the equation with the external force can be found by solving the homogeneous equations with the effected velocities, namely (17.6), and "summing" these solutions over the instances $t = s$.

We prove Duhamel's principle by direct substitution. The derivatives of v are

$$\begin{aligned}
v_t(x,t) &= u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds, \\
v_{tt}(x,t) &= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds = f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds, \\
v_{xx}(x,t) &= \int_0^t u_{xx}(x,t;s)\,ds,
\end{aligned}$$

where we used the initial conditions of (17.6). Substituting this into the wave equation gives

$$(\partial_t^2 - c^2\partial_x^2)v = \int_0^t [u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s)]\,ds + f(x,t) = f(x,t),$$

and v indeed solves the inhomogeneous wave equation. It is also clear that v has vanishing initial data.

Duhamel's principle gives an alternative way of proving that (17.2) solves the inhomogeneous wave equation. Indeed, from d'Alambert's formula for (17.6) and a time shift $t \mapsto t - s$, we have

$$u(x, t; s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s)\, dy.$$

Thus, the solution of the inhomogeneous wave equation with zero initial data is

$$v(x, t) = \int_0^t u(x, t; s)\, ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s)\, dy\, ds.$$

17.2 Conclusion

We defined the wave propagator as the operator that maps the initial velocity to the solution of the homogeneous wave equation with zero initial displacement. Using this operator, the solution of the inhomogeneous wave equation can be written in exactly the same form as the solution of the analogous inhomogeneous ODE in terms of its propagator. The significance of this observation is in the connection between the solution of the homogeneous and that of the inhomogeneous wave equations, which is the substance of Duhamel's principle. Hence, to solve the inhomogeneous wave equation, all one needs is to find the propagator operator for the homogeneous equation.

Problem Set 8

1. Solve the initial value problem for the following inhomogeneous heat equation

$$\begin{cases} u_t - \frac{1}{4}u_{xx} = e^{-t} & \text{in } -\infty < x < \infty, t > 0, \\ u(x,0) = x^2. \end{cases}$$

2. Solve the following Dirichlet problem for the inhomogeneous heat equation on the half-line

$$\begin{cases} v_t - kv_{xx} = \delta(t-1), & \text{for } 0 < x < \infty, 0 < t < \infty, \\ v(0,t) = 0; \quad v(x,0) = \delta(x-2). \end{cases}$$

Explain in terms of heat conduction how the external heat source effects the temperature in the rod.

3. (#3.3.3 in [Str]) Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{cases} w_t - kw_{xx} = 0, & \text{for } 0 < x < \infty, 0 < t < \infty, \\ w_x(0,t) = h(t); & w(x,0) = \phi(x), \end{cases}$$

by the subtraction method indicated in the text.

4. (#3.4.1 in [Str]) Solve $u_{tt} = c^2 u_{xx} + xt, \quad u(x,0) = 0, \quad u_t(x,0) = 0.$

5. (#3.4.3 in [Str]) Solve $u_{tt} = c^2 u_{xx} + \cos x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 1 + x.$

6. (#3.4.5 in [Str]) Let $f(x,t)$ be any function and let $u(x,t) = (1/2c) \iint_\triangle f$, where $\triangle$ is the triangle of dependence. Written as an iterated integral,

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\, dy\, ds.$$

Verify directly by differentiation that

$$u_{tt} = c^2 u_{xx} + f(x,t) \quad \text{and} \quad u(x,0) \equiv u_t(x,0) \equiv 0.$$

7. (#3.4.13 in [Str]) Solve the Dirichlet wave problem on the half-line

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } 0 < x < \infty, \\ u(0,t) = t^2; & u(x,0) = x, \quad u_t(x,0) = 0. \end{cases}$$

Earlier in the course we solved the Dirichlet problem for the wave equation on the finite interval $0 < x < l$ using the reflection method. This required separating the domain $(x, t) \in (0, l) \times (0, \infty)$ into different regions according to the number of reflections that the backward characteristic originating in the regions undergo before reaching the x axis. In each of these regions the solution was given by a different expression, which is impractical in applications, and the method does not generalize to higher dimensions or other equations. We now study a different method of solving the boundary value problems on the finite interval, called *separation of variables*.

18.1 Wave equation

Let us start by considering the wave equation on the finite interval with homogeneous Dirichlet conditions.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l, \\ u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = u(l, t) = 0. \end{cases} \tag{18.1}$$

The idea of the separation of variables method is to find the solution of the boundary value problem as a linear combination of simpler solutions (compare this to finding the simpler solution $S(x, t)$ of the heat equation, and then expressing any other solution in terms of the heat kernel). The building blocks in this case will be the *separated solutions*, which are the solutions that can be written as a product of two functions, one of which depends only on x, and the other only on t, i.e.

$$u(x, t) = X(x)T(t). \tag{18.2}$$

Let us try to find all the separated solutions of the wave equation. Substituting (18.2) into the equation gives

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing both sides of these identity by $-c^2 X(x)T(t)$, we get

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{c^2 T(t)} = \lambda. \tag{18.3}$$

Clearly λ is a constant, since it is independent of x from $\lambda = -T''/(c^2 T)$, and is independent of t from $\lambda = -X''/X$. We will shortly see that the boundary conditions force λ to be positive, so let $\lambda = \beta^2$, for some $\beta > 0$. One can then rewrite (18.3) as a pair of separate ODEs for $X(x)$ and $T(t)$

$$T'' + c^2\beta^2 T = 0, \qquad \text{and} \qquad X'' + \beta^2 X = 0.$$

The solutions of these ODEs are

$$T(t) = A\cos\beta ct + B\sin\beta ct, \qquad \text{and} \qquad X(x) = C\cos\beta x + D\sin\beta x, \tag{18.4}$$

where A, B, C and D are arbitrary constants. From the boundary conditions in (18.1), we have

$$X(0)T(t) = X(l)T(t) = 0, \quad \forall t \qquad \Rightarrow \qquad X(0) = X(l) = 0,$$

since $T(t) \equiv 0$ would result in the trivial solution $u(x, t) \equiv 0$ (our goal is to find all separated solutions). With this boundary condition for $X(x)$, we have from (18.4)

$$X(0) = C = 0, \qquad \text{and} \qquad X(l) = D\sin\beta l = 0.$$

The solution with $D = 0$ will again lead to the trivial zero solution, so we consider the case when $\sin\beta l = 0$. But this implies that $\beta l = n\pi$ for $n = 1, 2, \ldots$, and

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{l} \quad \text{for } n = 1, 2, \ldots$$

These formulas give distinct solutions for $X(x)$, and multiplying these by the $T(t)$ corresponding to λ_n, we find infinitely many separated solutions

$$u_n(x,t) = \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \quad \text{for } n = 1, 2, \dots,$$

where A_n, B_n are arbitrary constants as before. Since a linear combination of solutions of the wave equation is also a solution, any finite sum

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}, \tag{18.5}$$

will also solve the wave equation.

Returning to our boundary value problem (18.1), we would like to find the solution as a linear combination of separated solutions. However, finite sums in the form (18.5) are very special, since not every function is a finite sum of sines and cosines. Checking the initial conditions, we have

$$\begin{aligned} \phi(x) &= \sum_n A_n \sin \frac{n\pi x}{l}, \\ \psi(x) &= \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \end{aligned} \tag{18.6}$$

Obviously, not all initial data ϕ, ψ can be written as finite sums of sine functions. So instead of restricting ourselves to finite sums, we allow infinite sums, and ask the question whether any functions ϕ, ψ can be written as infinite sums of sine functions. This question was first studied by Fourier, and these infinite sums have the name of *Fourier series* (Fourier sine series in this case). It turns out that practically any function defined on $0 < x < l$ can be expressed in the form (18.6). Leaving the question of convergence of such sums, we see that if the initial data can be expressed in the form (18.6), then the solution is given by (18.5).

The coefficients of t inside the series (18.5), $\frac{n\pi c}{l}$, are called the frequencies. For a violin string of length l, we had $c^2 = \frac{T}{\rho}$, so the frequencies are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad n = 1, 2, \dots$$

The smallest frequency, $\frac{\pi\sqrt{T}}{l\sqrt{\rho}}$, is the fundamental note, while the double, triple, and so on of the fundamental note are the overtones. Notice that by shortening the length l of the vibrating portion of the string with a finger, a violinist produces notes of higher frequency.

18.2 Heat equation

For the Dirichlet heat problem on the finite interval,

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < l, \\ u(x,0) = \phi(x), \\ u(0,t) = u(l,t) = 0, \end{cases} \tag{18.7}$$

we similarly search for all the separated solutions in the form $u(x,t) = X(x)T(t)$. In this case the equation gives

$$-\frac{X''}{X} = -\frac{T'}{kt} = \beta^2,$$

and the resulting ODEs are

$$T' = -\beta^2 kT, \qquad \text{and} \qquad X'' + \beta^2 X = 0.$$

The solution for the T equation is then $T(t) = Ae^{-\beta^2 kt}$, while the function $X(x)$ satisfies the same equation and boundary conditions as before. This yields the same values $\beta_n = n\pi/l$. We thus have that

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l} \tag{18.8}$$

is the solution to problem (18.7), provided that the initial data is given as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \tag{18.9}$$

Notice that as t grows, all the terms in the series (18.8) decay exponentially, so the solution itself will decay, which makes sense in terms of heat conduction, since in the absence of a heat source, the temperatures in the rod will equalize with the zero temperature of the environment.

Example 18.1. Solve the following Dirichlet problem for the heat equation by separation of variables.

$$\begin{cases} u_t - ku_{xx} = 0, \quad \text{for } 0 < x < \pi/2, \\ u(x,0) = 3\sin 4x, \\ u(0,t) = u(\frac{\pi}{2},t) = 0, \end{cases}$$

In this problem $l = \pi/2$, so $\beta_n = 2n$. We can write the initial data in the form (18.9),

$$3\sin 4x = \sum_{n=1}^{\infty} A_n \sin 2nx,$$

which implies that $A_2 = 3$, and $A_n = 0$ for $n \neq 2$. But then from (18.8) the solution will be

$$u(x,t) = 3e^{-16kt} \sin 4x.$$

□

18.3 Eigenvalues

The numbers $\lambda = \left(\frac{n\pi}{l}\right)^2$ are called eigenvalues, and the functions $X_n(x) = \sin \frac{n\pi x}{l}$ are called eigenfunctions. Notice that we can think of the equation $-X'' = \lambda X$ as an eigenvalue problem for the operator $-\frac{d^2}{dx^2}$ in the space of functions that satisfy the Dirichlet conditions $X(0) = X(l) = 0$. An eigenfunction is then a solution of the equation which is not identically zero, i.e. $X(x) \not\equiv 0$.

However, unlike the operators in linear algebra, which have finitely many eigenvalues, in our case we have an infinite number of eigenvalues. This is due to the fact that the space of functions is infinite dimensional.

We return to the question of the sign of the eigenvalues. Suppose $\lambda = 0$, then we would have $X'' = 0$, which leads to $X(x) = C + Dx$. The boundary conditions then imply that $C = 0$, and $Dl = 0$, giving $X(x) \equiv 0$.

If, on the other hand, we assume that $\lambda < 0$, and write $\lambda = -\gamma^2$ for some $\gamma > 0$, then the equation for X becomes $X'' = \gamma^2 X$, which has the solution

$$X(x) = Ce^{\gamma x} + De^{-\gamma x}.$$

The boundary conditions then give

$$\begin{cases} C + D = 0 \\ Ce^{\gamma l} + De^{-\gamma l} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} C = -D \\ Ce^{2\gamma l} = C \end{cases} \quad \Rightarrow \quad C = D = 0,$$

which again results in the identically zero solution $X(x) \equiv 0$. So there are no nonpositive eigenvalues.

18.4 Conclusion

Returning to the Dirichlet problems for the wave and heat equations on a finite interval, we solved them with the method of separation of variables. That is, we looked for the solution in the form of an infinite linear combination of separated solutions. This lead to infinite series, which solve the appropriate initial-boundary value problems, as long as the initial data can be expanded in corresponding series, which in the case of Dirichlet conditions are the Fourier sine series. The question of convergence of such series will be discussed next quarter, while the case of Neumann conditions will be considered next time.

The same method of separation of variables that we discussed last time for boundary problems with Dirichlet conditions can be applied to problems with Neumann, and more generally, Robin boundary conditions. We illustrate this in the case of Neumann conditions for the wave and heat equations on the finite interval.

19.1 Wave equation

Substituting the separated solution $u(x,t) = X(x)T(t)$ into the wave Neumann problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \\ u_x(0,t) = u_x(l,t) = 0, \end{cases} \tag{19.1}$$

gives the same equations for X and T as in the Dirichlet case,

$$-X'' = \lambda X, \qquad \text{and} \qquad -T'' = c\lambda^2 T.$$

However, the boundary conditions now imply

$$X'(0)T(t) = X'(l)T(t) = 0, \quad \forall t \qquad \Rightarrow \qquad X'(0) = X'(l) = 0.$$

To find all the separated solutions, we need to find all the eigenvalues and eigenfunctions satisfying these boundary conditions. To do this, we need to consider the cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$ separately.

Assume $\lambda = 0$, then the equation for X is $X'' = 0$, which has the solution $X(x) = C + Dx$. The derivative is then $X'(x) = D$, and the boundary conditions imply that $D = 0$. So every constant function, $X(x) = C$, is an eigenfunction for the eigenvalue $\lambda_0 = 0$.

Next, we assume that $\lambda = -\gamma^2 < 0$, in which case the equation for X takes the form

$$X'' = \gamma^2 X.$$

The solution to this equation is $X(x) = Ce^{\gamma x} + De^{\gamma x}$, so $X'(x) = C\gamma e^{\gamma x} - D\gamma e^{-\gamma x}$. Checking the boundary conditions gives

$$\begin{cases} C\gamma - D\gamma = 0 \\ C\gamma e^{\gamma l} - D\gamma e^{-\gamma l} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} C = D \\ C\gamma(e^{2\gamma l} - 1) = 0 \end{cases} \quad \Rightarrow \quad C = D = 0,$$

since $\gamma \neq 0$, and hence, also $e^{2\gamma l} - 1 \neq 0$. This leads to the identically zero solution $X(x) \equiv 0$, which means that there are no negative eigenvalues.

For the remaining case, $\lambda = \beta^2 > 0$, the equation is $X'' = -\beta^2 X$, which as we saw last time when discussing Dirichlet boundary conditions, has the solution

$$X(x) = C\cos\beta x + D\sin\beta x.$$

The derivative of this function is

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x,$$

so the boundary conditions give

$$X'(0) = D\beta = 0, \qquad \text{and} \qquad X'(l) = -C\beta\sin\beta l = 0.$$

Since $\beta \neq 0$, and C and D cannot be both zero, we have $\sin\beta l = 0$, which implies that $\beta l = n\pi$ for $n = 1, 2, \ldots$. Then the eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \cos\frac{n\pi x}{l}, \qquad \text{for } n = 0, 1, 2, \ldots \tag{19.2}$$

Notice that we have also included $n = 0$, which gives the zero eigenvalue $\lambda_0 = 0$ with the eigenfunction $X_0 = 1$. The set of all eigenvalues, called the *spectrum*, for the Neumann conditions differs from that for the Dirichlet conditions by this additional eigenvalue.

In the case of $\lambda_0 = 0$ the T equation becomes $T'' = 0$, which has the solution $T(t) = \frac{1}{3}A_0 + \frac{1}{2}B_0 t$. The factors of $\frac{1}{2}$ are included for future convenience (to have a single formula for the Fourier coefficients).

The solutions T_n corresponding to $\lambda_n = (n\pi/l)^2$ for $n = 1, 2, \ldots,$ were found in the last lecture to be

$$T_n(t) = A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}.$$

Putting everything together gives the following series expansion for the solution of problem (19.1),

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}, \tag{19.3}$$

as long as the initial data can be expanded into cosine Fourier series

$$\begin{aligned} \phi(x) &= \frac{1}{2}A_0 + \sum_n A_n \cos\frac{n\pi x}{l}, \\ \psi(x) &= \frac{1}{2}B_0 + \sum_n \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}. \end{aligned} \tag{19.4}$$

These series for the data come from plugging in $t = 0$ into the solution formula (19.3), and its derivative with respect to t. We notice that in the case of the Neumann conditions we end up with cosine Fourier series for the data, while in the Dirichlet case we had sine Fourier series. This is in agreement with the reflection method, since one needs to take the odd extensions of the data in the case of Dirichlet conditions, and even extensions in the case of Neumann conditions. But odd functions (extended data) have only sines in their Fourier expansions, while even functions have only cosines.

Example 19.1. Solve the following Neumann problem for the wave equation by separation of variables.

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \quad \text{for } 0 < x < \pi, \\ u(x,0) = 3\cos x, \quad u_t(x,0) = 1 - \cos 4x \\ u_x(0,t) = u_x(\pi,t) = 0, \end{cases}$$

In this problem $l = \pi$, so $\beta_n = n$. Notice also that $c = 2$, and we can write the initial data in the form (19.4) as follows,

$$\begin{aligned} 3\cos x &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx, \\ 1 - \cos 4x &= \frac{1}{2}B_0 + \sum_{n=1}^{\infty} 2nB_n \cos nx. \end{aligned}$$

These identities imply that $A_1 = 3$, and $A_n = 0$ for $n \neq 1$, and $B_0 = 2, B_4 = -\frac{1}{8}$, and $B_n = 0$ for $n \neq 0, 4$. Then solution (19.3) will take the form

$$u(x,t) = t + 3\cos 2t \cos x - \frac{1}{8}\sin 8t \cos 4x.$$

□

19.2 Heat equation

For the Neumann heat problem on the finite interval,

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < l, \\ u(x,0) = \phi(x), \\ u_x(0,t) = u_x(l,t) = 0, \end{cases} \tag{19.5}$$

the equations for X and T factors of the separated solution $u(x,t) = X(x)T(t)$ are

$$X'' = -\lambda X, \qquad \text{and} \qquad T' = -\lambda kT.$$

The boundary conditions are the same as in the wave problem (19.1), so one gets the same eigenvalues and eigenfunctions (19.2). For the eigenvalue $\lambda_0 = 0$, the T equation is $T' = 0$, so $T_0(t) = \frac{1}{2}A_0$. For the positive eigenvalues we found the solutions for T in the last lecture to be

$$T_n(t) = A_n e^{-(n\pi/l)^2 kt}.$$

Thus, the solution to the heat Neumann problem is given by the series

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l},$$

as long as the initial data can be expanded into the cosine Fourier series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.$$

19.3 Mixed boundary conditions

Sometimes one needs to consider problems with mixed Dirichlet-Neumann boundary conditions, i.e. Dirichlet conditions at one end of the finite interval, and Neumann conditions at the other. Examples of such problems are vibrations of a finite string with one free and one fixed end, and the heat conduction in a finite rod with one insulated end and the other end kept at a constant zero temperature.

In such cases the method of separation of variables leads to the eigenvalue problem

$$\begin{cases} X'' = -\lambda X, \\ X(0) = X'(l) = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} X'' = -\lambda X, \\ X'(0) = X(l) = 0. \end{cases}$$

One can then show that the eigenvalues are $\lambda_n = [(n + \frac{1}{2})\pi/l]^2$, and the corresponding eigenfunctions for the respective problems are

$$X_n(x) = \sin\frac{(n+\frac{1}{2})\pi x}{l}, \qquad \text{and} \qquad X_n(x) = \cos\frac{(n+\frac{1}{2})\pi x}{l}, \qquad \text{for } n = 0, 1, 2, \ldots,$$

which is left as a homework exercise.

19.4 Conclusion

Similar to the case of the Dirichlet problems for heat and wave equations, the method of separation of variables applied to the Neumann problems on a finite interval leads to an eigenvalue problem for the $X(x)$ factor of the separated solution. In this case, however, we discovered a new eigenvalue $\lambda = 0$ in addition to the eigenvalues found for the Dirichlet problems. Then the general solutions of the Neumann problems for wave and heat equations can be written in series forms, as (infinite) linear combinations of all separated solutions, as long as the initial data can be expanded in cosine Fourier series. We will discuss in detail the questions on whether and how a given function can be expanded into Fourier series next quarter.

Problem Set 9

1. Find the solution to the following Dirichlet problem by separation of variables

$$\begin{cases} u_{tt} = u_{xx} \quad \text{for } 0 < x < \pi, \\ u(x,0) = \sin 3x, \quad u_t(x,0) = 0, \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

2. (#4.1.2 in [Str]) Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x,t)$ at later times. In this problem, *assume* the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right).$$

3. (#4.1.4 in [Str]) Consider waves in a resistant medium that satisfy the problem

$$\begin{gathered} u_{tt} = c^2 u_{xx} - r u_t \quad \text{for } 0 < x < l \\ u = 0 \quad \text{at both ends} \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \end{gathered}$$

where r is constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

4. (#4.1.6 in [Str]) Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0,t) = u(\pi,t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x,0) = 0$. So uniqueness is false for this equation!

5. Use separation of variables to solve the equation $u_{xx} = \frac{2}{k}u_t + u$, with boundary conditions $u(1,t) = 0$, $u_x(0,t) = -be^{-kt}$, where k, b are constants.

6. (#4.2.1 in [Str]) Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.

7. (#4.2.2 in [Str]) Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.

   (a) Show that the eigenfunctions are $\cos[(n + \frac{1}{2})\pi x/l]$.

   (b) Write the series expansion for a solution $u(x,t)$.

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