FANS AND LADDERS IN SMALL CANCELLATION THEORY

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1. Introduction

Small cancellation theory and its various generalizations have proven to be powerful tools in the study of infinite groups, particularly for the construction of examples of groups exhibiting specific properties. In this article we derive statements which are similar to but significantly stronger than the usual small cancellation formulations. These stronger results are presented using the notion of a fan, which is introduced here for the first time.

The main geometric conclusion in small cancellation theory is essentially that disc diagrams contain a 2-cell most of which lies on the very outside of the diagram, as illustrated on the left in Figure 1. In studying this situation, we found that it can be strengthened. Specifically, we show that disc diagrams satisfying small cancellation conditions have a sequence of consecutive cells all of which lie near the outside of the diagram. We call the union of these cells in the diagram a *fan*. The diagram on the right in Figure 1 contains a fan which is the union of four 2-cells. The first manner in which our results augment the traditional results of small cancellation theory is that when the small-cancellation conditions are sufficiently strict, the disc diagram will contain fans consisting of longer and longer chains of 2-cells.

Our second contribution to this subject is a classification result for disc diagrams. We show that either a disc diagram is small and round and contains fans in all directions, or it is long and relatively thin like a ladder and contains disjoint fans at each end, or in the generic case, it contains at least three sharp turns along three disjoint fans. This trichotomy is analogous to the trichotomy for finite trees: A finite tree either consists of a single point, or it is homeomorphic to a real interval, or it has at least three leaves.

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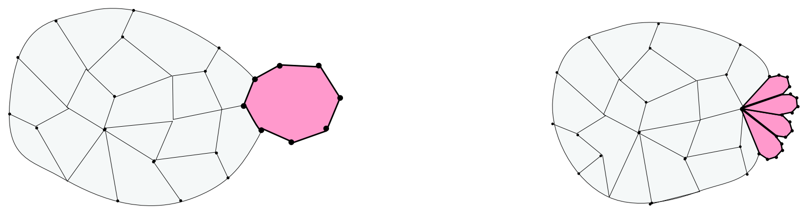


Figure 1.

We will state our two main results below but we first state an important special case, which is our refinement of the main theorem of small-cancellation theory as a trichotomy. All of the undefined terminology (namely: i-shell, spur, wheel of width k, ladder of width k, separate fans of type k) will be explained in the course of the article. Informally, a spur is a 1-cell which is not in the boundary of a 2-cell and which is attached to the rest of the diagram at only one end, and an i-shell is a 2-cell with exactly i maximal arcs of its boundary lying in the interior of the diagram.

Theorem 9.4. *If D is a $C(4)$-$T(4)$ [$C(6)$-$T(3)$] disc diagram, then one of the following holds:*

(1) *D contains at least three spurs and/or i-shells with $i \leq 2$ [$i \leq 3$].*
(2) *D is a ladder of width ≤ 1, and hence has a spur or 1-shell at each end.*
(3) *D consists of a single 0-cell or a single 2-cell.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a spur or an i-shell with $i \leq 2$ [$i \leq 3$] which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate such spurs and i-shells.

There is a similar theorem for annular diagrams. In this article we provide improved versions of both of these theorems. In order to concisely state our results we will use the following conventions.

Convention 1.2 (Restrictions)**.** Let D be a $C(p)$-$T(q)$ diagram and let k be a nonnegative integer. By *Euclidean restrictions* we will mean that p, q, and k satisfy one of the following sets of conditions:

(1) $p \geq 6$, $q = 3$, and k is odd
(2) $p \geq 4$, $q \geq 4$, and k is arbitrary
(3) $p = 3$, $q \geq 6$, and k is even

The Euclidean restrictions will typically be used for disc diagrams. The *hyperbolic restrictions* are slightly more stringent and will typically be used for annular diagrams. In particular, we will assume that p, q, and k satisfy one of the following sets of conditions:

(1) $p \geq 7$, $q = 3$, and k is odd
(2) $p \geq 5$, $q \geq 4$, and k is odd
(3) $p \geq 4$, $q \geq 5$, and k is even
(4) $p = 3$, $q \geq 7$, and k is even

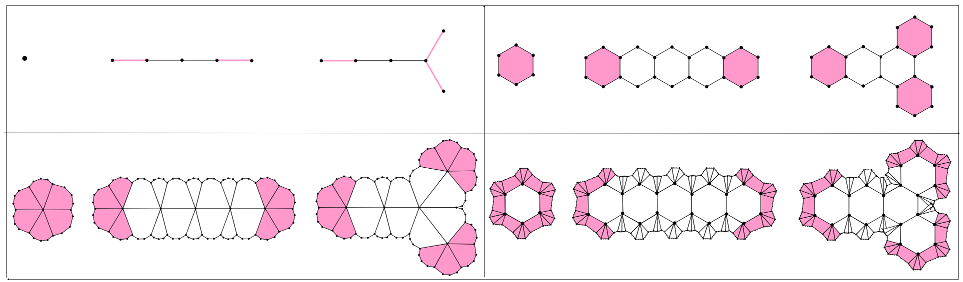


FIGURE 2. Each of the four collections of diagrams above contains a wheel of width k, a ladder of width k, and a disc diagram with three fans of type k.

Our two main results are as follows:

Theorem 9.2. *If D is a $C(p)$-$T(q)$ disc diagram and p, q, and k satisfy the Euclidean restrictions, then one of the following holds:*

1. *D contains at least 3 separate fans of type k.*
2. *D is a ladder of width $\leq k$.*
3. *D is a wheel of width $\leq k$.*

We refer the reader to Figure 2 for an accurate but idealized family of diagrams illustrating this theorem in the $C(6)$-$T(6)$ case. The annular version is as follows:

Theorem 10.6. *If A is a $C(p)$-$T(q)$ annular diagram and p, q, and k satisfy the hyperbolic restrictions then either A contains a fan of type k or A has width $\leq k+1$.*

Intuitively, a fan is an array of 2-cells along the boundary of a disc diagram, a ladder is a long thin diagram, and a wheel is a very small disc-like diagram. As the value of k increases, the fans of type k become more and more restricted and the trichotomy gives more detailed information about small cancellation diagrams. Finally, note that Theorem 9.4 is the special case of Theorem 9.2 with $k = 1$. Specializing to the case $k = 2$ yields the following small cancellation theorem for $C(3)$-$T(6)$ presentations:

Theorem 9.5. *If D is a $C(3)$-$T(6)$ disc diagram, then one of the following holds:*

1. *D contains at least three spurs, 1-shells, and/or pointed fans with two consecutive 2-shells.*
2. *D is a ladder of width ≤ 2.*
3. *D is a wheel of width ≤ 2. That is, D is either a single 0-cell, a single 2-cell, or a nonsingular diagram whose dual is a single 2-cell.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a spur, a 1-shell, or a pointed fan with two 2-cells which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate such spurs, 1-shells, and pointed fans.

The main application of traditional small cancellation theory has been to solve the word problem and conjugacy problem. Thus far, the main applications of the theorems presented in this paper are towards understanding the finitely generated subgroups of small cancellation groups. In [14] and [15], we have used these stronger results in conjunction with our perimeter reduction method to prove that many small cancellation groups are coherent, have the finitely generated intersection property, have decidable membership problem in finitely generated subgroups, or are even locally quasiconvex.

The local quasiconvexity theorem in [15] relies heavily upon the existence of ladders in the trichotomy, and could not be obtained using the traditional small cancellation theory. While the coherence proofs in [14] already work in conjunction with traditional small cancellation theory, the method applies to a significantly larger class of groups when combined with the results of this paper asserting the existence of fans. Furthermore, a family of examples produced in [15] demonstrates that the coherence and local quasiconvexity results proven using fans are asymptotically sharp. However, the theorems one obtains by combining the perimeter reduction method with the traditional small cancellation theory are not sharp. We are therefore convinced that fans and ladders are natural objects to consider towards understanding these subgroup properties. We expect that other group-theoretical applications will be found.

1.1. **History of small cancellation theory.** The small cancellation approach which was pioneered by Dehn and Tartakovskiĭ was sporadically developed in the papers of Schick, Britton, Greendlinger, Lyndon, and Weinbaum. Finally, the theory was fully developed in the book by Lyndon and Schupp [11] which is the standard reference on the subject. A concise and elegant treatment is given in Strebel's article in [6].

Various generalizations of small cancellation theory have appeared, such as those by Hill-Pride-Vella [8], Juhász [9], McCammond [12], Ol'shanskiĭ [16], and Rips [18]. Certainly, the most significant development has been Gromov's introduction of word-hyperbolic groups [7]. Other related developments which have drawn on small cancellation theory for inspiration include nonpositively curved groups, and automatic groups.

1.2. **Description of the sections.** Section 2 introduces a number of basic definitions about combinatorial 2-complexes and diagrams. Several of these definitions are new and several are variations on definitions which will be familiar to most readers. Because it is one of the starting points for small cancellation theory, we have also included a detailed proof that every null-homotopic path bounds a reduced disc diagram.

Section 3 presents small cancellation theory itself. We have chosen to recast the foundations of the theory in terms of 2-complexes to facilitate the results later in the article as well as for applications which extend beyond the scope of this article. Consequently, we have attempted to state the definitions and to prove the results in their most natural generality. For example, we show that a general 2-complex which admits a reduced map to a small cancellation complex is itself a small cancellation complex. This is a generalization of a corresponding statement for a reduced map from a disc diagram, but the proof of the more general statement requires little additional effort.

Section 4 contains a proof of the combinatorial Gauss-Bonnet theorem, followed by two easy applications of this result to disc diagrams. Section 5 is a short section on duals, cut-trees and their properties. Section 6 contains some of the primary objects introduced in this article: fans and ladders. These notions are introduced, their main properties are delineated, and several specific types of fans are defined and illustrated. The next section, Section 7, shows how fans in the dual of a diagram can be used to produce fans in the diagram itself. This is our main technical tool. Section 8 contains the inductive definitions of wheels and ladders of width k, which are used to state the main results. In Section 9 we prove our main result for disc diagrams and in Section 10 we prove our main result for annular diagrams. Section 11 defines the more general notion of a fan in a complex, and uses this to convert the above two main results into statements about subcomplexes. In section 12 we bound the number of minimal fans in a small cancellation complex, and in section 13 we study the lifts of minimal fans to the universal cover.

2. Complexes and diagrams

In this section we provide basic definitions and results about 2-complexes and diagrams. Some of the new definitions are designed to clarify the issues surrounding torsion elements in the fundamental group and how they behave under various maps between combinatorial 2-complexes.

Definition 2.1 (Combinatorial maps and complexes)**.** A map $Y \to X$ between CW complexes is *combinatorial* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X. A CW complex X is *combinatorial* provided that the attaching map of each open cell of X is combinatorial for a suitable subdivision.

It will be convenient to be explicit about the cells in a combinatorial 2-complex.

Definition 2.2 (Polygons)**.** A *polygon* is a 2-dimensional disc whose cell structure has n 0-cells, n 1-cells, and one 2-cell where $n \geq 1$ is a natural number. If X is a combinatorial 2-complex then for each open 2-cell $C \hookrightarrow X$ there is a polygon R, a combinatorial map $R \to X$ and a map $C \to R$ such that the diagram

$$\begin{array}{ccc} C & \hookrightarrow & X \\ \downarrow & \nearrow & \\ R & & \end{array}$$

commutes, and the restriction $\partial R \to X$ is the attaching map of C. In this article the term 2-*cell* will always mean a combinatorial map $R \to X$ where

R is a polygon. The corresponding *open* 2*-cell* is the image of the interior of R.

A similar convention applies to 1-cells. Let e denote the graph with two 0-cells and one 1-cell connecting them. Since combinatorial maps from e to X are in one-to-one correspondence with the characteristic maps of 1-cells of X, we will often refer to a map $e \to X$ as a 1*-cell* of X.

Definition 2.3 (Standard 2-complex)**.** In the study of infinite groups, the most commonly considered combinatorial 2-complexes correspond to presentations. Recall that the *standard* 2*-complex* of a presentation is formed by taking a unique 0-cell, adding a labeled oriented 1-cell for each generator, and then attaching a 2-cell along the closed combinatorial path corresponding to each relator.

Convention 2.4. Unless noted otherwise, all complexes in this article are combinatorial 2-complexes, and all maps between complexes are combinatorial maps. In addition, we will avoid certain technical difficulties by always assuming that all of the attaching maps for the 2-cells are immersions. For 2-complexes with a unique 0-cell, this is equivalent to allowing only cyclically reduced relators in the corresponding presentation.

Definition 2.5 (Paths and cycles)**.** A *path* is a map $P \to X$ where P is a subdivided interval or a single 0-cell. In the latter case, P is called a *trivial path*. A *cycle* is a map $C \to X$ where C is a subdivided circle. Given two paths $P \to X$ and $Q \to X$ such that the terminal point of P and the initial point of Q map to the same 0-cell of X, their concatenation $PQ \to X$ is the obvious path whose domain is the union of P and Q along these points. The path $P \to X$ is a *closed path* provided that the endpoints of P map to the same 0-cell of X. A path or cycle is *simple* if the map is injective on 0-cells. The *length* of the path P or cycle C is the number of 1-cells in the domain and it is denoted by $|P|$ or $|C|$. The *interior* of a path is the path minus its endpoints. In particular, the 0-cells in the interior of a path are the 0-cells other than the endpoints. A *subpath* Q of a path P [or a cycle C] is given by a path $Q \to P \to X$ [$Q \to C \to X$] in which distinct 1-cells of Q are sent to distinct 1-cells of P [C]. Notice that the length of a subpath is at most that of the path [cycle] which contains it. Finally, note that any nontrivial closed path determines a cycle in the obvious way. Finally, when the target space is understood we will often just refer to $P \to X$ as the path P.

Definition 2.6 (Diagrams)**.** A *diagram* D is a nonempty finite 2-complex each of whose connected components has a specific embedding in a distinct 2-sphere. A diagram which consists of a single 0-cell is *trivial*. A contractible diagram is a *disc diagram*. Notice that we allow the possibility of a component mapping onto a 2-sphere, thus contractible implies simply-connected, but not the reverse. A connected diagram with fundamental group $\mathbb{Z}$ is an *annular diagram*. Note that a disc diagram or an annular diagram will always be a strong deformation retraction of a topological disc or annulus, even though it may not be homeomorphic to a disc or annulus. A *boundary*

0-*cell* or *boundary* 1-*cell* is a 0-cell or 1-cell which lies in ∂D. A *boundary* 2-*cell* is a closed 2-cell which has nonempty intersection with ∂D. For example a 2-cell which intersects the boundary at a single 0-cell is a boundary 2-cell.

A disc diagram which is not homeomorphic to a disc is a *singular disc diagram*. In this case, D is either trivial, consists of a single 1-cell joining two 0-cells, or contains a *cut* 0-*cell* which is a 0-cell whose removal disconnects the remainder of the diagram. In general, a diagram D will be called a *singular diagram* whenever the topological boundary of D is not homeomorphic to the disjoint union of circles. Thus for singular diagrams there is a slight difference between the topological boundary of D and the set of cycles which we will refer to as its boundary cycles.

Definition 2.7 (Boundary cycles)**.** Intuitively a *boundary cycle* P of a nontrivial diagram D is a closed path around a component of the complement of D. More precisely, a boundary cycle can be described as follows: (1) start in the interior of any 1-cell in the boundary of D, (2) traverse the 1-cell in a direction so that the region to your left [right] is exterior to D, (3) after passing through a 1-cell and arriving at the 0-cell v, you should leave by the 1-cell which is next in the clockwise [counter-clockwise] ordering of the 1-cells incident at v, and (4) when you return to traversing the initial 1-cell in the same orientation, the boundary cycle is complete. Since the boundary cycle is a closed path we view it as a map $P \to D$ from a subdivided circle to the disc diagram itself. Each component of the complement of D (in the set of spheres in which D embeds) yields a clockwise and a counter-clockwise boundary cycle, and we will not distinguish between them. In particular, if the complement has n components then there are n boundary cycles $P_1 \to D, \ldots, P_n \to D$. For example, a disc diagram has one boundary cycle but a subdivided circle embedded in the sphere is an annular diagram with two boundary cycles. We note that there is a distinction between the (topological) boundary of a diagram D and its boundary cycles, and this distinction is more pronounced when D is singular. A path $Q \to D$ is a *boundary path* if Q is a subpath of a boundary cycle of D.

Definition 2.8 (Spurs)**.** Let D be a diagram. If a boundary cycle of D is not immersed, then there is a valence 1 0-cell v in the image of this boundary cycle. (The *valence* of a 0-cell v in the 2-complex X is the number of ends of 1-cells incident at v.) The 1-cell in D which contains v as an endpoint is a *spur*, the 0-cell v is the *tip* of the spur, and the other endpoint is its *base*. The leftmost illustration in Figure 9 (page 24) is a disc diagram with a spur.

Definition 2.9 (Equivalent and identical pairs)**.** Consider a pair of 2-cells R_1, R_2 in a 2-complex Y which meet along a nontrivial path $P \to Y$. More specifically, let $P \to R_i \to Y$ be a subpath of $\partial R_i \to Y$ such that the

$$\begin{array}{ccc} \text{Identical pair in } Y & \Rightarrow & \text{Identical pair in } X \\ \Downarrow & & \Downarrow \\ \text{Equivalent pair in } Y & \Rightarrow & \text{Equivalent pair in } X \end{array}$$

FIGURE 3. Relationships between identical and equivalent pairs in Y and X.

following diagram commutes:

$$\begin{array}{ccc} P & \rightarrow & R_2 \\ \downarrow & \searrow & \downarrow \\ R_1 & \rightarrow & Y \end{array}$$

Call this a *pair of 2-cells meeting along a path in Y*, or more briefly, a *pair in Y*. This pair in Y is an *equivalent pair* provided that there is a homeomorphism $\partial R_1 \to \partial R_2$ such that there is a commutative diagram of the form:

$$\begin{array}{ccc} P & \rightarrow & \partial R_2 \\ \downarrow & \nearrow & \downarrow \\ \partial R_1 & \rightarrow & Y \end{array}$$

An equivalent pair is an *identical pair* if the map $\partial R_1 \to \partial R_2$ extends to a map $R_1 \to R_2$ such that there is a commutative diagram of the form:

$$\begin{array}{ccc} P & \rightarrow & R_2 \\ \downarrow & \nearrow & \downarrow \\ R_1 & \rightarrow & Y \end{array}$$

Given a map $Y \to X$ and a pair in Y, there is a corresponding pair in X given by

$$\begin{array}{ccc} P & \rightarrow & R_2 \\ \downarrow & \searrow & \downarrow \\ R_1 & \rightarrow & X \end{array}$$

where $R_i \to X$ denotes the composition $R_i \to Y \to X$. It is clear from the definitions that if a pair in Y is an identical [or equivalent] pair, then the corresponding pair in X is an identical [equivalent] pair, and if a pair in Y [or X] is an identical pair, then it is also an equivalent pair in Y [X]. These relationships are summarized in Figure 3.

Definition 2.10 (Reduced maps)**.** A pair in Y is a *cancelable pair* relative to a map $Y \to X$ if the corresponding pair in X is an equivalent pair. The map $Y \to X$ is *reduced* if every cancelable pair in Y (relative to $Y \to X$) is an equivalent pair in Y. In other words, the map $Y \to X$ is reduced if for every pair in Y and corresponding pair in X, the pair in Y is an equivalent pair if and only if the corresponding pair in X is an equivalent pair.

Example 2.11 (Torsion)**.** Let X be the standard 2-complex of the presentation $\langle a | a^2 \rangle$ and let $\widetilde{X} \to X$ be its universal cover. Let $P \to \widetilde{X}$ be a length 1 path in $\widetilde{X}$ and let $R_1 \to \widetilde{X}$ and $R_2 \to \widetilde{X}$ be the two 2-cells of $\widetilde{X}$. This forms an equivalent pair in $\widetilde{X}$ which is not an identical pair in $\widetilde{X}$. The corresponding pair in X is an equivalent pair but not an identical pair in X, even though X only has a single 2-cell.

Lemma 2.12. *Suppose that $Z \to Y$ and $Y \to X$ are reduced maps. Then the composition $Z \to X$ is a reduced map.*

Proof. This follows from the definitions. □

Definition 2.13 (Immersions)**.** The map $Y \to X$ is an *immersion* if it is locally injective. The map $Y \to X$ is a *near-immersion* if $Y \setminus Y^{(0)} \to X$ is locally injective. Equivalently, a map is a near-immersion if for every pair in Y, the corresponding pair in X is an identical pair if and only if the original pair in Y is an identical pair.

Lemma 2.14. *If $Y \to X$ is an immersion then $Y \to X$ is reduced.*

Proof. Consider a pair of 2-cells, R_1 and R_2, in Y which meet along a path $P \to Y$, and assume that the corresponding pair in X is an equivalent pair. Since immersions have the unique lifting property, there is at most one lift of $\partial R_1 \to X$ to a map $\partial R_1 \to Y$ which extends the lift of $P \to X$ to $P \to Y$. Therefore, the diagram

$$\begin{array}{ccc} P & \to & \partial R_2 \\ \downarrow & \nearrow & \downarrow \\ \partial R_1 & \to & Y \end{array}$$

must commute. □

Lemma 2.15. *Let $Y \to X$ be a map between 2-complexes.*

(1) *If $Y \to X$ is a near-immersion and each equivalent pair of X is an identical pair, then $Y \to X$ is reduced.*
(2) *If $Y \to X$ is reduced and each equivalent pair of Y is an identical pair, then $Y \to X$ is a near-immersion.*

Proof. The proof is nearly immediate from Figure 3 and the definitions. In the first case, the assumptions are that the implications along the top and right side of Figure 3 are reversible. This clearly implies that the implication along the bottom is also reversible. In the second case, the assumptions are that the implications along the bottom and left side of Figure 3 are reversible. This clearly implies that the implication along the top is reversible. □

As expressed by Lemma 2.15, near-immersions are very similar to reduced maps, and these two notions are the same when we restrict ourselves to considering 2-complexes with the property that all their equivalent pairs are identical. While it is often the case that certain "redundant" copies of 2-cells can be removed without affecting the fundamental group, unfortunately, these types of 2-cells are unavoidable in general. This is because, especially within the context of small cancellation theory, it is natural to consider 2-cells whose attaching maps are proper powers (i.e. the attaching map is obtained by traversing a closed path in the 2-complex a number of times), and such 2-cells yield equivalent pairs that are not identical. Some pitfalls involving near-immersions in small cancellation theory will be illustrated in Examples 3.12 and 3.13.

Lemma 2.16 (Removing cancelable pairs)**.** *Let D be a diagram whose boundary cycles are $\{P_j \to D : j \in$ Components of Complement of $D\}$, and let $D \to X$ be a map from D to a 2-complex.*

(1) If $D \to X$ is not reduced, then there is a new diagram D' with fewer 2-cells, and a map $D' \to X$ such that there is a bijection between the components of the complements of D and D', and the boundary cycles of D and D' are the same in the sense that for each component j of the complement of D (in the union of spheres containing D) there is a commutative diagram

$$\begin{array}{ccc} P_j & \to & D \\ \downarrow & & \downarrow \\ D' & \to & X \end{array}$$

(2) There exists a diagram D'' and a reduced map $D'' \to X$ such that the boundary cycles of D'' are the same as the boundary cycles of D in the sense of (1).

Proof. We first prove (1). Since $D \to X$ is not reduced there exists a cancelable pair R_1, R_2 of 2-cells meeting along a 1-cell $e \to D$. Let C be the connected component of D which contains e and let Σ be the sphere containing C. The idea is to "cut out" these open cells and then to "sew up" the resulting hole. Let $k = |\partial R_1| - 1$. We first remove the open cells R_1, R_2, and e from the component C to obtain a new diagram D_k which has exactly one additional boundary cycle $S_k T_k^{-1}$ where S_k and T_k are the paths $\partial R_1 \setminus e \to D$ and $\partial R_2 \setminus e \to D$, so that $S_k e$ and $T_k e$ are the attaching maps of $R_1 \to D$ and $R_2 \to D$. Note that the composition $S_k \to D_k \to X$ is identical to $T_k \to D_k \to X$ and so the image of $S_k T_k^{-1}$ in X traces a path and then traces the same path in reverse.

We will now sew up the resulting hole by successively folding and removing 1-cells of the extra boundary cycle to obtain a sequence of diagrams D_i and maps $D_i \to X$, starting at $i = k$ and working our way down to $i = 0$. At each stage, D_i is a diagram whose boundary cycles are the same as those of D except for one additional boundary cycle of the form $S_i T_i^{-1}$, whose projection to X traces a path followed by the same path in reverse. Let C_i be the component of D_i containing S_i and T_i, and let Σ_i be the sphere containing C_i. Let $s_i \to D_i$ denote the restriction of S_i to its final 1-cell, and let S_{i-1} denote the initial subpath of S_i omitting this final 1-cell, so that $S_i = S_{i-1} s_i$. Define t_i and T_{i-1} similarly. Note that the composition $s_i t_i^{-1} \to D_i \to X$ has the form $ee^{-1} \to X$ for some 1-cell e in X. Our choice of D_{i-1} depends on the nature of the path $s_i t_i^{-1} \to D_i$. Typical diagrams for the three cases are illustrated in Figure 4.

Case 1: If $s_i t_i^{-1} \to D_i$ is not a closed path, then there is a quotient $C_i \to C_{i-1}$ obtained by identifying the two 1-cells s_i and t_i of C_i in Σ_i. The other components of D_i are unchanged and the resulting diagram is D_{i-1}.

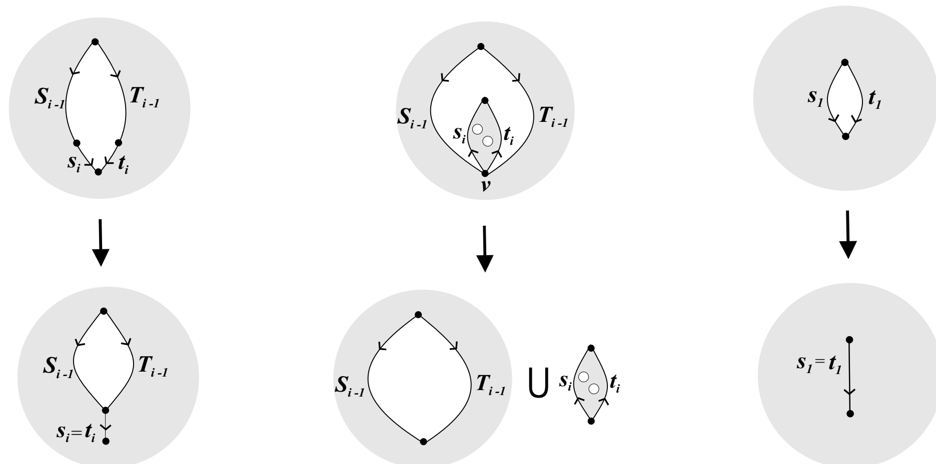


FIGURE 4. The three cases of Lemma 2.16 from left to right.

The new component C_{i-1} clearly embeds in $\Sigma_{i-1} = \Sigma_i$ in almost the exact same way as C_i. In particular, the boundary cycles of D_{i-1} are the same as the boundary cycles of D_i except that the boundary cycle $S_i T_i^{-1} \to D_i$ is replaced by the closed path $S_{i-1} T_{i-1}^{-1} \to D_{i-1}$ obtained by concatenating the compositions $S_{i-1} \to C_i \to C_{i-1}$ and $T_{i-1} \to C_i \to C_{i-1}$.

Case 2: If $s_i t_i^{-1} \to D_{i-1}$ is a closed path and $i > 1$, then $s_i t_i^{-1}$ must bound a connected subdiagram which has exactly two 0-cells on its boundary, one of which is a 0-cell v whose removal disconnects C_i. Let C_i' be the connected subdiagram bounded by $s_i t_i^{-1}$. If C_i' consists of the closure of a single 1-cell, then we simply remove this spur (retaining the 0-cell v) to form the next diagram D_{i-1}. Otherwise, we detach the subdiagram C_i' from the rest of C_i and embed each of the resulting connected components in its own separate sphere. Each connected component is then closed so that a copy of v occurs in C_i' and in the portion of the component of C_i which remained in Σ_i. In the sphere containing C_i' we can now identify s_i and t_i. Notice that if C_i' contains no other boundary cycles, this may create a complete sphere with no boundary cycles at all. Let C_{i-1} be the portion of C_i which remained in $\Sigma_i = \Sigma_{i-1}$. In either case, the boundary cycles of D_{i-1} are essentially the same as before except that the boundary cycle $S_i T_i^{-1} \to D_i$ is replaced by the path $S_{i-1} T_{i-1}^{-1} \to D_{i-1}$.

Case 3: When $i = 1$, $s_1 t_1^{-1} \to D_1$ is an entire boundary cycle of D_1, and there is a quotient $D_1 \to D_0$ obtained by identifying the two 1-cells s_1 and t_1 of D_1. The component C_0 clearly embeds in $\Sigma_0 = \Sigma_1$ in almost the exact same way as C_1, and we are done.

This procedure terminates at $i = 0$ after Case 3 has been applied. The final diagram has the same boundary cycles as the original one, and it has two fewer 2-cells as claimed. Finally, to prove (2) from (1), note that the procedure of (1) can be applied only finitely many times. $\square$

One immediate consequence of Lemma 2.16 is that the fundamental groups of 2-complexes can be studied via reduced disc diagrams. The following theorem, known as the Lyndon-van Kampen lemma, was first discovered in 1933 by E. R. van Kampen [20] and independently rediscovered in 1966 by R. C. Lyndon [10]. Alternative versions can be found in [2], [3], [11], or [12].

Lemma 2.17 (Disc version). *If X is a 2-complex and $P \to X$ is a null-homotopic closed path, then there exists a disc diagram D and a reduced map $D \to X$ such that $P \to D$ is the boundary cycle of D, and $P \to X$ is the composition $P \to D \to X$.*

Proof. Since $P \to X$ is null-homotopic, there exists a disc diagram $D \to X$ such that $P \to X$ factors as $P \to D \to X$, and such that the boundary cycle of D is $P \to D$. A proof of this can be found in [3, Section 2.2] or [11]. By Lemma 2.16, there is a diagram D' with the same boundary cycle as D. If $D'' \subset D$ is the component containing this boundary cycle, then $D'' \to X$ is the desired map. □

Another consequence is the existence of reduced annular diagrams for homotopic essential closed paths.

Lemma 2.18 (Annular version). *If X is a 2-complex and $P_1 \to X$ and $P_2 \to X$ are homotopic essential closed paths, then as in the proof of Lemma 2.17, there exists a reduced annular diagram $D \to X$ whose boundary cycles are $P_1 \to D \to X$ and $P_2 \to D \to X$.*

Proof. Since $P_1 \to X$ and $P_2 \to X$ are homotopic, there must exist an annular diagram $D \to X$ such that $P_i \to X$ factors as $P_i \to D \to X$, and such that the boundary cycles of D are $P_i \to D$ for $i \in \{1, 2\}$. By Lemma 2.16, there is a diagram D' with the same boundary cycles as D. Since $P_1 \to X$ is essential, the images of $P_1 \to D$ and $P_2 \to D$ must lie in the same component D' of D. This component must be an annular diagram because it has only two boundary cycles. □

3. Small cancellation theory

This section provides the basic definitions of small cancellation theory. Although small cancellation theory has traditionally been defined using group presentations (which are equivalent to 2-complexes with a unique 0-cell), the definitions can be adjusted to apply to arbitrary 2-complexes as we have done here.

Definition 3.1 (Piece). Let X be a combinatorial 2-complex. Intuitively, a piece of X is a path which is contained in the boundaries of the 2-cells of X in at least two distinct ways. More precisely, a nontrivial path $P \to X$ is a *piece* of X if there are 2-cells R_1 and R_2 such that $P \to X$ factors as $P \to R_1 \to X$ and as $P \to R_2 \to X$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$

such that there is a commutative diagram

$$\begin{array}{ccc} P & \rightarrow & \partial R_2 \\ \downarrow & \nearrow & \downarrow \\ \partial R_1 & \rightarrow & X \end{array}$$

In other words, P belongs to a pair in X which is not an equivalent pair. Excluding commutative diagrams of this form ensures that P occurs in ∂R_1 and ∂R_2 in essentially distinct ways. The definition can also be formulated in terms of reduced maps. Let $R_1 \cup_P R_2$ denote the diagram formed by gluing R_1 to R_2 along P. The path P is a piece of X if and only map $R_1 \cup_P R_2 \to X$ is reduced.

Example 3.2. Let $G = \langle a, b, c, d \mid abcabd \rangle$ and let X be the standard 2-complex for this presentation. The path $P \to X$ corresponding to the word ab factors through the unique 2-cell in two distinct ways. Since there is no map from the boundary of this 2-cell to itself which sends one instance of ab to the other, the path P is a piece of X. On the other hand, in the presentation $\langle a, b, c \mid abcabc \rangle$, the path corresponding to ab is not a piece, since there is a cyclic rotation of the boundary of the 2-cell which sends one occurrence of ab to the other.

Lemma 3.3. *If $Y \to X$ is reduced and $P \to Y$ is a piece then $P \to Y \to X$ is a piece.*

Proof. By definition, $P \to Y$ is a piece if and only if there are 2-cells R_1 and R_2 in Y such that $R_1 \cup_P R_2 \to Y$ is reduced. If such 2-cells exist, then by Lemma 2.12, $R_1 \cup_P R_2 \to Y \to X$ is reduced and thus $P \to X$ is also a piece. □

Definition 3.4 ($C(p)$-$T(q)$ complexes)**.** Let p be a natural number. A 2-complex X is a $C(p)$ *complex* provided that for each 2-cell $R \to X$, its attaching map $\partial R \to X$ is not the concatenation of fewer than p pieces in X. This is equivalent to requiring that for any diagram D and reduced map $D \to X$, if $R \to X$ is a 2-cell such that no 1-cell of ∂R is contained in ∂D, then the cycle $\partial R \to D$ passes through 0-cells of valence ≥ 3 at least p times.

There is a closely related metric condition depending on a positive real number α. The complex X is a $C'(\alpha)$ *complex* provided that for each 2-cell $R \to X$, and each piece $P \to X$ which factors as $P \to R \to X$, we have $|P| < \alpha|\partial R|$. Note that if X satisfies $C'(\alpha)$ and $n \leq \frac{1}{\alpha} + 1$ then X satisfies $C(n)$.

The final small cancellation condition concerns neighborhoods of 0-cells of X. We say X is a $T(q)$ *complex* if there does not exist a reduced map $D \to X$ where D is a disc diagram containing an internal 0-cell v, such that $2 < \text{valence}(v) < q$. If none of the 2-cells are "proper powers" then this is equivalent to requiring that there not exist a closed immersed path in the link of a 0-cell of length between 2 and q. When proper powers are present, various edges in the link need to be identified before searching for closed

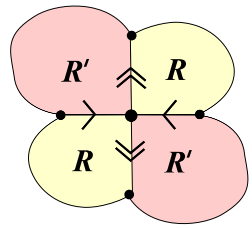


FIGURE 5. A piece of length 2 and its $T(5)$-violating consequence.

immersed paths. The first $T(q)$ condition which restricts the nature of the 2-complex is $T(4)$. A 2-complex which satisfies both $C(p)$ and $T(q)$ is a *$C(p)$-$T(q)$ complex.*

As mentioned above, the definitions presented here are a natural generalization of traditional small cancellation theory in the sense that a presentation $\langle a_1, a_2, \dots \mid R_1, R_2, \dots \rangle$ is a $C(p)$-$T(q)$ presentation (according to the usual definition) if and only if its associated standard 2-complex is a $C(p)$-$T(q)$ complex as defined here. The main distinction is that Definition 3.4 applies to 2-complexes with more than one 0-cell.

The following lemma was first observed by Pride (see [5]).

Lemma 3.5. *If X is a $T(q)$ complex with $q \geq 5$, then every piece in X has length 1.*

Proof. If there is a piece of length > 1, then there is a piece of length 2. In particular, there exist distinct 2-cells R and R' and a length 2 path P contained in both boundary cycles. Using these 2-cells and this path, it is possible to create a reduced disc diagram with four 2-cells and an internal 0-cell of valence 4. See Figure 5. Since this violates the $T(5)$ condition, no such piece exists. □

Convention 3.6. For convenience we will always assume that each 2-cell of a $C(p)$ complex X has the property that its boundary cycle is not the concatenation of fewer than p paths, each of which is either a piece or of length 1. In particular, the length of the boundary cycle of each 2-cell is $\geq p$. There is little loss of generality in making this assumption. Indeed, if X is a $C(p)$ complex, then X will have the above property after we subdivide p-times all of those 1-cells of X which are not pieces.

Reduced maps are the natural category of maps in small cancellation theory because of the following:

Theorem 3.7. *Let $Y \to X$ be reduced.*

1. *If X is $C(p)$ then Y is $C(p)$.*
2. *If X is $T(q)$ then Y is $T(q)$.*
3. *If X is $C'(\alpha)$ then Y is $C'(\alpha)$.*

Proof. Let $D \to Y$ be a reduced map of a diagram to Y. By Lemma 2.12, the composition $D \to Y \to X$ yields a reduced map $D \to X$. Parts (1) and (3) now follow from Lemma 3.3, and part (2) follows from the definition of $T(q)$. □

Theorem 3.7 has the following notable special cases:

Corollary 3.8. *A covering space of a $C(p)$-$T(q)$ complex is a $C(p)$-$T(q)$ complex.*

Proof. Since covering maps are immersions, this follows from Lemma 2.14 and Theorem 3.7. □

Corollary 3.9. *Let X be a $C(p)$-$T(q)$ complex, and let $D \to X$ be a reduced map of a disc diagram. Then D is a $C(p)$-$T(q)$ complex.*

As a consequence, $C(p)$-$T(q)$ disc diagrams exist for null-homotopic paths and annular $C(p)$-$T(q)$ diagrams exist for pairs of homotopic paths which are not null-homotopic.

Theorem 3.10 ($C(p)$-$T(q)$ disc diagrams)**.** *If X is a $C(p)$-$T(q)$ complex and $P \to X$ is a closed null-homotopic path, then there exists a $C(p)$-$T(q)$ disc diagram D and a reduced map $D \to X$ whose unique boundary cycle is the path $P \to X$. Consequently, the pieces of D are sent to pieces of X.*

Proof. This is immediate from Theorem 3.7 and Lemma 2.17. □

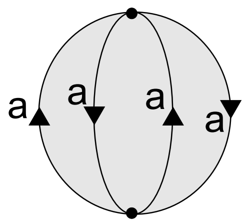
Theorem 3.11 ($C(p)$-$T(q)$ annular diagrams)**.** *If X is a $C(p)$-$T(q)$ complex and $P_1 \to X$ and $P_2 \to X$ are homotopic essential closed paths in X, then there exists an annular $C(p)$-$T(q)$ diagram D and a reduced map $D \to X$ whose boundary cycles are $P_1 \to D \to X$ and $P_2 \to D \to X$. Consequently, the pieces of D are sent to pieces of X.*

Proof. This is immediate from Theorem 3.7 and Lemma 2.18. □

The following examples show that Theorem 3.7 can fail for a near-immersion.

Example 3.12. Let X be the standard 2-complex of $\langle a \mid a^2 \rangle$ and observe that because the unique 1-cell of X is not a piece, X satisfies $C(p)$ for all p. Let D be the disc diagram illustrated on the left in Figure 6 which consists of two 0-cells, four 1-cells connecting the two 0-cells, and three 2-cells. Note that D only satisfies $C(2)$ since both internal 1-cells of Y are pieces and the boundary of the 2-cell between them is the concatenation of these two pieces. Finally, observe that there is a near-immersion $D \to X$ (although most of the combinatorial maps from D to X are not near-immersions.)

Example 3.13. Let X denote the standard 2-complex of $\langle a, b, c \mid (ab)^2, (bc)^2 \rangle$. Then it is easy to check that X satisfies $T(q)$ for each q. Let D denote the disc diagram illustrated on the right in Figure 6, which consists of four squares R_1, R_2, R_3, R_4 glued together along 1-cells around a single interior 0-cell. Note that D satisfies $T(4)$ but not $T(5)$. There is a near-immersion



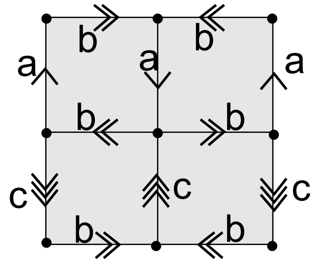


Figure 6. Domains of near-immersions which are not reduced. In both figures, the 1-cells are labeled and oriented so that they agree with their images.

$D \to X$ which sends R_1 and R_2 to the 2-cell corresponding to $(ab)^2$ and sends R_3 and R_4 to the 2-cell corresponding to $(bc)^2$.

4. Combinatorial Gauss-Bonnet

In this section, we state and prove a version of the combinatorial Gauss-Bonnet theorem, followed by two applications. It was first stated and proven for diagrams which embed in a sphere without boundary by Gersten in [4] and Pride in [17], thereby refining some earlier ideas of Lyndon's concerning (p, q)-maps (see [11]) as well as an idea of Sieradski's [19]. The Gauss-Bonnet theorem was stated for surfaces in [5]. In this article, we prove a generalization to arbitrary 2-complexes. Since first writing this article we have learned that this theorem was proven earlier for piecewise constant curvature 2-complexes by Ballmann and Buyalo [1]. The proof is the same.

Definition 4.1 (Links and perimeters)**.** Let X be a locally finite 2-complex and let x be a point in its 1-skeleton. The cells of X each have a natural partial metric obtained by making every 1-cell isometric to the unit interval and every n-sided 2-cell isometric to a Euclidean disc of circumference n whose boundary has been subdivided into n curves of length 1. In this metric, the set of points which are a distance equal to ϵ from x will form a finite graph. If ϵ is sufficiently small, then the graph obtained is independent of the choice of ϵ. This well-defined graph is the *link of x in X* and is denoted by $\text{Link}(x)$. If v is a 0-cell of X, then the graph $\text{Link}(v)$ is called the link of the 0-cell v. When X contains a single 0-cell v, then $\text{Link}(v)$ is the *star graph* or *coinitial graph* of the presentation X encodes. To avoid confusion, we will discuss $\text{Link}(v)$ using the language of *vertices* and *edges* and reserve the terms 0-cells and 1-cells for the 2-complex X containing v. Notice that the link of a 0-cell can be an arbitrary finite graph. In contrast, if x lies in the interior of a 1-cell e of X, then the link of x has a very particular form: $\text{Link}(x)$ will have exactly two vertices (corresponding to the two ends of e) and a finite number of edges connecting these two vertices. The number of edges in $\text{Link}(x)$ is called the *perimeter* of e and will be denoted $\mathbf{P}(e)$. The word "perimeter" is used because if each 1-cell is thought to have length 1, then this is the length of the boundary created when the 1-cell e is removed from X.

Definition 4.2 (Corners and sides)**.** Let X be a 2-complex, let v be a 0-cell in X, let $R \to X$ be a 2-cell in X, and let x be a point in the interior of a 1-cell e in X. If we regard the 2-cells of X as polygons, then the edges of $\text{Link}(v)$ correspond to the corners of these polygons attached to v. We will refer to a particular edge in $\text{Link}(v)$ as a *corner of R at v* if this edge comes from the polygon $R \to X$. Similarly, the edges of $\text{Link}(x)$ correspond to the sides of these polygons attached to e, and we will refer to a particular edge in $\text{Link}(x)$ as a *side of R at e* if this edge comes from $R \to X$.

Remark 4.3. It is immediate from the definition that the 2-cell $R \to X$ contributes exactly $|\partial R|$ corners at 0-cells of X and exactly $|\partial R|$ sides at 1-cells of X. Since the number of sides contributed by each polygon is the same as the number of corners, the total number of sides in X equals the total number of corners.

Definition 4.4 (Combinatorial curvature)**.** We say X is an *angled* 2-complex provided that every corner c of X has been assigned a real number $\angle c$ called the *angle of c*, and X is *positively angled* if all these angles are positive. If f is a 2-cell of X then the *curvature of f* is defined to be the sum of the angles assigned to its corners minus $(|\partial f| - 2)\pi$ (which is the expected Euclidean angle sum). In symbols we have

$$\text{Curvature}(f) = \left(\sum_{c \in \text{Corners}(f)} \angle c \right) - |\partial f|\pi + 2\pi$$

The curvatures of the 2-cells of X are its *2-cell curvatures*. If v is a 0-cell of X then the *curvature of v* is defined to be 2π minus $\pi \cdot \chi(\text{Link}(v))$ minus the sum of the angles assigned to corners at v. If X embeds in the plane and v is an interior 0-cell then $\text{Link}(v)$ is a circle and the curvature measures the difference between the expected Euclidean angle sum of 2π and the actual angle sum. This *0-cell curvature* equation gives the appropriate generalization of this idea to arbitrary 2-complexes. In symbols

$$\text{Curvature}(v) = 2\pi - \pi \cdot \chi(\text{Link}(v)) - \left(\sum_{c \in \text{Corners}(v)} \angle c \right)$$

Remark 4.5. Let D be a positively angled diagram and let v be a 0-cell of D. If $\text{Link}(v)$ is not a complete circle (but is not empty), then $\chi(\text{Link}(v)) \geq 1$ and thus $\text{Curvature}(v) \leq \pi$. Moreover, $\text{Curvature}(v) = \pi$ if and only if $\text{Link}(v)$ is a single vertex and v is the tip of a spur. Furthermore, if $\text{Link}(v)$ is disconnected, then $\chi(\text{Link}(v)) \geq 2$ and thus $\text{Curvature}(v) \leq 0$. In this case, $\text{Curvature}(v) = 0$ if and only if the $\text{Link}(v)$ consists of exactly two isolated vertices.

We can now state and prove the combinatorial Gauss-Bonnet theorem.

Theorem 4.6 (Combinatorial Gauss-Bonnet)**.** *If X is an angled 2-complex then the sum of the 2-cell curvatures and the 0-cell curvatures is 2π times the Euler characteristic of X.*

$$\sum_{f\in 2\text{-}cells(X)} Curvature(f) + \sum_{v\in 0\text{-}cells(X)} Curvature(v) = 2\pi\cdot\chi(X). \tag{1}$$

In particular, if X is a disc diagram then this sum will be 2π and if X is an annular diagram the sum will be 0.

Proof. For convenience we will define the following pair of constants.

$$C = \sum_{c\in\text{Corners}(X)} \angle c \qquad\qquad P = \sum_{e\in 1\text{-cells}(X)} \mathbf{P}(e)$$

The proof will follow from the following two equations:

$$\sum_{f\in 2\text{-cells}(X)} \text{Curvature}(f) = C - \pi P + 2\pi F \tag{2}$$

$$\sum_{v\in 0\text{-cells}(X)} \text{Curvature}(v) = 2\pi V - 2\pi E + \pi P - C \tag{3}$$

where the letters V, E, and F represent the number of 0-cells, 1-cells, and 2-cells in X, respectively. To prove the theorem, one simply adds Equations 2 and 3, and observes that the corner sums and the perimeter sums cancel, leaving $2\pi(V - E + F)$, which is exactly 2π times the Euler characteristic of X. The remainder of the proof justifies these two equations.

In the definition of the curvature of a 2-cell, there are three terms. The first term contributes C and the last term contributes $2\pi F$ towards the sum of all 2-cell curvatures. Observe that the sum of the lengths of the boundaries of the 2-cells of X is precisely the number of sides of 2-cells in X, and this is the total number of sides at 1-cells of X which is precisely P. Thus the middle term contributes $-\pi P$ towards the sum, and Equation 2 has been established.

Similarly, in the definition of the curvature of a 0-cell there are three terms. The first term contributes $2\pi V$ and the last term contributes $-C$ towards the sum of the 0-cell curvatures. Thus to establish Equation 3 it remains to show that the sum of the Euler characteristics of the links of the 0-cells is $2E - P$. We will consider the vertices and edges in the links separately. Note that the edges in the links of the 0-cells are in one-to-one correspondence with the corners of X. Since each 2-cell has as many corners as sides, the total number of edges occurring in the links of the 0-cells is P. On the other hand, the vertices in the links of the 0-cells correspond to the ends of the 1-cells of X. Since each 1-cell of X contributes two distinct vertices to the links of the 0-cells, the total number of vertices which occur in the links is $2E$. Finally, since the Euler characteristic of a graph is the number of vertices minus the number of edges, the sum of the

Euler characteristics of the links is $2E - P$. This establishes Equation 3 and completes the proof. $\square$

In the remainder of the section we present two quick applications of Theorem 4.6. The first application is perhaps the most surprising, and it is the source of essentially all of the results of small cancellation theory.

Theorem 4.7. *Let D be a positively angled disc diagram. Suppose that each 2-cell and each interior 0-cell of D has nonpositive curvature. Then one of the following holds:*

(1) *D is trivial.*
(2) *D is a subdivided interval.*
(3) *There are at least three 0-cells in ∂D with positive curvature.*

Proof. By the Combinatorial Gauss-Bonnet theorem, the total curvature of D is exactly 2π. Therefore there must be some 0-cells in ∂D with positive curvature. First observe that if there is a 0-cell v in ∂D with $\text{Curvature}(v) > \pi$ then $\text{Link}(v)$ is empty, and therefore D is trivial because it is connected and so $D = v$.

The other possibility is that no boundary 0-cell has curvature larger than π. Now if there are at most two sources of positive curvature, then both of these 0-cells must have curvature exactly equal to π and all other curvatures must equal 0. Let v_0 be one of the two 0-cells of positive curvature. By Remark 4.5, the link of v_0 must consist of a single 0-cell. If the 0-cell v_1 at the other end of the unique 1-cell emanating from v_0 is not the other 0-cell of positive curvature, then $\text{Curvature}(v_1) = 0$, and $\text{Link}(v_1)$ must be disconnected. By Remark 4.5 we conclude that $\text{Link}(v_1)$ has a specific structure: it consists of two disconnected vertices, and thus there is a unique additional 1-cell incident at v_1. Repeating this argument, we eventually reach the other 0-cell of positive curvature and the proof is complete. $\square$

As a consequence, diagrams satisfying $C(p)$-$T(q)$ for large p and q have restricted structures.

Theorem 4.8. *Let D be a $C(3)$-$T(6)$ [$C(4)$-$T(4)$] disc diagram. Then one of the following holds:*

(1) *D is trivial.*
(2) *D is a subdivided interval.*
(3) *There are at least three 0-cells in ∂D with connected links and valence ≤ 3 [≤ 2].*

Proof. We assign an angle of $\pi/3$ [$\pi/2$] to each corner of D. Let R be a 2-cell of D. By the $C(3)$ [$C(4)$] condition and Convention 3.6, R has at least 3 [respectively 4] corners and therefore has nonpositive curvature. By the $T(6)$ [$T(4)$] condition, each interior 0-cell has nonpositive curvature. Let v be a 0-cell in ∂D. If $\text{Link}(v)$ is disconnected then v will be nonpositively curved because $\chi(\text{Link}(v)) \geq 2$. On the other hand, if $\text{Link}(v)$ is connected and v has valence at least 4 [respectively 3], then v is nonpositively curved

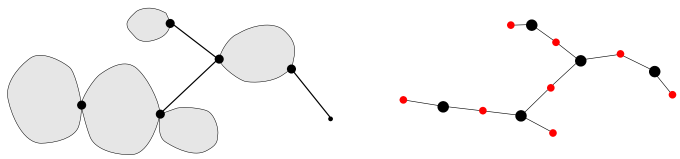


FIGURE 7. A disc diagram and its cut-tree. The bold 'black' 0-cells in the tree correspond to the cut 0-cells of the diagram and the smaller 'red' 0-cells correspond to the cut-components.

because of the way the angles are assigned. An application of Theorem 4.7 completes the proof. □

5. Cut-trees, basepoints, and duals

This section describes additional properties of diagrams which will be needed in the proofs of the main theorems. In particular, we associate to a disc diagram D a finite tree called a cut-tree which encodes the singularities of D and a new diagram called its dual. This section also introduces the language of doubly-based diagrams.

Definition 5.1 (Cut-tree)**.** Let D be a disc diagram. A 0-cell v is called a *cut* 0*-cell* of D provided that $D \setminus v$ is not connected. We now define a tree T, called the *cut-tree* of D, which encodes the arrangement of the cut 0-cells of D. Let V be the set of all cut 0-cells of D. A connected component of $D \setminus V$ will be called a *cut-component*. Let C be the set of cut-components of D. The tree T is constructed by adding a black 0-cell for each 0-cell $v \in V$ and a red 0-cell for each component $c \in C$. A 1-cell connects the 0-cell for v to the 0-cell for c if and only if v is in the closure of c. Since all 1-cells connect black 0-cells to red 0-cells, the graph is bipartite, and since each of the black 0-cells disconnects T, the graph is a tree. Note that since black 0-cells have valence at least 2, the leaves of T are red. This procedure is illustrated in Figure 7.

The following properties are immediate from the definition and they will be used to prove Theorem 9.2.

Lemma 5.2. *Let D be a disc diagram and let T be its cut-tree.*

(1) *T is trivial if and only if D is a single* 0*-cell or a single* 1*-cell or a nonsingular diagram.*
(2) *The leaves of T correspond to the nontrivial subdiagrams without cut* 0*-cells which are attached to the rest of D by a single* 0*-cell. In particular, if D is a nontrivial singular diagram with more than one* 1*-cell, then D contains at least two such subdiagrams.*
(3) *The subdiagram associated to a red* 0*-cell is singular if and only if it is a* 1*-cell which does not border a* 2*-cell in D. In particular, the subdiagram associated to a leaf of T is singular if and only if it is a spur.*

Definition 5.3 (Doubly-based diagrams)**.** A *doubly-based diagram* D is a disc diagram in which two (possibly identical) 0-cells, s and t, have been specified in the boundary cycle of D. These 0-cells are called the *startpoint* and the *endpoint* of D, respectively, and collectively they are the *basepoints* of D. Note that specifying a 0-cell in the boundary cycle of D is slightly different from specifying a 0-cell in the boundary of D. Recall from Definition 2.7 that the boundary cycle of D is a particular map from a subdivided circle into D. Thus specifying a startpoint and endpoint in the boundary cycle not only determines two 0-cells in ∂D, but it also determines a pair of paths $P_1 \to \partial D \to D$ and $P_2 \to \partial D \to D$ with s as their common startpoint and t as their common endpoint such that $P_1 P_2^{-1}$ is the boundary cycle of D. The paths $P_1 \to D$ and $P_2 \to D$ will be called *the boundary paths determined by the basepoints of* D. A pair of 0-cells in the boundary of D, on the other hand, do not uniquely determine such paths if either s or t is a cut 0-cell of D. Notice that when $s = t$, one of these paths will be a trivial path.

Definition 5.4 (Arcs)**.** A path $P \to X$ is an *arc* provided it is an embedding (except possibly at its endpoints) and each of its interior 0-cells is mapped to a 0-cell with valence 2 in X. An arc which is not a proper subpath of any other arc is a *maximal arc*. If X is a diagram, then we will further distinguish between internal arcs and boundary arcs. If the arc lies in the interior of the diagram (except possibly at its endpoints), then it is an *internal arc*; otherwise the arc must be a subpath of a boundary cycle of D and it is a *boundary arc*. Notice that a maximal internal arc in a diagram is the same as a maximal piece.

Definition 5.5 (Dual diagrams)**.** Let D be a connected diagram. The *dual* of D is a subspace of D which consists of

(1) a 0-cell at the center of each 2-cell of D,
(2) a 1-cell passing through each maximal internal arc of D connecting the centers of the 2-cells on either side, and
(3) a 2-cell for each interior 0-cell v of valence at least 3.

(What we have called the dual of D is what a graph theorist would typically call the "internal dual".) Notice that 0-cells of valence 2 play no role in this definition. The dual, however, may contain 0-cells of valence 2, particularly in its boundary. These 0-cells will be retained since they encode useful information about the original diagram, and they also ensure that the dual diagram satisfies Convention 3.6 in Lemma 5.6 below. Figure 11 (page 27) shows three diagrams and their duals. Finally, if D has a basepoint then a corresponding basepoint for E will be chosen so that the corresponding 2-cell of D contains the original basepoint in its boundary. Such a choice will be made for each basepoint of D. In practice, the actual choice of 0-cell will be unimportant.

Lemma 5.6. *Let D be a connected diagram and let E be its dual.*

(1) *If D is nonsingular then E is connected and $\pi_1 D \cong \pi_1 E$.*

(2) *If D is nonsingular, R is a boundary 2-cell of D, and v is the corresponding 0-cell in E, then the number of components of $\partial R \cap \partial D$ equals the number of components of* $\text{Link}(v)$.
(3) *If D is simply-connected then each component of E is simply-connected.*
(4) *If D is a $C(p)$-$T(q)$ diagram then E is a $C(q)$-$T(p)$ diagram.*

Proof. To prove (1) notice that when D is nonsingular, the portion of D which lies outside of E is a regular neighborhood of ∂D and that D can be viewed as a regular neighborhood of E. There is thus a strong deformation retraction from $D \to E$ and we are done. This perspective also makes the proof of (2) immediate.

To prove (3), note that if D is nonsingular and simply-connected then E is connected and simply-connected by (1). If, on the other hand, D is singular then there are cut 0-cells. By Lemma 5.2 the cut-components are either isolated 1-cells or nonsingular disc diagrams. The isolated 1-cells play no role in the formation of E, and the simply-connectedness of the remaining pieces now follows from (1).

Finally, to prove (4), notice that every 2-cell of E corresponds to an internal 0-cell of D and the number of sides of the 2-cell corresponds to the valence of the 0-cell. Similarly, every internal 0-cell of E corresponds to an internal 2-cell of D and the valence of the 0-cell corresponds to the number of maximal internal arcs into which the boundary cycle of the 2-cell is partitioned. The small cancellation conditions are therefore inherited as claimed. □

Remark 5.7. It is possible for a disc diagram D which satisfies Convention 2.4 to have a dual E which violates it. In particular, E may contain a spur which projects into the interior of the diagram, and this determines a 2-cell in E whose attaching map is not an immersion. The only way that this can happen, however, is if D violates the $C(2)$ condition. Since dual diagrams will only be used in the context of small cancellation restrictions, Convention 2.4 will always be preserved.

Lemma 5.8 (Double duals embed)**.** *Let D be a nonsingular $C(3)$ diagram whose dual E is also nonsingular and let D' be the dual of E. If D has no internal 0-cells of valence 2, then D' is a subdiagram of D. More generally, D' is a subdiagram of D after a suitable subdivision of its 1-cells.*

Proof. Every 0-cell of D' corresponds to a 2-cell of E which in turn corresponds to a 0-cell of D. Similarly, every 2-cell of D' corresponds to a 0-cell of E which corresponds to a 2-cell of D. The situation with the 1-cells of D' is slightly more complicated. Let e be a 1-cell of D' connecting 0-cells u and v (which we will prove to be distinct below). The 1-cell e corresponds to a maximal internal arc P_e in E which is contained in the boundaries of the 2-cells R_u and R_v. Each of the 1-cells in P_e corresponds to a maximal internal arc of D connecting u to v. If P_e contains more than one 1-cell, then D will contain a 2-cell bounded by these arcs, which violates the $C(3)$

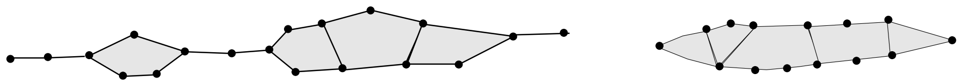


Figure 8. A singular and a nonsingular ladder

condition. Thus, P_e is a single 1-cell of E which corresponds to a maximal internal arc of D. If D has no 0-cells of valence 2 then this maximal internal arc is itself a single 1-cell, and D' embeds in D. Otherwise, an obvious subdivision enables the embedding. $\square$

6. Ladders and fans

In this section we introduce the general notions of a ladder and a fan as a prelude to the specific types of fans and ladders defined inductively in Sections 7 and 8.

Definition 6.1 (Ladders)**.** Let D be a nontrivial doubly-based disc diagram which is not a single 2-cell, and let $P_1 \to D$ and $P_2 \to D$ be the two boundary paths determined by the basepoints of D. Suppose that the cut-tree of D is either trivial or a subdivided interval, and that the basepoints of D are distinct and are not cut 0-cells. Suppose further that if the cut-tree is a subdivided interval then the basepoints lie in the cut-components corresponding to the endpoints of the interval. If every maximal internal arc of D begins at a 0-cell in the interior of $P_1 \to D$ and ends at a 0-cell in the interior of $P_2 \to D$, then D will be called a *ladder*. Notice that the basepoints of a ladder are distinct and so the paths P_1 and P_2 are nontrivial. A singular and a nonsingular ladder are illustrated in Figure 8. In each case, the basepoints of the ladder are the leftmost and rightmost 0-cells.

When D is nonsingular, this definition can be restated in terms of the dual of D. Specifically, a nonsingular doubly-based disc diagram D is a ladder if and only if its dual is a subdivided interval and the basepoints of D lie in the interiors of the boundary paths of the 2-cells corresponding to the endpoints of the interval. In Section 8, the ladders described in this definition will be called *ladders of width* ≤ 1.

Definition 6.2 (Fans)**.** A *fan* F is a nonsingular ladder or a doubly-based 2-cell in which a nontrivial path determined by the endpoints of F has been designated as its *outer path*. Call this path Q. The other path, denoted S, will be called the *inner path* of the fan.

A *fan in a diagram* is an embedding $F \hookrightarrow D$ such that the outer path of F projects to a boundary path $Q \to \partial D$, and the inner path $S \to F$ projects to a internal path $S \to D$. Note that $S \to F$ may be trivial if F is a doubly-based 2-cell, but it is always nontrivial if F is a ladder.

The following types of fans will be distinguished.

Definition 6.3 (i-shells)**.** Let D be a diagram. An *i-shell* of D is a 2-cell $R \hookrightarrow D$ whose boundary cycle is the concatenation of a boundary maximal

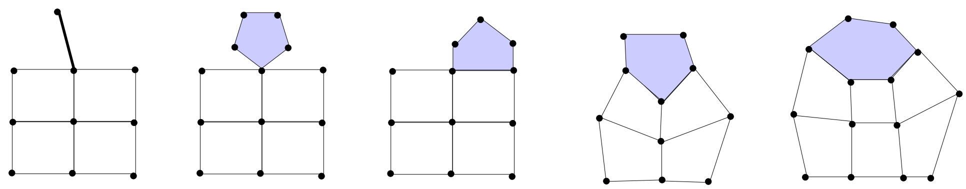


FIGURE 9. Spurs and i-shells

arc and an interior path which is the concatenation of i interior maximal arcs. More specifically, the boundary cycle $\partial R \to D$ is the concatenation of subpaths $P_0, P_1, \ldots, P_i$ where $P_j \to D$ is an interior maximal arc of D for all $j > 0$ and $P_0 \to D$ is the preimage of ∂D in ∂R. An i-shell is an embedding of a doubly based 2-cell fan in the following sense: Its outer path is $P_0 \to D$, and its inner path is the concatenation of the i consecutive pieces. Note that the outer path P_0 of the i-shell is nontrivial if D satisfies $C(i+1)$.

Illustrated from left to right in Figure 9 are disc diagrams containing a spur, a 0-shell, a 1-shell, a 2-shell, and a 3-shell. In each case, the 2-cell R is shaded, and the maximal boundary arc P_0 is $\partial R \cap \partial D$.

Definition 6.4 (Complement of n pieces)**.** Let D be a disc diagram (or more generally a 2-complex). Suppose that the boundary cycle of a 2-cell $R \to D$ is the concatenation of subpaths $P_0, P_1, \ldots, P_n$ where $P_i \to D$ is a nontrivial path which is a piece of D for all $i > 0$. The (possibly trivial) path $P_0 \to D$ is a *complement of n pieces of D*. For example, the outer path of an i-shell is a complement of i pieces of D.

Definition 6.5 (Pointed fans)**.** A *pointed fan* $F \hookrightarrow D$ is a fan whose 2-cells are 2-shells. If F has i distinct 2-cells then F contains $i+1$ pieces in D which have a common initial point and which terminate at various points on the outer path of F. The inner path $S \to D$ is the concatenation of the outer two of these maximal internal arcs, and the outer path $Q \to D$ is the concatenation of i maximal boundary arcs which are complements of two pieces. See the left side of Figure 10.

Definition 6.6 (Broad fans)**.** A *broad fan* $F \hookrightarrow D$ is a fan whose 2-cells are 2-shells and 3-shells. If F contains $i+j$ 2-cells, where i of them are 2-shells and j of them are 3-shells, then the outer path of F is the concatenation of $i+j$ maximal boundary arcs each of which is the complement of two or three pieces, and the rest of the 1-skeleton of F is the union of $i+2j+1$ pieces. The inner path S consists of $j+2$ of these pieces and the remaining $i+j-1$ of them are internal maximal arcs of the diagram F. A broad fan F is *k-separated* if from left to right, it has at least k 2-shells at the beginning, at least k 2-shells between every pair of 3-shells, and at least k 2-shells at the end.

The diagram on the left-hand side of Figure 10 contains a pointed fan with $i = 4$. The diagram in the middle contains a broad fan with $i = 0$ and

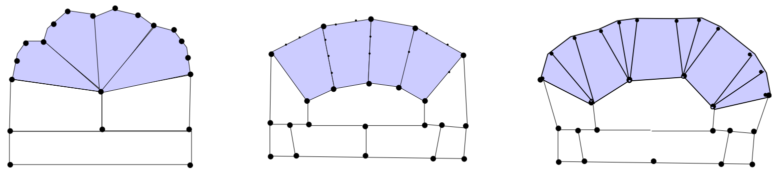


FIGURE 10. A pointed fan, a 0-separated broad fan, and a 2-separated broad fan

$j = 4$, and the diagram on the right-hand side contains a 2-separated broad fan with $i = 8$ and $j = 3$. Finally, we define a type of subdiagram which does not satisfy Definition 6.2, but which is similar enough in structure to merit the designation "fan".

Definition 6.7 (Degenerate fans)**.** Let D be a nontrivial connected diagram, and let v be a 0-cell on ∂D such that $\text{Link}(v)$ is connected. The *degenerate fan* F corresponding to v is the union of the closed cells of D which contain v. F is said to be a degenerate fan of valence p where p is the valence of v. The *outer path* of F is the length 2 path on ∂D which contains v in its interior. Note that when v has valence 1, F is a *spur*.

7. Fans of type k

In this section we prove an important technical result, Lemma 7.6, which describes the conditions under which a fan in the dual of a diagram D determines a fan in D itself. From this we deduce Corollary 7.13, which will be applied frequently in later sections.

Definition 7.1 (Determined subdiagrams)**.** Let D be a nonsingular diagram, let E be its dual, let $G \hookrightarrow E$ be a (possibly degenerate) fan in E, and let $V \subset E$ be the set of 0-cells in the image of the interior of the outer path of G. For each $v \in V$ let R_v denote the 2-cell in D corresponding to v. The subdiagram $F = \cup_{v \in V} R_v$ is the subdiagram of D *determined by* G.

Note that when 0-cells in V are connected by a 1-cell, the corresponding 2-cells have an arc in common. Thus the subdiagram F will always be connected. Actually, much more can be shown.

Lemma 7.2. *Let D be a nonsingular diagram, let E be its dual, let $G \hookrightarrow E$ be a (possibly degenerate) fan, and let P be the subpath of the outer path of G obtained by removing its first and last 1-cell. If F is the subdiagram of D determined by G, then the dual of F is P.*

Proof. Let V be the set of 0-cells in P. By construction, the set of 0-cells in the dual of F is V. Each 1-cell in the dual of F must connect two 0-cells of V and by the structure of the fan G, the 1-cells of P are the only candidates. On the other hand, each 1-cell of P corresponds to a maximal internal arc in the subdiagram F, and is thus included in the dual. Finally, the dual cannot contain any 2-cells since P does not contain a closed immersed path. $\square$

Under suitable conditions we will be able to prove that F is a fan of D. When this occurs, the structure of the fan F is closely tied to the structure of the fan G. This is explained in Lemma 7.3 and Example 7.4.

Lemma 7.3. *Let D be a nonsingular diagram, let E be its dual, let $F \hookrightarrow D$ be a fan of D, and let $G \hookrightarrow E$ be a fan of E which determines the fan F. If the fan G has exactly k 2-cells and an outer path of length l, then the fan F has $l-1$ 2-cells, an inner path consisting of $k+1$ pieces, and $l-2$ maximal internal arcs.*

Proof. We will use the notation of Definition 7.1. By definition, the 2-cells of F are in one-to-one correspondence with the 0-cells in the interior of the outer path of G. Since the outer path has length l, there are $l-1$ such 0-cells. The pieces of D which form the inner path of F are the maximal internal arcs which lie between a 2-cell in F and a 2-cell which is not in F. Therefore these pieces correspond to 1-cells in E which connect a 0-cell in V to a 0-cell which is not in V. The only such 1-cells are the first and last 1-cells in the outer path of G and the 1-cells in the interior of G. Since there are k 2-cells, there are exactly $(k-1)$ 1-cells in the interior of G. Thus there is a total of $(k+1)$ of these two types of 1-cells. Finally, the pieces of D which are in the interior of F are those maximal internal arcs which are in the boundary of two distinct 2-cells of F. Thus in E they correspond to the 1-cells which connect a 0-cell in V to another 0-cell in V. The only 1-cells of this type are those in the outer path of G which start and end in V, and there are $l-2$ of these. □

Example 7.4 (Determined fans)**.** Let D be a diagram, let E be its dual, let $G \hookrightarrow E$ be a fan, and let F be the subdiagram determined by G. If F is a fan, then the type of the fan F is determined by the type of the fan G as follows:

(1) If G is a degenerate fan with valence i then F is an i-shell.
In particular, if G is a spur then F is a 1-shell.
(2) If G is an i-shell then F is a pointed fan.
(3) If G is a pointed fan then F is a broad fan.

These three special cases are illustrated in Figure 11. The leftmost picture shows a spur which determines a 1-shell, the middle picture shows a 2-shell which determines a pointed fan (with $i = 4$), and the rightmost picture shows a pointed fan which determines a broad fan (with $i = 0$ and $j = 3$).

Definition 7.5 (Separate fans)**.** Let D be a diagram and let $F_1 \hookrightarrow D$ and $F_2 \hookrightarrow D$ be two fans in D whose outer paths are contained in the same boundary cycle of D. If F_1 and F_2 are both nondegenerate fans, then they are *separate* if their outer paths intersect at most at their endpoints. A degenerate fan F_1 is *separate* from a (possibly degenerate) fan F_2 if the interior 0-cell of Q_1 is not contained in the interior of Q_2.

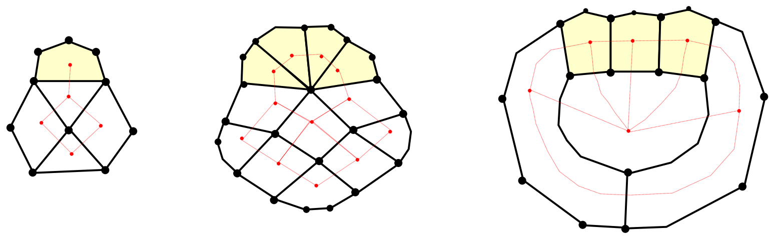


FIGURE 11. Determined Fans

The following result shows that a pair of separate fans in the dual of a nonsingular diagram determines a pair of separate fans in the diagram itself. This is one of our main technical tools for deriving new results.

Lemma 7.6 (Fan production)**.** *Let D be a nonsingular $C(p)$ diagram with $p \geq 3$, let E be its dual, let $G_1 \hookrightarrow E$ and $G_2 \hookrightarrow E$ be two separate, but possibly degenerate, fans in E whose outer paths lie in the same boundary cycle, and let P_i be the subpath of the outer path of G_i obtained by removing its first and last 1-cell. If the first and last 0-cells of both P_1 and P_2 have valence less than p, then the subdiagrams F_1 and F_2 determined by G_1 and G_2 are separate fans in D.*

Proof. To show that F_i is a fan we first define paths Q_i and S_i in the boundary and the interior of D, respectively, then show that $Q_i S_i^{-1}$ is an embedded closed path, and finally show that every maximal internal arc of F_i starts in the interior of Q_i and ends in the interior of S_i. Since $Q_i S_i^{-1}$ will be the entire boundary of F_i, these claims will show that F_i is either a doubly-based 2-cell or a nonsingular ladder which is embedded in D with outer path Q_i and inner path S_i. Note that by Lemma 7.2, the dual of F_i is P_i, and this induces a natural linear order on the 2-cells of F_i. Thus it makes sense to speak of the "previous" and the "next" 2-cell of F.

Outer paths: For each 0-cell $v \in P_i$, the 2-cell R_v is a boundary cell of D and Lemma 5.6.2 implies that the intersection $\partial D \cap \partial R_v$ is a single maximal boundary arc Q_v. Moreover, the boundary cycle of R_v is the concatenation of Q_v and valence(v) interior pieces. Thus when valence$(v) < p$, the path Q_v must be nontrivial. Next, notice that the path Q_v starts at the intersection of R_v with the previous 2-cell and ends at the intersection of R_v with the next 2-cell. Thus all of these Q_v paths can be concatenated into a single path in the boundary of D which we will call Q_i. By our assumption about valences, the first and the last Q_v are nontrivial and so the concatenation Q_i is also nontrivial. Moreover, since D is nonsingular, Q_i is a path in some particular boundary cycle of D, and since the outer paths of G_1 and G_2 were in the same boundary cycle of E, Q_1 and Q_2 lie in the same boundary cycle of D. Finally, the fact that G_1 and G_2 are separate implies that $P_1 \cap P_2 = \emptyset$, that $F_1 \cap F_2$ does not contain any 2-cells, and thus that $Q_1 \cap Q_2$ does not contain any 1-cells. Coupled with the fact that Q_1 and Q_2 are nontrivial, this shows that neither of them is a closed path, and that the paths Q_1 and Q_2 embed in a boundary cycle of D.

Inner paths: The portion of the boundary of F_i which lies in the interior of D is formed by the maximal interior arcs dual to the 1-cells of E with exactly one endpoint in P_i. When G_i is degenerate, P_i is a single 0-cell v, $Q_i = Q_v$, and the inner path of F_i is the complement of Q_i in ∂R_v. When G_i is a doubly-based 2-cell or a nonsingular ladder, the only such 1-cells are the first and the last 1-cells of the outer path of G_i and the 1-cells in the interior of G_i (if any). Observe that the dual of G_i is a simple path which is trivial when G_i is a doubly-based 2-cell, and the 1-cells in the interior of G_i correspond to the 1-cells of this dual path. By Lemma 5.8, this dual path, suitably subdivided, is a simple path T_i in D which is the concatenation of the internal maximal arcs of D that are dual to the 1-cells in the interior of G_i. The path S_i is the concatenation of the maximal internal arc of D dual to the first 1-cell of the outer path of G_i, followed by the path T_i just described, followed by the maximal internal arc of D dual to the last 1-cell of the outer path of G_i. Finally, notice that S_i and Q_i have the same endpoints, and consequently the path S_i is simple. This is because we are extending the simple path T_i by arcs to distinct 0-cells that do not lie on T_i.

Interior arcs: Since T_i is the concatenation of arcs whose endpoints lie in the interior of D, the interior of the path S_i lies in the interior of D and is thus disjoint from Q_i. This shows that $Q_i S_i^{-1}$ is an embedded closed path which is the entire boundary of F_i. It only remains to show that every maximal internal arc of F_i has one endpoint in the interior of Q_i and the other in the interior of S_i. Each maximal internal arc is dual to an 1-cell of P_i and therefore has one endpoint on a boundary 0-cell of D and one endpoint on an interior 0-cell of D. These 0-cells must lie on Q_i and S_i respectively. The fact that the endpoint in Q_i lies in the interior of Q_i follows from the nontriviality of the first and last Q_v subpaths. On the other hand, if there is a maximal internal arc of F_i which starts in the interior of Q_i and ends at an endpoint of S_i, then there is a 2-cell in F_i which is a 1-shell. The dual of a 1-shell is a spur, and thus G_i contains a spur. Since nondegenerate fans are nonsingular, this means that G_i is degenerate and so G_i is a spur. But in this case, P_i is trivial and F_i contains no internal arcs, which is a contradiction. Thus no internal arc of F_i ends at a basepoint of F_i, and so F_i is a ladder. □

Applying Lemma 7.6 requires two separate fans in the same boundary cycle of the dual along with restrictions on the valences of some 0-cells. The structure of the fan F_1, in contrast, only depends on the structure of G_1. It is independent of the structure of G_2 and largely independent of D. In order to extend Lemma 7.6 to singular disc diagrams, we need to clarify when a fan of D can be chosen to avoid a cut 0-cell.

Definition 7.7 (Avoiding and meeting 0-cells)**.** Let $F \subset D$ be a (possibly degenerate) fan, let Q be its outer path, and let d be a 0-cell in ∂D. We say

F *meets* d if the outer path Q contains d in its interior. Otherwise, we say that F *avoids* d.

The following corollary employs the hypotheses and notation of Lemma 7.6.

Corollary 7.8. *If G_i avoids a 0-cell $v \in \partial E$, then F_i avoids each 0-cell in each maximal boundary arc of the 2-cell R_v of D. Similarly, if G_i meets a 0-cell $v \in \partial E$, then F_i meets each 0-cell in the interior of the maximal boundary arc of the 2-cell R_v of D.*

Proof. If G_i avoids v then the 2-cell R_v is not a 2-cell of F_i since $v \notin V_i$. Observe that S_i separates the interior 0-cells of Q_i from the closure of each 2-cell not in F_i and the proof is complete. On the other hand, if G_i meets v, then R_v is contained in F_i and the result is clear. □

Using Lemma 7.6, we will now inductively describe collections of fan types for each p, q, and k satisfying the Euclidean restrictions (Convention 1.2).

Definition 7.9 (Fans of type k)**.** Let p, q, and k satisfy the Euclidean restrictions. There is a collection of fans denoted $\mathcal{F}^k_{pq}$ which is defined recursively starting with $k = 0$. Let $k = 0$ and let $F \hookrightarrow D$ be a fan in a $C(p)$-$T(q)$ disc diagram. If $p \geq 4, q \geq 4$ [$p \geq 6, q \geq 3$] and $F \hookrightarrow D$ is a degenerate fan of valence ≤ 2 [valence ≤ 3], then F is a fan of type $\mathcal{F}^0_{pq}$ in D. For $k \geq 1$, let $F \hookrightarrow D$ be a fan in a $C(p)$-$T(q)$ disc diagram. The fan F is a *fan of type $\mathcal{F}^k_{pq}$ in D* provided that either it is a spur, or it is a 0-shell, or it is determined by a fan F' of type $\mathcal{F}^{k-1}_{qp}$ in the dual E of D. When p and q are clear from context we refer to a fan in $\mathcal{F}^k_{pq}$ as a *fan of type k*.

To illustrate this definition we will explicitly describe the fans of type k for $k \in \{1, 2, 3\}$. The statement of each of the lists below follows immediately from Example 7.4 and Lemma 7.6.

Example 7.10 (Fan types for small values of k)**.** The fans of type 1, that is, the fans in the collection $\mathcal{F}^1_{pq}$, are described by the following lists:

$p \geq 4$, $q \geq 4$: {spurs, 0-shells, 1-shells, 2-shells}

$p \geq 6$, $q = 3$: {spurs, 0-shells, 1-shells, 2-shells, 3-shells}

Fans of type 2 are described by one of the following lists:

$p \geq 4$, $q \geq 4$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q - 3$}

$p = 3$, $q \geq 6$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q - 4$}

Fans of type 3 are described by the following lists:

$p \geq 4$, $q \geq 4$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q - 2$, $(q-3)$-separated broad fans with $j \geq p - 4$ and $i \geq (j+1)(q-3)$}

$p \geq 6$, $q = 3$: {spurs, 0-shells, 1-shells, pointed fans with $i \geq q - 2$, broad fans with $j \geq p - 5$ and $i \geq 0$}

Although there are no general inclusion relations among the collections of fans of various types, there are some fans which are of every type.

Lemma 7.11. *Let p, q, and k satisfy the Euclidean restrictions. A spur is a fan of type k for all $k \geq 0$, and 0-shells and 1-shells are fans of type k for all $k \geq 1$.*

Proof. By Definition 7.9, a spur is a fan of type k for all $k \geq 0$, and a 0-shell is a fan of type k for all $k \geq 1$. If D has a 1-shell, then the dual of D has a spur which determines it. Since every fan which is determined by a fan of type $k-1$ is a fan of type k, the proof is complete. □

See Lemma 8.3 for a similar result about pointed fans. We now show by induction that every fan of type k satisfies the valence requirements necessary in order to apply Lemma 7.6.

Lemma 7.12. *Let D be a $C(p)$-$T(q)$ diagram and let F be a fan of type k in D where F is not a spur, 0-shell, or 1-shell. Let U be the 0-cells in the interior of the outer path of F.*

(1) If $p \geq 4$, $q \geq 4$, and $k \geq 0$, then F is a 1-separated broad fan. That is, every 2-cell in F is a 2-shell or 3-shell, the first and last 2-cells are 2-shells, and there do not exist consecutive 3-cells. In addition, every 0-cell in U has valence ≤ 3, the first and last 0-cells in U have valence 2, and there do not exist consecutive 0-cells in U with valence 3.

(2) If $p = 3$, $q \geq 6$ and k is odd, then F is a 1-separated broad fan. That is, every 2-cell in F is a 2-shell or 3-shell, the first two and last two 2-cells are 2-shells, and there do not exist consecutive 3-shells. In addition, every 0-cell in U has valence ≤ 4, the first and last 0-cells have valence ≤ 3, and there do not exist consecutive 0-cells in U with valence 4.

(3) If $p \geq 6$, $q = 3$, and k is even, then every 2-cell in F is a 2-shell, 3-shell, or 4-shell, the first and last 2-cell is a 2-shell or 3-shell, and there do not exist consecutive 4-shells. In addition, every 0-cell in U has valence ≤ 3, the first two and last two 0-cells in U have valence 2, and there do not exist consecutive 0-cells in U with valence 3.

Proof. It is easy to check using Example 7.10 that all three assertions are true for $k \leq 1$, so assume that they are true for all $k < i$ ($i > 1$) and consider the case $k = i$. A fan of type i which is not a spur, 0-shell, or 1-shell is determined by a nondegenerate fan of type $i-1$ in the dual. If F is determined by a fan G of type $i-1$ in the dual, then by assumption the relevant conditions are true for G. Specifically, if F falls under case 1, then so does G, if F falls under case 2, then G falls under case 3, and if F falls under case 3, then G falls under case 2. Let V be the 0-cells in the interior of the outer path of G. In each case the restrictions on the 0-cells of V immediately imply the restrictions on the 2-cells of F.

We now prove the assertions about the valences of 0-cells in U. If a boundary 2-cell in a $C(p)$ diagram is a j-shell with $j < p$ then it must contain a nontrivial boundary arc (called Q_v in the notation of Lemma 7.6). Thus when F falls under case 1 or 3, each of the paths Q_v is nontrivial. As a result, each 0-cell in U lies in the boundary of at most two 2-cells and thus all of the 0-cells in U have valence at most 3. Moreover, in case 1, the boundary path Q_v has length ≥ 2 provided the 0-cell v has valence 2, and so the 0-cells of valence 3 in U are isolated from each other and from the ends of U by at least one 0-cell of valence 2. Similarly in case 3, the concatenated boundary paths Q_v have length at least 2 or at least 3, so the 0-cells of valence 3 in U are isolated from each other and from the ends of U by at least two 0-cells of valence 2.

Finally, when F falls under case 2, the 2-cells which can have trivial boundary path do not occur consecutively or at either end of F. As a result, each 0-cell in U lies in the boundary of at most three 2-cells and thus all of the 0-cells in U have valence at most 4. Moreover, since the first two 0-cells in V have valence 2, the first two 2-cells in F have nontrivial boundary paths, and the first 0-cell in U has valence at most 3. This completes the induction. □

Corollary 7.13. *Let D be a nonsingular $C(p)$-$T(q)$ diagram, and let E be its dual. If $G_1 \hookrightarrow E$ and $G_2 \hookrightarrow E$ are separate fans of type k and l whose outer paths lie in the same boundary cycle of E, then they determine separate fans $F_1 \hookrightarrow D$ and $F_2 \hookrightarrow D$ of type $k+1$ and $l+1$.*

Proof. This follows from Lemma 7.6 and Lemma 7.12. □

8. Ladders of width k

The main objective of this section is to provide inductive definitions of wheels and ladders of width k. At the end of the section we justify the description of these objects as having "width k".

Definition 8.1 (Wedges of type k)**.** Let D be a $C(p)$-$T(q)$ disc diagram and let $F \hookrightarrow D$ be a fan of type k in D. This fan of type k in D determines a wedge of type k in D which is defined inductively as follows: For $k \leq 2$, the wedge determined by F is just F itself. Similarly, if $k > 2$ and F is a spur, a 0-shell, or a fan determined by a spur or a 0-shell, then the wedge determined by F is just F itself.

Otherwise, F is a fan of type $k > 2$ which is determined by a fan which is determined by a fan in the double dual of D. Let D' be the double dual of D' and let F' be the fan of type $k-2$ in D' which determines the fan which determines F. By Lemma 5.8, D' can be viewed as a subdiagram of D after a suitable subdivision of its 1-cells. In particular, the wedge determined by F' in D', which has already been inductively defined, can be viewed as a subdiagram of D. Also notice that under this embedding the interior of the outer path of F' is contained in the inner path of F. The *wedge determined by F* is a subdiagram of D which is the union of the fan $F \hookrightarrow D$ and the wedge $W' \subset D'$ determined by F'. See Figure 12 for an illustration. Finally, notice that if F_1 and F_2 are separate nondegenerate fans of type k and l in D, then the wedges of type k and l in D corresponding to these fans are

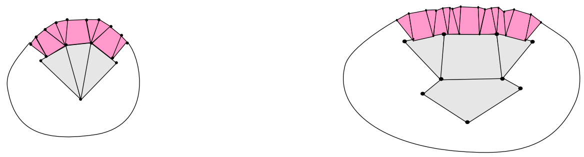


Figure 12. In each diagram above, the heavily shaded subdiagram is a fan, and the entire shaded subdiagram is its corresponding wedge. These wedges consists of two (or three) layers of fans from successive double duals. Note that the remaining arcs in the interior of the diagram are not drawn.

Figure 13. A wheel of width 2 and a wheel of width 3.

nonsingular and their interiors are disjoint. We will therefore call two such wedges *separate*.

Definition 8.2 (Wheels of width k)**.** Let p, q, and k satisfy the Euclidean restrictions. A *wheel of width* 0 is a single 0-cell. For $k \geq 1$, a *wheel of width* k is a nonsingular disc diagram whose dual is a wheel of width $k-1$. For example, a wheel of width 1 is a single 2-cell. Wheels of widths 2 and 3 are illustrated in Figure 13.

Lemma 8.3. *Let p, q, and k satisfy the Euclidean restrictions, let D be a $C(p)$-$T(q)$ diagram which is a wheel of width* 2*, let R be a* 2*-shell in D, and let F be a pointed fan in D which includes all of the* 2*-cells of D except R. If $k = 1$ then R is a fan of type $k = 1$ in D, and if $k \geq 2$ then F is a fan of type k in D.*

Proof. Let E be the single 2-cell which is the dual of D and let v be the 0-cell in ∂E which corresponds to R. When $k = 1$ we note that v is a degenerate fan of valence 2 and thus a fan of type 0. Since R is determined by this degenerate fan, R is a fan of type 1. On the other hand, the 0-cell v turns E into a 0-shell which by Lemma 7.11 is a fan of type k for all $k \geq 1$. Since F is determined by this 0-shell, it is a fan of type $k+1$ for all $k \geq 1$, and this completes the proof. □

Definition 8.4 (Ladders of width k and endfans)**.** A *ladder of width* 0 is a nontrivial subdivided interval. The *endfans* of a width 0 ladder are its two spurs. For $k \geq 1$, a nonsingular diagram D is a *ladder of width k* if its dual is a ladder of width $k-1$. In this case the *endfans* of D are the fans determined by the endfans of the dual of D. A singular diagram D is

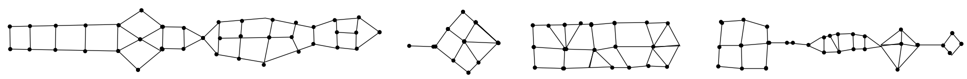


FIGURE 14. Four ladders of width 2.

a ladder of width k provided that its cut-tree is a subdivided interval and each of the following conditions hold:
(1) Each of its cut-components is either a ladder of width $\leq k$ or a nontrivial wheel of width $\leq k$, and at least one of these components has width strictly equal to k.
(2) Let C be a cut-component of D and let $v \in \partial C$ be a cut 0-cell of D. If C is a ladder, then v must meet one of the two endfans of C.
(3) If C is a ladder and ∂C contains two distinct cut 0-cells of D then these 0-cells meet distinct endfans of C.
The *endfans of* D are contained in the cut-components corresponding to the leaves of the cut-tree. Let C be a cut-component corresponding to a leaf of the cut-tree and let $v \in \partial C$ denote the unique cut 0-cell of D in C. The endfans of D are determined as follows: if C is a ladder of width k, then the endfan of D is the endfan of C which avoids v, and if C is a wheel of width k, then the endfan of D is a fan of type k in C which avoids v (that such a fan exists will be shown in Lemma 9.1).

A ladder of width 0 is a subdivided interval whose spurs are its endfans. The endfans of a ladder of width k are a generalization of these spurs. The ladders described in Definition 6.1 are ladders of width ≤ 1. Figure 14 depicts four width 2 ladders.

Definition 8.5 (Width of disc diagrams)**.** Let D be a doubly-based disc diagram with startpoint s and endpoint t and with corresponding boundary paths P_1 and P_2. If every 0-cell in P_1 is a 0-cell in P_2 and vice versa, then D is called a *disc diagram of width* 0. Notice that D has width 0 if and only if $P_1 = P_2$ and so D is either a subdivided interval or a single 0-cell.

Suppose that for some $k \geq 1$, every 0-cell in P_1 can be connected to some 0-cell in P_2 by a path of the form $Q_1 Q_2 \cdots Q_k$, where each Q_i is either trivial or a boundary path of a 2-cell of D, and every 0-cell of P_2 can be connected to some 0-cell of P_1 by a path of the same type. If D is not a disc diagram of width 0 and k is the smallest number with this property, then D is a *disc diagram of width* k.

Notice that a 0-cell v can be connected to a 0-cell u by a path of the above form if and only if there is a nonnegegative integer $\ell \leq k$, a sequence of 0-cells $v = v_0, v_1, \ldots, v_\ell = u$, and a sequence of 2-cells $R_1, R_2, \ldots, R_\ell$ such that v_{i-1} and v_i lie in ∂R_i for $1 \leq i \leq \ell$.

If D is a doubly-based disc diagram which is either a ladder of width 1 or a wheel of width 1, then D will be called a *layer*. The following lemmas show that ladders of width k and wheels of width k can be decomposed into at most k layers, and thus they are indeed diagrams of width k.

Lemma 8.6. *Let D be a nonsingular doubly-based disc diagram, let E be its dual, and let s and t be 0-cells in E which correspond to 2-cells of D which contain the basepoints of D in their boundaries. If E with basepoints s and t has width k, then D has width $k+1$.*

Proof. Let P and P' be the boundary paths of D determined by its basepoints, let Q and Q' be the boundary paths of E determined by s and t, and let u be a 0-cell in P. Since D is nonsingular, there is a 2-cell R with $u \in \partial R$, and since R is a boundary 2-cell, it corresponds to a 0-cell v in the boundary cycle of E. Furthermore, if u is a basepoint of D, then we require that R be chosen so that it corresponds to s or t. With this restriction, v lies on the boundary path Q from s to t. By hypothesis, E has width k, so for some $\ell \leq k$, there is a sequence of 0-cells $v = v_0, v_1, \ldots, v_\ell$ and a sequence of 2-cells $R_1, R_2, \ldots, R_\ell$ such that v_{i-1} and v_i are in ∂R_i for $i = 1, \ldots, \ell$, and such that v_ℓ lies in Q'. Let R'_i be the 2-cell in D which corresponds to v_i, let u_i be the 0-cell in D which corresponds to R_i, let $u_0 = u$, and let $u_{\ell+1}$ be a 0-cell in $\partial R'_\ell$ which lies in P'. Note that u_{i-1} and u_i are contained in $\partial R'_i$ either by definition of R and R' in D or since R_{i-1} and R_i in E contain v_i in their boundaries. Thus the desired sequences exist and hence D has width $k+1$. □

Lemma 8.7 (Wheels of width k)**.** *If D is a wheel of width k, and s and t are any 0-cells in ∂D, then the resulting doubly-based diagram has width k.*

Proof. If $k = 0$, then D is trivial and D has width 0. If $k \geq 1$ then D is nonsingular and its dual E is a wheel of width $k-1$ for any choice of basepoints in ∂E. Thus, by Lemma 8.6, D has width k. □

Lemma 8.8 (Ladders of width k)**.** *If D is a ladder of width at most k, and s and t are 0-cells which meet the endfans of D, then the resulting doubly-based diagram has width at most k.*

Proof. If $k = 0$, then D is a path and D really does have width 0. If $k > 0$ and D is nonsingular, then its dual E is a ladder of width at most $k-1$. By induction, E really does have width at most $k-1$ for any choice of basepoints in the endfans of E. Thus, by Lemma 8.6, D has width at most k. If $k > 0$ and D is singular, then by the definition of a singular ladder of width k, it is sufficient to show that each cut-component has width at most k. Since each cut-component is either a 1-cell, a wheel of width at most k, or a nonsingular ladder of width at most k, this follows either from the above argument or from Lemma 8.7. □

We close this section with a conjecture about the relationship between the lengths of the two sides of a width k ladder. We have proven the conjecture for $k \leq 2$, but a general proof has eluded us.

Conjecture 8.9 (Linear length)**.** *Let X be a compact $C(p)$-$T(q)$ complex and let p, q, and k satisfy the Euclidean restrictions. We conjecture that there exist constants K and ϵ which depend only on p, q, k, and X, such that the following holds:*

Let $D \to X$ be a reduced map from a ladder of width ℓ to X ($\ell \leq k$), and let P_1 and P_2 be the paths determined by the basepoints of D. Suppose that there does not exist a fan F of type k in X and an embedding $F \hookrightarrow D$ which sends the outer path of F to a subpath of either P_1 or P_2. Then the following inequalities hold:

$$\frac{1}{K}|P_1| - \epsilon < |P_2| < K|P_1| + \epsilon$$

9. Disc diagrams

In this section we prove our main result about disc diagrams, Theorem 9.2.

Lemma 9.1. *Let D be a nontrivial $C(p)$-$T(q)$ disc diagram and let p, q, and $k \geq 1$ satisfy the Euclidean restrictions. If D is a wheel of width $n \geq 1$, and v is a 0-cell in ∂D, then D contains a fan of type k which avoids v. Specifically, D contains two separate fans $F_1 \hookrightarrow D$ and $F_2 \hookrightarrow D$ such that F_1 has type k and avoids v and F_2 has type $\ell < k$.*

Proof. If $n = 1$ then D is a single 2-cell and thus for any 0-cell v in ∂D, there is a 0-shell which avoids v, and v yields a separate degenerate fan of valence 2. By Lemma 7.11, a 0-shell is a fan of type k for all $k \geq 1$. Suppose the statement holds for $n = i \geq 1$ and consider the case $n = i + 1$. The dual E of D is a wheel of width $i \geq 1$. Let v' be a 0-cell in ∂E corresponding to a 2-cell of D which contains v in its boundary. Since the statement is true for $n = i$, there are separate fans, G_1 of type i which avoids v' and G_2 of type $\ell < i$. Thus by Corollary 7.13 and Corollary 7.8, there are separate fans F_1 and F_2 of type $i + 1$, and $\ell + 1 < i + 1$ such that F_2 meets v, and hence F_1 avoids v. □

Theorem 9.2 (Main Theorem)**.** *Let D be a $C(p)$-$T(q)$ disc diagram where p, q, and k satisfy the Euclidean restrictions. One of the following holds:*

1. *D contains at least 3 separate fans of type k.*
2. *D is a ladder of width $\leq k$.*
3. *D is a wheel of width $\leq k$.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a fan of type k which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate fans of type k.

Proof. The proof is by induction on k, and the case $k = 0$ is Theorem 4.8. Suppose that the theorem is true for $k = n - 1$, and consider the case where $k = n$. If D is nonsingular, then by Lemma 5.6, the dual E of D is a disc diagram. We now apply the statement to E in the case $k = n - 1$. If E is a wheel of width $\leq n - 1$ then D is a wheel of width $\leq n$ by Definition 8.2. If E is a ladder of width $\leq n - 1$ then D is a ladder of width $\leq n$ by

Definition 8.4. Finally, if E has three separate fans of type $(n-1)$ then they determine three separate fans of type n by Corollary 7.13.

Now suppose that D is singular and consider its cut-tree. If its cut-tree is trivial, then D is a single 0-cell or a single 1-cell and D is thus a wheel of width 0 or a ladder of width 0. Otherwise the cut-tree is nontrivial and therefore has at least two leaves. Let C be the cut-component corresponding to a leaf and let $v \in \partial C$ be the unique cut 0-cell of D in ∂C. If C is singular, then C is a 1-cell and C contains a spur which avoids v (which is a fan of type k). If C is a wheel of width $\leq k$, then by Lemma 9.1, it contains a fan of type k which avoids v. If C is any other nonsingular diagram, then by the nonsingular case of the statement with $k = n-1$, C contains two separate fans of type k and thus at least one of them avoids v. Regardless, C contains a fan of type k which avoids v. Thus, if the cut-tree has $\ell \geq 3$ leaves, then D contains at least ℓ fans of type k and the result is true.

The only remaining case is where D has a nontrivial cut-tree which is a subdivided interval. By the argument above, D contains at least two fans of type k, since it contains one in each cut-component corresponding to an endpoint of the cut-tree. If D contains no other fans of type k which are separate from these two then D is a singular ladder with these two fans of type k as its endfans. This is because a violation of any of the conditions listed in Definition 8.4 leads immediately to the presence of a third fan of type k in D. □

The following theorem contains somewhat more information than Theorem 9.3 about the shape of the diagram but it is essentially equivalent.

Theorem 9.3. *Let D be a $C(p)$-$T(q)$ disc diagram where p, q, and k satisfy the Euclidean restrictions. One of the following holds:*

1. *D contains at least three separate wedges of type k.*
2. *D is a ladder of width $\leq k$.*
3. *D is a wheel of width $\leq k$.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a wedge of type k which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate wedges of type k.

Proof. This follows from Theorem 9.2 using the remarks in Definition 8.1 but it can also be proven in a manner similar to Theorem 9.2. □

For $k = 1$, Theorem 9.2 is a refinement of the main theorem of small cancellation theory.

Theorem 9.4. *If D is a $C(4)$-$T(4)$ [$C(6)$-$T(3)$] disc diagram, then one of the following holds:*

1. *D contains at least three spurs and/or i-shells with $i \leq 2$ [$i \leq 3$].*
2. *D is a ladder of width ≤ 1, and hence has a spur or 1-shell at each end.*
3. *D consists of a single 0-cell or a single 2-cell.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a spur or an i-shell with $i \leq 2$ [$i \leq 3$] which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate such spurs and i-shells.

Similarly explicit statements could be made for $k = 2$ and $k = 3$ using the explicit descriptions of the associated fans given in Example 7.10. We will just mention the case $k = 2$, $p = 3$, $q = 6$ since it has been overlooked by the traditional small cancellation theory, but it deserves to be stated in parallel with Theorem 9.4.

Theorem 9.5. *If D is a $C(3)$-$T(6)$ disc diagram, then one of the following holds:*

(1) *D contains at least three spurs, 1-shells, and/or pointed fans with two consecutive 2-shells.*
(2) *D is a ladder of width ≤ 2.*
(3) *D is a wheel of width ≤ 2. That is, D is either a 0-cell, a 2-cell, or a nonsingular diagram whose dual is a 2-cell.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a spur, a 1-shell, or a pointed fan with two 2-cells which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate such spurs, 1-shells, and pointed fans.

10. Annular diagrams

In this section we prove our main result about annular diagrams, Theorem 10.6. We begin by extending the definitions of fans and width to annular diagrams.

Definition 10.1 (Width of annular diagrams)**.** Let A be an annular diagram with boundary cycles P_1 and P_2. If every 0-cell in P_1 is a 0-cell in P_2 and vice versa, then A is an *annular diagram of width* 0. Notice that when A is $C(3)$, A has width 0 if and only if $P_1 = P_2$, and so A is a subdivided circle.

Next, suppose that for some $k \geq 1$, every 0-cell in P_1 can be connected to some 0-cell in P_2 by a path of the form $Q_1 Q_2 \cdots Q_k$ where each Q_i is either trivial or a boundary path of a 2-cell of A, and every 0-cell of P_2 can be connected to some 0-cell of P_1 by a path of the same type. If A is not an annular diagram of width 0 and k is the smallest number with this property, then A is an *annular diagram of width k*.

The proof of the following is essentially identical to that of Lemma 8.6 and will be omitted.

Lemma 10.2. *Let A be a nonsingular annular diagram and let E be its dual. If E has width k, then D has width $k + 1$.*

The next lemma is another easy application of the combinatorial Gauss-Bonnet theorem (Theorem 4.6).

Lemma 10.3. *Let A be an angled annular diagram. If all 2-cell curvatures and 0-cell curvatures are ≤ 0 and all internal 0-cell curvatures are < 0,* *then there are no internal* 0*-cells. Furthermore, if all angles are positive, then either A is a circle or A is a nonsingular annulus of width* 1*.*

Proof. By the combinatorial Gauss-Bonnet theorem, the total curvature of A is exactly 0. Therefore, there is no internal 0-cell since this would force the total curvature of A to be negative.

For the second statement, notice that if A is singular, then there is a 0-cell v on both boundary cycles. Since $\text{Link}(v)$ is disconnected but $\text{Curvature}(v) = 0$, Remark 4.5 implies that $\text{Link}(v)$ consists of exactly two vertices and no edges. This means that the 0-cells adjacent to v also have disconnected links and also lie on both boundary cycles. We can now repeat this argument for each successive 0-cell in a boundary cycle of A, thereby showing that A is a subdivided circle. □

As in the previous section, Theorem 10.6 generalizes a theorem of Lyndon and Schupp. In particular, a metric version of Theorem 10.4 was shown by Lyndon and Schupp [11, Theorem 5.5].

Theorem 10.4. *If A is a $C(3)$-$T(7)$ $[C(4)$-$T(5)]$ annular diagram, then either A contain a (degenerate) fan of type* 0 *or A has width* ≤ 1*.*

Proof. We assign an angle of $\pi/3$ $[\pi/2]$ to each corner in A. By Theorem 4.6, the sum of the resulting 0-cell curvatures and 2-cell curvatures is 0. Let R be a 2-cell of A. By the $C(3)$ $[C(4)]$ condition and Convention 3.6, R has at least three [respectively four] corners and therefore has nonpositive curvature. By the $T(7)$ $[T(5)]$ condition, each interior 0-cell has negative curvature. Let v be a 0-cell in ∂D. If $\text{Link}(v)$ is disconnected then v will be nonpositively curved because $\chi(\text{Link}(v)) \geq 2$. On the other hand, if $\text{Link}(v)$ is connected and v has valence at least 4 [respectively 3], then v is nonpositively curved because of the way the angles are assigned. Lemma 10.3 completes the proof. □

Remark 10.5 (Fans in annular diagrams)**.** According to Definition 6.2, a fan is a kind of *embedded* ladder or doubly-based 2-cell. For annular diagrams we will weaken this requirement slightly. Let $F \to A$ be a map from a nonsingular ladder or doubly-based 2-cell to an annular diagram A such that the outer path Q of F is sent to a boundary path of A and the inner path S is sent to a path which lies in the interior of A except at its endpoints. If the map $F \to A$ lifts to a fan in the double cover of the annulus A, then $F \to A$ is a *fan in A*. This allows the endpoints of Q to coincide in A so long as they lift to distinct 0-cells in the double cover of A.

Recall the hyperbolic restrictions on p, q, and k which were listed in Convention 1.2.

Theorem 10.6. *If A is a $C(p)$-$T(q)$ annular diagram and p, q, and k satisfy the hyperbolic restrictions then either A contains a fan of type k or A has width* $\leq k+1$*.*

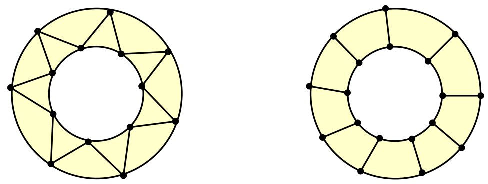


FIGURE 15. Possible thin annuli

Proof. The proof is by induction on k. The case $k = 0$ is Theorem 10.4. For $k > 0$ there are two possibilities. If A is singular, then after splitting the diagram at a 0-cell common to both boundary cycles, we obtain a doubly-based disc diagram D to which Theorem 9.2 can be applied. Specifically, if A is singular, there is a doubly-based disc diagram D with distinct basepoints s and t such that when s and t are identified the result is A. If D is a wheel of width $\leq k$ or a ladder of width $\leq k$, then D, and therefore A, has width $\leq k$. Otherwise, D has three separate fans of type k. Since at least one of these must avoid both s and t, D has a fan of type k which survives the identification of s and t. Notice, however, that if s and t are also the basepoints of the fan, then the weaker requirements discussed in Remark 10.5 are required for this to count as a fan in A.

On the other hand, if A is nonsingular, then by Lemma 5.6, its dual E is an annular diagram to which Theorem 10.6 can be applied for a smaller value of k. If E has width $\leq k - 1$ then by Lemma 10.2, A has width $\leq k$. If E has a fan of type k, then let $\hat{E}$ denote the double cover of E and note that $\hat{E}$ contains two separate fans of type k whose boundary paths lie in the same boundary cycle of $\hat{E}$. Thus Corollary 7.13 can be applied to the double cover $\hat{A}$ of A. Thus $\hat{A}$ contains a pair of separate, symmetrically placed fans of type k and so A contains a fan of type k. $\square$

Remark 10.7. More can be said about the structure of the possible annular diagrams of width k for small values of k. According to Theorem 10.4, if A has no fan of type 0 then either A has width 0 and so A is a circle, or A has width 1 and so A is an annulus of the type illustrated on the left [or right] in Figure 15. For $k = 1$ the only possibilities are a circle, one of the annuli in Figure 15, or a diagram whose dual is one of these annuli. Similarly for $k \geq 2$.

11. FANS OF TYPE k IN X, AND CLOSED SUBCOMPLEXES

We begin this section by defining fans of type k in a 2-complex X. Certain fans of type k in X will be examined closely in Sections 12 and 13. In the remainder of this section we record results which will be useful for applications in [13]. The two main diagrammatic results proven earlier in the paper are recast as statements about subspaces of small cancellation complexes which are 'closed' under the addition of certain types of fans. These statements will not be employed elsewhere in this paper.

Definition 11.1 (Fans of type k in X)**.** Let X be a $C(p)$-$T(q)$ complex and let p, q, and k satisfy the Euclidean restrictions. Let F be a fan. A map $F \to X$ is a *fan of type k in X* if it factors as $F \hookrightarrow D \to X$ where $D \to X$ is a reduced map of a disc diagram, and $F \hookrightarrow D$ is a fan of type k in D (Definition 7.9).

We will occasionally say that $F \to X$ is *a fan in X* if it factors as $F \hookrightarrow D \to X$ where $D \to X$ is a reduced map of a disc diagram and $F \hookrightarrow D$ is a fan in D. We can thus speak of a 3-shell in X, degenerate fan in X, pointed fan in X, etc. Note that saying $F \to X$ is a fan (of type k) in X implicitly designates its outer path $Q \to F$.

Observe that a fan of type k is defined using $C(p)$-$T(q)$ diagrams and that X plays no role in its definition. In particular, fans of type k in X are *not* determined inductively by fans of type $k-1$ in X. For example, let $D \to X$ be a reduced diagram, let E be the dual of D, let F' be a fan of type $k-1$ in E, and let F be a fan of type k in D which is determined by F'. The fan F is a fan of type k in X, but the fan F' is not a fan of type $k-1$ in X. Such a claim would not make sense, since E is a $C(q)$-$T(p)$ diagram, while X is a $C(p)$-$T(q)$ complex. The difficulty lies precisely in the fact that a $C(p)$-$T(q)$ complex does not in general have a $C(q)$-$T(p)$ complex which plays the role of a "dual".

Definition 11.2 (Closed subcomplexes)**.** Let X' be a subcomplex of a $C(p)$-$T(q)$ complex X and let k be a nonnegative integer. The subcomplex X' is *closed with respect to the addition of fans of type k* if for every fan $F \to X$ of type k whose outer path Q is a path in X', the entire fan F lies in X'. Note that every subcomplex is automatically closed with respect to the addition of spurs.

Theorem 11.3. *Let X be a $C(p)$-$T(q)$ complex, let X_1 and X_2 be subcomplexes of X which are closed with respect to the addition of fans of type k, and let p, q, and k satisfy the Euclidean restrictions. If s and t are 0-cells in $X_1 \cap X_2$ and $P_1 \to X_1$ and $P_2 \to X_2$ are paths from s to t which are homotopic in X relative to their endpoints, then there exists a ladder or wheel D of width $\leq k$ and a map $D \to X$ such that the basepoints of D are sent to s and t, one of the paths determined by the basepoints of D projects to a path in X_1 homotopic in X_1 to P_1, and the other path determined by the basepoints projects to a path in X_2 homotopic in X_2 to P_2.*

Proof. Since P_1 and P_2 are homotopic, $P_1P_2^{-1}$ is null-homotopic in X. Thus by Theorem 3.10, there is a $C(p)$-$T(q)$ disc diagram $D_1 \to X$ such that $P_1P_2^{-1}$ is the projection of its boundary cycle to X. If D_1 contains a fan of type k which avoids both basepoints then since both X_1 and X_2 are closed with respect to fans of type k, the interior of the fan and the outer path of the fan can be removed from D_1 to form a diagram D_2. Furthermore, the path P_1 (or P_2) is homotopic in X_1 (or X_2) to the corresponding path of D_2. These arguments can be repeated for D_2. Continuing in this way, we obtain a nested sequence of subdiagrams $D_1 \supset D_2 \supset D_3 \cdots$. Since D_1 is

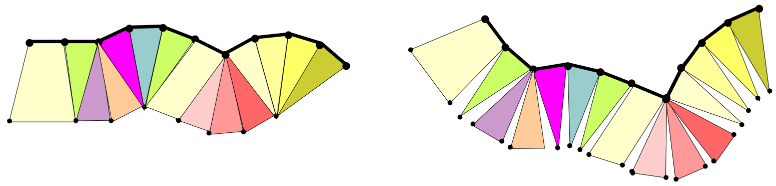


FIGURE 16. A fan and its essence.

finite, this sequence must terminate at a diagram D_t which does not contain a fan of type k avoiding both basepoints. By Theorem 9.2, D_t is either a ladder of width $\leq k$ or a wheel of width $\leq k$. Moreover, if D_t is a ladder, then the basepoints must meet the endfans of D_t. That D_t has width $\leq k$ now follows from either Lemma 8.7 or Lemma 8.8. □

A similar result can be obtained for annular diagrams. The proof is analogous and will be omitted.

Theorem 11.4. *Let X be a $C(p)$-$T(q)$ complex, let X_1 and X_2 be subcomplexes of X which are closed with respect to the addition of fans of type k, and let p, q, and k satisfy the hyperbolic restrictions. If $P_1 \to X_1$ and $P_2 \to X_2$ are closed paths which are homotopic but not null-homotopic in X, then there exist an annular diagram D of width $\leq k+1$ and a map $D \to X$ such that one of the boundary cycles of D projects to a cycle in X_1 homotopic in X_1 to P_1 and the other boundary cycle of D projects to a cycle in X_2 homotopic in X_2 to P_2.*

12. Minimal fans of type k

In this section we define minimal fans of type k in X, which are essentially the fans that do not contain another fan of type k as a subfan. The main objective of this section is to show that if X is compact then the set of all minimal fans of type k in X is finite. We will continue to study minimal fans in Section 13 where we prove that their outer paths lift to an embedding in the universal cover.

Definition 12.1 (Essence)**.** Let F be a fan. The *essence* E of F is the disc diagram obtained by cutting F along its interior arcs, but keeping each 2-cell glued to the outer path Q. The embedding $E \subset S^2$ keeps the 2-cells in the same order and orientation as they were in $F \subset S^2$. The essence of a degenerate fan is defined analogously. Note that the *essence* of a spur is the disc diagram corresponding to the outer path of the spur, which consists of the union of two edges along a vertex. Figure 16 contains a fan on the left and its essence on the right. Note that there is a map $E \to F$ and the outer path of F factors as $Q \to E \to F$.

Definition 12.2 (Subfan)**.** The fan $F_1 \to X$ is a *subfan* of the fan $F_2 \to X$ if there is an embedding $E_1 \hookrightarrow E_2$ of their essences such that:

(1) the sequence of consecutive 2-cells in E_1 is sent to a sequence of consecutive 2-cells in E_2 with the same linear ordering,
(2) the path Q_1 projects via $E_1 \to E_2$ to a subpath of Q_2, and
(3) the map $E_1^{(1)} \to X$ factors as $E_1^{(1)} \hookrightarrow E_2^{(1)} \to X$.

The subfan F_1 of F_2 is a *proper subfan* if Q_1 projects to a proper subpath of Q_2.

Suppose that F_1 is not a spur, and F_1 is a subfan of F_2, and let Y be the 2-complex obtained by gluing E_1 to E_2 along $E_1^{(1)} \hookrightarrow E_2^{(1)}$, and let $Y \to X$ be the induced map. Then for each 1-cell e of E_2, the pair of 2-cells meeting along e forms an equivalent pair relative to $Y \to X$.

Definition 12.3 (Minimal fans)**.** If $F \to X$ is a fan of type k in X which does not contain a proper subfan of type k, then F is *minimal fan of type* k. Note that the outer path of a nondegenerate minimal fan of type k is an immersed path, because a spur subfan would obviously be a proper subfan.

Lemma 12.4. *Let p, q, and k satisfy the Euclidean restrictions. If F is a fan of type k, then there are only a finite number of subfans of F. Consequently, every fan of type k has a minimal subfan of type k.*

Proof. First observe that if F_1 is a subfan of F_2, and F_2 is a subfan of F_3, then F_1 is a subfan of F_3. Now, each time we replace a fan of type k by a proper subfan of type k, the length of the outer path goes down. Therefore a minimal subfan must exist. □

Lemma 12.5. *Let X be a compact $C(p)$-$T(q)$ complex and let p, q, and $k \geq 2$ satisfy the Euclidean restrictions. Then there exists a constant $B = B(X)$ such that the following holds: If F is a pointed fan in X with more than B 2-cells then F contains a proper subfan of type k. Consequently if F is a minimal pointed fan of type k, then it has at most B 2-cells.*

Proof. Let B_1 be the maximum number of corners in X which occur at any one 0-cell, i.e. the maximum number of edges in the link of a 0-cell of X, and let $B = 2B_1 + 2$. Let F be a pointed fan which contains at least C 2-cells and let v be the common endpoint of all of the internal maximal arcs. Note that $\mathrm{Link}(v)$ is a subdivided interval of length at least B, and that the map $\phi : F \to X$ induces a map $\mathrm{Link}(v) \to \mathrm{Link}(\phi(v))$. The number B was chosen so that the path $\mathrm{Link}(v) \to \mathrm{Link}(\phi(v))$ must traverse the same edge twice in the same direction. This means that there are two 2-cells R and R' in F which are sent to the same 2-cell in X and that the corners of these 2-cells at v are lifts of the same corner at $\phi(v)$ in the same orientation.

Let F_1 be the subfan of F corresponding to the subcomplex extending from R to R' which includes R but excludes R'. This subfan as well as the sequence of diagrams that we will examine are illustrated in Figure 17. The conditions on the corners ensure that $F_1 \to X$ admits a fold along the length 2 subpath of ∂F_1 containing v in its interior. Folding F_1 along this subpath creates a wheel F_2 of width 2. Observe that for every pair of 2-cells

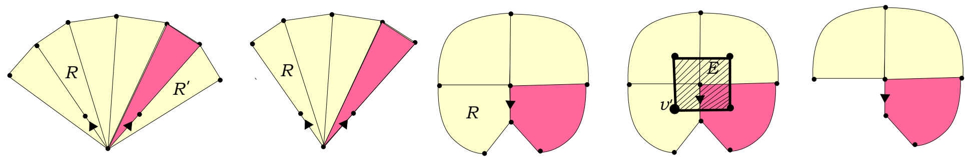


Figure 17. Illustrated from left to right are The pointed fan F, the subfan F_1, the wheel F_2, the dual E in F_2, and the fan F_3 determined by the 0-shell E.

in F_2 with a 1-cell in common, there is a pair of 2-cells in F with a 1-cell in common and these two pairs project to the same pair in X. Consequently, the hypothesis that $F \to X$ is reduced implies that $F_2 \to X$ is reduced. Note that the dual E of F_2 is a single 2-cell, and let v' be the 0-cell in E which corresponds to the 2-cell R of F_2. This choice of 0-cell v' gives E the structure of a 0-shell. Let F_3 be the pointed fan in F_2 which includes every 2-cell except R, and note that F_3 is the fan determined by the 0-shell just described. Finally, since 0-shells are fans of type $k-1$ for all $k \leq 2$, F_3 is a pointed fan of type k. This completes the proof. $\square$

Lemma 12.6. *Let X be a compact $C(p)$-$T(q)$ complex and let p, q, and $k \geq 2$ satisfy the Euclidean restrictions. Let L be a ladder of width 1, and let $L \to X$ be a reduced map. There is a constant $C = C(X)$ such that if L contains more than C maximal internal arcs with a common endpoint, then there is a fan $F \to X$ of type k which factors as $F \hookrightarrow L \to X$ such that the outer path of F is a proper subpath of ∂L which does not contain either basepoint of L in its interior.*

Proof. A set of $C+1$ internal maximal arcs with a common endpoint determines a subdiagram of L which looks like a pointed fan with C 2-cells. The rest of the proof is identical to that of Lemma 12.5. $\square$

Theorem 12.7. *Let X be a compact $C(p)$-$T(q)$ complex and let p, q, and k satisfy the Euclidean restrictions. The set of all minimal fans of type k in X is finite.*

Proof. The result is trivial for $k \leq 1$ because X is compact, so we will assume that $k \geq 2$. It follows from Definition 7.9 that a fan of type k is either a spur, a 0-shell, a 1-shell, a pointed fan, or F is determined by a fan of type $k-1$ in the dual which is itself determined by a fan of type $k-2$ in the double dual of D. Indeed, if F is a fan of type k which is not determined by a fan of type $k-1$, then it is either a spur or a 0-shell. If F is a fan of type k which is determined by a fan of type $k-1$, but this fan of type $k-1$ is not determined by a fan of type $k-2$, then F is a 1-shell determined by a spur or a pointed fan determined by a 0-shell. All other fans of type $k \geq 2$ are determined by fans which are determined by fans in the double dual.

Since X is compact, there are only a finite number of spurs, 0-shells, and 1-shells in X. Furthermore, by Lemma 12.5, there are only a finite number of pointed fans which are minimal fans of type k. Thus we only need to bound the number of minimal fans of type k which are determined by fans in the double dual.

Let F be a fan of type k in X which is 'doubly determined' by a fan F' of type $k-2$ in the double dual of some diagram D. If F' had a proper subfan of type $k-2$ in X, then that subfan would 'doubly determine' a fan in D which would be a proper subfan of F and a fan of type k in X. Consequently, if F is minimal then F' is minimal. By induction we can now assume that F' is one of a finite number of possibilities, and there is therefore an upper bound on the length of its outer path. Since the outer path of F' is almost the entire inner path of F (the two missing subpaths lie in the boundary cycles of the 2-cells at the beginning and the end of F), there is also an upper bound ℓ on the length of the inner path of F.

The fact that F is minimal, combined with Lemma 12.6, shows that the number of 2-cells in F is also bounded. Specifically, it is bounded by $(C+1)(\ell+1)$, where C is the constant used in Lemma 12.6. Since there are only a finite number of fans in X which contain at most $(C+1)(\ell+1)$ 2-cells, the proof is complete. □

13. Lifts and embeddings

It is well known that if X is a $C(p)$-$T(q)$ complex where p, q, and k satisfy the Euclidean restrictions, then every 2-cell $R \to X$ lifts to an embedding in the universal cover of X [21]. In this section we prove that the outer path of a minimal fan of type k lifts to a simple path in the universal cover. This result plays a critical role in the application of fans towards coherence and local quasiconvexity in [15]. We begin by defining minimal outer paths and showing that they lift to simple paths in the universal cover. The remainder of the section will establish a relationship between minimal outer paths and minimal fans.

Definition 13.1 (Minimal outer paths)**.** Let X be a $C(p)$-$T(q)$ complex where p, q, and k satisfy the Euclidean restrictions. A path $Q \to X$ is an *outer path in X of type k* if there exists a fan F in X of type k such that $Q \to X$ factors as $Q \to F \to X$, and $Q \to F$ is the outer path of F. The outer path $Q \to X$ is a *minimal outer path in X of type k* if no proper subpath of Q is an outer path of type k. Note that the outer path of a spur is a minimal outer path of type k for all $k \geq 0$, and thus all other minimal outer paths are immersions.

Lemma 13.2. *Let X be a $C(p)$-$T(q)$ complex, where p, q, and k satisfy the Euclidean restrictions. If $k \geq 1$ and $Q \to X$ is an immersed minimal outer path of type k, then the lift $Q \to \widetilde{X}$ is an embedding unless Q is the outer path of a* 0*-shell, in which case Q is a simple closed path.*

Proof. If $Q \to \widetilde{X}$ is not simple, then there is a not necessarily proper subpath Q' of Q which forms a closed path in $\widetilde{X}$. By Theorem 3.10, there is a $C(p)$-$T(q)$ disc diagram $D \to X$ with Q' as its boundary cycle. Let $v \in \partial D$ be the start/endpoint of the path $Q' \to D$. By Theorem 9.2 and Lemma 9.1, D contains a fan F of type k which avoids v. Since Q is immersed in X and spurs are the only degenerate fans of type $k \geq 1$, F must be a nondegenerate fan. Unless F is a 0-shell, the outer path of F embeds in D and is thus a proper subpath of $Q' \subseteq Q$. This is impossible because it contradicts the minimality of the outer path Q. Similarly, in the case where F is a 0-shell, the minimality condition implies that its outer path must equal Q. Hence $Q' = Q$ and we are done. □

The next few technical lemmas will enable us to prove in Lemma 13.7 that the outer path of a minimal fan of type k is a minimal outer path. Theorem 13.8, which is the main goal of this section, is proven by combining Lemmas 13.2 and 13.7.

Lemma 13.3. *Let $D \to X$ be a (not necessarily reduced) map of a disc diagram to 2-complex. Let v be a valence 3 interior 0-cell of D, and let R_1 and R_2 be 2-cells which meet along an edge e incident at v. Then R_1 and R_2 do not form a cancelable pair along e.*

Proof. Let R_3 be the third 2-cell incident with v. If R_1 and R_2 formed a cancelable pair along e, then the attaching map of R_3 would not be an immersion. This contradicts Convention 2.4. □

Lemma 13.4. *Let X be a $C(p)$-$T(q)$ complex, let $F \to X$ denote a nondegenerate fan of type k, let $F' \to X$ be an i-shell whose unique 2-cell is denoted by R', and suppose that the outer path of F' is a subpath of the outer path of F. Let $D \to X$ be the induced map of the disc diagram formed by attaching F' to F along Q'. Then each of the following conditions implies that there is a 2-cell R_u in F whose outer path Q_u contains Q' as a subpath, and R' and R_u form a cancelable pair along Q':*

1. *$p \geq 5$, $q \geq 4$, $k \geq 1$, and $0 \leq i \leq 3$.*
2. *$p \geq 7$, $q = 3$, k is odd, and $0 \leq i \leq 4$.*

For the proofs we use the following notation. Since F is a nonsingular ladder or a doubly based 2-cell, its dual is either a subdivided interval or a single 0-cell. Let V denote the set of 0-cells in the dual of F, let R_v denote the 2-cell of F corresponding to the 0-cell $v \in V$, and let Q_v denote the subpath of Q contained in ∂R_v. Finally, let $Q \to F$ and $Q' \to F'$ denote the outer paths of F and F'.

Proof of part 1. By Lemma 7.12, F is either a 0-shell, a 1-shell, or the concatenation of 2-shells and 3-shells, and the 0-cells in the interior of Q have valence ≤ 3. Since X satisfies $C(p)$ with $p \geq 5$, no path Q_v is the concatenation of fewer than two pieces of X. Similarly, Q' is not the concatenation of fewer than two pieces of X.

If the interior of the subpath Q' of Q contains a 0-cell v of Q, then $D \to X$ must contain a cancelable pair because D contains an internal 0-cell of valence 3 in violation of the $T(q)$ hypothesis. Since $F \to X$ is itself reduced, the cancelable pair must involve R' and a 2-cell of F, and therefore occurs along a path with an endpoint at an internal valence 3 0-cell. This contradicts Lemma 13.3.

On the other hand, if the interior of Q' does not contain a 0-cell of Q of valence 3 in F, then Q' is a subpath of Q_u for some $u \in V$. Since Q' is not a piece of X, this implies that $R_u \to X$ and $R' \to X$ form a cancelable pair along Q'. □

Proof of part 2. By Lemma 7.12, F is either a 0-shell, a 1-shell, or the concatenation of 2-shells, 3-shells, and 4-shells, and the 0-cells in the interior of Q have valence ≤ 3. Since X satisfies $C(p)$ with $p \geq 7$, no path Q_v is the concatenation of fewer than three pieces of X. Similarly, Q' is not the concatenation of fewer than three pieces of X.

If the interior of the subpath Q' of Q contains at least two distinct 0-cells of valence 3 in F, then Q' contains a path Q_u for some $u \in V$. Since Q_u is not a single piece, the 2-cells $R_u \to X$ and $R' \to X$ form a cancelable pair along Q_u. Furthermore, Q_u must equal Q' because if Q_u ended in the interior of Q' then Lemma 13.3 would be contradicted. If the interior of Q' contains only one valence 3 0-cell u, then Q' is the concatenation of two subpaths Q_1 and Q_2, each of which is contained in a path of the form $Q_u \subset \partial R_u$. Since Q' is not the concatenation of fewer than three pieces, one of these two is not a piece, so R_i and R' form a cancelable pair along Q_i for $i = 1$ or $i = 2$. Since u is an interior 0-cell of D, this contradicts Lemma 13.3. We conclude that Q' does not contain any valence 3 0-cell in its interior, and therefore Q' is a subpath of Q_u for some $u \in V$, and since Q' is not the concatenation of a single piece, R_u and R' form a cancelable pair along Q'. □

Definition 13.5 (Adjacent degenerate fans)**.** Let $F \to X$ be a degenerate fan of valence r and let $F' \to X$ be a degenerate fan of valence r'. Denote the 2-cells of F by $R_1, \ldots, R_{r-1}$ in the order they occur and denote the 2-cells of F' by $R'_1, \ldots, R'_{r'-1}$ in the order they occur. (Of course, a spur has no 2-cells.) These two degenerate fans are *adjacent* if their outer paths project to the same path in X. If the degenerate fans are adjacent and $r + r' - 2 > 2$, then we can glue F and F' together along their outer paths to form a disc diagram which looks like a wheel of width 2 with $(r + r' - 2)$ 2-cells, but which may or may not be reduced (see Figure 18).

Lemma 13.6. *Let X be a $C(p)$-$T(q)$ complex, and let F and F' be adjacent degenerate fans in X of valence r and r'. If $q \geq 5$ and $q > r + r' - 2$ then F' is a subfan of F.*

Proof. We begin by showing that if one of the fans, say F', is a spur, then the other fan F is an identical spur. This is obvious if $r = 1 = r'$, so we assume that $r + r' - 2 > 0$ and that $r' = 1$ and we will prove by induction

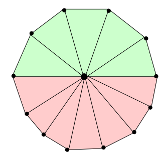


FIGURE 18. The disc diagram obtained by gluing F and F' together.

that this is impossible. As in Definition 13.5, we glue F and F' together along their outer paths to obtain a diagram $D \to X$ which looks like a wheel of width 2 with $(r + r' - 2)$ 2-cells. If $r = 2$ and $r' = 1$ and the 2-cell R_1 is adjacent to the spur F, then the attaching map of R_1 is not an immersion, which contradicts Convention 2.4. If $r = 3$ and $r' = 1$, then the 2-cells R_1 and R_2 meet along a length 2 path in D. By Lemma 3.5, this implies that R_1 and R_2 form a cancelable pair in F along this length 2 path, and thus form a cancelable pair in F which contradicts our assumption that F is reduced. Finally, if $r > 3$ and $r' = 1$, then we use our hypothesis that X satisfies $T(q)$ and that $q > r + r' - 2$, to conclude that the diagram D must contain a cancelable pair. The only possible location for a cancelable pair is between R_1 and R_{r-1}, but removing this pair creates a new adjacent pair of degenerate fans with valence $(r - 2)$ and valence 1, which is impossible by induction. We have therefore shown that if $r' = 1$ then $r = 1$ as well, and F and F' are identical spurs.

We now consider the case where $r \geq r' > 1$. Again, our hypotheses imply that the diagram D contains a cancelable pair. The only possible locations for the cancelable pair is between R_1 and R'_1 or between R_{r-1} and $R'_{r'-1}$. Removing either of these pairs of 2-cells creates a new pair of adjacent degenerate fans of valence $r-1$ and valence $r'-1$. Repeating this procedure eventually produces a pair with valence $(r-r'+1)$ and valence 1 and by the above result, $r - r' = 0$. Working backwards through the removals of the cancelable pairs reveals that each cancelable pair was between the 2-cells R_i and R'_i for some i. This determines the required isomorphism $E \to E'$ between the essences of F and F' and hence completes the proof. □

Lemma 13.7. *Let X be a $C(p)$-$T(q)$ complex and let p, q, and k satisfy the hyperbolic restrictions. If $F \to X$ is a minimal fan in X of type k, then its outer path $Q \to F$ is a minimal outer path of type k.*

Proof. Let F' be another fan of type k whose outer path Q' is a proper subpath of Q. To establish the claim we will show that F' is a subfan of F, and hence F is not minimal. If F' is a spur then F is obviously not minimal, so we shall proceed under the assumption that F' is not a spur. Let D be the diagram formed by attaching F to F' along Q'. The proof is divided into cases depending on the values of p, q, and k.

Case 1: Suppose $p \geq 5$, $q \geq 4$, and k is odd. By Lemma 7.12, every 2-cell R' in F' is an i-shell with $0 \leq i \leq 3$. By Lemma 13.4.1, the outer path Q'

of R' is a subpath of the outer path Q_u of a 2-cell R_u of F, and R' and R_u form a cancelable pair along Q'. This sequence of cancelable pairs induces the required map $E' \to E$ from the essence of F' to the essence of F. In particular, the sequence of consecutive 2-cells in F' is sent to a sequence of *consecutive* 2-cells in F in the same order, because each 2-cell of F has a nontrivial outer path, and so it is impossible to skip a 2-cell of F.

Case 2: Suppose $p \geq 7$, $q = 3$, and k is odd. The proof is the same as Case 1 except that Lemma 7.12 now implies that every 2-cell in F' is an i-shell with $0 \leq i \leq 4$, and Lemma 13.4.2 is used in place of Lemma 13.4.1

Case 3 [Case 4]: Suppose $p \geq 4$, $q \geq 5$, $[p = 3,\ q \geq 7]$ and k is even. It follows from Lemma 7.12 that Q' has length at least 2, and furthermore, every 2-cell of F' contains an interior 0-cell of Q' in its boundary. By Lemma 7.12, each 0-cell in the interior of Q or Q' has valence ≤ 3, $[\leq 4,]$ and consequently, every 0-cell in the interior of the path Q' has valence in D which is at most 4 [at most 6].

Each 0-cell v in the interior of Q' determines a degenerate subfan F_v of F and a degenerate subfan F'_v of F'. These degenerate subfans are obviously adjacent because their outer paths are glued together in D. By Lemma 13.6, F'_v is a subfan of F_v (and F_v is a subfan of F'_v). As v varies over the interior 0-cells of Q', the set of isomorphisms $E'_v \to E_v$ between the essences of F'_v and F_v induces an embedding $E' \to E$ between the essences of F' and F. This is well-defined because if u and v are 0-cells in the interior of Q' which are the endpoints of an edge e in Q', then $F'_u \cap F'_v$ consists of the 2-cell of F' whose outer path contains the edge e, and this 2-cell forms a cancelable pair with a corresponding 2-cell of F across e in a unique fashion. The sequence of consecutive 2-cells of F' is mapped to a sequence of consecutive of 2-cells of F because the same holds for each F'_v and F'_v. □

Theorem 13.8. *Let X be a $C(p)$-$T(q)$ complex, let $F \to X$ be a minimal fan of type k. If p, q, and k satisfy the hyperbolic restrictions, then its outer path $Q \to X$ lifts to a simple path in $\widetilde{X}$.*

Proof. This follows immediately from Lemma 13.2 and Lemma 13.7. □

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