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Computing reflection length in an affine Coxeter group

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Abstract. In any Coxeter group, the conjugates of elements in its Coxeter generating set are called reflections and the reflection length of an element is its length with respect to this expanded generating set. In this article we give a simple formula that computes the reflection length of any element in any affine Coxeter group. In the affine symmetric group, we have a combinatorial formula that generalizes Dénes' formula for reflection length in the symmetric group.

Keywords: Coxeter groups, affine reflection groups, reflection length

Introduction

In any Coxeter group W, the conjugates of elements in its standard Coxeter generating set are called *reflections* and we write R for the set of all reflections in W. The reflections generate W and the associated *reflection length* function $\ell_R \colon W \to \mathbb{Z}_{\geq 0}$ records the length of w with respect to this expanded generating set. When W is spherical, i.e., finite, reflection length can be given an intrinsic, geometric definition, as follows. Define a *root subspace* to be any space spanned by a subset of the corresponding root system and define the *dimension* $\dim(w)$ of an element w in W to be the minimum dimension of a root subspace that contains the move-set of w. (See Section 3 for the details.) The following result is due to Carter [2].

Theorem ([2, Lemma 2])**.** *Let W be a spherical Coxeter group and let $w \in W$. Then* $\ell_R(w) = \dim(w)$.

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When W is infinite, much less is known [8, 4, 9], even in the affine case. In this work, we give a simple, analogous formula that computes the reflection length of any element in any affine Coxeter group.[1]

Theorem A (Formula)**.** *Let W be an affine Coxeter group and let $p\colon W \twoheadrightarrow W_0$ be the projection onto its associated spherical Coxeter group. For any element $w \in W$, its reflection length is*

$$\ell_R(w) = 2 \cdot \dim(w) - \dim(p(w)) = 2d + e,$$

where $e = \dim(p(w))$ and $d = \dim(w) - \dim(p(w))$.

We call $e = e(w) = \dim(p(w))$ the *elliptic dimension* of w and we call $d = d(w) = \dim(w) - \dim(p(w))$ the *differential dimension* of w. Both statistics are geometrically meaningful. For example, $e(w) = 0$ if and only if w is a translation (that is, if it sends every point x to $x + \lambda$ for some fixed vector λ), and $d(w) = 0$ if and only if w is elliptic (that is, if it fixes a point) – see Proposition 3.4. In [7], we prove Theorem A by showing the claimed value is both a lower bound and an upper bound for the reflection length of w.

The translation and elliptic versions of our formula were already known (see [8] and [2], respectively) and these special cases suggest that the formula in Theorem A might correspond to a carefully chosen factorization of w as a product of a translation and an elliptic element. This is indeed the case.

Theorem B (Factorization)**.** *Let W be an affine Coxeter group. For every element $w \in W$, there is a translation-elliptic factorization $w = t_\lambda u$ such that $\ell_R(t_\lambda) = 2d(w)$ and $\ell_R(u) = e(w)$. In particular, $\ell_R(w) = \ell_R(t_\lambda) + \ell_R(u)$ for this factorization of w.*

The proof of Theorem B relies on a nontrivial technical result recently established by Vic Reiner and the first author [6, Corollary 1.4]. Elements in an affine Coxeter group typically have many different potential translation-elliptic factorizations and the most common way to find one is to view the group as a semidirect product. For an affine Coxeter group W that naturally acts cocompactly on an n-dimensional euclidean space, the set of translations in W forms a normal abelian subgroup T isomorphic to $\mathbb{Z}^n$, and the quotient $W_0 = W/T$ is the spherical Coxeter group associated with W. An identification of W as a semidirect product $T \rtimes W_0$ corresponds to a choice of an inclusion map $i\colon W_0 \hookrightarrow W$ that is a section of the projection map $p\colon W \twoheadrightarrow W_0$. There is a unique point x fixed by the subgroup $i(W_0)$ and every element has a unique factorization $w = t_\lambda u$ where t_λ is a translation in T and u is an elliptic element in the copy of W_0 that fixes x. However, for some elements w none of the translation-elliptic factorizations that come from an identification of W as a semidirect product satisfy Theorem B – see Example 2.4 of [7].

[1] This note is a condensed version of [7], in which we provide a simple uniform proof of Theorem A. Indeed, proofs of all results are omitted here but can be found in the full paper.

In the particular case of the symmetric group $\mathfrak{S}_n$ (the spherical Coxeter group of type A), reflection length also has a natural combinatorial characterization: $\ell_R(w) = n - c(w)$, where $c(w)$ is the number of *cycles* of the permutation w [3]. When W is the affine symmetric group $\widetilde{\mathfrak{S}}_n$, we show that the differential and elliptic dimensions of an element can also be given a combinatorial interpretation, leading to the very similar formula in Theorem 5.3.

In this brief note, we highlight the key definition at play in Theorem A, namely the notion of dimension of an element. We assume the reader is familiar with Coxeter groups at the level of Humphreys [5]. Full details can be found in [7].

1 Points, vectors, and affine Coxeter groups

Before we define affine Coxeter groups, it is helpful to establish notions for spherical Coxeter groups.

Definition 1.1 (Spherical Coxeter groups)**.** A *euclidean vector space* V is an n-dimensional real vector space equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$. A *crystallographic root system* Φ is a finite collection of vectors that span a real euclidean vector space V satisfying a few elementary properties – see [5] for a precise definition. (While there are non-crystallographic root systems as well, it is only the crystallographic root systems that arise in the study of affine Coxeter groups.) The elements of Φ are called *roots*. Each crystallographic root system corresponds to a finite (or *spherical*) Coxeter group W_0, as follows: for each α in Φ, H_α is the hyperplane through the origin in V orthogonal to α, and the unique nontrivial isometry of V that fixes H_α pointwise is a *reflection* that we call r_α. The collection $R = \{r_\alpha \mid \alpha \in \Phi\}$ generates the spherical Coxeter group W_0, and R is its set of reflections.

In the same way that doing linear algebra using fixed coordinate systems can obscure an underlying geometric elegance, working in affine Coxeter groups with a predetermined origin can have an obfuscating effect. One way to avoid making such a choice is to distinguish between *points* and *vectors*, as in [11] and [1].

Definition 1.2 (Points and vectors)**.** Let V be a euclidean vector space. A *euclidean space* is a set E with a uniquely transitive V-action, i.e., for every ordered pair of points $x, y \in E$ there exists a unique vector $\lambda \in V$ such that the λ sends x to y. When this happens we write $x + \lambda = y$. The preceding sentences illustrate two conventions that we adhere to throughout the paper: the elements of E are *points* and are denoted by Roman letters, such as x and y, while the elements of V are *vectors* and are denoted by Greek letters, such as λ and μ.

The main difference between E and V is that V has a well-defined origin, but E does not. If we select a point $x \in E$ to serve as the origin, then V and E can be identified by sending each vector λ to the point $x + \lambda$. We use this identification in the construction of the affine Coxeter groups.

Definition 1.3 (Affine Coxeter groups)**.** Let E be a euclidean space, whose associated vector space V contains the crystallographic root system Φ. An affine Coxeter group W can be constructed from Φ, as follows. Fix a point x in E, to temporarily identify V and E. For each $\alpha \in \Phi$ and $j \in \mathbb{Z}$, let $H_{\alpha,j}$ denote the (affine) *hyperplane* in E of solutions to the equation $\langle v, \alpha \rangle = j$, where the brackets denote the standard inner product (treating x as the origin). The unique nontrivial isometry of E that fixes $H_{\alpha,j}$ pointwise is a *reflection* that we call $r_{\alpha,j}$. The collection $R = \{r_{\alpha,j} \mid \alpha \in \Phi, j \in \mathbb{Z}\}$ generates the affine Coxeter group W and R is its set of reflections. A standard minimal generating set S can be obtained by restricting to those reflections that reflect across the facets of a certain polytope in E.

The affine Coxeter group W associated to a finite crystallographic root system Φ is closely related to the spherical Coxeter group W_0.

Definition 1.4 (Subgroups and quotients)**.** Let W be an affine Coxeter group constructed as in Definition 1.3. The map sending $r_{\alpha,j}$ in W to r_α in W_0 extends to a group homomorphism $p \colon W \twoheadrightarrow W_0$. The kernel of p is a normal abelian subgroup T, isomorphic to $\mathbb{Z}^n$, whose elements are called *translations*, and $W_0 \cong W/T$. When x is the point used in the construction of W, the map $i \colon W_0 \hookrightarrow W$ sending r_α to $r_{\alpha,0}$ is a section of the projection p, identifying W_0 with the subgroup of all elements of W that fix x. (Note also the composition $i \circ p$ sends reflection $r_{\alpha,j}$ to r_α to $r_{\alpha,0}$.) Thus W may be identified as a semidirect product $W \cong T \rtimes W_0$.

Of course, the identification of W_0 with a subgroup of W is not unique: conjugation by elements of T gives an infinite family of such subgroups.

We now define two fundamental affine subspaces associated to any element w.

Definition 1.5 (Move-sets)**.** The *motion* of a point $x \in E$ under a euclidean isometry w is the vector $\lambda \in V$ such that $w(x) = x + \lambda$. The *move-set* of w is the collection of all motions of the points in E; by [1, Proposition 3.2], it is an affine subspace of V. In symbols, $\mathrm{Mov}(w) = \{\lambda \mid w(x) = x + \lambda \text{ for some } x \in E\} \subset V$.

Definition 1.6 (Fixed space)**.** The *fixed space* $\textsc{Fix}(w)$ of an isometry w is the subset of points $x \in E$ such that $w(x) = x$. Equivalently, $\textsc{Fix}(w)$ consists of all points whose motion under w is the vector 0. When $\textsc{Fix}(w)$ is nonempty, it is an (affine) subspace of E.

When w is an element of a spherical Coxeter group, it is just an orthogonal transformation of V, so $\textsc{Fix}(w) = \textsc{Ker}(w-1)$ and $\mathrm{Mov}(w) = \textsc{Im}(w-1)$. In particular, $\textsc{Fix}(w)$ and $\mathrm{Mov}(w)$ are orthogonal complements in V. However, this complementarity between

$\text{Mov}(w)$ and $\text{Fix}(w)$ does not hold for isometries of euclidean space. It can be recovered if the fixed space is replaced with the min-set of points that are moved a minimal distance under w. Then the space of directions for the move-set and the space of directions for the min-set give an orthogonal decomposition of V – see [1, Lemma 3.6].

Move-sets for general euclidean plane isometries are given in Example 3.5.

2 Elliptics and translations

In this section, we record some basic facts about special kinds of elements in an affine Coxeter group.

Definition 2.1 (Elliptic elements and elliptic part)**.** An element w in an affine Coxeter group W is called *elliptic* if its fixed space is non-empty. Equivalently, these are exactly the elements of W of finite order. Given an arbitrary element $w \in W$, its *elliptic part* $w_e = p(w)$ is its image under the projection $p : W \twoheadrightarrow W_0$. (In particular, the elliptic part is an element of W_0, acting naturally on V rather than on E.)

Definition 2.2 (Coroots)**.** Let $\Phi \subset V$ be a crystallographic root system. For each root $\alpha \in \Phi$, its *coroot* is the vector $\alpha^\vee = \frac{2}{\langle \alpha,\alpha \rangle}\alpha$. The collection of these coroots is another crystallographic root system $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. The $\mathbb{Z}$-span $L(\Phi^\vee)$ of $\Phi^\vee$ in V is the *coroot lattice*; as an abelian group, it is also isomorphic to $\mathbb{Z}^n$ where $n = \dim(V)$.

Definition 2.3 (Translations)**.** For every vector $\lambda \in V$ there is a euclidean isometry t_λ of E called a *translation* that sends each point $x \in E$ to $x + \lambda$. Let W be an affine Coxeter group acting on E with root system $\Phi \subset V$. An element of W is a translation in this sense if and only if it is in the kernel T of the projection $p\colon W \twoheadrightarrow W_0$. Moreover, the set of vectors in V that define the translations in T is identical to the set of vectors in the coroot lattice $L(\Phi^\vee)$.

Definition 2.4 (Translation-elliptic factorizations)**.** Let W be an affine Coxeter group. There are many ways to write an element $w \in W$ as a product of a translation $t_\lambda \in W$ and an elliptic $u \in W$. We call any such factorization $w = t_\lambda u$ a *translation-elliptic factorization* of w. In any such factorization we call t_λ the *translation part* and u the *elliptic part*.

Definition 2.5 (Normal forms)**.** Let W be an affine Coxeter group acting cocompactly on a euclidean space E. An identification of W as a semidirect product $T \rtimes W_0$ corresponds to a choice of an inclusion map $i\colon W_0 \hookrightarrow W$ that is a section of the projection map $p\colon W \twoheadrightarrow W_0$. The unique point $x \in E$ fixed by the subgroup $i(W_0)$ serves as our origin and every element has a unique factorization $w = t_\lambda u$ where t_λ is a translation in T and u is an elliptic element in $i(W_0)$. In particular, $u = i(w_e)$ is the image under i of the elliptic part of w (Definition 2.1). For a fixed choice of a section i, we call this unique factorization $w = t_\lambda u$ the *normal form* of w under this identification.

If $w = t_\lambda u$ is a translation-elliptic factorization then $u_e = w_e \in W_0$ and t_λ is in the kernel of p. Some translation-elliptic factorizations come from an identification of W and $T \rtimes W_0$ as in Definition 2.5, but not all of them do: see Example 2.4 of [7].

Remark 2.6 (Maximal elliptics). Let W be an affine Coxeter group acting cocompactly on an n-dimensional euclidean space E. When u is an elliptic element of reflection length $n = \dim(E)$ (such as a Coxeter element of a maximal parabolic subgroup of W), its move-set $\mathrm{Mov}(u)$ is all of V, the elements $t_\lambda u$ are elliptic for all choices of translation $t_\lambda \in T$ and, in particular, $\ell_R(t_\lambda u) = \ell_R(u) = n$ for every $t_\lambda \in T$.

3 Dimension of an element

This section shows how to assign a dimension to each element in a spherical or affine Coxeter group. It is based on the relationship between move-sets and root spaces.

Definition 3.1 (Root spaces)**.** Let V be a euclidean vector space with root system Φ. A subset $U \subset V$ is called a *root space* if it is the span of the roots it contains. In symbols, U is a root space when $U = \textsc{Span}(U \cap \Phi)$. Equivalently, U is a root space when U is a linear subspace of V that is spanned by a collection of roots, or when U has a basis consisting of roots. Since Φ is a finite set, there are only finitely many root spaces; the collection of all root spaces in V is called the *root space arrangement* $\textsc{Arr}(\Phi) = \{U \subset V \mid U = \textsc{Span}(U \cap \Phi)\}$.

Definition 3.2 (Root dimension)**.** For any subset $A \subset V$, we define its *root dimension* $\dim_\Phi(A)$ to be the minimal dimension of a root space in $\textsc{Arr}(\Phi)$ that contains A. Since V itself is a root space, $\dim_\Phi(A)$ is defined for every subset A in V.

Definition 3.3 (Dimension of an element)**.** When w is an element of a spherical or affine Coxeter group, its move-set is contained in a euclidean vector space V that also contains the corresponding root system Φ. The *dimension* $\dim(w)$ of such an element is defined to be the root dimension of its move-set. In symbols, $\dim(w) = \dim_\Phi(\mathrm{Mov}(w))$. Let W be an affine Coxeter group acting on a euclidean space E and let $p \colon W \twoheadrightarrow W_0$ be its projection map. For each element $w \in W$ we can compute the dimension of w and the dimension of its elliptic part $w_e = p(w) \in W_0$. We call $e = e(w) = \dim(w_e)$ the *elliptic dimension* of w. Instead of focusing on the dimension of w itself, we focus on the number $d = d(w) = \dim(w) - \dim(w_e)$, which we call the *differential dimension* of w. Note that $\dim(w) = d + e$.

The statistics $d(w)$ and $e(w)$ carry geometric meaning.

Proposition 3.4 (Statistics and geometry)**.** *Let W be an affine Coxeter group. An element $w \in W$ is a translation if and only if $e(w) = 0$, and w is elliptic if and only if $d(w) = 0$.*

w	$\mathrm{Mov}(w)$	d	e	ℓ_R
identity	the origin	0	0	0
reflection	a root line	0	1	1
rotation	the plane	0	2	2
translation	an affine point	1 or 2	0	2 or 4
glide reflection	an affine line	1	1	3

Table 1: Basic invariants for the 5 types of elements in an affine Coxeter group acting on the euclidean plane.

Thus one way to interpret these numbers is that, roughly speaking, $d(w)$ measures how far w is from being an elliptic element and $e(w)$ measures how far w is from being a translation.

Example 3.5 (Euclidean plane)**.** The affine Coxeter groups that act on the euclidean plane have five different types of move-sets (see Table 1). Among the elliptic elements, the move-set of the identity is the point at the origin, the move-set of a reflection r with root $\alpha \in \Phi$ is the root line $\mathbb{R}\alpha$, and the move-set of any non-trivial rotation is all of $V = \mathbb{R}^2$. For these elements, $d(w) = 0$ and $\ell_R(w) = \dim(w) = \dim(w_e) = e(w)$ where this common value is 0, 1, or 2, respectively. The move-set of a non-trivial translation t_λ is the single nonzero vector $\{\lambda\}$. Its elliptic dimension is 0 and its differential dimension is either 1 (when λ is contained in a root line $\mathbb{R}\alpha$) or 2. By [8, Proposition 4.3], $\ell_R(t_\lambda)$ is twice its dimension. Finally, when w is a glide reflection, $\mathrm{Mov}(w)$ is a line not through the origin, so $e = \dim(w_e) = 1$, $\dim(w) = 2$, $d = 2 - 1 = 1$, and $\ell_R(w) = 3$.

We finish this section with a pair of remarks about computing e and d in general. Let W be an affine Coxeter group with a fixed identification of W with $T \rtimes W_0$ and let $w \in W$ be an element that is given in its semidirect product normal form $w = t_\lambda u$ for some vector λ in the coroot lattice $L(\Phi^\vee)$ and some elliptic element u. Computing the elliptic dimension $e(w)$ is straightforward.

Remark 3.6 (Computing elliptic dimension). To compute the elliptic dimension $e(w)$ it is sufficient to simply compute the dimension of the move-set of the elliptic part u of its normal form. Indeed, by Definition 3.3, $e(w) = \dim(w_e)$, but since $\mathrm{Mov}(w_e)$ is itself a root subspace, $\dim_\Phi(\mathrm{Mov}(w_e)) = \dim(\mathrm{Mov}(w_e))$. Finally, since $p(u) = p(w) = w_e$, $\mathrm{Mov}(w_e) = \mathrm{Mov}(u)$ and so $\dim(\mathrm{Mov}(w_e)) = \dim(\mathrm{Mov}(u))$.

Computing the differential dimension $d(w)$ is more complicated but it can be reduced to computing the dimension of a point in a simpler arrangement of subspaces in a lower dimensional space. How much lower depends on the elliptic dimension $e(w)$.

Remark 3.7 (Computing differential dimension). By Definition 3.3, to compute the differential dimension $d(w)$, we need to find the minimal dimension of a root subspace containing $\mathrm{Mov}(w)$ and then subtract $e(w)$ from this value. Since $w = t_\lambda u$, $\mathrm{Mov}(w) = \lambda + U$ where $U = \mathrm{Mov}(u) = \mathrm{Mov}(w_e)$. As shown in [7], we only need to consider root spaces that contain λ and U or, equivalently, root spaces that contain $\lambda + U$ and U. Let $q \colon V \twoheadrightarrow V/U$ be the natural quotient linear transformation whose kernel is U. Under the map q, the coset $\lambda + U$ is sent to a point in V/U that we call λ/U and the subspaces in $\textsc{Arr}(\Phi)$ containing U are sent to a collection of subspaces in V/U that we call $\textsc{Arr}(\Phi/U)$. Let $\dim_{\Phi/U}(\lambda/U)$ be the minimal dimension of a subspace in $\textsc{Arr}(\Phi/U)$ that contains the point λ/U. Since the dimensions involved have all been diminished by $e(w) = \dim(U)$, we have that $\dim_{\Phi/U}(\lambda/U) = d(w)$ is the differential dimension of w.

4 Local statistics

In a spherical Coxeter group $W = W_0$, Shephard and Todd [10, Theorem 5.3] showed that the generating function for reflection length has a particularly nice form:

$$f_0(t) = \sum_{u \in W_0} t^{\ell_R(t)} = \prod_{i=1}^{n} (1 + e_i t), \tag{4.1}$$

where the numbers e_i are positive integers called the *exponents* of W_0. In earlier work [8], the second and third authors asked whether there are similarly nice generating functions associated to an affine Coxeter group. In this section, we explore this question.

For an affine Coxeter group W, reflection length is bounded and $|W|$ is infinite, so the naive generating function is not defined. A natural fix is to consider only a finite piece of W.

Definition 4.1 (Local generating function)**.** Given an element λ of the coroot lattice $L(\Phi^\vee)$, define the bivariate generating function

$$f_\lambda(s,t) = \sum_{u \in W_0} s^{d(t_\lambda u)} \cdot t^{e(t_\lambda u)} = \sum_{u \in W_0} s^{\dim(t_\lambda u)} \cdot (t/s)^{\dim(u)}$$

that tracks the statistics of differential and elliptic dimension. By Theorem A, we have

$$f_\lambda(t^2, t) = \sum_{u \in W_0} t^{\ell_R(t_\lambda \cdot u)}.$$

By mild abuse of notation, we let $f_\lambda(t) = f_\lambda(t^2, t)$ denote this local reflection length generating function.

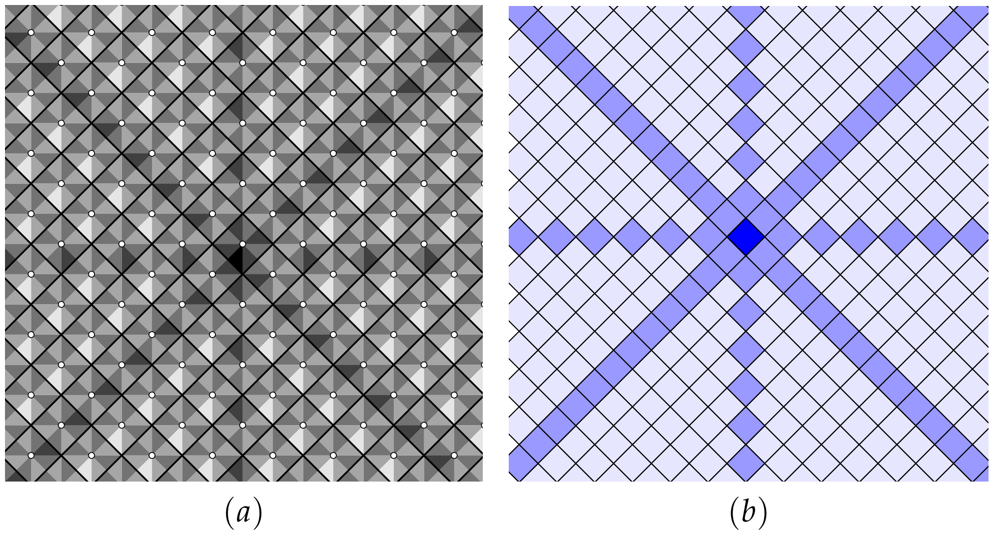

Figure 1: For the affine Coxeter group of type B_2, (a) alcoves shaded by reflection length and (b) the translates $t_\lambda \cdot P$ shaded by local generating function.

The term "local" here makes sense geometrically, once the elements of W have been identified with alcoves in the reflecting hyperplane arrangement for W. The alcoves neighboring the origin (each of which is identified with a unique element of W_0) form a W_0-invariant polytope P. The set $\{t_\lambda u \mid u \in W_0\}$ corresponds to the set the of alcoves in $t_\lambda \cdot P$, i.e., those alcoves neighboring λ. In Figure 1(a) we have shaded the alcoves in affine B_2 according to reflection length, while in Figure 1(b) we have colored the translates $t_\lambda \cdot P$ according to $f_\lambda(t)$. In Figure 1(a), the identity element is identified with the black alcove, and lighter colored cells have greater reflection length. The coroots are highlighted in white.

We collect here some easy facts about the local generating function.

Proposition 4.2 (Properties of local generating functions)**.** *Let λ be a vector in the coroot lattice $L(\Phi^\vee)$.*

1. *(The origin) If $\lambda = 0$, then $f_\lambda(s,t) = f_0(t)$ is the generating function for reflection length in W_0.*
2. *(Generic) If λ is generic, i.e., if it is not a member of any proper root subspace of V, then $f_\lambda(s,t) = s^n f_0(t/s) = \prod_{i=1}^{n}(s + e_i t)$, where the e_i are the exponents of W_0.*
3. *(Permutations) If λ and λ' belong to the same W_0-orbit (that is, there is some $w \in W_0$ such that $\lambda' = w(\lambda)$), then $f_\lambda(s,t) = f_{\lambda'}(s,t)$.*

From the pictures it appears that the local generating functions line up along faces of a hyperplane arrangement. The faces of the arrangement are intersections of maximal

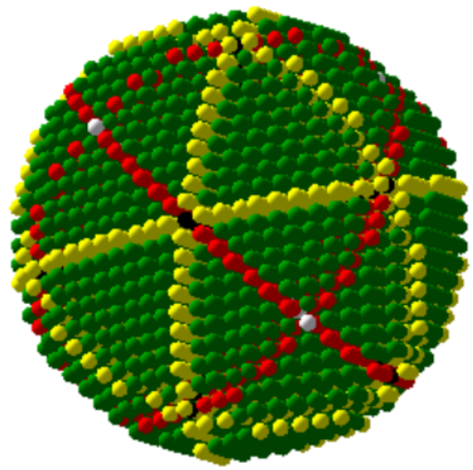

Figure 2: The translates $t_\lambda \cdot P$ in type A_3 colored according to the local distribution of reflection length.

root subspaces, and if two coroots lie in the same face, they have the same local generating function. However, note that while some of these intersections are themselves root subspaces, not all of them are. For example, the white balls in Figure 2 lie along lines that are not of the form $\mathbb{R}\alpha$ for any root α.

We make the phenomenon precise here.

Theorem 4.3 (Equality of local generating functions)**.** *Suppose that λ and μ are two elements of the coroot lattice L. Suppose furthermore that λ and μ belong to the same collection of root subspaces. Then $f_\lambda(s,t) = f_\mu(s,t)$.*

Unfortunately, while Theorem 4.3 and Proposition 4.2 imply bounds on the number of local generating functions in terms of the number of W_0-orbits of intersections of root subspaces, it is probably intractable to compute all $f_\lambda(s,t)$, or even all $f_\lambda(t)$, in general. We show in the Appendix of [7] that computing $d(t_\lambda)$ for an element λ of the type A_n coroot lattice is essentially equivalent to the NP-complete problem `SubsetSum`.

5 Affine symmetric groups

In this section, we restrict our attention to the affine symmetric groups and give simple combinatorial descriptions of the statistics $d(w)$ and $e(w)$ used to define reflection length. This provides an affine analog of the formula for the symmetric group given by Dénes [3].

First, we briefly recall the normal form of an element in the affine symmetric group $\widetilde{\mathfrak{S}}_n$. (See [7] for full details.) If $w \in \widetilde{\mathfrak{S}}_n$, we can write $w = t_\lambda u_\pi$, where $\pi \in \mathfrak{S}_n$ is a permutation and λ is a coroot, which we can express as a vector in $\mathbb{Z}^n$ whose entries sum to zero. For example, the pair $\lambda = (-2, -1, 3, 1, 1, -2, 0)$ and $\pi = (1, 5, 7)(2, 4)(3)(6)$ determines a unique element $w = t_\lambda u_\pi$ in $\widetilde{\mathfrak{S}}_7$. (Note we write π in cycle notation.) We will fix this element as an example throughout this section.

Now we consider the elliptic dimension. Our result generalizes Dénes's result that the reflection length of a permutation π is $n - |\textsc{cyc}(\pi)|$.

Proposition 5.1 (Elliptic dimension)**.** *Let $w \in \widetilde{\mathfrak{S}}_n$ be an affine permutation, with normal form $w = t_\lambda u_\pi$ (so that u_π is the elliptic part of w). Then $e(w) = n - |\textsc{cyc}(\pi)|$.*

The elliptic dimension of our running example is thus $e(w) = 7 - 4 = 3$.

In Remark 3.7, we have that $d(w) = \dim_{\Phi/U}(\lambda/U)$. In the more combinatorial setting of the affine symmetric group, this relative dimension boils down to a statement about partitioning the vector λ. We use the term *nullity*, denoted $\nu(\lambda)$, to describe the maximal size of a partition of the entries of λ whose blocks all sum to zero. For example, $\nu((-2, -1, 3, 1, 1, -2, 0)) = 3$ since the partition $\{\{-2, -1, 3\}, \{-2, 1, 1\}, \{0\}\}$ has all its blocks sum to zero, and no partition into four or more parts has all its parts sum to zero. (Note that the partition $\{\{-2, -2, 1, 3\}, \{-1, 1\}, \{0\}\}$ is also maximally refined in this way.) The *relative nullity*, denoted $\nu(\lambda/\pi)$, is the nullity of the vector λ' obtained by first coarsening λ according to the cycle structure of π. With our running example, $\pi = (1, 5, 7)(2, 4)(3)(6)$, so $\lambda' = (\lambda_1 + \lambda_5 + \lambda_7, \lambda_2 + \lambda_4, \lambda_3, \lambda_6) = (-1, 0, 3, -2)$. This vector has nullity 2, so the relative nullity is $\nu(\lambda/\pi) = 2$.

We now state the combinatorial result for differential dimension.

Proposition 5.2 (Differential dimension)**.** *An affine permutation $w \in \widetilde{\mathfrak{S}}_n$ with normal form $w = t_\lambda u_\pi$ has $d(w) = |\textsc{cyc}(\pi)| - \nu(\lambda/\pi)$.*

Thus in our example, the differential dimension is $d(w) = 4 - 2 = 2$.

Combining these propositions with the result Theorem A, that $\ell_R(w) = 2d(w) + e(w)$, gives us a combinatorial formula for the reflection length of an element in an affine symmetric group.

Theorem 5.3 (Formula)**.** *An affine permutation $w \in \widetilde{\mathfrak{S}}_n$ with normal form $w = t_\lambda u_\pi$ has $\ell_R(w) = n - 2 \cdot \nu(\lambda/\pi) + |\textsc{cyc}(\pi)|$.*

For our running example, we compute the reflection length as $\ell_R(w) = 2 \cdot 2 + 3 = 7$ or as $\ell_R(w) = 7 - 2 \cdot 2 + 4 = 7$.

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