AN INTRODUCTION TO GARSIDE STRUCTURES

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Abstract. Geometric combinatorialists often study partially ordered sets in which each covering relation has been assigned some sort of label. In this article we discuss how each such labeled poset naturally has a monoid, a group, and a cell complex associated with it. Moreover, when the labeled poset satisfies three simple combinatorial conditions, the connections among the poset, monoid, group, and complex are particularly close and interesting. Posets satisfying these three conditions are (roughly) equivalent to the notion of a Garside structure for a group as developed recently within geometric group theory by Patrick Dehornoy in [15]. The goal of this article is to provide a quick introduction to the combinatorial version of this notion of a Garside structure and, more specifically, to the particular combinatorial Garside structures which arise in the study of Coxeter groups and Artin groups. These are the labeled partially ordered sets that combinatorialists know as the generalized non-crossing partition lattices.

Over the past several years, geometric group theorists have developed a theory of Garside structures to help them better understand Artin's braid groups and their generalizations. See for example [1, 2, 3, 7, 9, 14, 15, 16, 17]. Groups with Garside structures are now emerging as a well-behaved class of groups worthy of study in their own right. Geometric combinatorialists might be interested in this theory because even though the original definition of a Garside structure was formulated strictly within group theory it can be recast in an essentially combinatorial form.

After presenting the combinatorial reformulation of a Garside structure as a particular type of labeled poset in §1, the benefits and consequences of having a Garside structure will be briefly touched upon. Next, a general technique for constructing Garside-like structures is presented in §2 that starts with a symmetric object such as a regular polytope or a Riemannian symmetric space. In §3 I describe the Garside structures associated with the finite reflection groups. Following the procedure outlined in §2, the symmetry group of the regular $(n-1)$-simplex, i.e. the symmetric group $\textsc{Sym}_n$, produces the non-crossing partition lattice NC_n. The labeled posets produced by the other finite Coxeter groups can thus be thought of as generalized noncrossing partition lattices. The final section of the article discusses some recent attempts to extend these ideas to the generalized noncrossing partition poset associated with an arbitrary Coxeter group.

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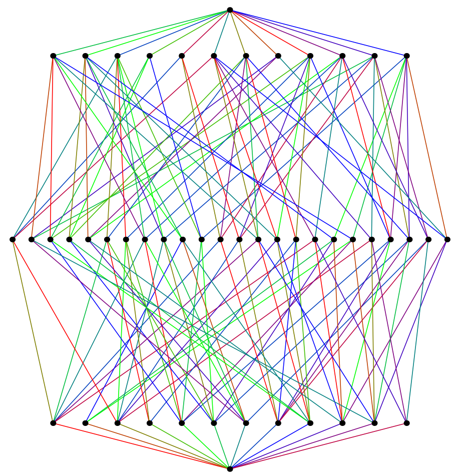

FIGURE 1. Hasse diagram for a bounded graded poset.

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1. POSETS, MONOIDS, GROUPS AND COMPLEXES

It is an underappreciated fact that associated to every labeled partially ordered set there is a monoid, a group and a cell complex. If these are denoted by P, M, G and K, respectively, then schematically, the process can be viewed as follows:

$$P \rightsquigarrow M,\ P \rightsquigarrow G \text{ and } P \rightsquigarrow K$$

These associated objects and the procedures needed to construct them are defined in this section. Since this article is aimed primarily at non-specialists I try to err on the side of too many definitions and too many details rather than too few. Nevertheless, some of the standard combinatorial terminology is left undefined and some of the given definitions are more intuitive than rigorous. The interested reader is referred to [25] for additional combinatorial definitions and remarks and to [6] for a more precise development of these ideas.

Definition 1.1 (Bounded graded posets)**.** Let P be a partially ordered set which is bounded, graded and has finite height. *Bounded* means that P contains a maximum element, usually denoted $\hat{1}$, and a minimum element $\hat{0}$. Finite height means that between any two elements $x \leq y$ there is a bound on the lengths of the finite chains which start at x and end at y. *Graded*, in this context, means that all of the maximal length chains from $\hat{0}$ to $\hat{1}$ have the same length. In particular, we can partition P into a finite number of levels based on where each element appears in one of these maximal chains. See, for example, the poset shown in Figure 1. Even though we are assuming that P has finitely many levels, we are *not* assuming that P has only a finite number of elements. Some levels might contain an infinite number of elements. The set $I(P) = \{(x,y) | x, y \in P \text{ with } x \leq y\}$ is called the set of *intervals* in P. The intervals of the form (x,x) are the *trivial intervals* and the ones where x and y belong to adjacent levels are called the *covering relations* of P.

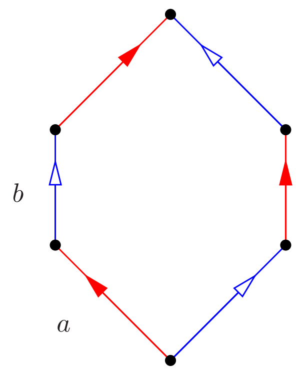

Figure 2. A simple edge-labeled poset in which the solid arrows are labeled a and the others are labeled b.

Definition 1.2 (Hasse diagrams)**.** The graph shown in Figure 1 is, technically speaking, the Hasse diagram of a poset rather than the poset itself. Hasse diagrams, by definition, have vertices indexed by the elements of the poset and an edge is drawn from x to y if and only if (x, y) is a covering relation. No confusion arises from this convention since the rest of the order relation in any finite height poset is implied by its covering relations. In addition, the elements are placed so that the lower end of an edge corresponds to the lesser element in the covering relation. Thus $\hat{0}$ is the bottom element in each diagram and $\hat{1}$ is the top element. As an object, the Hasse diagram, denoted $\textsc{Hasse}(P)$, can be thought of as a oriented graph where the edges are oriented so that they start at the lower end of an edge and end at its upper end.

Definition 1.3 (Labeled Posets)**.** Let P be a poset which is bounded, graded and has finite height. Because most of the posets considered in this article have all three of these properties, they should to be presumed from now on unless explicitly stated otherwise. An *edge-labeling* on P is a map from the covering relations of P (i.e. from the edges of its Hasse diagram) to some *labeling set* S. When edge labels are added to a Hasse diagram, it becomes a labeled oriented graph denoted $\textsc{Hasse}(P, S)$.

Although combinatorialists typically only assign labels to the covering relations, such a labeling can be easily extended to provide a label of a sort for each interval of P. In particular, each interval can be assigned a language. Recall from formal language theory that finite strings of elements are called *words* and arbitrary collections of words are called *languages*. Thus a word is an element of the free monoid S^* and a language is an element of its power set $\mathcal{P}(S^*)$.

In the context of an edge-labeled poset, each maximal length chain from x to y produces a word by concatenating the labels on its covering relations read, say, from bottom to top, and each interval (x, y) produces a language by collecting the words obtained from each maximal length chain from x to y. Using this approach, the trivial intervals are labeled by the language containing only the empty word, (i.e. the identity element of the free monoid) and the label assigned to a covering relation is the language $\{s\}$ where $s \in S$ is the label of this covering relation viewed as a word of length 1. To illustrate this idea with a slightly less trivial example, consider the labeled poset shown in Figure 2. The label on the interval $[\hat{0}, \hat{1}]$ would be the language $\{aba, bab\}$. When this labeling needs to be distinguished from the original edge-labeling, the extended labeling is called an *interval-labeling of P*.

Definition 1.4 (Monoids and groups)**.** Given an edge-labeled poset P, the monoid and the group naturally associated to P are defined via presentations. Define $M(P)$ / $G(P)$ to be the monoid / group generated by the set S of labels and subject to all of the relations obtained by equating the words associated to any two maximal length chains which start and end at the same elements. In other words, for each interval (x, y) all of the words in the language assigned to this interval are set equal to each other. For the poset shown in Figure 2, the monoid and group presentations are $M = M(P) = \textsc{Mon}\langle a, b \mid aba = bab\rangle$ and $G = G(P) = \textsc{Grp}\langle a, b \mid aba = bab\rangle$.

Because of the way in which these presentations are defined, there is a natural map from the Hasse diagram of P to the Cayley graphs of M and from the Cayley graph of M to the Cayley graph of G.

Definition 1.5 (Cayley graphs)**.** Recall that the *right Cayley graph* of a monoid M generated by a set S (usually denoted $\textsc{Cayley}(M, S)$) is a labeled oriented graph which has vertices indexed by the elements of M and for each $m \in M$ and for each $s \in S$ there is an edge labeled s starting at m and ending at $m \cdot s$. It is called the *right* Cayley graph because we are multiplying the generator s on the right. The right Cayley graph of a group G, $\textsc{Cayley}(G, S)$, is defined similarly.

Based on these definitions, it should be clear that for each labeled poset P and associated monoid M and group G there is a natural label-preserving map

$$\textsc{Hasse}(P, S) \to \textsc{Cayley}(M, S)$$

which sends $\hat{0}$ to the identity element of M. As remarked above this map is well-defined as a consequence of the way in which the relations defining M were chosen. Similarly there is a natural label-preserving map

$$\textsc{Cayley}(M, S) \to \textsc{Cayley}(G, S)$$

which sends the identity in M to the identity in G. Notice, however, that it is not at all obvious whether either map is injective and, in fact, when no further restrictions are placed on the poset P, the monoid M and the group G are generally hard to analyze and the connections among P, M and G are difficult to discern. The conditions defining a combinatorial Garside structure are explicitly designed to overcome these difficulties. Before turning our attention to these conditions, there is one final construction to introduce.

Definition 1.6 (The complex K)**.** Given any labeled poset P we can construct a cell complex K whose fundamental group is equal to G. We begin with the *geometric realization* (or *order complex*) of P, usually denoted $\Delta(P)$. Recall that $\Delta(P)$ is defined as the simplicial complex whose simplices correspond to the chains in P. Since P has a minimal element $\hat{0}$, $\Delta(P)$ is a cone over the geometric realization of $P \backslash \{\hat{0}\}$ and thus $\Delta(P)$ is contractible. To create a space with interesting topology, a quotient of $\Delta(P)$ is defined using the labeling of P as a guide. Using the extended interval-labeling of P, every interval (x, y) has both a well-defined label and (except for the trivial intervals) an orientation. Since the 1-cells in $\Delta(P)$ are in one-to-one correspondence with the non-trivial intervals of P, each edge in the 1-skeleton of $\Delta(P)$ can be viewed as having an induced label and orientation. The identifications to be made are as follows. If σ and τ are two simplices in $\Delta(P)$ of the same

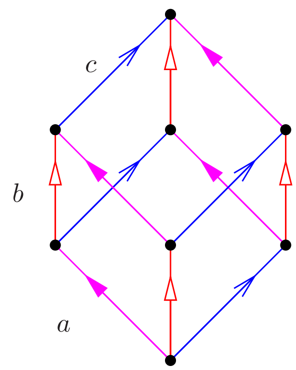

FIGURE 3. Finite Boolean lattices are Garside structures.

dimension and if $f : \sigma \to \tau$ is a label and orientation preserving isometry between them, then σ and τ are identified using the map f. Because the simplices have acyclic orientations, there is at most one such map f for each fixed pair of simplices. Notice that since every 0-cell received the same language as a label, all of the vertices of $\Delta(P)$ are identified under this procedure. The complex which results from all of these identifications is called $K = K(P)$. It is not too hard to see, by examining its 2-skeleton, that the fundamental group of K is precisely the group G.

Since arbitrary finitely presented groups are notoriously difficult to work with, it is not clear whether the complex K contains anything more than the most basic information about G. When P is a combinatorial Garside structure, however, the connection between G and K is as close as one could possibly expect. A precise definition of a combinatorial Garside structure will be given below once the resulting benefits have been made more precise. The following theorem answers the question: "What are combinatorial Garside structures good for?"

Theorem 1.7 (Consequences)**.** *Let P be an edge-labeled poset which is bounded, graded and has finite height, and let M, G and K be the monoid, group and complex derived from P. If P satisfies the combinatorial definition of a Garside structure given below then*

- HASSE(P, S) *embeds into* CAYLEY(M, S)*,*
- CAYLEY(M, S) *embeds into* CAYLEY(G, S)*,*
- *G is the group of fractions of M,*
- *The word problem for G is solvable*[1]*, and*
- *The universal cover of K is contractible.*

This final property makes K an Eilenberg-MacLane space for G. It thus follows that the cohomology of G is equal to that of K, the cohomological dimension of G is bounded above by the dimension of K which in turn is equal to the height of P. It also follows that the group G is torsion-free.

[1]When the poset P is infinite, extra hypotheses and conditions need to be added in order to make this statement precise, but there is still a sense in which is remains fundamentally true. In particular, P needs to be understood well enough so that the elements of P can be recursively described, equality in P can be algorithmically tested and meets and joins of elements in P can be algorithmically produced. When hypotheses of this sort are assumed the usual solution to the word problem given in the case where P is finite can be readily extended to cover the general case.

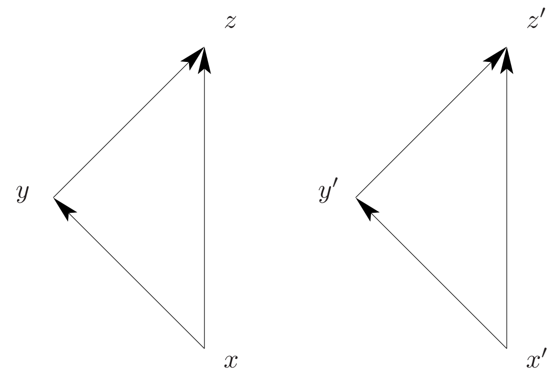

FIGURE 4. The group-like condition involves pairs of 3 element chains such as the ones shown.

As one can see from these definitions and consequences, when P is a combinatorial Garside structure, the connections among P, M, G and K are indeed remarkably strong. An elementary and fundamental example of a Garside structure in the combinatorial sense is a finite Boolean lattice with the natural edge-labeling.

Example 1.8 (Boolean lattices)**.** Let $\mathcal{B}_n$ denote the Boolean lattice viewed as all subsets of an n-element set under inclusion. If we label each covering relation by the unique element added, then it turns out that this labeled poset satisfies the Garside structure conditions. See Figure 3 for an illustration of $\mathcal{B}_3$. For this labeled poset, the monoid M is the free abelian monoid on n generators, the group G is the free abelian group on n generators, $\Delta(P)$ is a simplicial structure on the n-cube, and the complex K is a triangulation of an n-torus. Since all of these objects are well-known, it is easy to see that the consequences described in Theorem 1.7 really do hold in this case.

So what is a combinatorial Garside structure? It is labeled poset satisfying three simple conditions.

Definition 1.9 (Combinatorial Garside structures)**.** Let P be an edge-labeled poset which is bounded, graded and has finite height. If P is group-like, balanced, and a lattice then P is called a *combinatorial Garside structure*. The precise definitions of group-like, balanced and lattice are given below. If P is merely group-like and balanced (but not necessarily a lattice) then P is said to be *Garside-like*. Technically speaking, the poset need not be graded, but merely *weakly-graded*, in a sense not defined here. Grading has been assumed here for ease of exposition. See [6] for a more detailed development. I should also note that the combinatorial structure associated with Garside groups have traditionally been required to be finite. This requirement has not been incorporated into the definition given above, but the reader should be aware that it is presumed by many of the researchers working in the area.

As the definitions of "group-like", "balanced" and "lattice" are given, I will try to explain how each of the three restrictions arise naturally from the desired consequences. For example, if there is any hope of the Hasse diagram $\textsc{Hasse}(P,S)$ injecting into the Cayley graph $\textsc{Cayley}(G,S)$ then the labeling of P should be

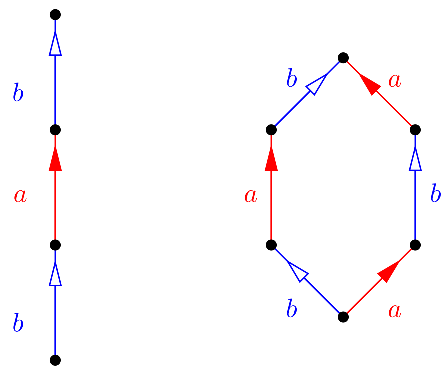

FIGURE 5. A non-balanced poset and a balanced poset

both multiplicative and left and right cancellative. This leads immediately to the definition of a group-like labeling.

Definition 1.10 (Group-like)**.** An interval-labeled poset P is called *group-like* if whenever two 3 element chains $x \leq y \leq z$ and $x' \leq y' \leq z'$ have two pairs of corrsponding labels in common, then the third pair of labels are also equal. See Figure 4. The three possible configurations of common labels ensure that the labeling is left cancellative, right cancellative and multiplicative.

Even if we were somehow able to ensure that the Hasse diagram of P embeds in the Cayley graph of M and the Cayley graph of G, getting the Cayley graph of M to embed into the Cayley of G (which is equivalent to getting M to embed in G) is a stronger restriction and a problem which has been thoroughly studied. If M is to embed in G then it is clearly necessary that M be left and right cancellative. On the other hand, it is well-known that when M is left and right cancellative and every pair of elements in M have a right common multiple (known as the *right Ore condition*), then M embeds in its right group of fractions.[2] An element $m \in M$ is a *right common multiple* of elements m_1 and m_2 in M if there exist elements n_1 and n_2 in M with $m = m_1 \cdot n_1 = m_2 \cdot n_2$. Notice that this is merely a common multiple (on the right) rather than a *least* common multiple. Right common multiples in M are easy to produce when the labeling on P is group-like and balanced.

Definition 1.11 (Balanced)**.** Let P be an interval-labeled poset and let $\lambda(x, y)$ denote the label assigned to the interval $(x, y) \in I(P)$. Further, define $L(P) = \{\lambda(\hat{0}, p) : p \in P\}$, $C(P) = \{\lambda(p, q) : p, q \in P\}$, and $R(P) = \{\lambda(p, \hat{1}) : p \in P\}$. The poset P is called *balanced* if $L(P) = R(P)$.

If the interval labeling under consideration was derived from an edge labeling, then the languages in $L(P)$ contain all of the prefixes of the words labeling maximal chains from $\hat{0}$ to $\hat{1}$, the languages in $R(P)$ contain all of the suffixes of these words and the languages in $C(P)$ contain all of the subwords of these words. More colloquially, the languages in $L(P)$, $R(P)$ and $C(P)$ contain the subwords from the left, right and center of the words labeling the maximal chains, hence the notation.

[2]A technical list of conditions on M which are both necessary and sufficient was established by Mal'cev [22], but the easy sufficient condition given above is all that is needed here.

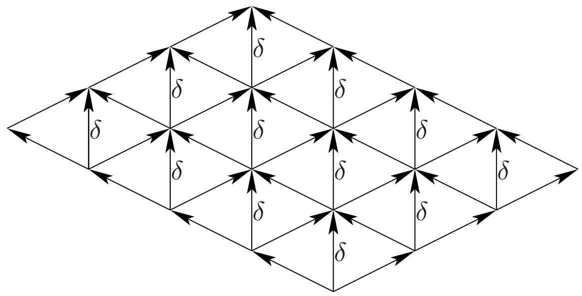

FIGURE 6. The type of diagram used to prove that right common multiples always exist.

Remark 1.12 (Balanced vs. Symmetric)**.** Notice that the notion of being balanced is quite different from that of having a symmetric labeling. For example, the labeled poset on the lefthand side of Figure 5 is *not* balanced because $\{ab\} \in R(P)$ but $\{ab\} \notin L(P)$. On the other hand, the poset on the righthand side is balanced.

When P is an edge-labeled poset that is Garside-like (i.e. both balanced and group-like), it is easy to show that $L(P) = C(P) = R(P)$. Thus every label ever used in P occurs as the label on some interval starting at $\hat{0}$ and as the label on some interval ending at $\hat{1}$. As was mentioned above, the other fact that is easy to prove is that the monoid M derived from a Garside-like poset P must have right common multiples. This is the main reason for introducing the balanced property.

Lemma 1.13 (Right common multiples exist)**.** *Let P be an edge-labeled poset and let M be the monoid derived from P. If P is Garside-like then M has right common multiples.*

Proof. The proof is essentially contained in the diagram shown in Figure 6. Let δ be the label on the interval $(\hat{0}, \hat{1})$ and let m_1 and m_2 be any two elements of M. Being elements of M, both can be written in terms of its generating set, each of which is a label used in P. Depending on the number of generators needed to write m_1 and m_2 draw a figure similar to the one shown with the factorization of m_1 into generators written along the edges on the lower lefthand side and the factorization of m_2 into generators written along the lower righthand side. The diagram shown assumes m_1 is a product of four generators and m_2 is a product of three generators. Next write δ on each of the vertical arrows. Finally, we add labels to each of the other arrows, labeling one column at a time, working our way from the bottom to the top. If any two of the sides of a triangle are already labeled (which must be the bottom edge and the side edge) then there is a canonical way to add a label to the top edge, namely, find the element $x \in P$ so that the label on the bottom edge matches the label on $(\hat{0}, x)$ and then label the top edge by the label on the interval $(x, \hat{1})$. This works because every label assigned to an interval of P also labels an interval of the form $(\hat{0}, x)$ for some x. Once the diagram is completely labeled, let $n_1 \in M$ be the product of the labels on the upper lefthand edges and let $n_2 \in M$ be the product of the labels on the upper righthand edges. Because each triangle represents a relation in M, the diagram itself represents a proof that $m_1 \cdot n_1$ and $m_2 \cdot n_2$ represent the same element in M which is thus a right common multiple of m_1 and m_2. $\square$

The last definition is the easiest to give, but generally the hardest to establish.

Definition 1.14 (Lattice)**.** Let P be a poset. The *meet* of x and y—if it exists—is the unique largest element among those elements below both x and y in P. Similarly the *join* of x and y is the unique smallest element among those elements above both x and y. When a meet and a join exists for every pair of elements in P, then P is called a *lattice*.

Unfortunately, our supply of techniques for proving that a poset is a lattice is rather limited. For example, is the poset in Figure 1 a lattice?

Remark 1.15 (Why lattice?)**.** The group-like condition is needed if the Hasse diagram of P is to have any chance of embedding into the Cayley graph of G and the balanced condition is a quick way to ensure that right common multiples exists in M. By comparison, the reasons for requiring P to be lattice are rather opaque at first. In the standard proofs of the consequences listed in Theorem 1.7, the lattice condition is used:

(1) to show that P cancellative implies M cancellative,
(2) to efficiently solve the word problem in G, and
(3) to show that the universal cover of K is contractible.

Without going into too many details, the proofs of all these consequences will now be sketched.

Sketching the proof of Theorem 1.7. Using the group-like and balanced properties, it is relatively easy to show that the Hasse diagram of P embeds into the Cayley graph of M. To get the Cayley graph of M to embed into the Cayley graph of G it is sufficient to know that M is left and right cancellative and that it has right common multiples. The latter is true by Lemma 1.13, but the former is not quite as immediate as it might seem at this point. A priori, M and G are rather mysterious, so extrapolating cancellativity in M from the fact that P is left and right cancellative is no easy task. It can be accomplished, however, with an inductive argument which relies on the existence of meets and joins in P. Once this gap is filled, the standard sufficient conditions listed above show that M embeds into its right group of fractions and the fact that M and G share a presentation shows that this group of fractions can be identified with G.

Next, because G is the group of fractions of M, the word problem in G can be reduced to the word problem in M, and the word problem of a finitely generated monoid with a length-preserving presentation is trivial to solve. Notice that the presentation of M is length-preserving as a consequence of the fact that the poset P is graded. When P is infinite, or merely weakly-graded, the word problem for G can still be reduced to the word problem for M. More care needs to be taken in this case, but the word problem in M can still be solved using only basic properties of the poset P. See Footnote 1.

Finally, the universal cover of K is tiled by copies of the order complex $\Delta(P)$ which is contractible. Using the fact that P is a lattice, the overlaps between these fundamental domains will contain cone points making their overlaps contractible. At this point, a Quillen-type argument can be used to show that the universal cover can be built up is a systematic way by adding contractible pieces one at a time to a contractible portion of the universal cover with a contractible intersection at each stage, thereby showing that the final result is a contractible space. For a full proof of this result, see the article by Ruth Charney, John Meier and Kim Whittlesey [14]. The rest of the conclusions are standard consequences of having a finite dimensional Eilenberg-MacLane space. □

We end this section by briefly describing the traditional definition of a Garside structure within geometric group theory (due to Patrick Dehornoy [15]) and commenting on how it compares with the combinatorial definition presented here.

Definition 1.16 (Garside structures in groups)**.** When Garside structures are defined by geometric group theorists, they typically start with a group G generated as a group by some set S. Inside G there is a natural submonoid M generated by S and inside M they select an element δ. If the choice of G and S are restricted so that for every element m in M there is an upper bound on the length of factorizations m over S (a property which makes M an *atomic monoid*), then the various factorizations of δ in M can be used to construct an interval-labeled finite-height poset P. By construction the Hasse diagram of this poset P embeds in $\textsc{Cayley}(M, S)$ which in turn embeds in $\textsc{Cayley}(G, S)$. If the labeled poset P is both balanced and a lattice, then δ is called a *Garside element* and the collection of relevant information, (G, S, δ), defines a *Garside structure* on G. Notice that the poset P need not be graded under this definition. If it is then G will be called a *graded Garside group.* Alternatively, a Garside group G is *graded* if the function sending each element in S to 1 extends to a group homomorphism from G to $\mathbb{Z}$. It is called *weakly-graded* if there is some function from S to the positive integers which extends to a group homomorphism from G to $\mathbb{Z}$.

As should be clear from the definition, if G is a group with a graded Garside structure, then the poset P representing factorizations of δ over the generating set S will a combinatorial Garside structure. It is then a standard result (due primarily to Patrick Dehornoy) that the submonoid M is the same as the natural monoid derived from P (i.e. that the relations which occur inside P are sufficient to present the submonoid M), that G is the group of fractions for M and that the word problem for G is decidable. That the complex K derived from P is an Eilenberg-MacLane space for G was shown by Charney, Meier and Whittlesey in [14]. In short, if a labeled poset P was derived from a Garside structure on a group G, then P is all that is needed to reconstruct G.

A more complicated question is how to characterize those labeled posets that arise in this way. The definition of a combinatorial Garside structure given here is one attempt to do just that, but a partial converse has only been established when the poset P is at least weakly-graded. The reason the (weakly) graded condition is imposed is because a (weakly) graded poset P leads naturally to a (weakly) graded monoid M (which is necessarily atomic) but a poset P that is merely of finite height might not lead to an atomic monoid. As will become clear latter in the article, the graded case is sufficient for most applications and, in fact, the proofs of many of the standard properties of Garside structures simplify significantly when a grading is presumed. The following theorem explicitly states this bijective correspondance between weakly-graded Garside structures in the geometric group theory sense and the combinatorial notion of a Garside structure defined here. For a proof, see [6].

Theorem 1.17 (Equivalence)**.** *There is a natural bijection between combinatorial Garside structures and weakly-graded Garside groups.*

2. Constructing Garside-like posets

At this point, it should be clear that combinatorial Garside structures are a good way to find posets, monoids, groups and complexes where the interconnections among them are particularly strong and useful. What might not be so clear is whether these structures are easy to find in practice. The goal of this section is show that Garside-like posets are relatively abundant "in nature" by describing a fairly general procedure which can be used to construct lots of examples. The first step is to create bounded, graded posets that have finite height and a labeling that is group-like. Of course, if one is looking for a directed graph with a group-like labeling, a good place to search is inside the Cayley graph of a group.

Definition 2.1 (Factor posets)**.** Let W be a group and let S be a generating set for W that is not necessarily finite.[3] Recall that the *word length* of $w \in W$ with respect to S is the nonnegative integer $\ell_S(w)$ that represents the length of the shortest word in S^* which is equal to w in the group. We say that a factorization $u \cdot v = w$ is a *minimal factorization in* W, and that $u \leq w$ in the *factor order*, if $\ell_S(u) + \ell_S(v) = \ell_S(w)$. Geometrically, $u \leq w$ in the factor order if and only if the vertex u lies on some minimal length path from the identity to w in the right Cayley graph $\textsc{Cayley}(W, S)$. The elements below w in the factor order form a bounded, graded poset of finite height called the *factor poset of* w *factored over* S. This poset is denoted $\textsc{Factor}(w, S)$. The bounding elements are w and the identity, it is graded by word length, and its height is $\ell_S(w)$.

Notice that when (x, y) is a covering relation in a factor order, then x and y must be connected by some edge in the Cayley graph. Thus there is an $s \in S$ such that $x \cdot s = y$ or $x = y \cdot s$. By orienting the edge from x to y (and relabeling the edge by s^{-1} if need be) we can assign each covering relation an element of $S \cup S^{-1}$. Because $\textsc{Factor}(w, S)$ lives in the Cayley graph of a group, this labeling is automatically group-like. The Boolean lattices already examined in Example 1.8 can now be reinterpreted as factor posets.

Example 2.2. Consider the finite group $W = (\mathbb{Z}/2\mathbb{Z})^n$ with the obvious minimal n-element generating set S. In $\textsc{Cayley}(W, S)$ there happens to be a unique element w furthest away from the identity which corresponds to the product of the elements in S (in any order since W is abelian). The factor poset of w in this case is just the Boolean lattice $\mathcal{B}_n$ discussed in Example 1.8 with its natural labeling.

These examples of factor posets happen to be Garside structures, but that is definitely not true in general. There is no reason to expect that factor posets would be balanced or lattices without further restrictions on W and S. There is, however, a simple condition on S which ensures that the factor posets are balanced.

Theorem 2.3 (Constructing Garside-like posets)**.** *If W is a group generated by a set S which is closed under conjugation, then every factor poset in* $\textsc{Cayley}(W, S)$ *is Garside-like and locally Garside-like.*

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[3] We are not calling our group G to emphasize the fact that the group whose Cayley graph is used to *find* a group-like labeled poset need not be the same as the group *derived from* the labeled poset once it has been found. Moreover, our choice of the letter W foreshadows the fact that the generalized noncrossing partitions focused on in the next section are created from Coxeter groups, and arbitrary Coxeter groups are conventionally denoted W.

Proof. When S is closed under conjugation it is straightforward to show that the length function ℓ_S is constant on conjugacy classes. As a consequence, $u \cdot v = w$ is a minimal factorization if and only if $v' \cdot u = w$ is a minimal factorization where $v' = u \cdot v \cdot u^{-1}$. Thus the set of left factors is equal to the set of right factors and $\textsc{Factor}(w, S)$ is balanced. Since the same argument applies to every interval inside $\textsc{Factor}(w, S)$, it is also locally balanced. Recall that combinatorialists say a poset *locally* has some property if every interval has that property. Since each interval is itself a factor poset, it is also balanced and group-like. Thus the poset $\textsc{Factor}(w, S)$ is Garside-like and locally Garside-like. □

Example 2.4 (Symmetric groups)**.** Inside a symmetric group, the conjugacy classes are determined by cycle type. Thus the set of all transpositions is closed under conjugacy and the factor poset of any permutation with respect to the generating set of all transpositions is Garside-like.

Theorem 2.3 makes it easy to create many concrete examples of Garside-like posets. In order for these examples to be full-fledged combinatorial Garside structures, we would need to be able to show that well-defined meets and joins exist. This turns out to be difficult in general, but it is the only obstacle standing in the way of the creation of lots of combinatorial Garside structures. Luckily, the main class of examples we are interested in have additional geometric aspects which make proving the lattice property more tractable. First of all, the main class of examples we wish to investigate are ones where the Hasse diagram of the factor order in W is very closely related to its Cayley graph. In fact, every example we have in mind is a group which is "signed" in the following sense.

Definition 2.5 (Signed groups)**.** Let W be a group generated by a set S. If there is a group homomorphism $W \to \mathbb{Z}/2\mathbb{Z}$ such that every element of S is sent to the non-identity element of $\mathbb{Z}/2\mathbb{Z}$, then we say that (W, S) is *signed*. The elements in the kernel of this map are called the *even* elements and the others are called the *odd* elements. The terminology is meant to extend the standard notion of an even/odd permutation to a more general context.

Our primary examples are signed groups that are constructed from the isometries of some well-behaved geometric objects. Although the following class of examples might seem very specialized at first, it is flexible enough to include all of the generalized noncrossing paritition lattices introduced in the next section.

Example 2.6 (Signed groups from Isometries)**.** A geometric way to produce lots of signed groups is to start with a fairly symmetric geometric object X which has some notion of an orientation and then to look at a group W of isometries generated by some collection S of orientation-reversing isometries. For example, if X is a regular $(n-1)$-simplex, and S is the collection of isometries which fix $n-2$ of the vertices and switch the remaining two, then S generates the full group of isometries of X, which is the symmetric group on n elements in this particular case. For more complicated examples, let X denote one of the simply-connected constant curvature Riemannian manifolds, such as the n-sphere $\mathbb{S}^n$, Euclidean n-space $\mathbb{R}^n$ or hyperbolic n-space $\mathbb{H}^n$, and let S denote any collection of reflections (where a reflection is any non-trivial isometry which fixes a codimension 1 subspace). If W is the group of isometries generated by S, then the pair (W, S) is always signed.

Remark 2.7 (Even Presentations)**.** One quick way to identify whether a group W generated by a set S is signed is to check whether there is a presentation of W that is generated by S where each relator has even length. The existence of a presentation with this property is easily seen to be equivalent to being signed.

We remarked above that the covering relations in the factor order of a group come from edges of its Cayley graph. One reason for introducing signed groups is that in the Cayley graph of a signed group the converse holds: *every* edge in the Cayley graph leads to a covering relation in the factor order.

Remark 2.8 (Cayley graphs of signed groups)**.** Suppose x and y are connected by an edge in $\textsc{Cayley}(W,S)$. With no assumptions on W or S, $\ell_S(x)$ and $\ell_S(y)$ can differ by at most 1. When (W,S) is signed, multiplying by a generator changes the parity of the length, ensuring that $\ell_S(x)$ and $\ell_S(y)$ cannot be equal. Thus they differ by exactly 1 and either (x,y) or (y,x) is a covering relation in the factor order. In other words, when (W,S) is signed and the edges in $\textsc{Cayley}(W,S)$ are oriented away from the identity—with the labels suitably modified—the result is the Hasse diagram of the factor order. As a minor detail, when a generator in S is an involution, the edges in the Cayley graph labeled by this generator come in pairs, one of which can safely be removed.

Let (W,S) be signed and let T be the closure of S under conjugation in W. Because the elements of S are odd, their conjugates are also odd. Thus (W,T) is another example of a signed group. As an immediate corollary of Theorem 2.3 we have the following.

Corollary 2.9 (Constucting Garside-like posets)**.** *If W is a signed group generated by S and T is the closure of S under conjugation in W, then every factor poset in $\textsc{Cayley}(W,T)$ is Garside-like and locally Garside-like.*

The main advantage of using signed groups, and particularly signed groups which are defined using isometries of some object X is that it is often quite easy to determine which isometries are conjugate to each other and the geometry of X can be used to establish the lattice property. For example, here is one situation where the factor posets have been shown to be lattices.

Theorem 2.10 (Brady-Watt [10])**.** *Let W be the group of all isometries of $\mathbb{R}^n$ and let S be the set of all Euclidean reflections. If α is an isometry of $\mathbb{R}^n$ which fixes at least one point, then the factor poset $\textsc{Factor}(\alpha,S)$ is isomorphic as a poset to the poset of affine subspaces of $\mathbb{R}^n$ that contain $\textsc{Fix}(\alpha)$ ordered by reverse inclusion. Moreover, the isomorphism between these posets is the function that sends each isometry β below α in the factor order to the affine subspace $\textsc{Fix}(\beta)$ that it fixes. As a consequence, these factor posets are lattices.*

Their proof uses only fairly elementary geometric arguments but in a rather elegant way. As a corollary we know that these factor posets are Garside structures.

Corollary 2.11. *If W is the group of all isometries of $\mathbb{R}^n$ which fix the origin and S is the set of all Euclidean reflections that fix the origin, then every factor poset in W is a (typically infinite) combinatorial Garside structure.*

Proof. The set S is closed under conjugation and a well-known generating set for W. By Theorem 2.3 $\textsc{Factor}(w,S)$ is Garside-like and by Theorem 2.10 it is a lattice. $\square$

Notice that these combinatorial Garside structures are far from finite. The size of the generating set S is that of the continuum, so even though there is a monoid, a group and a finite-dimensional Eilenberg-MacLane space naturally associated with this poset, I doubt that they are objects that have been studied previously. At this point, it might be natural to conjecture (and at one point, I thought I could prove) the following.

Conjecture 2.12. *If S is an arbitrary set of Euclidean reflections that fix the origin, W is the group of Euclidean isometries generated by S, and T is the closure of S under conjugation in W, then for each $w \in W$, the factor poset* $\textsc{Factor}(w,T)$ *is a lattice and thus a combinatorial Garside structure.*

Unfortunately this conjecture is provably false because there exist explicit counterexamples, even in the case where T is finite. A more careful statement, which has the added benefit of being true, is the following.

Theorem 2.13 (Brady-Watt [8])**.** *If W is a finite Coxeter group, T is its full set of reflections and c is any Coxeter element of W then the factor poset* $\textsc{Factor}(c,T)$ *is a lattice.*

Their proof is uniform (i.e. not case-by-case) and uses the geometry of real reflection groups in an essential way. The explicit counterexample alluded to above can be found in the finite Coxeter group of type D_4. There is a non-Coxeter element w such that the poset $\textsc{Factor}(w,T)$ is finite poset that is not a lattice. The question of when factor posets are lattices is thus more subtle than one would hope.

3. Coxeter groups lead to Garside structures for Artin groups

The finite groups which both (1) act on Euclidean n-space faithfully by isometries and (2) are generated by elements which act by Euclidean reflections are called the *finite reflection groups*. The finite reflection groups have, of course, been completely classified and one special property that they all share is that they admit presentations of a similar form. More specifically, if W is a finite reflection group then there is a special subset S of elements in W which act by Euclidean reflections, and W can be presented by adding relations which merely record the order of the product of any two elements in S. The surprising aspect here is that these relations are sufficient to present W. Arbitrary groups presented in this way are called *Coxeter groups* and the list of Coxeter groups which turn out to be finite is exactly the list of finite reflection groups. Although this section focuses primarily on the *finite* Coxeter groups, similar ideas are applied to arbitrary Coxeter groups in the final section.

Definition 3.1 (Coxeter groups)**.** Let $m : S\times S \to \mathbb{N}\cup\{\infty\}$ be a symmetric function with $m(s,t) \geq 1$ and equal to 1 if and only if $s = t$. The *Coxeter presentation* based on m is the presentation $\langle\, S \mid (st)^{m(s,t)} = 1,\ \forall s,t \in S\,\rangle$. When $m(s,t) = \infty$, this is interpreted to mean that no relation involving s and t should be added to the presentation. A *Coxeter group* is a group W which admits a Coxeter presentation.

Remark 3.2 (Coxeter diagrams)**.** The information contained in m can certainly be re-encoded in other ways. One common version is to use a finite graph whose vertices are labeled by the elements of S. Start with a complete graph (i.e. one in

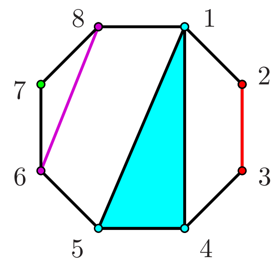

FIGURE 7. A noncrossing partition of the set [8]

which edges connect all pairs of distinct vertices) and label the edge connecting s and t with $m(s,t)$. Such a representation is crowded and can be greatly simplified with additional conventions. In finite Coxeter groups, no edge is ever labeled ∞, "most" of the edges are labeled 2 and "most" of the other edges are labeled 3. As a result, people who primarily study finite Coxeter groups usually remove the edges labeled 2 and remove the labels equal to 3. An alternative convention which is used by many geometric group theorists is to remove the edges labeled ∞ and to keep all of the labels on the remaining edges. Both conventions have their advantages: disconnected components in the first notation indicate groups which split as direct products, while disconnected components in the second notation indicate groups which split as free products.

Because of their identification with the finite reflection groups, every finite Coxeter group has a faithful linear representation in which its standard generating set acts by Euclidean reflections. The following result is a restatement of Theorem 2.13.

Theorem 3.3 (Garside structures from finite Coxeter groups)**.** *Let W be a finite Coxeter group, let S be its standard generating set, and let T be the closure of S under conjugacy in W. For any Coxeter element $w \in W$, the factor poset of w factored over T is a combinatorial Garside structure.*

The first example of this phenomenon predates the general construction by some years and even predates the general definition of a Garside structure.

Example 3.4 (Garside structures in symmetric groups)**.** The symmetric group on n letters, viewed as the isometry group of the $(n-1)$-simplex, is clearly an example of a finite reflection group. Call this group W. Its standard generating set S is the set of all adjacent transpositions, i.e. the transpositions (ij) with $j = i+1$, and the closure of S under conjugation in W is the set T consisting of all transpositions. By Theorem 3.3, the factor poset of w factored over T is a Garside structure for every $w \in W$. If we set w equal to the n-cycle $(123\cdots n)$, then its poset of factors is naturally isomorphic with the lattice that combinatorialists call the noncrossing partition lattice. The isomorphism sends each permutation to the partition determined by its disjoint cycle decomposition.

Definition 3.5 (Classical noncrossing partitions)**.** Following traditional combinatorial practice, let $[n]$ denote the set $\{1,\ldots,n\}$. Recall that a *partition* of a set is a collection of pairwise disjoint subsets whose union is the entire set and that the subsets in the collection are called *blocks*. A *noncrossing partition* is a partition σ

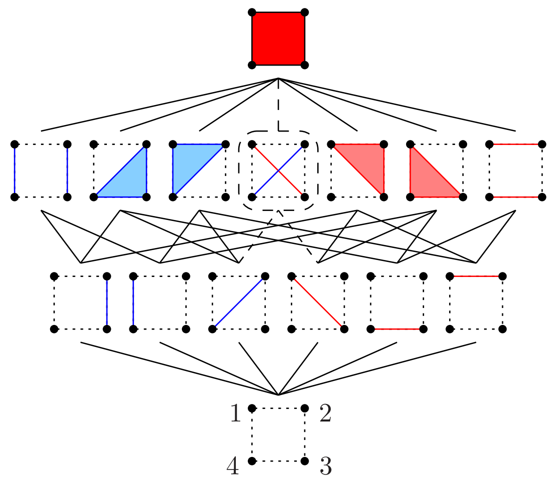

FIGURE 8. The figure shows the partition lattice for $n = 4$. If the vertex surrounded by a dashed line is removed, the result is the noncrossing partition lattice for $n = 4$.

of the vertices of a regular n-gon (labeled by the set $[n]$) so that the convex hulls of its blocks are pairwise disjoint. Figure 7 illustrates the noncrossing partition $\{\{1,4,5\},\{2,3\},\{6,8\},\{7\}\}$. The partition $\{\{1,4,6\},\{2,3\},\{5,8\},\{7\}\}$ would be crossing.

Given partitions σ and τ we say $\sigma \leq \tau$ if each block of σ is contained in a block of τ. This ordering on the set of all partitions defines a partially ordered set called the *partition lattice* and is usually denoted Π_n. When restricted to the set of noncrossing partitions on $[n]$, it called the *noncrossing partition lattice* and denoted NC_n. The poset Π_4 is shown in Figure 8. For $n = 4$, the only difference between the two posets is the partition $\{\{1,3\},\{2,4\}\}$ which is not noncrossing.

We should note that the group G derived from this Garside structure is the n-string braid group, but that the presentation used is not the standard one given by Emil Artin in 1925. This new presentation, which uses all of the transpositions as its generating set, is the one introduced by Birman, Ko and Lee in [3].

In order to talk in more detail about arbitary finite Coxeter groups, we need to introduce more of the standard notations.

Remark 3.6 (The standard notations)**.** In a finite Coxeter group W, it is traditional to use S to denote the standard generating set and T for the closure of S under conjugation in W. The size of S, usually denoted n, is equal to the dimension of the standard faithful linear representation of W. The size of T, usually denoted N, counts the number of elements of W which act on $\mathbb{R}^n$ by Euclidean reflections. There are certain elements in W that play a special role. In CAYLEY(W,S) there is always a unique element w_0 which is a maximal distance from the identity. It is called, naturally enough, the *longest element* in W. In CAYLEY(W,T) there are many elements that are a maximal distance from the identity. Some of these elements can be found by taking the product of the elements in S in any order. These particular elements are called the *Coxeter elements* of W. They do not, in general, exhaust the elements at a maximal distance from the identity, but they are all conjugate to one another. Thus, the Coxeter element of W, usually denoted c, is well-defined at least up to conjugacy in W. Finally, the order of c—which does not depend on the representative chosen—is called the *Coxeter number* and is denoted by h.

Using these notations there are certain formulae that hold in all finite Coxeter groups such as $nh = 2N$. In the symmetric group on n-letters, a Coxeter element is an n-cycle. Thus, it makes sense to define a general noncrossing partition lattice as follows.

Definition 3.7 (Generalized noncrossing partitions)**.** Let W be an arbitary Coxeter group generated by S. Using the standard notations described above, $\textsc{Factor}(c, T)$ is a Garside structure called the *generalized noncrossing partition lattice*. It is usually denoted NC^W. When a different Coxeter element for W is chosen, the result is another labeled poset that has the same underlying poset structure and the same partition of the covering relations according to their labels, but the particular label used to mark a particular equivalence class of the covering relations might be different.

Example 3.8 (Commutative Coxeter groups)**.** The only commutative Coxeter groups are the ones where $m(s,t) = 2$ for all distinct $s, t \in S$. If $|S| = n$, then W is the group $(\mathbb{Z}/2\mathbb{Z})^n$ and the noncrossing partition lattice NC^W is just the Boolean lattice with its natural labeling. Thus, the Garside structures described in Example 1.8 and Example 2.2 are examples of generalized noncrossing partition lattices.

The poset shown in Figure 1 is an additional example. This poset is the noncrossing partition lattice associated with the finite Coxeter groups of type D_4. In particular, it turn out to be a lattice.

Remark 3.9 (Proving the lattice property)**.** In the development outlined above, we have relied on Theorem 2.13 to give a uniform proof that the posets NC^W are lattices. Historically, the first proof of this result used the classification of finite Coxeter groups and involved a brute force computer check for the exceptional groups, i.e. those groups that do not belong to one of the infinite families of finite Coxeter groups. The three infinite families were shown to be lattices by Tom Brady [7] and Tom Brady and Colum Watt [9]. The nontrivial exceptional cases were computer tested by David Bessis [2]. The new proof by Brady and Watt eliminates the case-by-case analysis and also points towards an interesting connection with the generalized associahedra defined by Fomin and Zelevinsky. See [8] for details.

Let W be a finite Coxeter group and let $P = NC^W$ be the corresponding combinatorial Garside structure. From the general theory described in §1 it is clear that associated to P is a monoid M, a group G and complex K such that all of the consequences listed in Theorem 1.7 hold. The group G, however, cannot be the same as the group W since G is torsion-free and thus infinite. In the case where W is the symmetric group on n letters, we have already mentioned that G is the braid group on n strings. For an arbitrary finite Coxeter group W, the group G is a generalized braid group more commonly known as an Artin group of finite-type. The basic idea is that Artin groups generalize braid groups in the same way that Coxeter groups generalize symmetric groups. Artin groups were first defined and studied using hyperplane arrangements.

Remark 3.10 (Coxeter groups and arrangements)**.** Let W be a finite Coxeter group and let T be its set of reflections. If the hyperplane H_t fixed by the Euclidean reflection $t \in T$ is removed from $\mathbb{R}^n$ for each $t \in T$, then W acts freely on the resulting disconnected space. Moreover, the action of W is transitive on the set of connected components and the components are in bijection with the elements of W. When the situation is complexified, the topology is more interesting. Let W act by isometries on $\mathbb{C}^n$ by extension of scalars and remove the complex hyperplanes from $\mathbb{C}^n$ which satisfy the same linear equations as the hyperplanes H_t in $\mathbb{R}^n$. This time the complement remains connected and it has a nontrivial fundamental group. Moreover, because the group W acts freely and properly discontinuously on the resulting complexified hyperplane complement, it is a covering of the space obtained by quotienting out by the action of W. The fundamental group of this quotient is called the *Artin group* A associated with W. There is a natural group homomorphism from A to W and the kernel of this map is called the *pure Artin group*. The pure Artin group is the fundamental group of the original complexified hyperplane complement.

Example 3.11 (The braid arrangement)**.** If W is a symmetric group acting on $\mathbb{R}^n$ by permuting the coordinates, then the corresponding hyperplane arrangement (consisting of the hyperplanes $x_i = x_j$ for all $i \neq j$) is called the *braid arrangement*. As expected, the fundamental group of the complement of the complexified version of this hyperplane arrangement is the *pure braid group* which is a finite index subgroup in the braid group itself and it is the kernel of the natural map from the braid group to the symmetric group.

The fundamental groups of the complements of these complex hyperplane arrangements were investigated by Jacques Tits [26], Egbert Brieskorn [12], Pierre Deligne [18], and Brieskorn and Kyoji Saito [13]. What they found was that like the finite Coxeter groups used to define them, all of the finite-type Artin groups can be defined using similar presentations. This leads to the following general definition of an Artin group.

Definition 3.12 (Artin groups)**.** Let W be a Coxeter group defined by a function $m : S \times S \to \mathbb{N} \cup \{\infty\}$. The Artin group A corresponding to W is the group defined by the following presentation. The generating set is S and a relation is added to the presentation for each $m(s,t)$ that is finite (with $s \neq t$). The relation added equates the two words of length $m(s,t)$ that alternate between s and t. Thus if $m(s,t) = 2$ the relation added is $st = ts$, if $m(s,t) = 3$ the relation added is $sts = tst$ and if $m(s,t) = 4$ the relation added is $stst = tsts$.

As an example, if W is the Coxeter group defined by the presentation

$$\left\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^4 = 1 \right\rangle$$

then the corresponding Artin group is the group defined by the presentation

$$\langle a, b, c \mid aba = bab,\ ac = ca,\ bcbc = cbcb \rangle$$

Notice that the presentation introduced above to define an Artin group is generated by S while the presentation of the group derived from the poset $P = NC^W$ is generated by T. Consider the simple example of the 3-string braid group as the finite-type Artin group associated with the symmetric group on 3 letters. The presentation defined above is $\langle a, b \mid aba = bab \rangle$ while the presentation from the corresponding noncrossing partition lattice is $\langle a, b, c \mid ab = bc = ca \rangle$. It is not too hard to prove in this case that both presentations define the same group, but something deeper is going on. The presentation used to define an Artin group is a presentation derived from a *second* Garside structure on W. The easiest way to define these additional Garside structures in general is through the use of W-permutahedra.

Definition 3.13 (W-permutahedra)**.** Let W be a finite Coxeter group, let S be its standard generating set and let T be the closure of S under conjugacy in W. In its standard action on Euclidean space by isometries, the elements in T are precisely those which act by Euclidean reflections and W acts freely on the complement of the hyperplanes fixed by the elements of T. In particular, if we pick a point x which does not lie on any of the hyperplanes, then the orbit of x under the action of W results in $|W|$ distinct points. The convex hull of these $|W|$ points is called a *W-permutahedron*. It is important to note that the combinatorial type of the polytope Q which results is independent of the point x chosen, so that Q is reasonably well-defined. There is one specific choice of x which has particularly nice properties. Let C be some connected component of $\mathbb{R}^n$ once all the fixed hyperplanes have been removed and consider the minimal list of linear inequalities needed to define C. This connected component is called a *chamber* of W and the hyperplanes associated with its defining linear inequalities are called its *walls*. There is a unique point in C where the sphere of radius $1/2$ around x is tangent of the each of the walls that bound C. Think about dropping a ball of radius $1/2$ into the hyperplane arrangement. When this point x is reflected around by W, then convex hull of the result is a polytope where every edge in its 1-skeleton has length 1. Call this the *unit length W-permutahedron*.

The W-permutahedra defined above have two special properties.

Remark 3.14 (W-permutahedra as Cayley graphs)**.** Let W be a finite Coxeter group and let Q be its W-permutahedron. The 1-skeleton of Q can be labeled by the elements of S so that the resulting graph is the Cayley graph of W with respect to S (at least so long as we adopt the convention that generators that are involutions add unoriented edges to the Cayley graph rather than pairs of oriented edges).

The second main property is that W-permutahedra are zonotopes.

Remark 3.15 (W-permutahedra as zonotopes)**.** For the readers familar with both roots systems in Coxeter groups and the theory of polytopes, the W-permutahedron can be viewed as the Minkowski sum of the vectors in the root system of W. From this point of view it is clear that these polytopes are *zonotopes*, since a zonotope can be defined as a Minkowski sum of line segments. One consequence of this fact is that every face of the polytope is centrally symmetric and, in particular, every 2-cell has an even number of sides.

Using these two aspects, the 1-skeleton of a W-permutahedron can be turned into a combinatorial Garside structure.

Definition 3.16 (The standard Garside structure)**.** Let W be a finite Coxeter group and let Q be the W-permutahedron. If a linear functional is chosen on $\mathbb{R}^n$ so that no edge of the W-permutahedron is horizontal (i.e. a height function), then

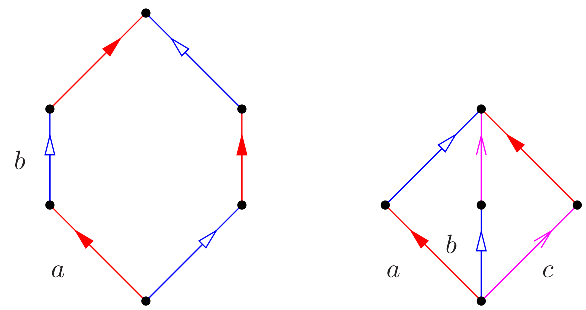

FIGURE 9. Two Garside structures derived from the symmetric group on 3 letters.

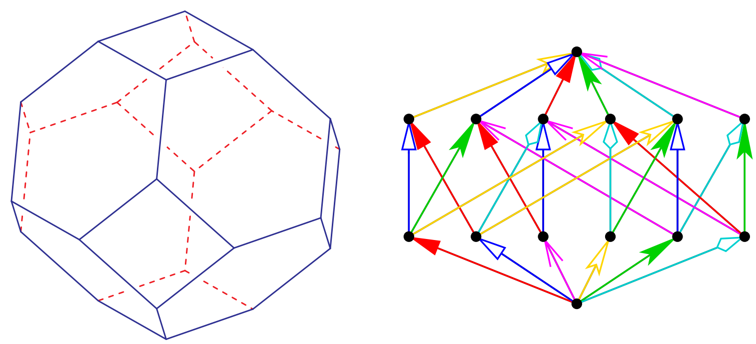

FIGURE 10. The two Garside structures derived from the symmetric group on 4 letters. The figure on the left (courtesy of Frank Sottile) has been drawn so that its polytopal nature is visible, even though this required suppressing the edge-labeling and obscuring the grading.

this induces an orientation on the edges in the 1-skeleton of Q. Let P be the poset corresponding to this oriented graph and give it the edge-labeling which turns this graph into the Cayley graph of W with respect to S. The resulting partial order on the elements of W is called the *weak order* (or *weak Bruhat order*). It turns that for every finite Coxeter group W, this labeled poset P is a combinatorial Garside structure. In fact, this was the combinatorial Garside structure originally investigated by F.A. Garside in [20]. Without going into the details, the fact that these posets are bounded, graded and have finite height follows quickly from the geometry of zonotopes; that they are group-like and balanced follows from the fact that each poset is the complete Cayley graph for a group generated by involutions, and finally, the fact that they are lattices is a consequence of one of the key properties of Coxeter groups: the Deletion-Contraction condition. The combinatorial Garside structure thus defined is called the *standard Garside structure* associated with W.

The two Garside structures derived from the symmetric group on 3 letters are shown in Figure 9 and the two Garside structures derived from the symmetric group on 4 letters are shown in Figure 10. The fact that these two Garside structures lead to isomorphic groups has been known for some time, although the first proof used the classification theorem for finite Coxeter groups and a case-by-case analysis. There is now a uniform proof of this fact that extends to arbitrary Coxeter groups. See Remark 4.6 in the next section for details.

Geometric group theorists tend to call the combinatorial Garside structure coming from the 1-skeleton of the W-permuhedron the *standard Garside structure* from W because it was introduced first and the noncrossing partition lattice NC^W is called the *dual Garside structure* from W. The reason for the latter terminology is explained by the following observations.

Remark 3.17 (Why "dual"?)**.** The two combinatorial Garside structures derived from a finite Coxeter group W satisfy a strange sort of "duality" that was first observed by David Bessis [2]. It should be noted, however, that the duality is merely observational in the sense that no theory currently exists to explain why the observed duality occurs. Table 1 summarizes the relevant observations using the standard notations for Coxeter groups. As the observations in the table are described, the two posets derived from the symmetric group on 4 letters (Figure 10) will be used to illustrate these points.

	Weak Bruhat order	Noncrossing partitions
Other names	Classical Garside Str.	Dual Garside Str.
Set of atoms	**S**	**T**
Height	N	n
$\lambda(\hat{0},\hat{1})$	w_0	c
Order of $\lambda(\hat{0},\hat{1})$	2	h
Number of atoms	n	N
Product of atoms	c	w_0
Regular degree	h	2

TABLE 1. The numerology which justifies the use of the word "dual".

In a poset containing a minimal element $\hat{0}$ the elements that cover $\hat{0}$ are called its *atoms*. The first observation is that the height of one Garside structure is equal to the number of atoms in the other. In Figure 10 the poset on the left has height 6 and 3 atoms while the poset on the left has height 3 and 6 atoms. Since the intervals from $\hat{0}$ to the atoms are bijectively labeled with the elements of S or T, respectively, the numbers which switch are n and N. The second duality involves the special elements c and w_0. One of these elements is the label on the interval $(\hat{0},\hat{1})$, i.e. $\lambda(\hat{0},\hat{1})$, while the other is the product of the atoms in some particular order. On the lefthand side of Figure 10, the product of the three atoms is a Coxeter element c and the label on the interval $(\hat{0},\hat{1})$ is the longest element w_0. On the righthand side of Figure 10 the label on the interval $(\hat{0},\hat{1})$ is a Coxeter element, and the product of its 6 atoms (in an appropriate order) is w_0. In the third and final duality, the order of the label on the interval $(\hat{0},\hat{1})$ and the associated regular degree switch roles from one poset to the other. Rather than define the concept of regular degree here, the interested reader is directed to the original article [2] and the sources it cites.

4. Garside-like structures for arbitrary Artin groups

This final section contains a few observations highlighting those parts of the previous discussion that extend readily from finite Coxeter groups to arbitrary Coxeter groups and those that cannot be easily extended. In an infinite Coxeter group W generated by S, for example, it still makes sense to define a Coxeter element c as the product of the elements in S in some order. It does not makes sense to define an element w_0 since the Cayley graph of W with respect to S has infinite diameter and a "longest element" no longer exists. Other differences are less obvious. It turns out that the proof that all Coxeter elements belong to the same conjugacy class relies heavily on the finiteness of W. In fact, in many infinite Coxeter groups there are several qualitatively distinct Coxeter elements. The first step is to review those aspects of the general theory that do generalize and that help to make arbitary Coxeter groups relatively easy to work with. The polytopal W-permutahedron is replaced by a polytopal complex (called the Davis complex) and the action by reflections on a sphere around the origin is replaced by an action by reflections on some other highly symmetric space (via the Tits representation).

Definition 4.1 (The Davis complex of a Coxeter group)**.** Let W be a Coxeter group with standard generating set S and consider the Cayley graph of W with respect to S following the convention that involutive generators add unoriented edges to the Cayley graph. Add a metric to this graph so that each edge has length one. If some subset S_0 of S generates a finite Coxeter group W_0, then inside $\textsc{Cayley}(W,S)$ will be a copy of the 1-skeleton of the (unit) W_0-permutahedron. Moreover, every vertex v in $\textsc{Cayley}(W,S)$ belongs to some coset of W_0 and thus there is a copy of the 1-skeleton of the W_0-permutahedron that includes v as one of its vertices. In each case attach the unit length W_0-permutahedron along its 1-skeleton. If this is done for every subset of S which generates a finite subgroup (and the obvious identifications are made when one such finite subgroup is contained in another), then the resulting complex D_W, called the *Davis complex* (or more technically, the Davis complex with the Moussong metric), is a contractible piecewise Euclidean space that is CAT(0) in the sense of [11]. The CAT(0) property is a general form of non-positive curvature. In addition, W acts properly cocompactly by isometries on D_W so that the group structure of W is intimately connected with the geometry of D_W. As an example, a portion of the Davis complex for the Coxeter group W defined by the presentation $\langle a,b,c \mid a^2=b^2=c^2=(ab)^2=(ac)^3=1\rangle$ is shown in Figure 11.

Definition 4.2 (The Tits' representation of a Coxeter group)**.** Without going into the details, the procedure that produces a faithful linear representation of a finite Coxeter group, readily extends to an arbitrary Coxeter group. The resulting faithful representation is called the *Tits' representation* of W. The action of W in this representation preserves a symmetric bilinear form that can be used to classify Coxeter groups into four basic types according to the signature of this form. For example, the form is positive definite if and only if the Coxeter group W is finite which is true if and only if W acts faithfully on $\mathbb{S}^n$ by reflections. The other three types of forms correspond roughly to the Coxeter groups that act faithfully by reflections on $\mathbb{R}^n$, $\mathbb{H}^n$ or a space related to the higher rank Lie groups $SO(p,q)$. These Coxeter groups are called the *affine*, *hyperbolic* and *higher rank* Coxeter groups, respectively. See [21] or [24] for further details.

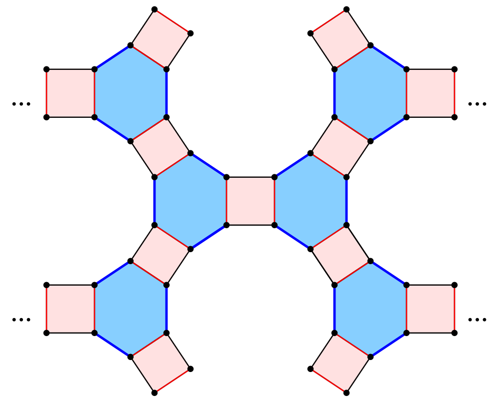

FIGURE 11. A portion of the Davis complex for an infinite Coxeter group.

Arbitrary Coxeter groups have many wonderful properties, most of which follow quickly from the basic properties of the Davis complex, the Tits representation, or both. For example, every Coxeter group has a decidable word problem (because groups that act properly discontinuously and cocompactly on piecewise Euclidean CAT(0) spaces have decidable word problems) and every Coxeter group has a torsion-free subgroup of finite index (because every matrix group has this property by Selberg's lemma). The general class of Artin groups, i.e. those groups defined by a finite presentation of the type given in Definition 3.12, are much more difficult to work with.

Remark 4.3 (Artin groups are natural yet mysterious)**.** Artin groups are "natural" in the sense that they are closely tied to the complexified version of the hyperplane arrangements for Coxeter groups. But they are "mysterious" in the sense that many basic questions about them remain open. For example, it is still not known whether an arbitary Artin group has a decidable word problem. This is known to be true in certain special situations, such as for the Artin groups of finite type, but a proof that works for a generic Artin group remains out of reach. Once it is understood that the word problem is open, the fact that it is unclear whether all Artin groups are torsion-free (or even contain a torsion-free subgroup of finite index) becomes easier to understand. And, of course, it is not known whether they have finite dimensional Eilenberg-MacLane spaces or faithful linear representations since either of these would imply properties already listed as unknown. Even the Artin groups corresponding to the affine Coxeter groups have been somewhat mysterious until very recently.

Notice that if A is an Artin group that can be shown to be the derived from a combinatorial Garside structure, then A has a decidable word problem, has a finite-dimensional Eilenberg-MacLane space and thus is torsion-free. The obvious candidate for such a Garside structure is the factor poset of a Coxeter element. By Theorem 2.3, the following is immediate.

Theorem 4.4 (Garside-like structures in Coxeter groups)**.** *Let W be an arbitary Coxeter group and let c be one of its Coxeter elements. The factor poset* $\textsc{Factor}(c,T)$ *where T is the set of all "reflections" in W is both Garside-like and locally Garside-like.*

As in the finite case, these factor posets are called *generalized noncrossing partitions* but the poset structure now implicitly depends on the choice of Coxeter element c used to define it. The notation NC_c^W is used to indicate this dependence. Thus each infinite Coxeter group produces not just one, but rather a finite list of noncrossing paritition posets, each of which is Garside-like and locally Garside-like. Although Artin groups, by definition, are finitely presented groups with presentations of a very specific form, the groups derived from any of these NC_c^W posets are typically infinitely generated and infinitely presented. One question that immediately leaps to mind is whether all of these presentations derived from W lead to isomorphic groups. This is in fact the case as my coauthors and I have recently been able to show.

Theorem 4.5 (Brady-Crisp-Kaul-McCammond [5])**.** *Let W be a Coxeter group, let A denote the corresponding Artin group and let c be any choice of a Coxeter element in W. If $P = NC_c^W$ denotes the resulting factor poset and G and K are the group and complex derived from P, then the Artin group A, the group G, and the fundamental group of the complex K are all naturally isomorphic groups and the Hasse diagram of P embeds into the Cayley graph of A with respect to the set T of all reflections.*

Remark 4.6 (Isomorphic groups)**.** Since Theorem 4.5 covers the case when W is finite, it shows that the group associated with the noncrossing partitions NC_c^W is isomorphic to the associated finite-type Artin group. Moreover, since it is straightforward to show that group associated with the standard Garside structure (which is only defined when W is finite) is isomorphic to the associated Artin group, we can conclude that the groups associated with the two Garside structures derived from a finite Coxeter group W are themselves isomorphic.

Part of what makes this theorem slightly surprising in the general case is that a solution to the word problem for an arbitrary Artin group has not yet been found. Our proof is somewhat indirect and uses certain locally noncrossing arcs traced around inside a punctured disc. See [5] for details. At this point there are well-defined Garside-like posets that define the right groups and they are defined using Coxeter groups so they can be worked with concretely, effectively and efficiently. All that is missing is a proof of that they satisfy the lattice condition.

Unfortunately, it is not always true that these posets are lattices. Francois Digne has shown in [19] that, in particular affine examples, for some choices of a Coxeter element, the noncrossing partition poset is a lattice and for other choices of a Coxeter element *in the same group* the noncrossing partition poset is not a lattice. The difficulty here lies not in the affine Coxeter group, but in the full group of isometries of Euclidean space.

Theorem 4.7 (Brady-Crisp-Kaul-McCammond [4])**.** *Let $W = \textsc{Isom}(\mathbb{R}^n)$ be viewed as a group generated by the set S of all Euclidean reflections and note that S is closed under conjugation. When n is at least 3 there exist isometries $\alpha \in W$ such that the poset $\textsc{Factor}(\alpha, S)$ is not a lattice. It can, however, be extended to a lattice in a minimal, canonical, and understandable way.*

These complicated posets with their continuous set of generators play a crucial role in the non-crossing partition posets derived from the affine Coxeter groups. As asserted in the theorem, although these posets are not always lattices, they are

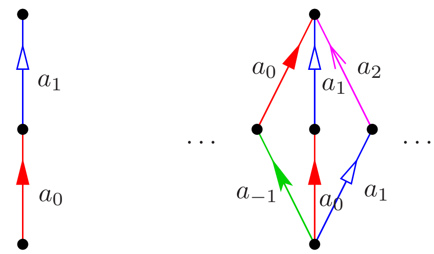

FIGURE 12. A free Garside structure.

always "close" to being lattices. The key aspect of Theorem 4.7 is that completion of $\text{Isom}(\mathbb{R}^n)$ to a lattice (technically known as its Dedekind-MacNeille completion) is well understood and tractable enough to be useful in practice. In particular, the completion is close enough to the original that we can recover many of the consequences of having a Garside structure even though no single poset satisfying all of the properties of a combinatorial Garside structure is ever found [4].

As a fitting conclusion to this brief tour of the Garside-like structures found in infinite Coxeter groups, consider the case of a free Coxeter group. A free Coxeter group is one where $m(s,t) = \infty$ for all $s \neq t$. The Artin group associated to this Coxeter group has no relations and thus is the free group F_n where n is the size of the standard generating set. The noncrossing partition poset in this case is, in fact, a lattice, thus showing that each finitely generated free group is a Garside group. A topological proof of this fact was discovered independently by David Bessis [1] and John Crisp [5] during the summer of 2003. In order to see the benefits of the topological approach, consider the free noncrossing partition posets purely from an algebraic perspective.

Example 4.8 (Garside structure for F_2)**.** The free Coxeter group on two generators is $W = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a_0, a_1 \mid a_0^2 = a_1^2 = 1\rangle$ and the factor poset of the Coxeter element $c = a_0 a_1$ is shown on the righthand side of Figure 12. The elements a_i for each integer i can be recursively defined in terms of a_0 and a_1. In the end, the group derived from NC_c^W is given by the presentation $G = \langle a_i | a_i a_{i+1} = a_j a_{j+1}\rangle$ where i and j are arbitrary integers. Despite appearances, this is in fact a rather unusual presentation for the free group F_2 and the associated complex K (which is locally infinite and 2-dimensional) is an Eilenberg-MacLane space for F_2. More specifically, the Artin group defined by this poset is the free group and the construction in this case leads to a universal cover which is an infinitely branching tree cross the reals with a vertex-transitive free F_2 action.

This particular example is easily seen to be a lattice, but the proof of the lattice condition for the noncrossing partiton poset in the free 3-generated Coxeter group is already far from obvious. Here is more topologically defined poset that turns out to be closely related.

Definition 4.9 (Poset of cut-curves)**.** Let $\mathbf{D}^*$ denote the closed unit disc with n punctures and 4 distinguished boundary points, N, S, E and W as shown in Figure 13. A *cut-curve* is an isotopy class (in $\mathbf{D}^*$) of a path from E to W (relative to its endpoints, of course). Notice that cut-curves divide $\mathbf{D}^*$ into two pieces, one containing S and the other containing N. Its *height* is the number of puncture in

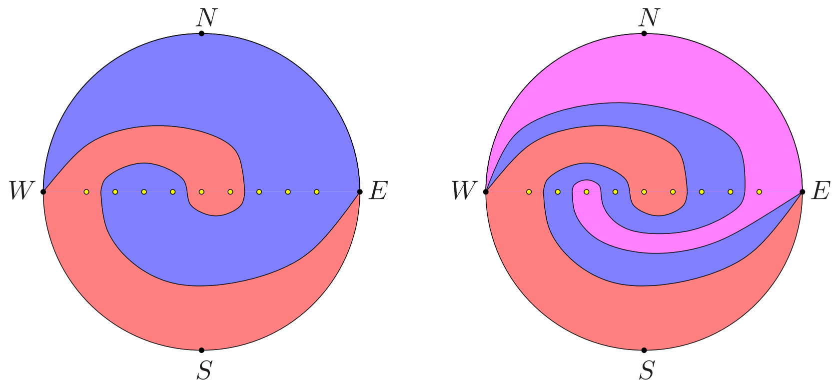

FIGURE 13. A cut-curve and a pair of comparable cut-curves.

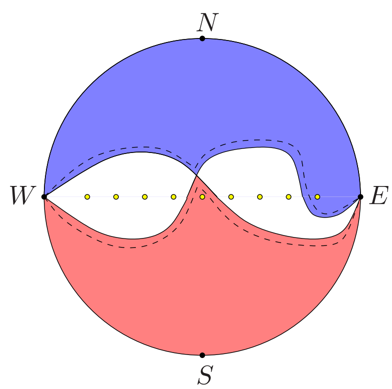

FIGURE 14. Illustration of the proof that the cut-curve poset is a lattice.

the "lower" piece, i.e. the piece containing the point S. Let $[c]$ and $[c']$ be cut-curves. We write $[c] \leq [c']$ if there are representatives c and c' that are disjoint (except at their endpoints) and c is "below" c'. A representative of a cut-curve is shown on the lefthand side of Figure 13 while a pair of comparable cut-curve representatives are shown on the right.

Notice that if a representative c is given, then one can tell whether $[c] \leq [c']$ by keeping c fixed and isotoping c' into a "minimal position" with respect to c (i.e. no football shaped regions without punctures). This topological observation leads to the following lemma.

Lemma 4.10. *The poset of cut-curves is a lattice.*

A sketch of the proof. Suppose $[c]$ is above $[c_1]$ and $[c_2]$. Place representatives c_1 and c_2 in minimal position with respect to each other (i.e. no puncture-free football regions) and then isotope c so that it is disjoint from both. By way of example, the new curve c will be above the dotted line shown in Figure 14. Thus this dotted line represents a least upper bound for $[c_1]$ and $[c_2]$. $\square$

Finally, using the fact that the fundamental group of $\mathbf{D}^*$ is F_n, the cut-curve lattice can be identified with the factor poset of a Coxeter element inside F_n. See

[5] or [1] for details. Using a modification of these noncrossing curve diagrams, my coauthors and I can extend this argument to all Coxeter groups with no "short" relations. In particular, we can prove the following.

Theorem 4.11 (Brady-Crisp-Kaul-McCammond [5])**.** *If W is a Coxeter group where every relation is long (in the sense that $m(s,t) \geq 6$ for all $s \neq t$) then for each Coxeter element c, the factor poset NC_c^W is a lattice and thus a combinatorial Garside structure.*

At this point, the obvious next steps are to investigate the factor posets in the groups $\text{Isom}(\mathbb{H}^n)$ and $\text{Isom}(SO(p,q))$ factored over the set of all reflections in each case. These posets will most likely play a crucial role in any attempt to understand the noncrossing partition posets associated with the hyperbolic and higher rank Coxeter groups just as the noncrossing partition posets for finite and affine Coxeter groups used the geometry of the factor posets inside $\text{Isom}(\mathbb{S}^n)$ and $\text{Isom}(\mathbb{R}^n)$. Once these posets (and their lattice completions) are well understood, the task of bringing the class of Artin groups into the realm of well understood groups can begin in earnest.

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