Math 117: Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem. (The Archimedean Property of $\mathbb{R}$) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Note: We will use the completeness axiom to prove this theorem. Although the Archimedean property of $\mathbb{R}$ is a consequence of the completeness axiom, it is weaker than completeness. Notice that $\mathbb{N}$ is also unbounded above in $\mathbb{Q}$, even though $\mathbb{Q}$ is not complete. We also have an example of an ordered field for which the Archmidean property does not hold! $\mathbb{N}$ *is* bounded above in $\mathbb{F}$, the field of rational polynomials!

Proof by contradiction. If $\mathbb{N}$ were bounded above in $\mathbb{R}$, then by ____________________ ________________ $\mathbb{N}$ would have a ________________________. I.e., there exists $m \in$ ______ such that $m =$ ____________. Since m is the ________________________________, ________ is not an upper bound for $\mathbb{N}$. Thus there exists an $n_o \in \mathbb{N}$ such that $n_0 >$ ________. But then $n_o + 1 >$ ________, and since $n_o + 1 \in \mathbb{N}$, this contradicts __________ ________________________.

The Archimedean property is equivalent to many other statements about $\mathbb{R}$ and $\mathbb{N}$.

12.10 Theorem. Each of the following is equivalent to the Archimedean property.

(a) For every $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$.

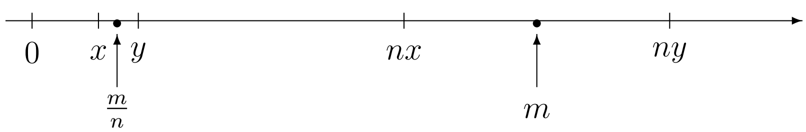
(b) For every $x > 0$ and for every $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$.

(c) For every $x > 0$, there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

The proof is given in the book. The idea is that (a) is the same as the Archimedean property because (a) is essentially the statement that "For every $z \in \mathbb{R}$, z is *not* an upper bound for $\mathbb{N}$." Then, it is fairly easy to see why (b) and (c) follow.

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$). For every $x, y \in \mathbb{R}$ such that $x < y$, there exists a rational number r such that $x < r < y$.

Notes: The idea of this proof is to find the numerator and denominator of the rational number that will be between a given x and y. To do this, we first find a natural number n for which nx and ny will be more than one unit apart (this will require the Archimedian property!) Notice that the closer together x and y are, the bigger this n will need to be! Picture (assuming $x > 0$):



Since nx and ny are far enough apart, we expect that there exists a natural number m in between nx and ny. Finally, $\frac{m}{n}$ will be the rational number in between x and y!

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$ be given. We will first prove the theorem in the case $x > 0$. Since $y - x > 0$, __________ $\in \mathbb{R}$. Then, by the Archimedean property, there exists an $n \in \mathbb{N}$ such that $n >$ __________. Therefore, __________ $< ny$. Since we are in the case $x > 0$, ______ > 0 and there exists $m \in \mathbb{N}$ such that $m - 1 \leq$ ______ $< m$ (The proof that such an m exists uses the well-ordering property of $\mathbb{N}$; see Exercise 12.9.) Then, $ny >$ __________ $\geq$ ______. Thus $nx < m < ny$. It then follows that the rational number $r = \frac{m}{n}$ satisfies $x < r < y$.

Now, in the case $x \leq 0$, there exists $k \in \mathbb{N}$ such that $k > |x|$. Since $k - |x| = k + x$ is positive and $k + x < k + y$, the above argument proves that there is a rational number r such that $k + x < r < k + y$. Then, letting $r' = r - k$, r' is a rational number such that $x < r' < y$. □

It is also true that for every $x, y \in \mathbb{R}$ such that $x < y$, there exists an *irrational* number w such that $x < w < y$. Combining these facts, it follows that for every $x, y \in \mathbb{R}$ such that $x < y$ there are in fact infinitely many rational numbers and infinitely many irrational numbers in between x and y!