Single pixel X-ray Transform and Related Inverse Problems

Ru-Yu Lai[*] Gunther Uhlmann[†] Jian Zhai[‡] Hanming Zhou[§]

July 30, 2022

Abstract

In this paper, we analyze the nonlinear single pixel X-ray transform K and study the reconstruction of f from the measurement Kf. Different from the well-known X-ray transform, the transform K is a nonlinear operator and uses a single detector that integrates all rays in the space. We derive stability estimates and an inversion of the linearization at zero. We also consider the case where we integrate along geodesics of a Riemannian metric. Moreover, we conduct several numerical experiments to corroborate the theoretical results.

Key words. X-ray transform, single pixel X-ray transform, inverse problems

AMS subject classifications 2010. 35R30.

1 Introduction

In this paper, we study the *single pixel X-ray transform* K defined by

$$Kf(x) := \int_{\mathbb{S}^{n-1}} e^{-Xf(x,\theta)}\, d\theta, \tag{1.1}$$

whose exterior integral integrates over the entire unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $n \geq 2$. The inner function consists of an exponential function and the conventional X-ray transform X defined by

$$Xf(x,\theta) := \int_\ell f\, ds = \int_{\mathbb{R}} f(x+s\theta)\, ds, \qquad (x,\theta) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \tag{1.2}$$

where $\ell = x + s\theta$ is a line passing through a point $x \in \mathbb{R}^n$, in the direction $\theta \in \mathbb{S}^{n-1}$. Notice that the single pixel X-ray transform K in (1.1) is different from the single pixel imaging. In the

[*]School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA; rylai@umn.edu

[†]Department of Mathematics, University of Washington, Seattle, WA 98195, USA; HKUST Jockey Club Institute for Advanced Study, HKUST, Clear Water Bay, Kowloon, Hong Kong; gunther@math.washington.edu

[‡]School of Mathematical Sciences, Fudan University, Shanghai 200433, China; jianzhai@fudan.edu.cn

[§]Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA 93106, USA; hzhou@math.ucsb.edu

single pixel imaging, weighted integrals of the Radon transform are assumed. However, from the definition of $Kf(x)$ in (1.1), we see that it integrates all lines through the point x, which results in collapse of an image into a scalar value. This distinguishes the K transform from the usual single pixel imaging.

Inversion of the standard X-ray transform consists of recovering a function supported in a bounded domain from its integrals along straight lines through this domain. In dimension two ($n = 2$), it coincides with the Radon transform [10], which provides the theoretical underpinning for several medical imaging techniques such as Computed Tomography (CT) and Positron Emission Tomography (PET). The X-ray transform has been extensively studied, including its uniqueness, stability estimates and reconstruction formula, see for example, the book [7]. Generalizations of the standard X-ray transform include integrals of tensor fields or along curved lines. We refer to recent survey papers [3, 8] and the references therein for more details.

A notable difference between the conventional X-ray transform X and the single pixel X-ray transform K is the nonlinearity due to the exponential function. Another difference comes from the fact that for instance, in $\mathbb{R}^3$, the X-ray transform X gives a two-dimensional image $Xf(x, \theta)$, $\theta \in \mathbb{S}^2$ for a fixed x, while $Kf(x)$ gives a scalar value. In this sense, the available data for the single pixel X-ray transform is much less than that of the conventional X-ray transform. In practice, it can be used in the scenario when one wants to validate the structure of an object without revealing much detail. For more details on the applications, we refer to [9].

The objective of this paper is to recover f from the data Kf by establishing a reconstruction formula and deriving stability estimates. The second author of this paper has previously demonstrated the global uniqueness of the inverse problem. This proof can be found in the supporting information of [9] by applying the monotonic property of K and the known injectivity of the X-ray transform X [1, 7]. Due to the special structure of the transform K, however, it is not clear that if the same technique in [9] can be directly applied to the study of stability estimates and reconstruction formulas.

1.1 Motivation

The single pixel X-ray transform finds applications in protection of information in highly sensitive systems. The nonlinearity of the transform K acts as a shield against the disclosure of such information. Here the exponential function is chosen to be the nonlinear function in K since attenuation is naturally exponential in space. Specifically, this nonlinearity ensures that there is no one-to-one correspondence between the density f and the true mass $\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} f(x + s\theta) ds d\theta$ and, therefore, f cannot be estimated from a single projection. This thus protects the detailed information of the system such as its structure and composition. For interested readers, we refer to [9] for more discussions on applications of the single pixel X-ray transform.

1.2 Main results

The transform K is nonlinear and a monotone decreasing map due to the exponential function in the definition. In particular, the nonlinearity of the transform helps secure information; on the other hand, it also introduces difficulties to the mathematical and practical reconstruction of f. Moreover, the monotonicity of K implies that when f is increasing, the measurement Kf becomes decreasing and could be eventually very small, which makes it challenging to distinguish the true measurement from noise if the noise does exist.

As mentioned earlier, the global uniqueness of K was proved in [9]. However, the inversion formula, to the best of the authors' knowledge, has not been derived and stability estimate has not yet been investigated. The first result we study here is to establish a reconstruction formula of the linearization of K at $f = 0$.

Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 2$. We define the space

$$\mathcal{M}_k(\Omega) := \{f \in C^k(\mathbb{R}^n) :\ f \text{ is supported in } \overline{\Omega}\}.$$

Through out the paper, we denote by $\overline{\Omega}$ the closure of the domain Ω in $\mathbb{R}^n$. When linearizing around zero, we have the following inversion formula and the stability estimate for the linearized setting.

Theorem 1.1 (Inversion of the linearization at $f = 0$)**.** *Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary. For any $h \in \mathcal{M}_0(\Omega)$, assume that we know $K(\varepsilon h)$ for all $\varepsilon \in \mathbb{R}$ sufficiently close to zero, then*

$$h = -c_n|D|(\partial_\varepsilon|_{\varepsilon=0}K(\varepsilon h)),$$

where $c_n = (2\pi|\mathbb{S}^{n-2}|)^{-1}$ with $|\mathbb{S}^{n-2}|$ the measure of the unit sphere $\mathbb{S}^{n-2}$, and $|D| = (-\Delta)^{1/2}$ is the square root of Laplacian, which is a pseudo-differential operator. Here we define the first derivative of the function $K(\varepsilon h)$ with respect to ε at $\varepsilon = 0$ by $\partial_\varepsilon|_{\varepsilon=0}K(\varepsilon h)$.

Theorem 1.1 states that a function h can be reconstructed through this formula based on the linearized data. It also immediately implies the estimate for h, see Corollary 1.1. The proof of Corollary 1.1 follows directly from Proposition 2.2 and the inversion of the linearization of K at zero.

Corollary 1.1 (Stability estimate for the linearized problem)**.** *Let Ω be an open bounded domain in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary and let Ω_1 be a larger open and bounded domain satisfying $\overline{\Omega} \subset \Omega_1$. There exists a constant $C > 0$ depending on n, Ω, and Ω_1 so that*

$$C^{-1}\|h\|_{L^2(\mathbb{R}^n)} \leq \|\partial_\varepsilon|_{\varepsilon=0}K(\varepsilon h)\|_{H^1(\Omega_1)} \tag{1.3}$$

for $h \in \mathcal{M}_0(\Omega)$.

Then we return to the nonlinear problem. We state stability estimate results for small f in two different settings. Notice that K is nonlinear, and $K(0) \equiv |\mathbb{S}^{n-1}| \neq 0$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n > 2$ be an open and bounded domain and let Ω_1 be a larger open and bounded domain so that $\overline{\Omega} \subset \Omega_1$. Let $f \in \mathcal{M}_1$. For any $L > 0$, there exists $\varepsilon > 0$ such that for any f satisfying*

$$\|f\|_{C^1(\mathbb{R}^n)} < \min\{\varepsilon, 1\}, \quad \|f\|_{H^t(\mathbb{R}^n)} < L, \quad t > n + 2,$$

one has the conditional stability estimate

$$\|f\|_{C^1(\mathbb{R}^n)} \leq C\|K(f) - |\mathbb{S}^{n-1}|\|^{\mu}_{H^1(\Omega_1)}, \tag{1.4}$$

for some $\mu \in (0, 1)$, where the constant $C > 0$ depends on K and L only.

In addition, we also study continuity estimate for K as well as another stability estimate under suitable assumptions of f_j, $j = 1, 2$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n > 2$ be an open and bounded domain and let Ω_1 be a larger open and bounded domain so that $\overline{\Omega} \subset \Omega_1$. Then for any f_1, $f_2 \in \mathcal{M}_1(\Omega)$ with $\|f_j\|_{C^1(\mathbb{R}^n)} \leq M$, $j = 1, 2$, for some fixed positive constant M, we have the continuity estimate*

$$\|Kf_1 - Kf_2\|_{H^1(\Omega_1)} \leq C(1 + e^{CM} + Me^{CM})\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} + C(e^{CM} - 1)\|\nabla(f_1 - f_2)\|_{L^2(\mathbb{R}^n)},$$

where the positive constant C is independent of M.

Moreover, assume that f_1, f_2 satisfy

$$\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} \geq c\|\nabla(f_1 - f_2)\|_{L^2(\mathbb{R}^n)} \tag{1.5}$$

for a fixed constant $c > 0$, if M is sufficiently small, then we have

$$\widetilde{C}\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} \leq \|Kf_1 - Kf_2\|_{H^1(\Omega_1)}, \tag{1.6}$$

where the positive constant $\widetilde{C}$ depends on c, M.

Notice that Theorem 1.2, 1.3, hold for $n > 2$ since their proofs rely on Proposition 2.2, which is valid when $n > 2$.

The estimates (1.4) and (1.6) imply that the data Kf in Ω_1 is sufficient to give local stability estimate. Under the assumption (1.5), the left hand side of (1.6) can be replaced by $\|f_1 - f_2\|_{H^1(\mathbb{R}^n)}$ with another constant $\widetilde{C}$.

We apply the linearization scheme to investigate this nonlinear inverse problem, namely, reconstructing f from the measurement Kf. Indeed, to study nonlinear inverse problems, it is classical to utilize the linearization scheme and then reduce it to the problem of their linearization, where the existing results, such as injectivity, are utilized to identify the unknown property [4]. We would also like to note that a general result is proved in [13] when linearizing nonlinear inverse problems. It gives Hölder type estimates for the nonlinear problem under some conditions. In Section 2, we linearize the transform K around zero function. This is motivated by the following observation. Since the first nonconstant term of Taylor's expansion of Kf is the normal operator of the X-ray transform X, linearizing Kf then reveals this term while the remaining higher order terms vanish. Additionally, thanks to the previously established results for the X-ray transform, we can derive a local reconstruction formula of K and also stability estimates for small enough f. For a general function f (not necessary small), however, it would be more challenging to stably recover f since the higher order terms dominate the behavior of f. We do not consider this issue here.

In this paper, we also study the single pixel X-ray transform K in the Riemannian case. We establish the uniqueness of K on compact manifolds with boundary, on which the geodesic X-ray transform X is injective. Our proof is a generalization of the argument for the Euclidean case [9], by applying the Santalo's formula.

Besides the above theoretical results, we conduct numerical reconstructions for the single pixel X-ray transform K by an optimization method. These experiments provide numerical evidence for our stability estimates of K. In particular, if the magnitude of f is small, the reconstruction of f from Kf works quite well, even in the presence of mild noise. While in the case of large f, the optimization approach could fail, which suggests that the estimates (1.4), (1.6) might not hold when M is large.

2 Inverse problems

2.1 Preliminary results

We introduce several known results for the X-ray transform. For a function $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space, the X-ray transform

$$Xf(x,\theta) = \int_{\mathbb{R}} f(x+s\theta)\,ds$$

is well-defined and is constant along lines in direction θ, that is, $Xf(x,\theta) = Xf(x+t\theta,\theta)$ and, moreover, $Xf(x,\theta) = Xf(x,-\theta)$ for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$. By the Fubini's theorem, X can be extended to $L^1(\mathbb{R}^n)$. Let $\Sigma := \{(z,\theta) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : z \in \theta^\perp\}$, where $\theta^\perp := \{z \in \mathbb{R}^n : z \perp \theta\}$ is the orthogonal complement of θ, be the parameter set of straight lines, then

$$X : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\Sigma)$$

is a continuous map. In particular, Xf is compactly supported in Σ if f is compactly supported. We denote the adjoint of X, under the L^2 inner product, by X' and the normal operator by $X'X$. Then $X' : C_c^\infty(\Sigma) \to C^\infty(\mathbb{R}^n)$ has the expression

$$X'\psi(x) := \int_{\mathbb{S}^{n-1}} \psi(x-(x\cdot\theta)\theta,\theta)\,d\theta.$$

Then we have the following results, see [7, 14] for detailed discussions and proofs.

Lemma 2.1. *For $f \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space),*

$$X'Xf(x) = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} f(x+s\theta)ds\,d\theta = 2\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\,dy.$$

An inversion formula of the X-ray transform is stated below.

Proposition 2.1. *For $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$f = c_n|D|X'Xf,$$

where $c_n = (2\pi|\mathbb{S}^{n-2}|)^{-1}$, $|D| = (-\Delta)^{1/2}$ is a non-local pseudo-differential operator and $|\mathbb{S}^{n-2}|$ is the measure of the unit sphere $\mathbb{S}^{n-2}$.

This proposition implies that, up to a positive constant, $X'X$ is the Fourier multiplier $|\xi|^{-1}$. Note that the inversion formula of Proposition 2.1 remains true for any distribution f with compact support.

Throughout this paper, we use C to denote positive constants, which may change from line to line.

Next proposition shows the stability of the inversion.

Proposition 2.2. *Let Ω be an open bounded domain in $\mathbb{R}^n$, $n > 2$ with smooth boundary and let Ω_1 be a larger open and bounded domain satisfying $\overline{\Omega} \subset \Omega_1$. For any nonnegative integer s, there is a constant $C > 0$ so that*

$$C^{-1}\|f\|_{H^s(\mathbb{R}^n)} \le \|X'Xf\|_{H^{s+1}(\Omega_1)} \le C\|f\|_{H^s(\mathbb{R}^n)} \tag{2.1}$$

for $f \in H^s(\mathbb{R}^n)$ supported in $\overline{\Omega}$.

Similar stability estimates in the Riemannian setting have been established in [12]. Notice that the estimate (2.1) is associated with the normal operator $X'X$, which has strong connections to the operator K. There also exist stability results regarding the transform X itself, see e.g. [7, Section II.5]. Since Proposition 2.2 is crucial in the derivation of our main results later, we provide the proof here following [14].

Proof of Proposition 2.2. The second inequality of (2.1) follows from the fact that $X'X$ is the Fourier multiplier $c_n^{-1}|\xi|^{-1}$, which can be directly derived by applying Proposition 2.1. To this end, we first estimate

$$\begin{aligned}\|X'Xf\|^2_{H^{s+1}(\mathbb{R}^n)} &= \|(1+|\xi|^2)^{(s+1)/2}\mathcal{F}(X'Xf)(\xi)\|^2_{L^2(\mathbb{R}^n)} = \|c_n^{-1}(1+|\xi|^2)^{(s+1)/2}|\xi|^{-1}\widehat{f}(\xi)\|^2_{L^2(\mathbb{R}^n)}\\ &\le C\left(\|(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\|^2_{L^2(|\xi|>1)} + \||\xi|^{-1}\widehat{f}(\xi)\|^2_{L^2(|\xi|\le 1)}\right).\end{aligned}$$

Here we denote the Fourier transform of a function f by $\hat{f}$ or $\mathcal{F}(f)$. It remains to estimate the second term $\||\xi|^{-1}\widehat{f}(\xi)\|^2_{L^2(|\xi|\le 1)}$. We have

$$|\widehat{f}(\xi)| = \left|\int e^{-ix\cdot\xi}f(x)\,dx\right| \le \|f\|_{H^s(\mathbb{R}^n)}\|\phi_\xi\|_{H^{-s}(\mathbb{R}^n)},$$

where $\phi_\xi(x) := e^{-ix\cdot\xi}\chi_\Omega(x)$ with $\chi_\Omega \in C_0^\infty(\mathbb{R}^n)$ equals 1 in a neighborhood of $\overline{\Omega}$. Then

$$\begin{aligned}\||\xi|^{-1}\widehat{f}(\xi)\|^2_{L^2(|\xi|\le 1)} &= \int_{|\xi|\le 1}|\xi|^{-2}|\widehat{f}(\xi)|^2\,d\xi\\ &\le \left(\int_{|\xi|\le 1}|\xi|^{-2}\,d\xi\right)\|f\|^2_{H^s(\mathbb{R}^n)}\max_{|\xi|\le 1}\|\phi_\xi\|^2_{H^{-s}(\mathbb{R}^n)}.\end{aligned}$$

Note that $\|\phi_\xi\|_{H^{-s}(\mathbb{R}^n)} \le C$, where the constant $C>0$ depends on n, s for each $|\xi|\le 1$. Moreover, $\int_{|\xi|\le 1}|\xi|^{-2}\,d\xi < \infty$ for $n\ge 3$. Hence, $\||\xi|^{-1}\widehat{f}(\xi)\|^2_{L^2(|\xi|\le 1)} \le C\|f\|^2_{H^s(\mathbb{R}^n)}$ also holds. This proves the second inequality of (2.1) by observing that

$$\|X'Xf\|_{H^{s+1}(\Omega_1)} \le \|X'Xf\|_{H^{s+1}(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}.$$

To show the first inequality of (2.1), we begin by applying Proposition 2.1 to get that

$$\|f\|^2_{H^s(\mathbb{R}^n)} \le C\|X'Xf\|^2_{H^{s+1}(\mathbb{R}^n)} = C\left(\|X'Xf\|^2_{H^{s+1}(\Omega_1)} + \|X'Xf\|^2_{H^{s+1}(\mathbb{R}^n\setminus\Omega_1)}\right). \tag{2.2}$$

The operator $X'X : H_0^s(\Omega) \to H^{s+1}(\mathbb{R}^n\setminus\Omega_1)$ (note that f is supported in $\overline{\Omega}$) has a smooth kernel, so it is compact. Together with the fact that $X'X : H_0^s(\Omega) \to H^{s+1}(\Omega_1)$ is injective, one can remove the term $\|X'Xf\|_{H^{s+1}(\mathbb{R}^n\setminus\Omega_1)}$ from the estimate (2.2), see [14] and [15, Proposition V 3.1]. This proves the first inequality of (2.1). □

Remark 2.1. *The proof of Proposition 2.2 shows that the first inequality of* (2.1) *also works when* $n = 2$.

To conclude this section, we establish the following mapping properties of the operator K.

Proposition 2.3. *Let Ω and Ω_1 be two open bounded domains in $\mathbb{R}^n$ with $\Omega \subset \overline{\Omega} \subset \Omega_1$. Then there exists a positive constant C, depending on n, Ω, and Ω_1, such that*

$$\|Kf\|_{H^1(\Omega_1)} \leq Ce^{C\|f\|_{C^0(\mathbb{R}^n)}}(1 + \|f\|_{H^1(\mathbb{R}^n)})$$

for any $f \in C^1(\mathbb{R}^n)$ supported in $\overline{\Omega}$.

This implies that Kf is well-defined in H^1 norm. Notice that $K[0] = |\mathbb{S}^{n-1}|$ and, therefore, $Kf \neq 0$ even if $f \equiv 0$. Due to this fact, it is worth emphasizing that Kf in general is not in $L^2(\mathbb{R}^n)$.

Proof. Since $f \in C^1(\mathbb{R}^n)$ with support in $\overline{\Omega}$, one can denote $M := \|f\|_{C^0(\mathbb{R}^n)}$ for some finite constant $M \geq 0$, which implies that $|e^{-Xf}|$ is bounded by e^{CM}. Then

$$\|Kf\|^2_{L^2(\Omega_1)} = \int_{\Omega_1} \left| \int_{\mathbb{S}^{n-1}} e^{-Xf(x,\theta)}\, d\theta \right|^2 dx \leq \int_{\Omega_1} \left| \int_{\mathbb{S}^{n-1}} e^{CM}\, d\theta \right|^2 dx \leq Ce^{2CM},$$

where the constant C depends on n, Ω, and Ω_1. Moreover, we have

$$\begin{aligned}
\|\nabla Kf\|^2_{L^2(\Omega_1)} &= \int_{\Omega_1} \left| \nabla_x \int_{\mathbb{S}^{n-1}} e^{-Xf(x,\theta)}\, d\theta \right|^2 dx = \int_{\Omega_1} \left| \int_{\mathbb{S}^{n-1}} -e^{-Xf} \left(\nabla_x \int f(x+s\theta)\, ds \right) d\theta \right|^2 dx \\
&\leq e^{2CM} \int_{\Omega_1} \left(\int_{\mathbb{S}^{n-1}} \int |\nabla_x f(x+s\theta)|\, ds d\theta \right)^2 dx = e^{2CM} \int_{\Omega_1} (X'X|\nabla f|)^2\, dx \\
&= e^{2CM} \|X'X(|\nabla f|)\|^2_{L^2(\Omega_1)} \leq Ce^{2CM} \|\nabla f\|^2_{L^2(\mathbb{R}^n)} \leq Ce^{2CM} \|f\|^2_{H^1(\mathbb{R}^n)},
\end{aligned}$$

where we applied Proposition 2.2 with $s = 0$ to derive the second last inequality. Combining the two estimates for Kf together yields the result. □

2.2 A reconstruction formula for linearization of K at $f = 0$

To study the inverse problem, we replace $e^{-Xf(x,\theta)}$ in Kf by its Taylor expansion and then obtain

$$\begin{aligned}
Kf(x) &= \int_{\mathbb{S}^{n-1}} (1 - Xf(x,\theta) + Rf(x,\theta))\, d\theta \\
&= |\mathbb{S}^{n-1}| - X'Xf(x) + \int_{\mathbb{S}^{n-1}} Rf(x,\theta)\, d\theta,
\end{aligned} \tag{2.3}$$

where the higher order terms are denoted by

$$Rf(x,\theta) := \sum_{m=2}^{\infty} \frac{(-1)^m}{m!} (Xf)^m(x,\theta)$$

and then $\int_{\mathbb{S}^{n-1}} Rf(x,\theta)\, d\theta$ is finite for $f \in \mathcal{M}_0(\Omega)$.

We linearize the transform K around the zero function so that the problem is reduced to the inverse problem for the X-ray transform.

Proof of Theorem 1.1. Now we take $f = \varepsilon h \in \mathcal{M}_0(\Omega)$ and let $\varepsilon > 0$ be a sufficiently small real number. We differentiate $K(\varepsilon h)$ with respect to ε at $\varepsilon = 0$, denoted by $\partial_\varepsilon|_{\varepsilon=0} K(\varepsilon h)$, and then obtain

$$\begin{aligned}
\partial_\varepsilon|_{\varepsilon=0} K(\varepsilon h)(x) &= \lim_{\varepsilon\to 0} \varepsilon^{-1}(K(\varepsilon h)(x) - K(0)(x)) \\
&= -X'Xh(x) + \lim_{\varepsilon\to 0} \int_{\mathbb{S}^{n-1}} \sum_{m=2}^{\infty} \varepsilon^{m-1} \frac{(-1)^m}{m!} (Xh)^m(x,\theta)\, d\theta \\
&= -X'Xh(x),
\end{aligned} \tag{2.4}$$

where we used the fact that X and $X'X$ are linear operators and also $K(0) = |\mathbb{S}^{n-1}|$.

Applying non-local operator $c_n|D|$ to both sides of (2.4), the reconstruction formula of X-ray transform in Proposition 2.1 yields the following reconstruction formula of linearization at $f = 0$:

$$c_n|D|(\partial_\varepsilon|_{\varepsilon=0} K(\varepsilon h)) = -c_n|D|X'Xh = -h. \tag{2.5}$$

Hence the proof of Theorem 1.1 is complete. □

Remark 2.2. *By Proposition 2.2 with $s = 0$, the formula* (2.5) *also leads to the stability estimate of $h \in \mathcal{M}_0(\Omega)$ in Corollary 1.1.*

2.3 Stability estimate

We are ready to show that the reconstruction of f from the data Kf is stable under suitable assumptions. Below we show the stability estimates with two different approaches.

In the first result, we deduce the stability estimate for small f in the trivial background ($f_0 = 0$). To begin, we first note that for a fixed $f,\ f_0 \in \mathcal{M}_1$, we have

$$K(f) = K(f_0) + K'_{f_0}(f - f_0) + \int_{\mathbb{S}^{n-1}} e^{-Xf_0(x,\theta)} R(f - f_0)(x,\theta)\, d\theta,$$

where $K'_{f_0}(f - f_0) = -\int_{\mathbb{S}^{n-1}} e^{-Xf_0(x,\theta)} X(f - f_0)(x,\theta)\, d\theta$.

Proof of Theorem 1.2. We will check the conditions in Theorem 2 in [13] in order to apply it to deduce the desired stability estimate at $f_0 \equiv 0$. To this end, we take the Banach spaces

$$\mathcal{B}_1'' = H^t_{\overline{\Omega}}(\mathbb{R}^n), \quad \mathcal{B}_1 = \mathcal{M}_1, \quad \mathcal{B}_1' = L^2_{\overline{\Omega}}(\mathbb{R}^n),$$

and

$$\mathcal{B}_2'' = \mathcal{B}_2' = \mathcal{B}_2 = H^1(\Omega_1).$$

Then we have $\mathcal{B}_1'' \subset \mathcal{B}_1 \subset \mathcal{B}_1'$. Here we define the spaces $H^s_{\overline{\Omega}}(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) : \operatorname{supp}(u) \subset \overline{\Omega}\}$.

By the definition of K'_{f_0}, we have $K'_0(f) = -X'Xf$. It is clear that $K'_0 : \mathcal{B}_1'' \to \mathcal{B}_2''$ is a continuous linear map by Proposition 2.2 and the estimate

$$\left\| \int_{\mathbb{S}^{n-1}} Rf(x,\theta)\, d\theta \right\|_{\mathcal{B}_2} \le C\|f\|^2_{\mathcal{B}_1}$$

holds for some constant C independent of f. Next, to check the conditional stability of linearization at $f_0 = 0$, we apply Proposition 2.2 to derive

$$\|K_0'(f)\|_{\mathcal{B}_2'} \geq C^{-1}\|f\|_{\mathcal{B}_1'}.$$

Finally we only need to check the two interpolation estimates. Applying the Sobolev embedding theorem, we have for $s > \frac{n}{2} + 1$,

$$\|f\|_{\mathcal{B}_1} \leq C\|f\|_{H^s(\Omega)}.$$

Moreover, the standard interpolation estimate (see for instance Remark 1 in [13]) implies that

$$\|f\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}^{\mu_1}\|f\|_{H^t(\mathbb{R}^n)}^{1-\mu_1}, \quad \mu_1 = 1 - \frac{s}{t} \geq 0.$$

Hence it follows that

$$\|f\|_{\mathcal{B}_1} \leq C\|f\|_{\mathcal{B}_1'}^{\mu_1}\|f\|_{\mathcal{B}_1''}^{1-\mu_1}.$$

Since $\mathcal{B}_2'' = \mathcal{B}_2' = \mathcal{B}_2$, it is clear that

$$\|f\|_{\mathcal{B}_2'} = \|f\|_{\mathcal{B}_2}^{\mu_2}\|f\|_{\mathcal{B}_2''}^{1-\mu_2}.$$

for any $\mu_2 \in (0, 1]$. Taking $\mu_2 = 1$. If we choose $t > 2s$, then $\mu_1\mu_2 = 1 - \frac{s}{t} > 1/2$. Therefore all conditions required in Theorem 2 in [13] are satisfied, the estimate (1.4) follows. $\square$

Then we show the continuity estimate of K together with another conditional stability estimate.

Proof of Theorem 1.3. Suppose that $\|f_j\|_{C^1(\mathbb{R}^n)} \leq M$ for some constant $M > 0$. For any $x \in \Omega_1$, from (2.3), we have

$$Kf_1(x) - Kf_2(x) = (X'Xf_2 - X'Xf_1)(x) + X'(Rf_1 - Rf_2)(x). \tag{2.6}$$

By direct computations, the remainder term $X'(Rf_1 - Rf_2)(x)$ satisfies

$$\begin{aligned}|X'(Rf_1 - Rf_2)(x)| &\leq \left(CM + \frac{(CM)^2}{2!} + \frac{(CM)^3}{3!} + \cdots\right)(X'X|f_1 - f_2|)(x) \\ &\leq (e^{CM} - 1)(X'X|f_1 - f_2|)(x),\end{aligned}$$

which yields the following two estimates:

$$|Kf_1(x) - Kf_2(x)| \leq |X'X(f_1 - f_2)(x)| + (e^{CM} - 1)(X'X|f_1 - f_2|)(x)$$

and

$$|Kf_1(x) - Kf_2(x)| \geq |X'X(f_1 - f_2)(x)| - (e^{CM} - 1)(X'X|f_1 - f_2|)(x).$$

Here C is a positive constant depending on Ω. Based on these, we can derive the L^2 norm estimate

$$\begin{aligned}&\|X'X(f_1 - f_2)\|_{L^2(\Omega_1)} - (e^{CM} - 1)\|X'X|f_1 - f_2|\|_{L^2(\Omega_1)} \\ &\leq \|Kf_1 - Kf_2\|_{L^2(\Omega_1)} \leq e^{CM}\|X'X|f_1 - f_2|\|_{L^2(\Omega_1)}.\end{aligned} \tag{2.7}$$

Next we estimate $\|\nabla(Kf_1 - Kf_2)\|_{L^2(\Omega_1)}$ by differentiating (2.6):

$$\nabla(Kf_1 - Kf_2)(x) = \nabla(X'Xf_2 - X'Xf_1)(x) + \nabla(X'Rf_1 - X'Rf_2)(x). \tag{2.8}$$

It is sufficient to estimate the remainder term $\nabla X'Rf_1 - \nabla X'Rf_2$. Note that for any $m \geq 2$,

$$
\begin{aligned}
&\frac{1}{m!}|\nabla(X'(Xf_1)^m - X'(Xf_2)^m)(x)| \\
&= \frac{1}{(m-1)!}|X'((Xf_1)^{m-1}\nabla_x Xf_1 - (Xf_2)^{m-1}\nabla_x Xf_2)| \\
&\leq \frac{1}{(m-1)!}|X'(((Xf_1)^{m-1} - (Xf_2)^{m-1})\nabla_x Xf_1)| + \frac{1}{(m-1)!}|X'((Xf_2)^{m-1}(\nabla_x Xf_1 - \nabla_x Xf_2))| \\
&\leq \frac{1}{(m-2)!}(CM)^{m-1}X'X|f_1 - f_2| + \frac{1}{(m-1)!}(CM)^{m-1}X'X|\nabla(f_1 - f_2)|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|\nabla(X'Rf_1 - X'Rf_2)(x)| \leq &\left(CM + (CM)^2 + \frac{(CM)^3}{2!} + \cdots\right) X'X|f_1 - f_2|(x) \\
&+ \left(CM + \frac{(CM)^2}{2!} + \frac{(CM)^3}{3!} + \cdots\right) X'X|\nabla(f_1 - f_2)(x)|,
\end{aligned}
$$

which leads to

$$
\|\nabla X'Rf_1 - \nabla X'Rf_2\|_{L^2(\Omega_1)} \leq CMe^{CM}\|X'X|f_1 - f_1|\|_{L^2(\Omega_1)} + (e^{CM} - 1)\|X'X|\nabla f_1 - \nabla f_2|\|_{L^2(\Omega_1)}.
$$

Now since $f_1 - f_2$ is compactly supported in $\overline{\Omega}$, we apply Proposition 2.2 to get

$$
\|X'X|f_1 - f_2|\|_{L^2(\Omega_1)} \leq \|X'X|f_1 - f_2|\|_{H^1(\Omega_1)} \leq C\|f_1 - f_2\|_{L^2(\mathbb{R}^n)}
$$

and similarly,

$$
\|X'X|\nabla f_1 - \nabla f_2|\|_{L^2(\Omega_1)} \leq C\|\nabla(f_1 - f_2)\|_{L^2(\mathbb{R}^n)}.
$$

Thus, the above inequalities imply that

$$
\begin{aligned}
&\|\nabla X'Rf_1 - \nabla X'Rf_2\|_{L^2(\Omega_1)} \\
&\leq CMe^{CM}\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} + C(e^{CM} - 1)\|\nabla(f_1 - f_2)\|_{L^2(\mathbb{R}^n)} =: \mathcal{G}.
\end{aligned}
\tag{2.9}
$$

Note that Proposition 2.2 yields that

$$
C^{-1}\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} \leq \|X'Xf_2 - X'Xf_1\|_{H^1(\Omega_1)} \leq C\|f_1 - f_2\|_{L^2(\mathbb{R}^n)}
$$

Combining with the second inequality of (2.7) and also (2.8), (2.9), we now have

$$
\begin{aligned}
&\|Kf_1 - Kf_2\|_{H^1(\Omega_1)} \\
&\leq e^{CM}\|X'X|f_1 - f_2|\|_{L^2(\Omega_1)} + \|\nabla(X'Xf_2 - X'Xf_1)\|_{L^2(\Omega_1)} + \mathcal{G} \\
&\leq C(1 + e^{CM} + Me^{CM})\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} + C(e^{CM} - 1)\|\nabla(f_1 - f_2)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

On the other hand, combining with the first inequality of (2.7) and also (2.8), (2.9), we obtain the lower bound

$$
\begin{aligned}
&\|Kf_1 - Kf_2\|_{H^1(\Omega_1)} \geq \|X'X(f_1 - f_2)\|_{H^1(\Omega_1)} - (e^{CM} - 1)\|X'X|f_1 - f_2|\|_{L^2(\Omega_1)} - \mathcal{G} \\
&\geq (C^{-1} - C(e^{CM} - 1) - CMe^{CM})\|f_1 - f_2\|_{L^2(\mathbb{R}^n)} - C(e^{CM} - 1)\|\nabla(f_1 - f_2)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
\tag{2.10}
$$

Note that when M is decreasing to zero, so is $e^{CM}-1$. It implies that $C^{-1}-C(e^{CM}-1)-CMe^{CM}$ is nonnegative if M is small enough. Therefore, if we fix some constant $c>0$, given any f_1 and f_2 satisfying $\|f_1-f_2\|_{L^2(\mathbb{R}^n)} \geq c\|\nabla(f_1-f_2)\|_{L^2(\mathbb{R}^n)}$, then we can derive from (2.10) that

$$\widetilde{C}\|f_1-f_2\|_{L^2(\mathbb{R}^n)} \leq \|Kf_1-Kf_2\|_{H^1(\Omega_1)}, \tag{2.11}$$

where the positive constant $\widetilde{C}=C^{-1}-C(1+1/c)(e^{CM}-1)-CMe^{CM}$, depending on c, M with M sufficiently small. $\square$

Remark 2.3. *We note that the stability estimate improves when the magnitude of f becomes smaller. To explain this, we observe from* (2.11) *that when M is decreasing, the term $C^{-1}\|f_1-f_2\|_{L^2(\mathbb{R}^n)}$ on the right-hand side will dominate. Hence, the whole estimate becomes slightly stabler since the coefficient $\widetilde{C}$ is increasing.*

Finally we make a comment on the connection between the stability estimate (1.3) in Corollary 1.1 and the lower bound (2.10). Indeed (2.10) implies (1.3) when either one of f_j is zero.

More precisely, we take $f_1=\varepsilon h\in\mathcal{M}_1(\Omega)$ with ε, $h\geq 0$, and $f_2\equiv 0$ with $\widetilde{M}:=\|\varepsilon h\|_{C^1(\mathbb{R}^n)}$. The estimate (2.10) yields that

$$\|K(\varepsilon h)-K(0)\|_{H^1(\Omega_1)} \geq (C^{-1}-C(e^{C\widetilde{M}}-1)-C\widetilde{M}e^{C\widetilde{M}})\|\varepsilon h\|_{L^2(\mathbb{R}^n)} - C(e^{C\widetilde{M}}-1)\|\nabla(\varepsilon h)\|_{L^2(\mathbb{R}^n)}.$$

Dividing by ε and letting $\varepsilon\to 0$ (then $\widetilde{M}\to 0$), one has

$$\|\partial_\varepsilon|_{\varepsilon=0}K(\varepsilon h)\|_{H^1(\Omega_1)} \geq C^{-1}\|h\|_{L^2(\mathbb{R}^n)}.$$

2.4 Single pixel transform on Riemannian manifolds

As mentioned in the introduction, the second author [9] proved the uniqueness of the single pixel X-ray transform K in the Euclidean space $\mathbb{R}^n$. In this section, we show that the proof for the Euclidean case can be generalized to the case of non-trivial geometries.

Let (M,g) be an n-dimensional, $n\geq 2$, compact non-trapping Riemannian manifold with smooth strictly convex boundary ∂M. Here non-trapping means that every geodesic exits the manifold in both directions in finite times. Let SM be the unit sphere bundle consisting of all unit vectors on M, so any $(x,v)\in SM$ satisfying $\|v\|_{g(x)}=1$. Given any $(x,v)\in SM$, let $\gamma_{x,v}$ be the geodesic with initial conditions $\gamma_{x,v}(0)=x$, $\dot\gamma_{x,v}(0)=v$. We define the X-ray transform of a function f on (M,g) as

$$Xf(x,v)=\int f(\gamma_{x,v}(t))\,dt, \quad (x,v)\in SM.$$

Since (M,g) is non-trapping, the above integral is indeed over a finite interval. Moreover, $Xf(x,v)=Xf(\gamma_{x,v}(t),\dot\gamma_{x,v}(t))$ for all t. Then the single pixel X-ray transform on (M,g) is defined by

$$Kf(x)=\int_{S_xM} e^{-Xf(x,v)}\,dv, \quad x\in M.$$

Let $C(M)$ be the space of continuous functions on M.

Theorem 2.1. *Let (M,g) be a compact non-trapping Riemannian manifold with smooth strictly convex boundary. Assume that the X-ray transform X is injective on $C(M)$, then the single pixel X-ray transform K is injective on $C(M)$, in other words, $Kf = Kh$ implies that $f = h$ for f, $h \in C(M)$.*

Proof. We denote ∂_+SM the set of all unit inward pointing vectors on the boundary ∂M, i.e. $(x,v) \in \partial_+SM$ if and only if $x \in \partial M$, and $\langle v, \nu(x)\rangle_g \geq 0$ where $\nu(x)$ is the unit inward normal vector at x. Let $d\mu(x,v) = \langle v, \nu(x)\rangle_g dxdv$ be a measure on ∂_+SM which vanishes in directions tangent to ∂M. Consider

$$\begin{aligned}\langle Kf - Kh, f-h\rangle &= \int_M (Kf-Kh)(x)(f-h)(x)\,dx\\ &= \int_{SM} (e^{-Xf(x,v)} - e^{-Xh(x,v)})(f-h)(x)\,dxdv\\ &= \int_{\partial_+SM} d\mu(x,v)\int (e^{-Xf(\gamma_{x,v}(t),\dot\gamma_{x,v}(t))} - e^{-Xh(\gamma_{x,v}(t),\dot\gamma_{x,v}(t))})(f-h)(\gamma_{x,v}(t))\,dt.\end{aligned}$$

The last equality is a direct application of the Santalo's formula, see e.g. [11, Lemma 3.3.2].

Notice that $Xf(\gamma_{x,v}(t),\dot\gamma_{x,v}(t))$ is invariant with respect to t, thus

$$\begin{aligned}\langle Kf - Kh, f-h\rangle &= \int_{\partial_+SM} (e^{-Xf(x,v)} - e^{-Xh(x,v)})\,d\mu(x,v)\int (f-h)(\gamma_{x,v}(t))\,dt\\ &= \int_{\partial_+SM} (e^{-Xf(x,v)} - e^{-Xh(x,v)})(Xf(x,v) - Xh(x,v))\,d\mu(x,v).\end{aligned}$$

The integrand in the last integral has the form $(e^{-u} - e^{-v})(u-v)$, which is non-positive, and it equals zero if and only if $u = v$. Therefore, if $Kf = Kh$, we have that $Xf(x,v) = Xh(x,v)$ for all $(x,v) \in \partial_+SM$. By our assumption, the X-ray transform X is injective, thus $f = h$. □

It is known that the X-ray transform is injective on *simple* manifolds [5, 6], which are compact non-trapping manifolds with strictly convex boundary and free of conjugate points. In dimension ≥ 3, X is injective on compact non-trapping manifolds with strictly convex boundary, which admit convex foliations [16]. The convex foliation condition allows the existence of conjugate points.

In the current paper, we only consider the uniqueness of K on Riemannian manifolds. It's reasonable to expect that stability estimates similar to Theorem 1.2, 1.3 will hold on simple manifolds as well. In particular, stability estimates of the normal operator $X'X$ on simple manifolds can be found in [12].

3 Numerical experiments

In this section, we conduct several numerical experiments to corroborate our theoretical results above. We use the Shepp-Logan phantom with 101×101 pixels for illustrations.

Assume k is the measured data. More precisely, $k(x)$ is the single pixel X-ray transform of f at the point x. For all the numerical tests below, k has the same resolution as f. To reconstruct f from the data k_0, we use Gauss-Newton method to minimize the following functional:

$$\arg\min_f \frac{1}{2}\|Kf - k_0\|^2_{L^2}.$$

To compute the gradient of the above functional, note that for

$$Kf(x) = \int_{\mathbb{S}^{n-1}} e^{-Xf(x,\theta)} \mathrm{d}\theta,$$

one can see that the Frechét derivative of K at f is given as

$$K'_f(h)(x) = \int_{\mathbb{S}^{n-1}} e^{-Xf(x,\theta)}(-Xh(x,\theta))\mathrm{d}\theta$$

for any function h. For computation of the X-ray transform Xf, we adapt the code provided in Carsten Høilund's lecture notes [2].

We first reconstruct f when the data k is not noisy. The data k is generated using a finer grid with 201×201 pixels and then downsampled by averaging into 101×101 pixels. This is because that using the same discretization would make the inverse problem look less ill-posed. The results are shown in Figure 1. The true image of Shepp-Logan is in the middle. The image of Kf is on the left.

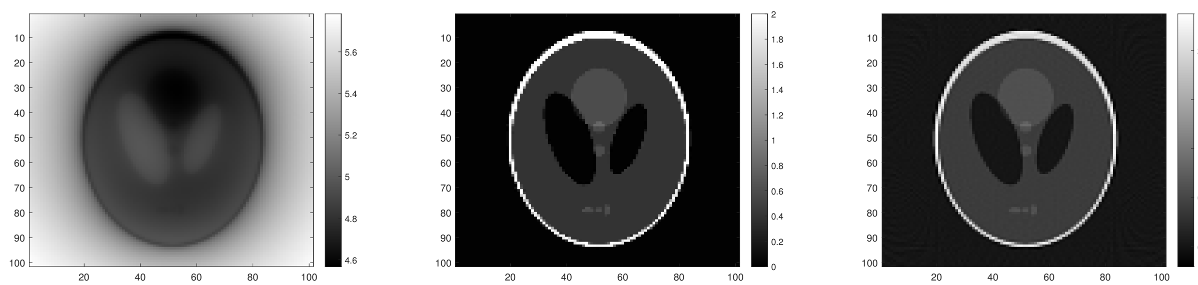


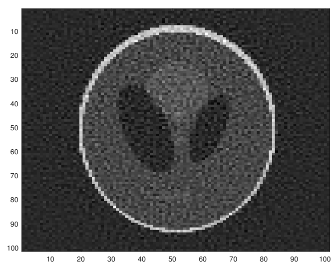
Figure 1: left: true data; middle: true image, right: reconstructed image

For the following numerical tests, the data k is generated with the same grid as for the reconstruction, but the data k is polluted by Gaussian random noise. The noise level is compared with $Kf - 2\pi$, not Kf itself, since when $f = 0$, $Kf = 2\pi$ in $\mathbb{R}^2$. The results are presented in Figure 2, where the reported error is measured in L^2 norm. One can see that the magnitude of error is almost linearly dependent on the level of noise. This confirms the Lipschitz stability result in Theorem 1.3. Notice that even if the noise level is low when compared with Kf, real information for small f's could be completely lost.

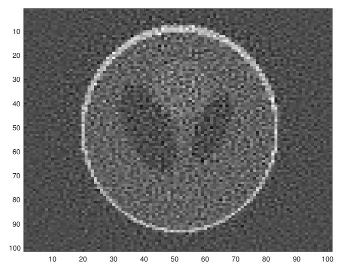
Finally, we reconstruct the same Shepp-Logan phantom with different magnitudes of f. For these numerical experiments, we generate data on a finer mesh to avoid "inverse crimes". Although we do not manually add noise, discretization itself generates noise or artifacts. The results are displayed in Figures 3. One can see that for both $0.1f$ and $1f$, the reconstructions perform quite well. Moreover, Figure 3 shows that the quality of image deteriorates if the magnitude of f becomes larger. For $20f$, the reconstruction is already a total failure. This result suggests that the Lipschitz stability derived for small f's no longer holds for large ones. One can also see that some regularization techniques need to be adopted for the reconstruction, when the ill-posedness becomes worse as the magnitude of f becomes larger.

(a) the reconstruction with no noise

(b) the reconstruction with 0.1% relative noise; the error is 0.7289



(c) the reconstruction with 0.5% relative noise; the error is 3.5906



(d) the reconstruction with 1% relative noise; the error is 7.1819

Figure 2: Recovery with different noises

Acknowledgements

RL is partially supported by the National Science Foundation (NSF) through grant DMS-2006731. GU is partly supported by NSF, a Simons Fellowship, a Walker Family Professorship at UW, and a Si Yuan Professorship at IAS, HKUST. Part of this research was performed while GU was visiting the Institute for Pure and Applied Mathematics (IPAM) in Fall 2021. HZ is partly supported by NSF grant DMS-2109116.

References

[1] S. Helgason. *Integral Geometry and Radon Transforms*. Springer, 2010.

[2] C. Høilund. The radon transform. *Aalborg University*, 12, 2007.

[3] J. Ilmavirta and F. Monard. Integral geometry on manifolds with boundary and applications. *The Radon Transform: the first 100 years and beyond*, pages 43–114, 2019.

[4] V. Isakov. On uniqueness in inverse problems for semilinear parabolic equations. *Archive for Rational Mechanics and Analysis*, 124(1):1–12, 1993.

[5] R. G. Mukhometov. The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. *Dokl. Akad. Nauk SSSR (in Russian)*, 232:32–35, 1977.

[6] R. G. Mukhometov and V. G. Romanov. On the problem of finding an isotropic Riemannian metric in an n-dimensional space. *Dokl. Akad. Nauk SSSR (in Russian)*, 243:41–44, 1978.

[7] F. Natterer. *The Mathematics of Computerized Tomography.* SIAM, 2001.

[8] G.P. Paternain, M. Salo, and G. Uhlmann. Tensor tomography: progress and challenges. *Chinese Annals of Math. Ser. B*, 35:399–428, 2014.

[9] R. R. Macdonald R. S. Kemp, A. Danagoulian and J. R. Vavrek. Physical cryptographic verification of nuclear warheads. *PNAS*, 113 (31):8618–8623, 2016.

[10] J. Radon. über die bestimmung von funktionen durch ihre integralwerte lngs gewisser mannigfaltigkeiten. *Berichte über die Verhandlungen der Königlich-Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse*, 69:262–277, 1917.

[11] V. Sharafutdinov. Ray transform on Riemannian manifolds. Eight lectures on integral geometry. *http://math.nsc.ru/ sharafutdinov/files/Lectures.pdf.*

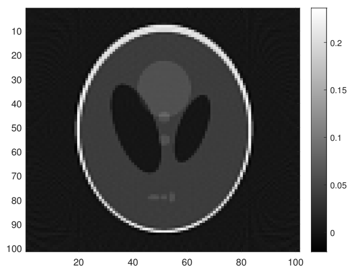
[12] P. Stefanov and G. Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123:445–467, 2004.

[13] P. Stefanov and G. Uhlmann. Linearizing non-linear inverse problems and an application to inverse backscattering. *Journal of Functional Analysis*, 256(9):2842–2866, 2009.

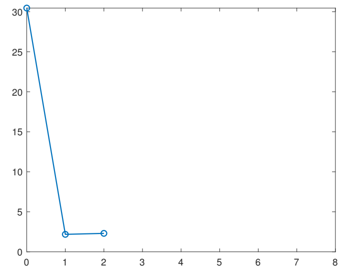
[14] P. Stefanov and G. Uhlmann. *Microlocal Analysis and Integral Geometry (working title).* draft version, 2018.

[15] M. Taylor. *Pseudodifferential Operators*, volume 34, Princeton Mathematics Series. Princeton, NJ: Princeton University Press, 1981.

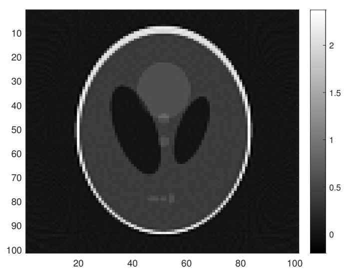
[16] G. Uhlmann and A. Vasy. The inverse problem for the local geodesic ray transform. *Inventiones Mathematicae*, 205:83–120, 2016.



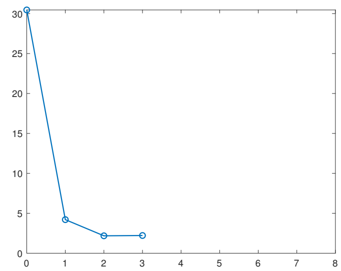
(a) reconstruction for $0.1f$



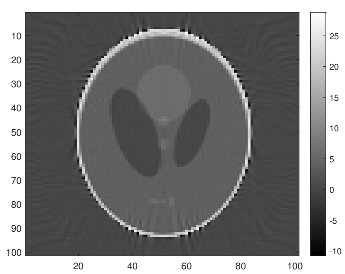
(b) errors$\times$10 versus steps



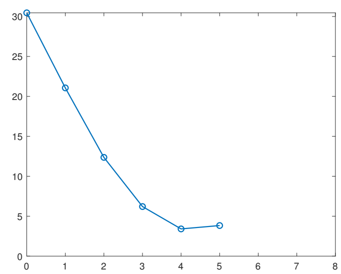
(c) reconstruction for f



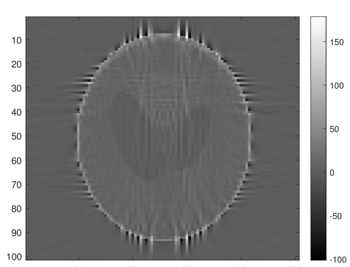
(d) errors versus steps



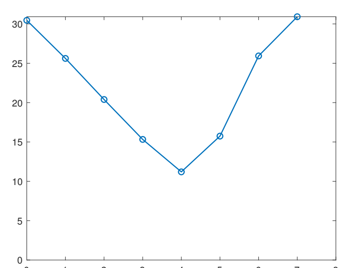
(e) reconstruction for $10f$



(f) errors/10 versus steps



(g) reconstruction for $20f$



(h) errors/20 versus steps

Figure 3: Noiseless reconstruction for different magnitudes of f. The errors are scaled to be in the same range.