THE $0/1$-BORSUK CONJECTURE IS GENERICALLY TRUE FOR EACH FIXED DIAMETER

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Abstract. In 1933 Karol Borsuk asked whether every compact subset of $\mathbf{R}^d$ can be decomposed into $d+1$ subsets of strictly smaller diameter. The 0/1-Borsuk conjecture asks a similar question using subsets of the vertices of a d-dimensional cube. Although counterexamples to both conjectures are known, we show in this article that the 0/1-Borsuk conjecture is true when d is much larger than the diameter of the subset of vertices. In particular, for every k, there is a constant n which depends only on k such that for all configurations of dimension $d > n$ and diameter $2k$, the set can be partitioned into $d-2k+2$ subsets of strictly smaller diameter. Finally, Lásló Lovász's theorem about the chromatic number of Kneser's graphs shows that this bound is in fact sharp.

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1. Introduction

In 1933 Karol Borsuk asked whether every compact subset of $\mathbf{R}^d$ can be decomposed into $d+1$ subsets of strictly smaller diameter. The 0/1-Borsuk conjecture asks a similar question using subsets of the vertices of a d-dimensional cube. More specifically, given a subset $S \subset \{0,1\}^d$ of diameter k, can S be partitioned into $d+1$ subsets of strictly smaller diameter? Both of these conjectures are known to be true in lower dimensions and false in higher dimensions. The original conjecture is known to be true in dimension $d \leq 3$ ([]) and false in dimension $d = 560$ ([2]). The 0/1-Borsuk conjecture is true in dimension $d \leq 9$ ([]) and false in dimension $d = 561$ ([4]). See [1] for a detailed discussion of these counterexamples.

Despite the existence of these counterexamples, in this article we show that the 0/1-Borsuk conjecture is true whenever d is much larger than the diameter of the subset S. In particular, we prove the following:

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Theorem 7.2. *For every $k > 0$ there is a constant n depending only on k such that the 0/1-Borsuk conjecture is true for every configuration of dimension $d > n$ and diameter $2k$. Moreover, every configuration of dimension $d > n$ and diameter $2k$ can be partitioned into $d - 2k + 2$ configurations of strictly smaller diameter, and this bound is sharp.*

The sharpness of the lower bound uses Lásló Lovász's theorem about the chromatic number of Kneser's graphs.

2. Configurations

We begin with some basic results about configurations.

Definition 2.1 (Configurations)**.** A subset $S \subset \{0,1\}^d$ will be called a *configuration*. Since the elements of $\{0,1\}^d$ can be viewed as the vertices of the unit d-cube in d-dimensional space, every configuration comes equipped with a natural distance function, although to avoid square roots, we will use the Hamming distance function instead of the usual Euclidean one. The Hamming distance between two points $u, v \in S$ is simply the square of the Euclidean distance, which is also the number of coordinates in which u and v differ. Since the 0/1-Borsuk conjecture will be true using the Hamming distance function if and only if it is true using the Euclidean distance function, the difference between the two is insignificant. The *diameter* of a configuration is the maximum (Hamming) distance between two of its elements. Two configurations will be called *isomorphic* if they are isometric as metric spaces.

Definition 2.2 (Set systems)**.** Let $[d]$ denote the set $\{1, 2, \ldots, d\}$. A *set system* is a collection of subsets of a fixed set such as $[d]$. Notice that every configuration corresponds to a set system in an obvious way, namely, send each element $v \in S$ to the set of coordinates which are equal to 1. For example, $(1, 0, 1, 1)$ corresponds to the set $\{1, 3, 4\} \subset [4]$. The distance between two elements of a configuration corresponds to the cardinality of the symmetric difference of the corresponding sets in the set system. We will use a lowercase letter to indicate an element of a configuration and an uppercase letter for the corresponding set. Thus, if $v = (1, 0, 1, 1)$, then $V = \{1, 3, 4\}$. The *radius* of a configuration S is the size of the largest set in its set system.

Definition 2.3 (Rotating a configuration)**.** For each $V \subset [d]$ there is an automorphism of the unit d-cube obtained by switching the values in the coordinates in V. This automorphism is clearly distance preserving and sends a configuration to an isomorphic configuration. The operation of replacing a configuration by its image under such an automorphism will be called *rotating a configuration*. Notice that the diameter of a configuration does not change if the configuration is rotated, but the radius of a configuration might change.

Lemma 2.4. *Every configuration S of diameter k can be rotated so that it has radius at most k.*

Proof. Given any $v \in S$, rotate S by changing all of the values of the coordinates in V. This sends v to the origin, and since S has diameter k, all of the other elements in S are sent to elements within k of the origin. □

Lemma 2.5. *If S is a configuration of radius r and a rotation is applied which switches the values of exactly s coordinates, then the new configuration will have radius at most $r + s$.*

Proof. This is obvious when phrased in terms of sets. Initially the largest set has size r and the rotation changes the size of each set by at most s. □

3. Borsuk graphs

In this section we show that the 0/1-Borsuk conjecture is true for a particular configuration if and only if its Borsuk graph has a $(d+1)$-coloring.

Definition 3.1 (Graphs)**.** We will use the following terminology from graph theory. The *clique number* of a graph Γ is the size of the largest complete subgraph. It is usually denoted $\omega(\Gamma)$. A subset V of vertices in a graph is an *independent set* if for all $u, v \in V$, u and v are not connected by an edge. A *k-coloring* of a graph is a way of partitioning the vertex set into k (possibly empty) independent subsets. The *valence* of a vertex is the number of other vertices it is connected to by an edge.

Definition 3.2 (Borsuk graphs)**.** Let $S \subset \{0,1\}^d$ be a configuration of diameter k. The *Borsuk graph of S* is defined as follows. Its vertex set is indexed by the elements of S, and two vertices are connected by an edge if and only if the distance between them is k. More generally, $\mathbf{B}_i(S)$ has S as its vertex set and two vertices are connected by an edge if and only if the distance between them is i. Thus the Borsuk graph of S is $\mathbf{B}_k(S)$ where k is the diameter of S. A configuration S is a *connected configuration* if and only if its Borsuk graph is connected.

The following lemma and its corollary are essentially immediate.

Lemma 3.3. *A configuration of diameter k can be partitioned into l configurations of strictly smaller diameter if and only if its Borsuk graph is l-colorable. In particular, the 0/1-Borsuk conjecture is true for a particular configuration $S \subset \{0,1\}^d$ if and only if its Borsuk graph is $(d+1)$-colorable.*

We can also quickly reduce to the case where the diameter is even.

Lemma 3.4. *The 0/1-Borsuk conjecture is true for configurations of odd diameter.*

Proof. If the diameter of the configuration is odd, then the Borsuk graph is bipartite with all of the $v \in S$ with $|V|$ even forming one set of vertices and the $v \in S$ with $|V|$ odd forming the other set. □

We can also dispense with the case where the largest valence of a vertex in the Borsuk graph is small.

Lemma 3.5. *If each vertex has valence less than r, then the graph is r-colorable.*

Proof. The proof is by induction. If $r = 1$, then the result is trivial, so suppose that $r > 1$ and let V be a maximal independent set of vertices. All of these vertices can be assigned the same color. When they are removed (along with all of the edges incident to them) all of the remaining vertices have valence at most $r - 1$ since V was maximal. We are now done by induction. □

Corollary 3.6. *If S is a configuration of diameter $2k$ whose Borsuk graph has no vertex of valence at least $k+2$, then S is $(k+2)$-colorable.*

4. Shells

This section contains two basic definitions and a lemma about distances in configurations.

Definition 4.1 (Shells)**.** If S is a configuration, the *i-shell* of S is the set of elements which have exactly i coordinates equal to 1. Equivalently, these are the elements $v \in S$ with $|V| = i$. Notice that the i-shell of a configuration might change if the configuration is rotated.

Definition 4.2 (Centers)**.** A configuration S of diameter $2k$ is said to have a *center* if there is a rotation of S such that S' lies completely in the k-shell of $\{0,1\}^d$. If no such rotation exists, then S is a *centerless configuration*.

Lemma 4.3. *Let S be a configuration of diameter $2k$, and let u and v be elements of S with $|U| = m$ and $|V| = n$.*

1. *If $m + n < 2k$ then u and v are not connected in $\mathbf{B}_{2k}(S)$.*
2. *If $m + n \geq 2k$, then $2|U \cap V| \geq m + n - 2k$*
3. *If $m + n \geq 2k$, then u and v are connected in $\mathbf{B}_{2k}(S)$ if and only if $2|U \cap V| = m + n - 2k$.*
4. *If S is connected, then m and n have the same parity.*

Proof. Parts (1), (2), and (3) are nearly immediate. In part (4), the connectedness of S implies that u can be connected to v by a sequence of elements $u = v_0, v_1, \ldots, v_r = v \in S$ where V_{i-1} and V_i have a symmetric difference of size $2k$ for all $i \in [r]$. This implies that $|V_{i-1}|$ and $|V_i|$ have the same parity and thus that $|U| = m$ and $|V| = n$ have the same parity. □

5. Kneser graphs

In this section we examine configurations with a center.

Definition 5.1 (Kneser graphs)**.** For $d \geq 2k$, let S be the k-shell of the full configuration $\{0,1\}^d$. That is, let S be the collection of all $v \in \{0,1\}^d$ with $|V| = k$. Since $d \geq 2k$, S has diameter $2k$. In particular, there exist $u, v \in S$ with $|U| = |V| = k$ and $|U \cap V| = 0$. The *Kneser graph*, denoted $\mathbf{KG}(d,k)$, is the Borsuk graph of this S. Alternatively, $\mathbf{KG}(d,k)$ is the graph whose vertices are indexed by the subsets of $[d]$ of size k and two vertices are connected by an edge if and only if the corresponding subsets are disjoint. Finally, notice that if T is a configuration of diameter $2k$ with a center, then after a rotation $T \subset S \subset \{0,1\}^d$ and thus $\mathbf{B}_{2k}(T)$ is a subgraph of $\mathbf{KG}(d,k)$.

Theorem 5.2 (Lovász)**.** *The chromatic number of $\mathbf{KG}(d,k)$ is exactly $d - 2k + 2$.*

Proof. We will prove that at most $d-2k+2$ colors are needed. That this coloring is optimal is a major theorem by Lásló Lovász [3]. Let $[d-2k-2]$ be our set of colors. For each $V \subset [d]$ of size k assign it the color i if $i \leq d-2k+2$ is the smallest number occuring in V, or the color $d - 2k + 2$ otherwise. This is a valid coloring since two sets assigned the same color $i < d - 2k + 2$ will have the element i in common and thus they will not be connected by an edge in $\mathbf{KG}(d,k)$. Similarly, two sets assigned the color $d-2k+2$ will have an element in common since they each contain k elements from the $2k - 2$ elements in the list $d - 2k + 2, d - 2k + 3, \ldots, d$. □

As a consequence, the conjecture is true for configurations with a center.

Corollary 5.3. *If S is a configuration of diameter $2k$ with a center, then $\mathbf{B}_{2k}(S)$ is isomorphic to a subgraph of $\mathbf{KG}(d,k)$ and thus it is $(d-2k+2)$-colorable. In particular, the $0/1$-Borsuk conjecture is true for every configuration with a center.*

Proof. Since rotating a configuration does not change the isomorphism type of its Borsuk graph, this is immediate from Theorem 5.2. □

For later purposes, we record the following improvement of Corollary 5.3.

Lemma 5.4. *Let S be a configuration of diameter $2k$. If S has a center. and the clique number of its Borsuk graph is r, then $\mathbf{B}_{2k}(S)$ is rk-colorable.*

Proof. By rotating, we may assume that the center of S is the origin and that S lies completely in the k-shell of $\{0,1\}^d$. If $v_1, v_2, \ldots, v_r \in S$ are the vertices of an r-clique in $\mathbf{B}_{2k}(S)$, then $V_1, V_2, \ldots, V_r$ are pairwise disjoint sets in $[d]$. Let V be their union and note that $|V| = rk$. The set V will be our set of colors. Since $\mathbf{B}_{2k}(S)$ has clique number r, there does not exist an element $v_{r+1} \in S$ such that V_{r+1} is disjoint from V. Thus for each element $u \in S$, $U \cap V$ is nonempty and there is some smallest element in the intersection. This will be the color of u. As in the proof of Theorem 5.2, if two elements u_1 and u_2 are assigned the same color then U_1 and U_2 have an element in common and thus u_1 and u_2 are not joined by an edge in $\mathbf{B}_{2k}(S)$. This proves that this coloring is valid. □

Lemma 5.5. *If S is a connected configuration of radius r and diameter $2k$ and $T \subset S$ is its k-shell, then either $\omega(\mathbf{B}_{2k}(T)) \leq r$, or $S = T$.*

Proof. Assume the clique number of $\mathbf{B}_{2k}(T)$ is at least $r+1$, and choose elements $v_1, v_2, \ldots, v_{r+1} \in T$ such that $V_1, V_2, \ldots, V_{r+1}$ are pairwise disjoint. If $S \neq T$, then since S is connected, there is an element $u \in S$ which is not in T, but which is distance $2k$ from an element of T. Assume u is in the l-shell of S, $l \neq k$. Note that $l < k$ is impossible since the maximum distance between two such points is at most $k + l < 2k$. Thus $l > k$. Since the diameter of S is $2k$ and $k + l > 2k$, U and V_i have a nonempty intersection for each $i \in [r+1]$. But since the sets V_i are disjoint, this would imply that U has size at least $r+1$, contradicting the assumption on the radius of S. Thus S lies completely in the k-shell and $S = T$. □

Corollary 5.6. *If S is a connected, centerless configuration of radius r and diameter $2k$, then the k-shell of S is rk-colorable.*

6. Central sets

If a configuration is centerless and has a vertex with valence at least $k+2$, then it will have a central set. Central sets are then used to provide an explicit method (Theorem 6.4) of coloring an arbitrary connected centerless configuration.

Definition 6.1 (Central sets)**.** Let S be a configuration of diameter $2k$ and let $v \in S$ be a vertex of valence at least $k+2$ in the Borsuk graph of S. After a possible rotation, we may assume that v is at the origin and that the $2k$-shell contains at least $k+2$ elements. A *central set* for S will be the union of the pairwise intersections of the $k+2$ subsets of $[d]$ corresponding to these elements. Call this set C and let c denote $|C|$. In particular, if $v_1, v_2, \ldots, v_{k+2}$ are $k+2$ elements of the $2k$-shell of S, then there is a central set

$$C = \bigcup_{i \neq j \in [k+2]} (V_i \cap V_j)$$

The elements v_i, $i \in [k+2]$ are said to *generate* C. Alternatively, notice that listing all of the elements of each V_i would be a list of $2k(k+2)$ elements and C would be exactly the elements that occur at least twice in the list. Incidentally, this shows that $c \leq k(k+2)$.

Lemma 6.2. *Every configuration S of diameter $2k$ which is not $(k+2)$-colorable can be rotated so that it has a central set of size at most $k(k+2)$.*

Proof. This follows from Corollary 3.6 and Definition 6.1. □

Lemma 6.3. *Let S be a connected configuration of diameter $2k$ and let u be an element of S with $|U| = 2i$, $i \leq k$. If C is a central set for S, then $|U \cap C| \geq i$.*

Proof. Let $v_1, v_2, \ldots, v_{k+2}$ be the elements of S which generate C. Since $|V_j| = 2k$ and $|U| = 2i$, by part 2 of Lemma 4.3, $|U \cap V_j| \geq i$ for all $j \in [k+2]$. If $U \cap V_j \subset C$ for any $j \in [k+2]$, then we are done. If not, then let l be the size of the largest such intersection and notice that U contains at least $i - l \geq 1$ elements in $V_j - (V_j \cap C)$ for each $j \in [k+2]$. Since these complements are disjoint by the definition of C, U contains at least $(k+2)(i-l)+l$ elements. But since

$$(k+2)(i-l)+l = (k+1)(i-l)+i \geq k+1+i \geq 2i+1$$

this contradicts our assumption that $|U| = 2i$. □

Theorem 6.4. *Let S be a connected centerless configuration of radius $2k$ and diameter $2k$. If S has a central set C of size c, then there exists a constant n which depends on k and c, but independent of d, such that the Borsuk graph of S can be n-colored.*

Proof. The strategy will be to split $B_{2k}(S)$ into 5 subgraphs each of which is colored with its own distinct set of colors. In particular, we define the following subsets which by Lemma 6.3 partition S:

1. $S_1 = \{v \in S \mid |V| < k\}$
2. $S_2 = \{v \in S \mid |V| = k\}$
3. $S_3 = \{v \in S \mid k < |V| < 2k\}$
4. $S_4 = \{v \in S \mid |V| = 2k \text{ and } |V \cap C| > k\}$
5. $S_5 = \{v \in S \mid |V| = 2k \text{ and } |V \cap C| = k\}$

Once we show that $B_{2k}(S_i)$ requires a bounded number of colors in terms of k and c for each $i \in [5]$, the proof will be complete.

Step 1: By part 1 of Lemma 4.3, the elements in S_1 form an independent set in $\mathbf{B}_{2k}(S)$ and thus they can be assigned a single color.

Step 2: Since S does not have a center, and since S_2 is the k-shell of S, by Lemma 5.5, $\omega(B_{2k}(S_2)) \leq 2k$ and by Corollary 5.6, $\mathbf{B}_{2k}(S_2)$ is $2k^2$-colorable.

Step 3: Let U be a subset of C of size i, where $k < 2i < 2k$, and let S_U be the set of all $v \in S_3$ such that $|V| = 2i$ and $U \subset V$. Notice that by part 3 of Lemma 4.3, the elements in S_U form an independent set since for each $v_1, v_2 \in S_U$, $|V_1 \cap V_2| \geq |U| = i$, but i is greater than the $i + (i-k)$ required for v_1 and v_2 to be connected by an edge in $\mathbf{B}_{2k}(S)$. By Lemma 6.3 every element of S_3 is contained in S_U for some subset $U \subset C$ with $k < |U| < 2k$. Thus S_3 can be partitioned into

$$\sum_{i>k/2}^{k-1} \binom{c}{i} \leq \sum_{i=0}^{k} \binom{c}{i} = 2^c$$

independent sets and as a consequence $\mathbf{B}_{2k}(S_3)$ can be 2^c-colored.

Step 4: Let U be a subset of C of size $k+1$ and let S_U be the set of all $v \in S_4$ such that $U \subset V$. By part 3 of Lemma 4.3, the elements in S_U form an independent set since for each $v_1, v_2 \in S_U$, $|V_1 \cap V_2| \geq |U| = k+1$ which is greater than the k required for v_1 and v_2 to be connected by an edge in $\mathbf{B}_{2k}(S)$. By Lemma 6.3 every element of S_4 is contained in S_U for some subset $U \subset C$ with $|U| = k+1$. Thus S_4 can be partitioned into $\binom{c}{k+1}$ independent sets and as a consequence $\mathbf{B}_{2k}(S_3)$ can be $\binom{c}{k+1}$-colored.

Step 5: For the configuration S_5 we will distinguish two possibilities. Suppose there exists a set $U \subset C$ with $|U| = k$ such that for all $v \in S_5$, $U \subset V$. In this case, there is a rotation which sends the elements of S_5 into the k-shell, namely, the rotation which switches all of the values of the k coordinates in U. For all $v \in S_5$, $|V| = 2k$ and since $U \subset V$, the rotation V' of V will have $|V'| = k$. Let S_5' denote the rotated configuration. If this rotation were applied to all of S, then by Lemma 2.5 the result would be a configuration S' of radius r where $r \leq 3k$. Since the original configuration did not have a center, and since S_5' is at least contained in the k-shell of S', by Lemma 5.5, $\omega(B_{2k}(S_5')) \leq r \leq 3k$. Thus by Corollary 5.6, $\mathbf{B}_{2k}(S_5')$ is rk-colorable. Since $\mathbf{B}_{2k}(S_5')$ is isomorphic to $\mathbf{B}_{2k}(S_5)$ and $r \leq 3k$, $\mathbf{B}_{2k}(S_5)$ is $3k^2$-colorable.

If, on the other hand, such a set $U \subset C$ does not exist, then there exist $v_1, v_2 \in S_5$ with $V_1 \cap C \neq V_2 \cap C$. Let U be a subset of C of size k and suppose that $U \neq V_1 \cap C$. As above, let S_U be the set of all $v \in S_5$ such that $U \subset V$. For each $v \in S_U$, the set $V \cap V_i$ has size k even though $V \cap V_i \cap C$ has size at most $k-1$. In particular, for every element $v \in S_U$, V contains an element of $V_i - C$. Let a be an element of $V_i - C$ and let $S_{(U,a)}$ be the set of all $v \in S_U$ with $a \in V$. Since the elements of $S_{(U,a)}$ have sets with at least $k+1$ elements in common, they form an independent set (by part 3 of Lemma 4.3) and by the above argument, the union of the sets $S_{(U,a)}$ as a varies over $V_i - C$ is all of S_U. Since $|V_i \cap C| = k$, the size of $V_i - C$ is also k, and thus $\mathbf{B}_{2k}(S_U)$ is k-colorable. Finally, by Lemma 6.3, every element $v \in S_5$ is contained in S_U for some $U \subset C$ of size k. The above argument shows that if $U \neq V_1 \cap C$, then $\mathbf{B}_{2k}(S_U)$ is k-colorable. If $U = V_1 \cap C$, then since $V_1 \cap C \neq V_2 \cap C$, we have instead $U \neq V_2 \cap C$, and a similar argument shows that $\mathbf{B}_{2k}(S_U)$ is k-colorable in this instance as well. Since there are only $\binom{c}{k}$ such sets U, $\mathbf{B}_{2k}(S_5)$ is $k\binom{c}{k}$-colorable. This completes the proof. □

Remark 6.5 (Upper bounds)**.** Examining the various bounds given in the proof of Theorem 6.4 gives an upper bound for the constant n mentioned in the theorem. Depending on which part of step 5 is applied, the bound on the number of colors needed is either

$$1 + 2k^2 + \left(\sum_{i>k/2}^{k-1} \binom{c}{i} \right) + \binom{c}{k+1} + 3k^2 \leq 5k^2 + 2^c$$

or

$$1 + 2k^2 + \left(\sum_{i>k/2}^{k-1} \binom{c}{i} \right) + \binom{c}{k+1} + k\binom{c}{k} \leq 2k^2 + (k+1)2^c$$

Finally, we recall that by Lemma 6.2, c itself is at most $k(k+2)$, and thus all of these estimates depend on k alone.

7. Main Theorems

In this final section we collect together our main results.

Theorem 7.1. *Let $S \subset \{0,1\}^d$ be a connected configuration of diameter $2k$. If S has a center, then its Borsuk graph can be $d-2k+2$ colored. If it does not have a center then its Borsuk graph can be n-colored, where n is a constant which depends only on k and not on d.*

Proof. If S has a center this follows from Corollary 5.3. If S does not have a center, then by Lemma 6.2 either $\mathbf{B}_{2k}(S)$ is $(k+2)$-colorable or S has a rotation which has a central set and the result follows from Theorem 6.4. □

Theorem 7.2. *For every $k > 0$ there is a constant n depending only on k such that the 0/1-Borsuk conjecture is true for every configuration of dimension $d > n$ and diameter $2k$. Moreover, every configuration of dimension $d > n$ and diameter $2k$ can be partitioned into $d - 2k + 2$ configurations of strictly smaller diameter, and this bound is sharp.*

Proof. Apply Theorem 7.1 to each connected component of $\mathbf{B}_{2k}(S)$. The sharpness of the bound is shown by Kneser's graphs and Theorem 5.2. □

Theorem 7.3. *If $S \subset \{0,1\}^d$ is a configuration of diameter $2k$ and*

$$d > (k+1)2^{k(k+2)} + 5k^2 + 2k - 2$$

then $\mathbf{B}_{2k}(S)$ is $(d-2k+2)$-colorable and the 0/1-Borsuk conjecture is true for S.

Proof. This is immediate from Theorem 7.2 and Remark 6.5. □

One final remark. No attempt has been made to make the number in Theorem 7.3 as small as possible since the most interesting fact is that such a bound even exists.

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