THE EXCEPTIONAL SYMMETRY

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ABSTRACT. This note gives an elementary proof that the symmetric groups possess only one exceptional symmetry. I am referring to the fact that the outer automorphism group of the symmetric group SYM_n is trivial unless $n = 6$ and the outer automorphism group of SYM_6 has a unique nontrivial element.

When we study symmetric groups, we often invoke their natural faithful representation as permutations of a set without a second thought, but to what extent is this representation intrinsic to the structure of the group and to what extend is it one of several possible choices available? Concretely, suppose I am studying the permutations SYM_X of a set $X = \{1, 2, 3, 4, 5, 6\}$ and you are studying the permutations SYM_A of a set $A = \{a, b, c, d, e, f\}$ and suppose further that we know an explicit isomophism ϕ between my group SYM_X and your group SYM_A. Does this means that there is a way to identify my set X with your set A which gives rise to the isomorphism ϕ? In other words, must my transpositions correspond to your transpositions? Must my 3-cycles correspond to your 3-cycles? Or might it be possible that the transposition $(1, 2)$ in my group is sent by the isomorphism ϕ to the element $(a, b)(c, d)(e, f)$ in your group? The goal of this note is to give an elementary proof of the fact that yes there is an isomorphism ϕ between these two specific groups sending $(1, 2)$ to $(a, b)(c, d)(e, f)$, but that this is essentially the only unexpected isomorphism among all of the symmetric groups. In the language of outer automorphism groups (which we recall below) we give a proof of the following well-known and remarkable fact.

Theorem. $\text{OUT}(\text{SYM}_n)$ *is trivial for* $n \neq 6$ *and* $\mathbb{Z}/2\mathbb{Z}$ *when* $n = 6$.

Recall that the set of all isomorphisms from a group G to itself form a group $\text{AUT}(G)$ under composition called its *automorphism group*. Moreover, in any group we can create an automorphism by conjugating by a fixed element of G. Such automorphisms are called *inner automorphisms* and they form a subgroup $\text{INN}(G)$ which is normal in $\text{AUT}(G)$. These are what one might call the "expected" automorphisms. Note that in the case of the symmetric groups, conjugating by a permutation corresponds to relabeling the elements of the set on which it acts. The quotient group $\text{OUT}(G) := \text{AUT}(G)/\text{INN}(G)$ is the group of *outer automorphisms*. When

the outer automorphism group is trivial it means that there are no unexpected automorphisms. When it is non-trivial, each non-trivial element represents an equivalence class of unexpected automorphisms which differ from each other by composition with an inner automorphism. It is in this sense that the unique non-trivial element in $\mathrm{OUT}(\mathrm{SYM}_6)$ represents the only unexpected symmetry that the symmetric groups possess. Our proof naturally splits into two parts: restrictions and a construction. Following the proof we make a few remarks about the structure of these exceptional automorphisms and we conclude with pointers to the literature that the interested reader can pursue.

1. RESTRICTIONS

The restrictions follow from two easy lemmas about involutions in symmetric groups. Recall that the conjugacy classes of elements in the symmetric group are determined by their *cycle type* and that the order of a permutation is the least common multiple of the lengths of the disjoint cycles used to represent it. In particular, if we let $\mathcal{C}_j$ denote the elements of SYM_n with cycle structure $1^i 2^j$ (with, of course, $i + 2j = n$), then these are precisely the conjugacy classes of order 2 elements in SYM_n. The set $\mathcal{C}_1$ is the conjugacy class of transpositions. Because automorphisms must preserve order and conjugacy, they end up permuting the conjugacy classes of each fixed order. Thus the image of $\mathcal{C}_1$ under an automorphism of SYM_n must be one of the classes $\mathcal{C}_j$. Our first lemma is already an enormous restriction.

Lemma 1. *Any automorphism that sends $\mathcal{C}_1$ to $\mathcal{C}_1$ is inner.*

Proof. When x and y are noncommuting elements in $\mathcal{C}_1$ and $z = xyx = yxy$ we call $\{x, y, z\}$ a *dependent* set of transpositions. Consider the maximal independent noncommuting subsets of $\mathcal{C}_1$. In other words, consider the maximal subsets $S \subset \mathcal{C}_1$ such that for all distinct elements $x, y \in S$: (1) x and y do not commute, and (2) xyx is not in S. The key observation is that the set $S_i = \{(i, j) | j \neq i\}$ has these properties for each i and there are no others. To see this note that noncommuting transpositions must share exactly one number, say $x = (i, j)$ and $y = (i, k)$, and that the only transpositions that do not commute with either x or y are those of the form (i, l) with $l \neq j, k$ or the exceptional case (j, k)—which is ruled out since $(j, k) = xyx$. Since the subsets S_i are the only subsets satisfying these algebraic conditions, an automorphism ϕ sending $\mathcal{C}_1$ to $\mathcal{C}_1$ must permute the subsets S_i among themselves, say $\phi(S_i) = S_{\pi(i)}$. Conjugating ϕ by the permutation π produces a conjugate automorphism ψ that fixes each S_i setwise. In fact, ψ must fix each S_i pointwise since (i, j) is the unique element in the intersection $S_i \cap S_j$. Finally, since it fixes a generating set, ψ is the identity and ϕ is inner. $\square$

One consequence of Lemma 1 is that any two automorphisms ϕ and ψ that send $\mathcal{C}_1$ to $\mathcal{C}_j$ differ by an inner automorphism since $\phi^{-1} \circ \psi$ sends $\mathcal{C}_1$ to
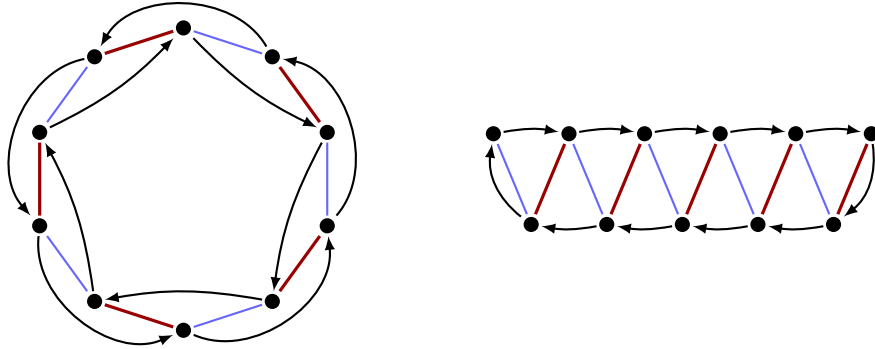
FIGURE 1. The example on the left shows that there are elements in $\mathcal{C}_j$ whose product is two j-cycles. The example on the right shows that when $n > 2j$, there are elements in $\mathcal{C}_j$ whose product is a single $(2j+1)$-cycle. Both examples use $j = 5$ with 5 thick dark edges representing one element of $\mathcal{C}_5$ and 5 thin light edges representing the other element.

$\mathcal{C}_1$. The converse also holds: if ϕ and ψ differ by an inner automorphism then both send $\mathcal{C}_1$ to the same conjugacy class $\mathcal{C}_j$ since conjugation preserves cycle type. This means that the size of $\mathrm{Out}(\mathrm{Sym}_n)$ is completely determined by the list of places that $\mathcal{C}_1$ can be sent. The next lemma shows that this list is very short.

Lemma 2. *If an automorphism sends $\mathcal{C}_1$ to $\mathcal{C}_j$ with $j > 1$ then $n = 2j = 6$.*

Proof. The key observation is that for all $x, y \in \mathcal{C}_1$ the order of xy is either 1, 2, or 3 so that an automorphism sending $\mathcal{C}_1$ to $\mathcal{C}_j$ is only possible when $\mathcal{C}_j$ also has this property. It is easy to find $x, y \in \mathcal{C}_j$ whose product has order j (so j is at most 3), and when $n > 2j$ it is also easy to find two elements $x, y \in \mathcal{C}_j$ whose product has order $2j + 1 > 3$ (thus n must equal $2j$). Examples of both types of products are shown in Figure 1. Finally, when $j = 2$ and $n = 4$, there are elements in $\mathcal{C}_1$ whose product has order 3, but the three elements in $\mathcal{C}_2$ pairwise commute. Therefore the only possibility is $j = 3$ and $n = 6$. $\square$

As a consequence of Lemma 2 we know that $\mathrm{Out}(\mathrm{Sym}_n)$ is trivial for $n \neq 6$ and that $\mathrm{Out}(\mathrm{Sym}_6)$ has at most two elements. The only remaining question is whether or not an exceptional automorphism of Sym_6 sending $\mathcal{C}_1$ to $\mathcal{C}_3$ actually exists.

2. A CONSTRUCTION

An exceptional automorphism of Sym_6 that sends $\mathcal{C}_1$ to $\mathcal{C}_3$ can be constructed using labeled icosahedra. A regular icosahedron has twelve vertices that come in 6 antipodal pairs. Consider all $6! = 720$ ways to label these
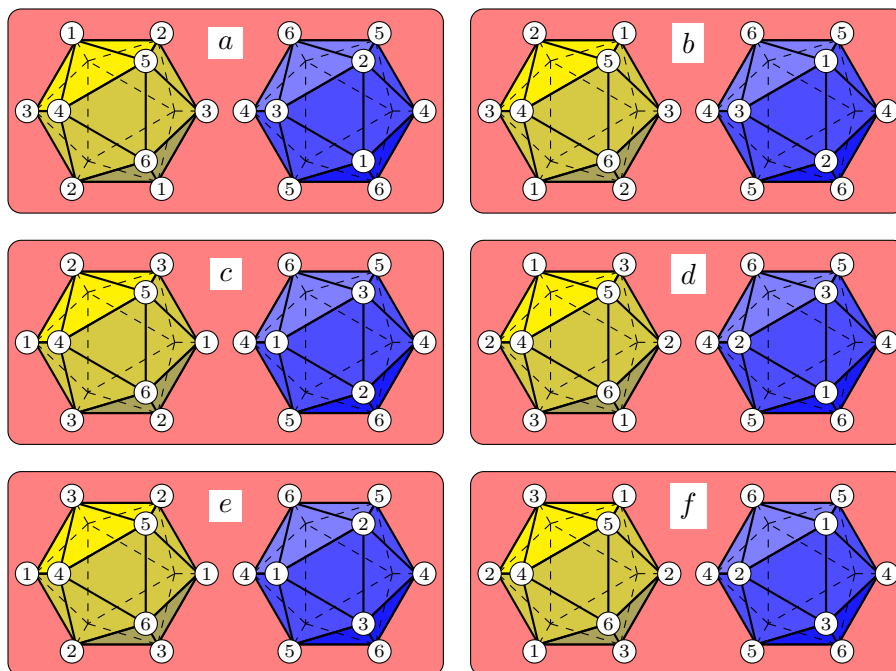
FIGURE 2. The twelve antipodal labelings of an icosahedron up to isometry grouped into six dual pairs. The six dual pairs are labeled a through f.

antipodal pairs by the numbers 1 through 6. If we identify labelings that differ by a rigid motion than the number of labelings drops to 12. See Figure 2. This is true whether we include reflection symmetries or we restrict our attention to rigid motions that are possible in $\mathbb{R}^3$. Icosahedra have 120 symmetries but because we have restricted our attention to antipodal labelings, the antipodal map acts trivially on labelings. Thus only 60 distinct labeled icosahedra arise under rigid motions. Moreover, the antipodal map, being orientation-reversing, can be composed with any orientation-reversing isometry to produce an orientation-preserving one that performs the same modification.

Next, the 12 antipodal labelings of an icosahedron up to isometry can be grouped into 6 pairs. To see this note that a single labeled icosahedron contains 20 labeled triangles but since antipodal triangles receive the same labels, exactly 10 out of the possible $\binom{6}{3} = 20$ labeled triangles actually occur. It turns out that the 10 unused labeled triangles glue together to form one of the other labeled icosahedra. An alternative way to see that such a pairing exists is to consider the complete graph on the 12 vertices of an icosahedron with the edges color-coded based on combinatorial distance in the 1-skeleton. The edges representing vertices distance 1 apart are the original 1-skeleton of the icosahedron. The edges representing vertices distance 3

apart are a perfect matching, i.e. 6 disjoint edges connecting antipodal vertices. The remaining edges, representing vertices distance 2 apart form the 1-skeleton of what one might call the *dual icosahedron*. This is analogous to the way that the diagonals of a regular pentagon form another (nonconvex) regular pentagon whose side length has been multiplied by the golden ratio. The diagonals of an icosahedron that connect non-adjacent non-antipodal vertices are the 1-skeleton of another (nonconvex) icosahedron.

The 12 antipodally labeled icosahedra are shown in Figure 2 as 6 pairs of labeled dual icosahedra that we identify by the letters a through f. Note that every possible labeled triangle occurs in one of the two icosahedra in the pair. We have colored the icosahedron yellow when it contains a triangle labeled 456 and blue when it contains a triangle labeled 123.

The symmetry group of the set $X = \{1, 2, 3, 4, 5, 6\}$ acts on this set of labeled icosahedra by permuting the vertex labels. And since this action of Sym_X respects rigid motions and the dual pairing, every permutation in Sym_X induces a permutation in Sym_A where $A = \{a, b, c, d, e, f\}$. In particular we get a homomorphism ϕ from Sym_X to Sym_A. As an illustration, consider the transposition $(1, 2)$. It is easy to see from Figure 2 that switching 1 and 2 in the labeled icosahedra swaps the dual pair a and the dual pair b, it swaps the dual pair c and the dual pair d and it swaps the dual pair e and the dual pair f. In other words, the image of the transposition $(1, 2)$ under the map ϕ is the permutation $(a, b)(c, d)(e, f)$ of the labeled dual pairs.

To see that this homomorphism ϕ from Sym_X to Sym_A is an isomorphism, we note that Sym_6 has very few normal subgroups. In fact, the only non-trivial normal subgroup is Alt_6 and the resulting quotient has size 2. Since ϕ sends the elements $(1, 2)$, $(1, 3)$ and $(2, 3)$ to the permutations $(a, b)(c, d)(e, f)$, $(a, e)(b, c)(d, f)$ and $(a, d)(b, f)(c, e)$ respectively, the image has size bigger than 2, the kernel must be trivial and, since both groups have the same size, the map must be onto and therefore an isomorphism. In short, this natural construction produces an isomorphism ϕ of Sym_6 that sends $\mathcal{C}_1$ in Sym_X to $\mathcal{C}_3$ in Sym_A. Moreover, its inverse, which is also an isomorphism of Sym_6 must send $\mathcal{C}_1$ in Sym_A to a conjugacy class other than $\mathcal{C}_1$ in Sym_X. By Lemma 2 its image can only be $\mathcal{C}_3$. In other words, ϕ sends the conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_3$ in Sym_X to the conjugacy classes $\mathcal{C}_3$ and $\mathcal{C}_1$ in Sym_A, respectively.

Finally, to turn this isomorphism into an automorphism we simply identify the letters a through f with the numbers 1 through 6 sending Sym_A back to Sym_X in a more traditional fashion. Note that the various ways of identifying A and X differ from each other by an inner automorphism of Sym_X so as we run through the 6! possibilities this procedure actually produces *all* of the outer automorphisms representing the unique nontrivial element of $\text{Out}(\text{Sym}_6)$.
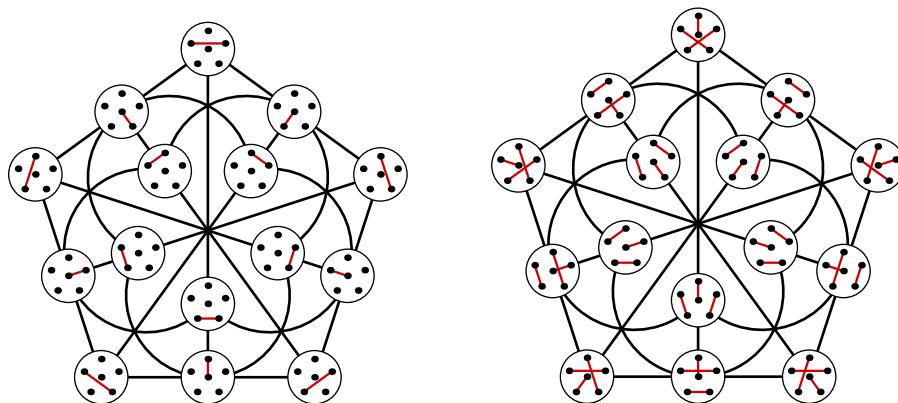
FIGURE 3. The edge and factor versions of the doily. The labels of the "points" are inscribed in the small discs and the line segments and circular arcs represent the "lines".

3. STRUCTURE

The exceptional symmetry of SYM_6 has a lot of interesting structure. Following Cameron and van Lint (among others) we describe the various aspects of SYM_6 using terminology from graph theory [CvL91, Chapter 6]. If we use X (or A) to label the 6 vertices of a complete graph K_6, then the transpositions in $\mathcal{C}_1$ are its *edges*. An involution in $\mathcal{C}_3$ corresponds to three disjoint edges which graph theorists would call a *perfect matching* or a 1-*factor* or simply a *factor*. The 6 sets S_i of 5 edges with a common endpoint that we used in the proof of Lemma 1 as an algebraic replacement for points are called *stars* and the 6 ways to partition the 15 edges of K_6 into 5 disjoint factors are called *factorizations*. An exceptional automorphism of SYM_6 swaps the 15 edges and the 15 factors and it swaps the 6 stars and the 6 factorizations. Composing this automorphism with itself produces an inner automorphism, but the result is not necessarily the identity map.

There are, however, some exceptional automorphisms whose square is the identity (36 of them to be precise) and we demonstrate their existence with the help of an auxillary graph. The edges and factors can be used to define an example of a partial geometry known as a *generalized quadrangle* and this particular example is called $GQ(2,2)$. It uses the edges as points and the factors as lines (or the other way around). Both versions are shown in Figure 3 in a representation that Stan Payne dubbed "the doily". The incidence graph of this geometry is a bipartite graph with 15 white vertices representing edges and 15 black vertices representing factors known as *Tutte's 8-cage*. A white vertex is connected to a black vertex if and only if the corresponding edge belongs to the corresponding factor. See Figure 4. The automorphism group of the Tutte graph is precisely the group $\mathrm{AUT}(\mathrm{SYM}_6)$ of size 1440. In particular, the outer automorphisms of SYM_6 correspond to symmetries of
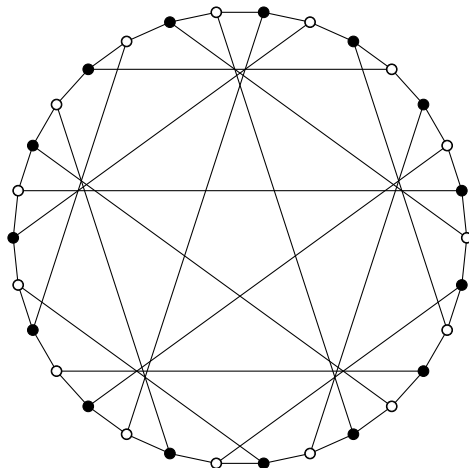
FIGURE 4. The incidence graph of the doily is known as Tutte's 8-cage. A reflection across the vertical axis of symmetry illustrates the duality between edges and factors.

this graph that switch the white and black vertices. One such symmetry is the reflection across the vertical axis of Figure 4 and this clearly corresponds to an exceptional automorphism of Sym_6 whose square is the identity. An exceptional automorphism that is equal to its own inverse reminds one of a polarity in projective geometry that establishes a bijection between points and lines and there are ways to make this resemblance precise.

4. CONNECTIONS

There is much more that can be said about the exceptional symmetry of Sym_6, but in this final section I merely make a few remarks about the connections this symmetry has with other exceptional objects coupled with a few pointers to some standard references in the literature. For those wishing to read more about the exception symmetry of Sym_6 at an accessible level, I highly recommend Cameron and van Lint's book "Designs, Graphs, Codes and their Links" [CvL91], especially Chapter 6, which is called "A property of the number six". In that chapter, the authors construct the exceptional symmetry of Sym_6 and use these automorphisms to construct the unique projective plane of order 4, the 50 vertex graph known as the Hoffman-Singleton graph its with many remarkable properties, and the $S(5,6,12)$ Steiner system whose automorphism group is the Mathieu group M_{12}, one of the smallest and simplest of the sporadic finite simple groups. Another good source for some of the same material is the book on "Algebraic Graph Theory" by Godsil and Royle [GR01]. For explicit details of the automorphisms themselves and for many references to the early literature (going back to Sylvester in 1844), I recommend two articles by H.S.M. Coxeter that are collected as Chapters 6 and 7 in his book "The beauty of geometry:

twelve essays" [Cox99]. Online there is a post written by John Baez in 1992 called "Some thoughts on the number 6" [Bae] which is similar in spirit to the material presented here and the labeled icosahedra construction is one of several constructions given in the recent article by Howard, Millson, Snowden and Vakil [HMSV08]. Finally, for the truly adventurous, I recommend Conway and Sloane's book on "Sphere packings lattices and groups" (particular Chapter 10 called "Three lectures on exceptional groups") [CS99] and the entry for ALT_6 in the "Atlas of finite groups" [CCN$^+$85]. Both contain a wealth of material that place the outer automorphism of SYM_6 in a much, much larger context.

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