CURVATURE AND COMPUTATION

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ABSTRACT. When undergraduates ask me what geometric group theorists study, I describe a theorem due to Gromov which relates the groups with an intrinsic geometry like that of the hyperbolic plane to those in which certain computations can be efficiently carried out. In short, I describe the close but surprising connection between negative curvature and efficient computation. This theorem was one of the clearest early indications that applying a metric perspective to traditional group theory problems might lead to new and important insights.

The theorem I want to discuss asserts that there is a close relationship between two collections of groups: one collection is defined geometrically and the other is defined computationally. The first section describes the relevant geometric and topological ideas, the second discusses the key algebraic and computational concepts, and the short final section describes the relationship between them. An informal style, similar to the one I use when answering this question face-to-face, is maintained throughout.

1. GEOMETRY AND TOPOLOGY

The first thing to highlight is that there is a close relationship between groups and topological spaces. More specifically, to each connected topological space X there is an associated group G called its *fundamental group* and absolutely every group arises in this way (in the sense that for each group G one can construct a topological space X whose fundamental group is isomorphic to G). Because of this connection and because spaces with isomorphic fundamental groups share many key properties, we can use the topology of the space X to understand the algebraic structure of its fundamental group G.

Fundamental groups. In order to make this discussion as accessible as possible, here is a quick sketch of the basic idea behind the notion of a fundamental group. As an initial attempt, one could try to form a group out of a space by using the paths in the space as our elements

and the operation of concatenation as our multiplication, but there are problems that arise. First, we want to be able to "multiply" (i.e. concatenate) any two paths, but to do so we need the first path to end where the second path begins. To fix this we select a point x in our space and consider only those paths that start and end at x. The role of an identity element is played by the trivial path that starts at x and stays at x.

But now we come to the second problem. Concatenating paths only makes them longer so that nontrivial paths do not yet have inverses. It seems intuitive that traveling along a path in the opposite direction should count as its inverse but in order to make this work, we need to replace individual paths with equivalence classes of paths. We call two paths equivalent when we can continuously deform one to the other without moving its endpoints and the multiplication of equivalence classes of paths is defined as the equivalence class of the concatenation of representative paths. It is relatively easy to check that this multiplication is well-defined and that the resulting algebraic structure is a group. Moreover, so long as the space X is path connected, the algebraic structure of this group does not depend on our choice of basepoint up to isomorphism. In other words, the group G is a invariant of the space X itself independent of our choice of basepoint.

The standard illustration of this procedure and in many ways the most crucial one is the following: the fundamental group of the unit circle is isomorphic to the integers. The nontrivial group elements come from paths that wrap around the circle and, in fact, the equivalence classes essentially collect together those paths that wrap around the circle the same number of times in the same direction. A second example, closely related to the first, is that the fundamental group of the torus is $\mathbb{Z} \oplus \mathbb{Z}$.

Covering spaces. There is another connection between the space X and its fundamental group G. So long as the space X is sufficiently nice, it can be completely unwrapped in the following sense. There is another space called its *universal cover* with trivial fundmamental group and a projection map back to X that is locally a homeomorphism. In the case of the circle, its universal cover is an infinite spiral, continuing forever in both directions. Topologically this space looks like the real line, it is contractible and it is easy to believe that its fundamental group is trivial. It is also easy to see that the natural projection from the spiral to the circle is a local homeomorphism. The universal cover of the torus is the euclidean plane and the projection map from the plane to

the torus is the one that first wraps it up into an infinite cylinder in one direction and then wraps it up in the other direction into a torus.

One useful fact is that the fundamental group of X acts on its universal cover by homeomorphisms. In our examples, the integers act on the infinite spiral by rigidly shifting it up or down and the group $\mathbb{Z} \oplus \mathbb{Z}$ acts on the plane by translating by vectors with integer coordinates. In fact, the action of the fundamental group on the universal cover is always transitive on the preimages of a point and these preimages are in one-to-one correspondence with the elements of the fundamental group. From the early twentieth century to the present day group theorists have used this relationship to study infinite discrete groups. In particular, the topology of the universal cover on which the fundamental group G acts can be used to extract information about the algebraic structure of G.

Gestures. This is probably as good as place as any to mention one key aspect of my interactional style that is difficult to replicate in a written text. Gestures are an important aspect of how I communicate mathematics orally and this is especially true when I am talking to students. In particular, throughout this entire discussion I usually employ a collection of gestures in specific locations to focus attention and to illustrate what is going on. The result is something like Stravinsky's orchestration that accompanies the story of Peter and the Wolf. Recurring characters (mathematical concepts) have musical themes (stylized gestures) that are repeated every time they reappear. Whenever I mention the geometric and topological aspects of groups I gesture to my lefthand side (and the algebraic and computational aspects involve gestures to my righthand side). The space X is located on the lower left and its universal cover is directly above it. The gesture associated to X, its theme, is the miming of the shape of a torus. For its universal cover, I start with the torus on the lower left and then raise my arms and spread out my hands to indicate the euclidean plane. The action of G on the universal cover of X is indicated by moving both hands (in euclidean plane position) in small syncronized circles with the hands themselves always pointing in the same direction and maintaining a rigid relationship between them. The reader should visualize these gestures as they read along.

Metrics on groups. Returning to our discussion of the action of the fundamental group on the universal cover, suppose we add a metric to the original space X. In our example, rather than imagining a space that is merely a topological torus, imagine a space with a precise metric so that we can calculate distances, angles and areas. This local metric

information induces a metric on the unwrapped version and we can use this metric on the universal cover to turn the group G itself into a metric space. I should also point out that the action of G on the universal cover is one that preserves this local metric information. In other words, it acts on the universal cover by isometries.

To turn G into a metric space we to pick a point $\tilde{x}$ in the universal cover and then for each group element $g \in G$ record the distance between this point and its image under g. We call this the distance from the identity element to g. More generally, given two group elements g and g', we define the distance between them to be the distance in the universal cover between the images of $\tilde{x}$ under g and g'. It is now relatively easy to convince yourself that this distance function defines a metric on the elements of G, i.e. that it is symmetric, nonzero on distinct pairs of elements and satisfies the triangle inequality.

Intrinsic Metrics. The other thing that is obvious is that the precise values of the metric on G very much depend on the specific metric we added to our space X and the point $\tilde{x}$ that we selected. It turns out, however, that when X satisfies certain minimal conditions (such as being compact) altering the metric on X or choosing a different point $\tilde{x}$ does not significantly alter the induced metric on G. More specifically, given two distinct metrics on X and two different selected points, the metrics they induce on G are related by linear inequalities. In other words, there exist constants so that for every pair of elements in G, their distance in the first metric is bounded above by a linear function of their distance in the second metric and vice versa. Two metrics that are related in this way are said to be quasi-isometric and the notion of quasi-isometry partitions all metrics into quasi-isometry classes. In this language, the result I'm alluding to is that for any reasonable space X with fundamental group G, the possible metrics on X induce metrics on G that all belong to the same quasi-isometry class. In fact, this remains true even if we replace our reasonable space X with any other reasonable space Y with the same fundamental group G, a result known as the Milnor-Svarc Theorem. This means that if G is the kind of group that can be the fundamental group of a reasonable metric space X, then the quasi-isometry class of the metric induced on G through its action on the universal cover of X is completely independent of the space X used to produce this metric. This is what geometric group theorists mean when they say that (reasonable) groups come equipped with an intrinsic metric that is well-defined up to quasi-isometry.

Differential geometry. We are now going to restrict our attention to a special class of groups but the motivation for this restriction involves

a short digression into the history of a different part of mathematics. Differential geometry is an area that studies spaces called Riemannian manifolds that are locally homeomorphic to n-dimensional euclidean space and which come equipped with a nice smooth metric that allows them to be investigated using the standard tools of multivariable calculus. Early on, differential geometers defined various notions of curvature and they proved that Riemannian manifolds that are negatively curved, in a suitable sense, have very nice properties such as a contractible universal cover. Initially their proofs used the full force of the analytic tools available to them, but as they simplified the proofs to extract the essential of why these results were true, they soon discovered that they could assume much less about the original space and still produce significant consequences. In fact, all that was really necessary was that certain inequalities hold involving points on the sides of geodesic triangles. Once reformulated in this way, their ideas could be applied to a much larger class of metric spaces which did not necessarily locally look like euclidean space and where the ordinary operations of multivariable calculus could not be applied. One of these differential geometers was Misha Gromov and he soon realized that these distilled ideas from differential geometry could be applied to infinite discrete groups.

Thin triangles. The key definition is inspired by the properties of triangles in the hyperbolic plane. If you have ever studied the geometry of the hyperbolic plane, you have probably learned that there are important differences between triangles in the hyperbolic plane and triangles in the euclidean plane. In a euclidean triangle, the sum of its three angles is π but in a hyperbolic triangle, the sum of its angles is always strictly less than π. The more relevant fact about hyperbolic triangles for our discussion is one that does not always make it into a first course on hyperbolic geometry, namely, that all triangles in the hyperbolic plane are uniformly thin.

In the euclidean plane, some triangles are fat. What I mean by this is that for every constant r we can find a euclidean triangle and a point in its interior so that the distance from this point to any point on its boundary is at least r. Just pick the center of a large equilateral triangle. In the hyperbolic plane you can do this for small values of r but not for large values of r. Let me give an equivalent reformulation of this property where I actually know the exact value of the constant where the behavior changes. In the euclidean plane for any constant r it is easy to find a triangle and a point p on one of its sides so that the distance from p to any point on either of the other two sides is at least

r. In the hyperbolic plane it turns out that given any triangle and any point p on one of its sides, there is a point q on one of its other sides so that the distance from p to q is less than $\log(1 + \sqrt{2})$. This exact value is relatively easy to establish but the interesting point is that such a value even exists.

Hyperbolicity. Gromov turned the uniform thinness of triangles in the hyperbolic plane into a defining characteristic of hyperbolic spaces and hyperbolic groups. A space is called δ-*hyperbolic* when all geodesic triangles in this space are δ-thin for a fixed constant δ. In other words, given any three points and any three length-minimizing paths connecting them into a triangle and given any point p on one of these paths, there is a point q on one of the other two paths so that the distance from p to q is less than δ. A group G is called *word hyperbolic* or *Gromov hyperbolic* when it is the fundamental group of a reasonable metric space X whose universal cover is δ-hyperbolic for some constant δ. It turns out that being Gromov hyperbolic really is an intrinsic property of G in the sense that it is independent of our choice of X and of our choice of a metric on X. Concretely, if X and Y are reasonable spaces with fundamental group G and one of them has a δ-hyperbolic universal cover then the other universal cover is δ'-hyperbolic for a possibly different constant δ'. In short, groups that are hyperbolic in the sense of Gromov are those where the geometry of its intrinsic metric shares a key property possessed by triangles in the hyperbolic plane.

2. ALGEBRA AND COMPUTATION

And now for something completely different. Set the geometric and topological properties of groups aside for the moment and consider their algebraic and computational properties. The first thing to note is that the infinite groups which are the easiest to work with from a computational perspective are those that have some sort of finite description.

Descriptions of groups. The classical method of describing an infinite group is to list a set of elements that are sufficient to generate the entire group and then to list some relations satisfied by these elements that are sufficient to generate all of the relations that hold in the group. Such a *presentation* is said be finite when both the set of generators and the set of relations are finite and the group it described is called *finitely presented*. The classical example of a finite presentation is the group generated by a and b and subject only to the relation that $ab = ba$. This is a finite description of the group $\mathbb{Z} \oplus \mathbb{Z}$. There are other ways to characterize the class of finitely presented groups that make clear

that this is an important and interesting class of groups to study. For example, the class of finitely presented groups is exactly the same as the class of groups that are fundamental groups of compact manifolds and it is exactly the same as the class of groups that are fundamental groups of finite simplicial complexes.

The word problem. It is traditional to use a language metaphor when working with a finitely presented group. Individual generators are called *letters* and finite products of generators and their inverses are called *words*. One problem that immediately arises is that because the generators satisfy relations, there are typically many different words that represent the exact same element of the group. In the standard presentation of the group $\mathbb{Z} \oplus \mathbb{Z}$, for example, both *aaabb* and *ababa* represent the same element even though they are distinct words. The key question, first identified by Max Dehn in 1912, is the *word problem*: For a fixed finite presentation, is there an algorithm that takes as input two words written as products of the generators and outputs whether or not they represent the same element of the group after a finite amount of time. For $\mathbb{Z} \oplus \mathbb{Z}$ the answer is yes, there does exist such an algorithm. One such algorithm goes as follows. Systematically move all the a's and a^{-1}'s to the left and all the b's and b^{-1}'s to the right and then simplify until the final result is a word of the form $a^i b^j$ for some integers i and j. Two words that have the same *normal form* represent the same group element and two words that have distinct normal forms represent distinct group elements. This works for $\mathbb{Z} \oplus \mathbb{Z}$ but the general situation is much more complicated.

Some problems cannot be solved. In the early twentieth century mathematicians were beginning to learn that there is an important distinction between what is true and what can be proved. In the same way that a statement such as "This is a lie" cannot consistently be assigned a truth value, Gödel showed how one could construct, for any finite self-consistent axiomatic system for the natural numbers, a sentence that was true but which cannot be proved from the axioms. When translated into the language of the emerging theory of computation, this means that there are problems that cannot be solved algorithmically and one can prove that they cannot be solved algorithmically. One example of such an unsolvable problem is the halting problem: Does there exist a computer program which takes as input an arbitrary computer program and outputs, after a finite amount of time, whether or not the inputted program will run forever? The answer is that no such generic program analyzing software can exist since there will always be some program that it cannot successfully analyze.

Once mathematicians realized that certain problems cannot be solved, they used that fact to prove that other problems cannot be solved. They did this by showing that a solution to the second problem leads to a solution to the first problem, which contradicts the fact that we know the first problem cannot be solved. Working along these lines, Boone and Novikov, working independently, showed that a single algorthm that solves the word problem in an arbitrary finitely presented group cannot and does not exist. In fact, there are explicit finite presentations for which it is known that there is no algorithm to solve the word problem for this specific group.

Efficient solutions. The fact that some finitely presented groups have word problems that cannot be solved merely prompts mathematicians to shift their attention to those groups with word problems that can be solved. Going one step further, we can divide groups with solvable word problems into classes based on how hard their word problems are to solve. One indication of the level of difficulty is how long it takes for the algorithm to work: how many units of time does it take as a function of the total length of the two words under consideration? In other words, let n denote this total length of the two words given as input and describe the time bound as a function of n. Is it a linear function of n? quadratic? polynomial? exponential?

Clearly the best possible algorithms cannot run in less than linear time since in order to correctly answer whether or not two words represent the same element in the group, the program must, at the very least, read the two words which takes a linear amount of time. In our example of the standard presentation of $\mathbb{Z} \oplus \mathbb{Z}$, the process described that places words in normal form $a^i b^j$ can take a quadratic amount of time since one of the initial inputs might be $b^m a^m$. Each b must be moved past each a which involves m^2 local modifications. These can be visualized in the plane as pushing across single squares from one pair of sides of a large rectangle to the other pair of sides.

Isoperimetric inequalities. In fact, the best possible time bound for a solution to the word problem in a particular finitely presented group G is quite closely related to the isoperimetric inequality satisfied by closed loops in the universal cover of any reasonable metric space X with fundamental group G. By isoperimetric inequality we mean the following. For each closed loop in a simply connected metric space, we can measure the minimal area of a disc mapped isometrically into the space so that its boundary is the specified closed loop. We measure the ratio of this minimal area to the length of the curve and then find the largest such ratio as the curves vary over all curves up to a

specified length. As this bound grows we get an increasing function that measures how hard it is to fill loops of a given bounded size. It turns out that the rate of growth of this function is independent of X and is an invariant of G alone. It is bounded above by a recursive function iff the word problem for G can be solved and for our example of $\mathbb{Z} \oplus \mathbb{Z}$ the growth rate of this function is always quadratic. The finitely presented groups with an isoperimetric inequality that grows linearly are the ones whose word problem can be solved in linear time. Thus these are the groups with the fastest possible solution to their word problem, the groups with the best possible computational properties.

3. Gromov's theorem

And now for the surprising conclusion. In the 1980s Gromov proved that the groups with the best possible computational properties correspond *exactly* to those with an intrinsic geometry that is negatively curved in the sense described earlier [Gro87].

Theorem 1. *A finitely presented group has a linear time solution to its word problem* if and only if *it is hyperbolic in the sense of Gromov.*

This is an amazing result because the implication is not merely in one direction; it is an exact correspondence. As a consequence of this result, geometric group theorists tend to view hyperbolic geometry as the best possible geometry for a group to have since it corresponds to the group having the best possible computational properties. This is the kind of result that makes researchers sit up and take notice, and it prompted a thorough-going review of the foundations of the subject. It also immediately prompts a large number of follow-up questions. How strong is this bridge between geometry and topology on the one hand and algebra and computation on the other? In particular, what happens when we expand the class of groups under consideration? Are there geometric consequences when a finitely presented group has a quadratic, cubic or polynomial time solution to its word problem? Are there computational consequences when a finitely presented group has an intrinsic geometry that is non-positively curved in some sense (rather than negatively curved)? Over the past 30 years these types of questions have led to the development of several general theories such as the theory of automatic and biautomatic groups [ECH$^+$92] and the theory of groups that act on nonpositively-curved spaces [BH99].

A second natural set of questions involves taking the various classes of groups traditionally investigated by combinatorial group theorists (such as outer automorphisms of free groups, mapping class groups of closed surfaces, braid groups, Coxeter groups, Artin groups, one-relator

groups, etc., etc.) and asking which of the various general theories of curvature and computation apply in each case. To my mind, this single theorem of Gromov is like the Big Bang and it played a major role in the creation of a new subfield called geometric group theory.

References

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