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An inverse kinematic problem with internal sources

Leonid Pestov¹, Gunther Uhlmann² and Hanming Zhou^{3,4}

¹Immanuel Kant Baltic Federal University, Russia

²Department of Mathematics, University of Washington, USA and IAS, Hong Kong University of Science and Technology, Hong Kong

³Department of Mathematics, University of Washington, USA

E-mail: LPEstov@kantiana.ru, gunther@math.washington.edu and hzhou@math.washington.edu

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Abstract

Given a bounded domain M in \mathbb{R}^n with a conformally Euclidean metric $g = \rho dx^2$, we consider the inverse problem of recovering a semigeodesic neighborhood of a domain $\Gamma \subset \partial M$ and the conformal factor ρ in the neighborhood from the travel time data (defined below) and the Cartesian coordinates of Γ . We develop an explicit reconstruction procedure for this problem. The key ingredient is the relation between the reconstruction procedure and a Cauchy problem of the conformal Killing equation.

Keywords: inverse kinematic, internal data, conformal Killing vector fields

1. Introduction

The inverse kinematic problem arises in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. The Earth is generally isotropic; thus one can assume that the geometry of the Earth is conformally Euclidean. In applications the conformal factor corresponds to $1/c^2(x)$, where $c(x)$ is the sound speed (index of refraction). This problem goes back to [6, 19], who considered the case of a radial metric conformal to the Euclidean metric. The geometric version of the problem is related to the boundary rigidity problem and its linearization, namely the geodesic ray transform (see [15, 18] for recent surveys).

The usual inverse kinematic problem takes into account only the travel times between boundary points. In the current paper we also make use of the internal data (travel times from internal sources to the boundary) so that we can reconstruct the geometry explicitly. The

⁴ Author to whom any correspondence should be addressed.

problem of determining a sound speed of a medium from the corresponding Dirichlet-to-Neumann map has been solved using the boundary control method (see [7]). An ingredient in the method is to recover the boundary distance function between interior points and the boundary. Our paper gives an explicit reconstruction in the case conformal to the Euclidean metric. We describe now the mathematical setting of the problem.

Let (M, g) be a bounded domain in \mathbb{R}^n , $n \geq 2$ with smooth boundary ∂M . We assume g is conformal to the Euclidean metric i.e., $g = \rho dx^2$, where ρ is a positive smooth function on M and $dx^2 = (dx^1)^2 + \dots + (dx^n)^2$ is the Euclidean metric. Let Γ be a domain in ∂M (in particular, Γ could be ∂M). From each $x' \in \Gamma$, there is a unique geodesic $\gamma_{x'}(t)$ with $\gamma_{x'}(0) = x'$, $\dot{\gamma}_{x'}(0) = \nu(x')$, where $\nu(x')$ is the inward unit normal vector to ∂M at x' w.r.t. the metric g . Moreover, since γ is a geodesic of unit speed w.r.t. the metric g , we have $\rho(\gamma) |\dot{\gamma}|^2 = 1$, where $|\cdot|$ is the Euclidean norm.

There is a positive smooth function $T(x')$ on Γ such that for each $x' \in \Gamma$, the geodesic $\gamma_{x'}$, which is orthogonal to ∂M at x' , is defined on the interval $[0, T(x')]$. Let

$$D := \{(x', t) : x' \in \Gamma, 0 \leq t < T(x')\}.$$

We consider the map $\gamma : D \rightarrow \gamma(D)$, $\gamma(x', t) := \gamma_{x'}(t) = x$. Generally such a map is not a diffeomorphism; for example, given the Euclidean disk with radius r , let Γ be a domain of the boundary; then γ is not a diffeomorphism if $T(x') > r$. Thus we modify $T(x')$, i.e., D , so that $\gamma : D \rightarrow \gamma(D)$ is a diffeomorphism. Under this assumption, D actually provides a semigeodesic coordinate system (or boundary normal coordinates) for $\gamma(D)$.

Now given a point $x \in \gamma(D)$, if $x' \in \Gamma$ such that $x = \gamma(x', t)$ for some $t \in [0, T(x')]$, let $U(x') \subset \Gamma$ be a neighborhood of x' . Moreover, we fix $U(x')$ for all $x \in \gamma_{x'}([0, T(x')])$, and $U(x')$ can be arbitrarily small. Notice that the choice of $U(x')$ depends on *a priori* knowledge of x ; i.e., we need to know the geodesic projection $x'(x) \in \Gamma$ of any $x \in \gamma(D)$. We define the *travel time data* with respect to D by

$$\Omega(D) := \{(\tau(x, x''), x'(x)) : x \in \gamma(D), x'' \in U(x')\},$$

where $\tau(x, x'') := \text{dist}_g(x, x'')$. In this paper, we consider the problem of recovering the neighborhood $\gamma(D)$ and the conformal factor ρ in $\gamma(D)$ from the travel time data $\Omega(D)$. To solve the problem, we need some extra inverse data; namely, we assume the Cartesian coordinates of Γ are known. This is a reasonable assumption, since any rigid transformation of the domain M does not change the travel time data. We call the problem the *inverse kinematic problem with internal sources*. It is worth pointing out that we do not make any assumption on the convexity of the boundary (or Γ).

Theorem 1.1. *Let M be a bounded domain in \mathbb{R}^n , $n \geq 2$, and $g = \rho dx^2$ be a conformally Euclidean metric on M . Let Γ be a domain in ∂M ; then there exists a semigeodesic coordinate system D such that $\gamma : D \rightarrow \gamma(D) \subset \mathbb{R}^n$ is a diffeomorphism, and $\Gamma = \gamma(\{t = 0\})$. From the travel time data $\Omega(D)$ and the Cartesian coordinates of Γ one can recover the diffeomorphism γ and the conformal factor ρ in $\gamma(D)$.*

Uniqueness for this inverse problem was proved by Anikonov [1]. In this paper we give a reconstruction procedure based on conformal Killing vector fields.

Generally one cannot expect to reconstruct ρ on the whole manifold, as the necessary assumption that γ is a diffeomorphism. However, if $M \setminus \gamma(D)$ has an empty interior, we reconstruct the domain M and the conformal factor ρ globally, since points in $M \setminus \gamma(D)$ are

limit points of $\gamma(D)$. The example mentioned above satisfies the assumption, and it is easy to see that $M \setminus \gamma(D)$ is the center of the disk if $\Gamma = \partial M$, $T(x') \equiv r$.

In [7, 8] an isometric copy of a compact Riemannian manifold was recovered from the set of boundary distance functions $\{r_x(y) := (x, y) : x \in M, y \in \partial M\}$, which is also internal data. Such internal data is also related to the broken geodesic flow, which consists of two geodesic segments sharing a common end point inside the manifold (see, e.g., [7, 9]). A related reconstruction problem with different assumptions was considered in [5] by reducing the travel time data to measurements of the shape operators of the wave fronts of waves diffracted from interior points. Different from our method, their approach treated the case of two dimensions and the case of three and higher dimensions separately.

Notice that the statement of theorem 1.1 actually shows that this is a local problem; i.e., for a point $x = \gamma(x', t)$, we only need the travel time data from x to an arbitrarily small neighborhood $U(x') \subset \Gamma$. If the function $T(x')$ is also uniformly small, the problem can be formulated just near one boundary point. Similar to the local version of the problem, the local boundary rigidity problem and local geodesic ray transform were considered in [16, 17], and a generalization to local ray transforms along arbitrary smooth curves was studied in [20].

As mentioned above, our arguments give a reconstruction procedure for the diffeomorphism γ and the conformal factor ρ . The reconstruction procedure consists of two steps: step 1 (section 2) is devoted to the recovery of a semigeodesic (isometric) copy of the metric g and the boundary restriction of the conformal factor from the inverse data. In step 2 (section 3), we reconstruct γ and ρ by studying the relation between γ and conformal Killing vector fields on the semigeodesic copy D .

As noticed, the inverse dynamical problem for the wave equations (with boundary data) may be reduced by the boundary control method to the inverse kinematic problem with internal sources and then to the Yamabe problem [3]. It gives in our case the Cauchy problem for the Laplace operator. We use another approach (using conformal Killing vector fields, which are determined by a linear system of first-order partial differential equations) that gives stability for dimensions $n > 2$ [12].

2. Recovery of the semigeodesic copy of the metric

Given $x \in \gamma(D)$, there is a unique geodesic $\gamma_{x'}$ (normal to ∂M) and $0 \leq t < T(x')$ such that $x = \gamma(x', t)$. Here $x'(x)$ is the geodesic projection of x on Γ , so $t(x) = \tau(x, x'(x))$ is the distance from x to the boundary. Thus the travel time data $\Omega(D)$ uniquely determines the semigeodesic coordinates of points in $\gamma(D)$.

Let the pair $(x'(x), t(x))$ be the semigeodesic coordinate of the point x . Thus for any point $y = (x', t) \in D$, the function $\tau(\gamma(x', t), x'')$ is known. Define

$$\lambda((x', t), x'') := \tau(\gamma(x', t), x''),$$

then $\lambda(y, x'')$, $y = (x', t)$ is the distance between points $y \in D$ and $x'' \in U(x')$ in the metric $\tilde{g} := \gamma^*(g)$. Note that we identify Γ with $\Gamma \times \{0\}$. We call \tilde{g} the *semigeodesic copy* of the metric g .

We first recover the metric \tilde{g} . Notice that γ , now as an isometry, sends geodesics to geodesics. This implies

$$\tilde{g}_{kn}(x', 0) = \delta_{kn}, \quad x' \in \Gamma, \quad 1 \leq k \leq n,$$

where δ_{ij} is the Kronecker delta. In local coordinates, $y = (y^1, \dots, y^n)$, where $y^n = t$. One has the following eikonal equation

$$1 = \left| \nabla \tau(x, x'') \right|_g^2 = \left| \tilde{\nabla} \lambda(y, x'') \right|_{\tilde{g}}^2 = \tilde{g}^{ij}(y) \frac{\partial \lambda(y, x'')}{\partial y^i} \frac{\partial \lambda(y, x'')}{\partial y^j}, \quad x'' \in U(x'). \quad (1)$$

Note that if y' is close enough to y , and $x'' \in U(x'(y))$ is sufficiently close to $x'(y)$, then $x'' \in U(x'(y'))$ too. (1) gives a family of linear algebraic equations with respect to the contravariant components $\tilde{g}^{ij}(y)$ of the metric \tilde{g} . For $y \in D$ with $y^n > 0$, it is known that $\tilde{g}^{ij}(y)$ can be recovered by the knowledge of $\tilde{g}^{ij}(y) \partial_{y^i} \lambda(y, x_k'') \partial_{y^j} \lambda(y, x_k'')$ for $N = n(n+1)/2$ ‘generic’ points $x_k'' \in U(x'(y))$, $k = 1, 2, \dots, N$ (see, e.g., [13, 14]). Such N generic points always exist in a neighborhood of $x'(y)$ in Γ . Thus $\tilde{g}^{ij}(y)$ is determined for $y \in D$, $y^n > 0$. By the smoothness of \tilde{g} , it also recovers $\tilde{g}^{\alpha\beta}(x', 0)$, $1 \leq \alpha, \beta \leq n-1$, $x' \in \Gamma$. So the metric \tilde{g} is uniquely determined.

Notice that the boundary restriction of the isometry γ is given; i.e., $x' = x'(y^1, \dots, y^{n-1}, 0) = (x'^1, \dots, x'^n)$ as a point in \mathbb{R}^n is known. We recover the conformal factor ρ on Γ . Since $\tilde{g} = \gamma^*(g)$ is the pullback, we have

$$\tilde{g}_{\alpha\beta}(x', 0) = \frac{\partial \gamma^k}{\partial y^\alpha} \frac{\partial \gamma^l}{\partial y^\beta} \rho(x') \delta_{kl} = \rho(x') \sum_{k=1}^n \frac{\partial x'^k}{\partial y^\alpha} \frac{\partial x'^k}{\partial y^\beta}, \quad \alpha, \beta = 1, \dots, n-1. \quad (2)$$

Thus equation (2) and the knowledge of $\tilde{g}|_{l=0}$ together determine $\rho|_\Gamma$.

3. Recovery of the isometry and the conformal factor

To recover the conformal factor ρ , we need to solve the *pullback problem*, i.e., to find the map γ . To this end, we need some knowledge of conformal Killing vector fields. Recall that a vector field u is called a conformal Killing vector field if it satisfies the conformal Killing equation (in covariant form)

$$Ku := \sigma \nabla u - g \delta u / n = 0,$$

where $\sigma \nabla$ is the symmetric part of the covariant derivative ∇ , and δ is the divergence. They are exactly those vector fields whose flows preserve the conformal structures of the manifolds. In local coordinates, the conformal Killing equation has the form (for covariant components)

$$\frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - 2\Gamma_{ij}^k u_k \right) - \frac{1}{n} g_{ij} \left(g^{kl} \frac{\partial u_l}{\partial x^k} - g^{kl} u_m \Gamma_{kl}^m \right) = 0.$$

Thus the conformal Killing equation is equivalent to a linear system of first-order partial differential equations.

However, not all of the metrics admit conformal Killing vector fields. Actually for $n \geq 3$, a ‘generic’ metric does not possess any non-trivial conformal Killing vector fields (see, e.g., [2, 10]). On the other hand, note that the metric g is conformal to the Euclidean metric; thus they share the same set of conformal Killing vector fields. When $n = 2$, in the Cartesian coordinates (x^1, x^2) all conformal Killing vector fields of the Euclidean metric have the form $u = (u^1, u^2)$, where u^1 and u^2 are conjugate harmonic functions. In the case $n > 2$, the contravariant components of u in the Cartesian coordinates (x^1, \dots, x^n) are given by

$$u^i(x) = a_0 x^i + (Ax)^i - b^i |x|^2 + 2x^i(b, x) + c^i,$$

where a_0 is a real constant, A is an $n \times n$ skew-symmetric constant matrix, and b and c are vectors in \mathbb{R}^n .

Now we are in a position to recover the map γ and the conformal factor ρ . Let $e_{(j)} = \frac{\partial}{\partial x^j}$, $j = 1, \dots, n$ be the standard basis vectors in \mathbb{R}^n . It is easy to see that they are conformal Killing vector fields in the Euclidean metric; thus also conformal Killing vector fields for g . Then $u_{(j)} = \gamma^* e_{(j)}$, $j = 1, \dots, n$ are conformal Killing vector fields for the metric $\tilde{g} = \gamma^* g$. This implies $u_{(j)}$, $j = 1, \dots, n$ satisfies the conformal Killing equation

$$Ku_{(j)}(y) = 0, \quad y \in D, \quad j = 1, \dots, n.$$

It is known that $u_{(j)}$ is uniquely determined by the Cauchy data $\{u_{(j)}(x', 0): x' \in \Gamma\}$ (see, e.g., [4, 11]). Thus we calculate the Cauchy data of $u_{(j)}$ first.

Since we have already recovered the semigeodesic copy \tilde{g} , we denote the dual vector of $u_{(j)}$ by $u^{(j)}$. In the meantime, we denote the dual vector of $e_{(j)}$ under the metric $g = \rho dx^2$ by $w^{(j)}$; then $w^{(j)} = \rho dx^j$. In local coordinates, the equality $u_{(j)} = \gamma^* e_{(j)}$ means (for covariant components)

$$u_i^{(j)}(y) = w_k^{(j)} \frac{\partial x^k}{\partial y^i} = \rho \frac{\partial \gamma^j(y)}{\partial y^i}.$$

This observation is crucial in our reconstruction procedure. It relates the isometry γ to the conformal Killing vector fields on the semigeodesic copy D . Thus at $y^n = t = 0$,

$$u_\alpha^{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial y^\alpha}(x', 0) = \rho(x') \frac{\partial x^j}{\partial y^\alpha}, \quad \alpha = 1, \dots, n - 1.$$

Since $\rho(x')$ and $\gamma(x', 0)$ are known for $x' \in \Gamma$, $u_\alpha^{(j)}(x', 0)$ are determined. To determine the value of $u_n^{(j)}$ at $t = 0$, notice that $\dot{\gamma}_{x'}(0) = \nu(x')$, so

$$u_n^{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial t}(x', 0) = \rho(x') \nu^j(x').$$

However, ν is a unit vector w.r.t. metric $g = \rho dx^2$, if we denote the inward unit normal vector on ∂M w.r.t. the Euclidean metric by ν_0 . Then $\nu = \frac{1}{\sqrt{\rho}} \nu_0$; i.e.,

$$u_n^{(j)}(x', 0) = \sqrt{\rho(x')} \nu_0^j(x').$$

Fortunately, the Cartesian coordinates of the hypersurface Γ are given; thus ν_0 as the normal vector to Γ is known. Together with the knowledge of $\rho|_\Gamma$, we recover $u_n^{(j)}|_{t=0}$ too.

From the Cauchy data, we uniquely recover the conformal Killing vector fields $u_{(j)} = \gamma^* e_{(j)}$ and equivalently the dual vector $u^{(j)}$. Now by defining $v = (v^1, \dots, v^n)$,

$$v^j(x', t) := u_n^{(j)}(x', t) = \rho(\gamma(x', t)) \frac{\partial \gamma^j(x', t)}{\partial t},$$

we have $v = \rho(\gamma) \dot{\gamma}$. In the meantime, notice that

$$|v|^2 = \rho^2(\gamma) |\dot{\gamma}|^2 = \rho(\gamma) \quad (\text{since } |\dot{\gamma}|_g = 1); \tag{3}$$

we obtain

$$\dot{\gamma} = \frac{v}{|v|^2}.$$

This implies

$$\gamma(x', t) = \int_0^t \dot{\gamma}_{x'}(t) dt + x' = \int_0^t \frac{v}{|v|^2}(x', t) dt + x';$$

i.e., we recover the geodesics $\gamma_{x'}(t)$ and therefore the diffeomorphism $\gamma : D \rightarrow \gamma(D)$, namely, the range $\gamma(D)$. Moreover, by (3)

$$\rho(\gamma(x', t)) = |v(x', t)|^2,$$

the conformal factor $\rho|_{\gamma(D)}$ is determined, and this finishes the proof of the main theorem.

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