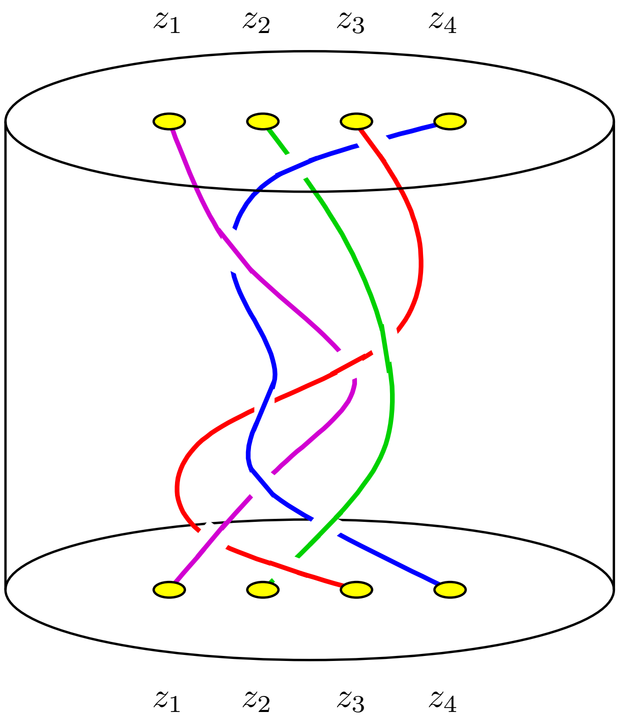
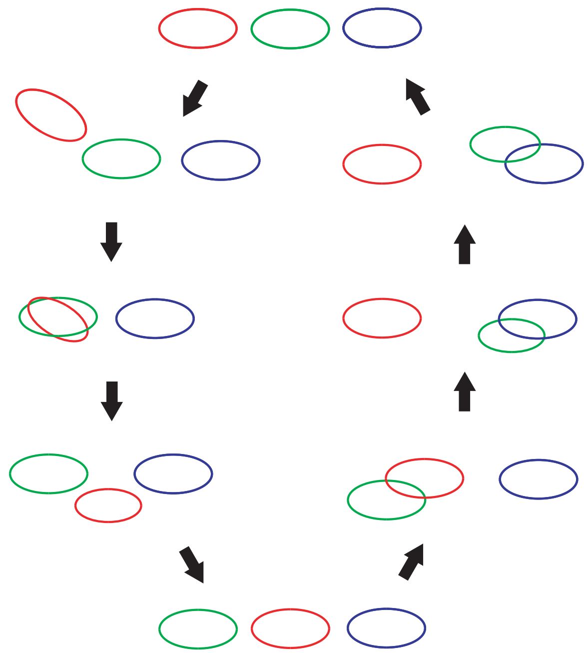
Points in the plane and loops in space

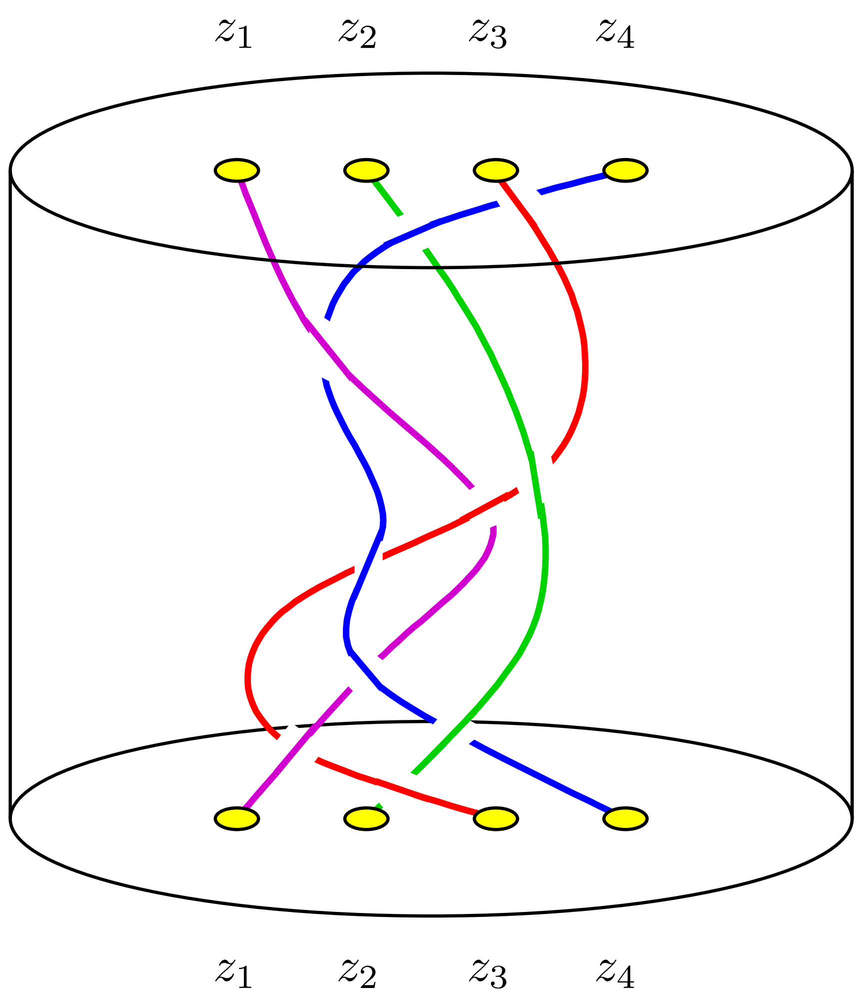




Jon McCammond

U.C. Santa Barbara

I. Points in the Plane



Geometric group theory

Geometric group theorists like it when groups act on metric spaces because, so long as the action is “nice”, the geometry of the space tells you a lot about the group.

Nice usually means, by isometries, with a compact quotient, and the action should be free or proper or, at the very least, have understandable stabilizers.

A classic example is the fundamental group of a compact metric space acting freely on its universal cover by isometries.

The symmetric group

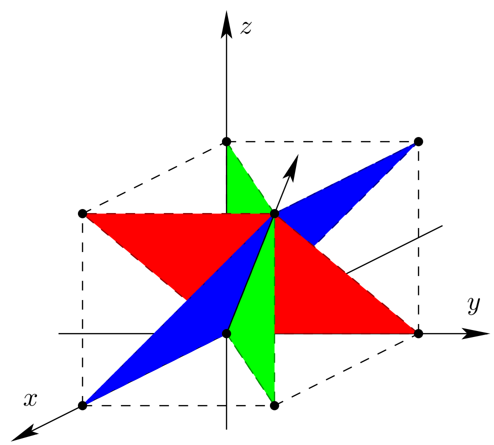
Here is a simple example of a group acting on a space. For any field $\mathbb{K}$, the symmetric group Sym_n acts on the vector space $\mathbb{K}^n$ by permuting coordinates.

The action is not free since vectors with repeated coordinates are fixed by non-trivial permutations, but it is free on the complement of the $\binom{n}{2}$ hyperplanes defined by the equations $x_i = x_j$.

This is called the *braid arrangement*.

The real braid arrangement

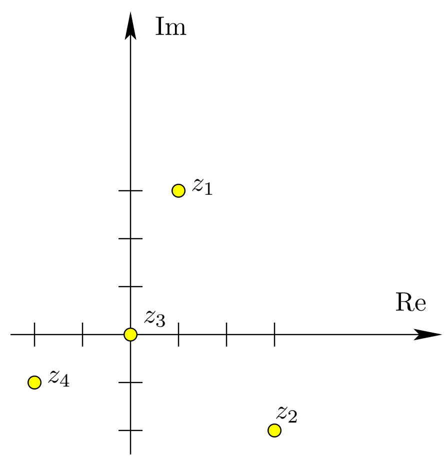
The *real braid arrangement* is the space of all n-tuples of distinct real numbers $(x_1, x_2, \ldots, x_n)$.



In $\mathbb{R}^3$ it consists of 6 connected pieces separated by the planes $x = y$, $x = z$ and $y = z$. In $\mathbb{R}^n$ it has $n!$ connected pieces separated by the hyperplanes $\{x_i = x_j\}$. The convex hull of the orbit of a point is the *permutahedron*.

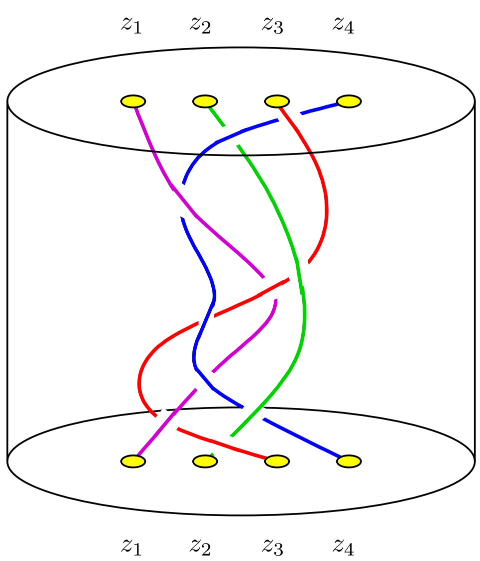
The complex braid arrangement

The *complex braid arrangement* is the space of all n-tuples of distinct complex numbers $(z_1, z_2, \ldots, z_n)$. There is a trick that enables us to visualize this space. The point $(z_1, z_2, z_3, z_4) = (1 + 3i, 3 - 2i, 0, -2 - i)$ is encoded in the figure.



The braid group

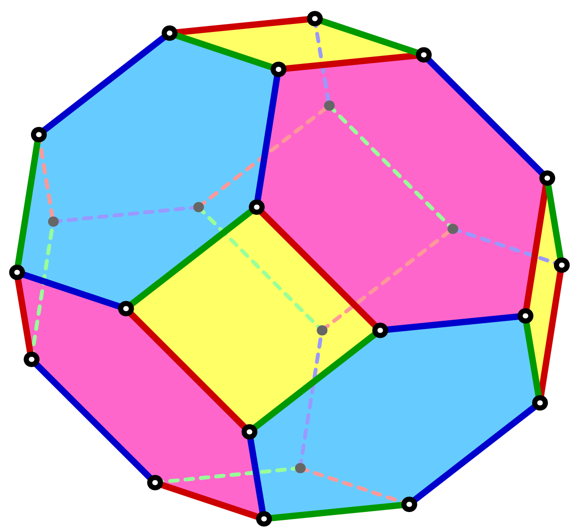
Moving around in the complex braid arrangement corresponds to moving the labeled points in $\mathbb{C}$ without letting them collide. Tracing out what happens over time produces braided strings.



The fundamental group of the complex braid arrangement is the *pure braid group*. The fundamental group of the quotient by the Sym_n action is the (full) *braid group*.

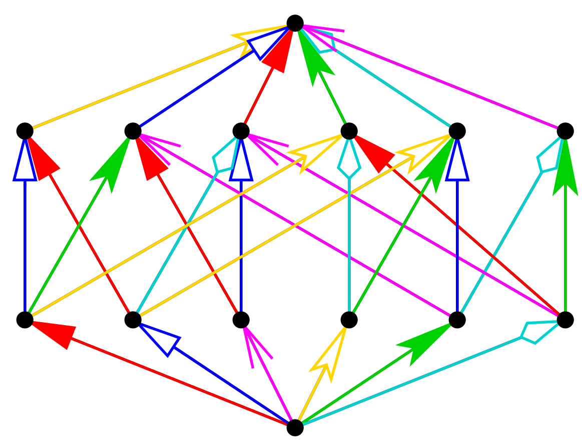
The Salvetti complex

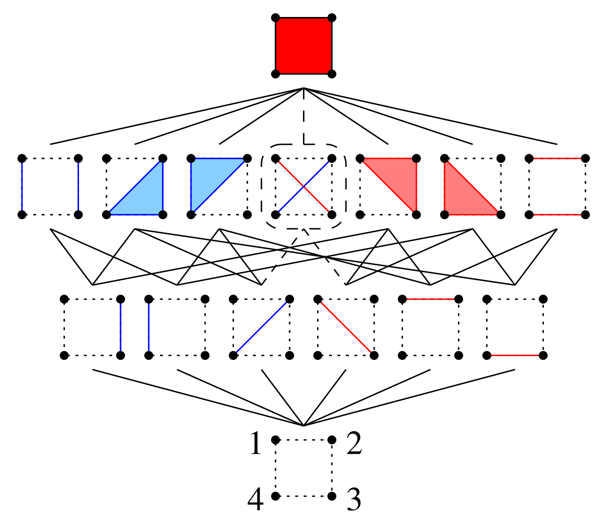
Neither quotient is compact, but they deformation retract onto compact subspaces that can be given cell structures. The *Salvetti complex* for the braid group Braid_n is obtained from a permutahedron with an edge orientation induced from a Morse function and an edge coloring invariant under reflections orthogonal to edges. Glue faces with matching labels and orientations.



The dual Garside structure

Alternatively, here is a very different construction. Minimally factor an n-cycle in Sym_n into transpositions (closely related to non-crossing partitions). Geometrically realize the resulting poset. Finally, glue facets with matching labels and orientations.





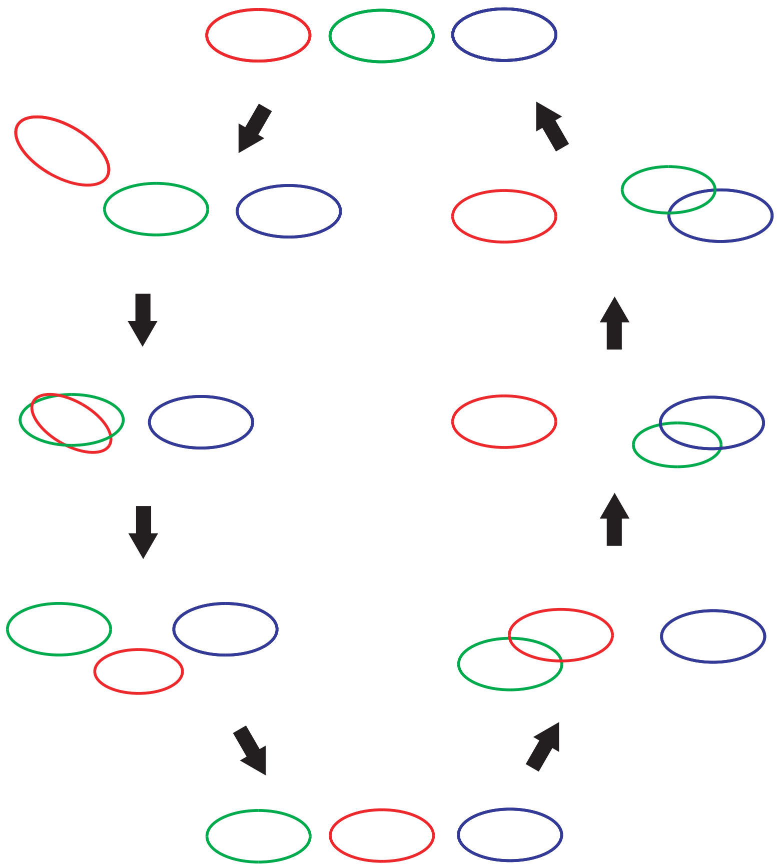
The upshot

These two rather different complexes are both Eilenberg-Maclane spaces for the braid groups and either one can be used to calculate the homology and cohomology of Braid_n.

Geometrically, the braid group acts freely and cocompactly by isometries on either universal cover with the permutahedron or the order complex of the noncrossing partition lattice as a fundamental domain for the action.

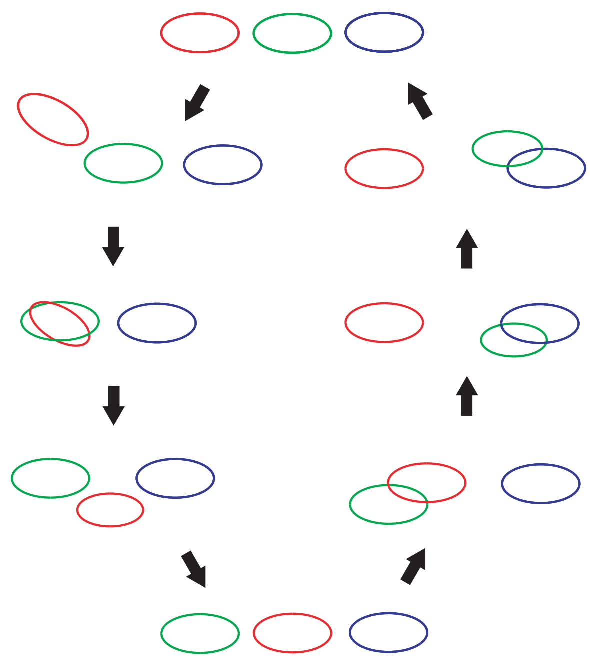
As a result, there is a close connection between (co)homology calculations for the braid groups, the combinatorics of the permutahedron and/or the lattice of non-crossing partitions.

II. Loops in Space



Σ_n and $P\Sigma_n$

Our second example is the group of motions of the trivial n-link. Σ_n is the group of motions of L_n in $\mathbb{S}^3$ and $P\Sigma_n$ is the index $n!$ subgroup of motions where the n components of L_n return to their original positions. (This is the *pure* motion group.)



Motion groups

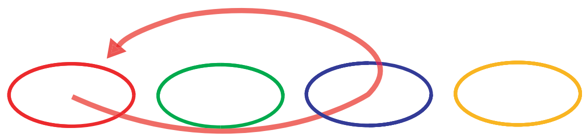
Let L_n be the trivial n-link in $\mathbb{S}^3$, let $H(\mathbb{S}^3)$ be the space of all self-homeomorphisms of the 3-sphere in the compact-open topology, and let $H(\mathbb{S}^3, L_n)$ be the subspace of homeomorphisms with $\phi(L_n) = L_n$ (preserving circle orientations) for a fixed embedding $L_n \hookrightarrow \mathbb{S}^3$.

A *motion* of L_n is a path $\mu : [0,1] \to H(\mathbb{S}^3)$ such that $\mu(0) =$ the identity and $\mu(1) \in H(\mathbb{S}^3, L_n)$. Two motions μ and ν are *equivalent* if $\mu^{-1}\nu$ is homotopic to a stationary motion, that is, a motion contained in $H(\mathbb{S}^3, L_n)$.

Introduced by Fox $\Rightarrow$ Dahm $\Rightarrow$ Goldsmith $\cdots$

Representing $P\Sigma_n$

Thm(Goldsmith, Mich. Math. J. '81) There is a faithful representation of $P\Sigma_n$ into Aut $(\mathbb{F}(x_1, \ldots, x_n))$ induced by sending the generators of $P\Sigma_n$ to automorphisms



$$\alpha_{ij}(x_k) = \begin{cases} x_k & k \neq i \\ x_j^{-1} x_i x_j & k = i \end{cases}$$

The image in Aut($\mathbb{F}_n$) is referred to as the group of *pure symmetric automorphisms* since it is the subgroup of automorphisms where each generator is sent to a conjugate of itself.

Thinking of $P\Sigma_n$ as a subgroup of Aut($\mathbb{F}_n$) we can form the image of $P\Sigma_n$ in Out($\mathbb{F}_n$), denoted $OP\Sigma_n$.

A group by any other name...

Four papers, four names, same group.

- "The **pure symmetric automorphisms of a free group** form a duality group" (with N. Brady, J. Meier, and A. Miller) *J. Algebra* (2001)

- "The hypertree poset and the ℓ_2-Betti numbers of the **motion group of the trivial link**" (with J. Meier) *Math. Annalen* (2004)

- "The integral cohomology of the **group of loops**" (with C. Jensen and J. Meier) *Geometry and Topology* (2006)

- "The Euler characteristic of the **Whitehead automorphism group of a free product**" (with C. Jensen and J. Meier) *Trans. AMS* (2007)

McCullough-Miller Complex

The computations in these papers are done via an action of $OP\Sigma_n$ on a contractible simplicial complex MM_n, constructed by McCullough and Miller (*MAMS*, '96).

The complex MM_n is a space of $\mathbb{F}_n$-actions on simplicial trees, where the actions all take seriously the decomposition of $\mathbb{F}_n$ as a free product $\mathbb{F}_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ copies}}$.

Each action in this space can be described by a marked hypertree.

Properties of MM_n

The McCullough-Miller space, MM_n, is the geometric realization of a poset of marked hypertrees. The marking is similar (and related) to the marked graph construction for outer space.

Some Useful Facts:

• MM_n admits $P\Sigma_n$ and $OP\Sigma_n$ actions.

• The fundamental domain for either action is the same, it's finite and isomorphic to the order complex of HT_n (also known as the *Whitehead* poset).

• The isotropy groups for the $OP\Sigma_n$ action are free abelian; the isotropy groups are free-by-(free abelian) for the action of $P\Sigma_n$.

Good News/Bad News

The cohomology and/or asymptotic topology of a group G is same as that of the universal cover of a $K(G, 1)$.

Good News: We have a contractible, cocompact $P\Sigma_n$-complex.

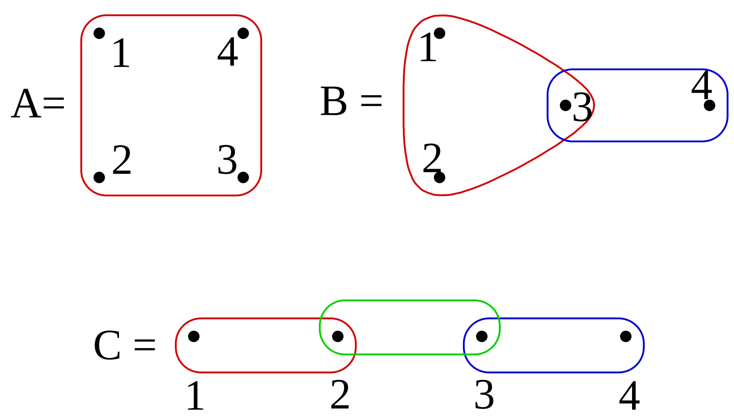
Bad News: The action isn't free or even proper.

Good News: The stabilizers are well understood.

Punch Line: The cohomology and/or asymptotic topology of $P\Sigma_n$ cannot be directly understood from the cohomology and/or asymptotic topology of MM_n because of the bad stabilizers. Instead we plug the combinatorics of HT_n and the isotropy groups into arguments involving spectral sequences.

Hypertrees

A *hypertree* is a connected hypergraph with no hypercycles. In hypergraphs, the "edges" are subsets of the vertices, not just pairs of vertices. The growth is quite dramatic: The number of hypertrees on $[n]$ (due to Smith and Warme,Kalikow), for $n \geq 3$ is $= \{4, 29, 311, 4447, 79745, 1722681, 43578820, \ldots\}$. The formula is $|\mathsf{HT}_n| = \sum_k n^{k-1} S(n-1, k)$ where $S(n, k)$ are Stirling numbers of the second kind.



Exponential generating functions

Define the edge weight of a hypertree on $[n]$ as $u_2^{\lambda_2} \cdots u_n^{\lambda_n}$ where λ_i counts the number of edges of size i.

Let T_n be the sum of all the weights of hypertrees on $[n]$.
Let R_n be the sum of all the weights of rooted hypertrees on $[n]$.
Let $T = \sum_n T_n \frac{t^n}{n!}$ and let $R = \sum_n R_n \frac{t^n}{n!}$

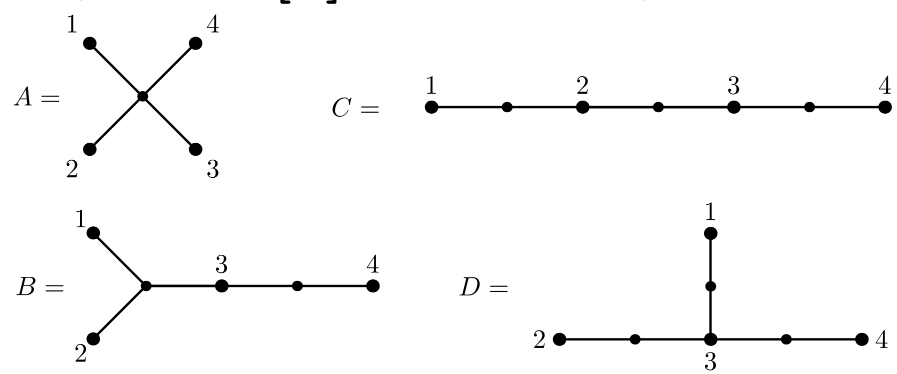
$$T_3 = u_3 + 3u_2^2 \qquad R_3 = 3 \cdot T_3$$
$$T_4 = u_4 + 12u_2u_3 + 16u_2^3 \qquad R_4 = 4 \cdot T_4$$

Thm(Kalikow) R solves the functional equation $R = te^y$ where

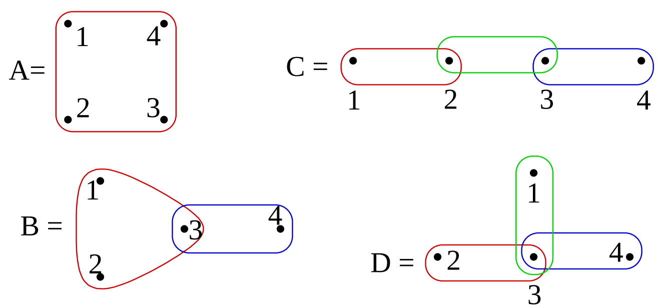
$$y = \sum_{j \geq 1} u_{j+1} \frac{R^j}{j!}$$

Drawing conventions

Examples of [4]-labelled bipartite trees.

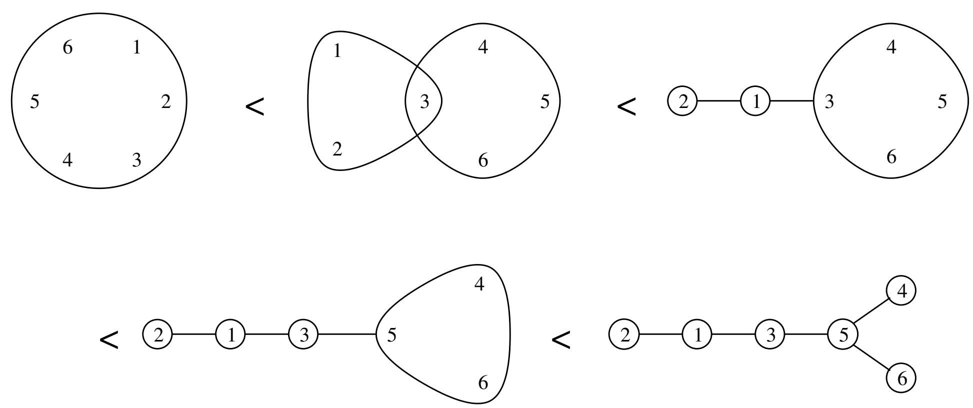


Examples of hypertrees on [4].



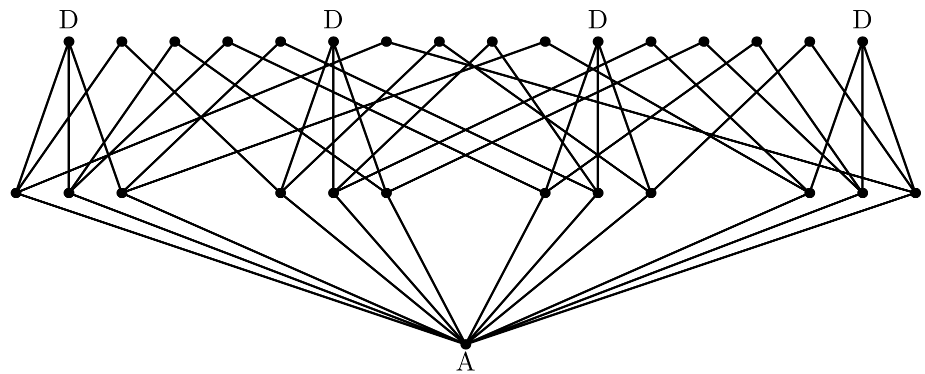
The hypertree poset

The hypertrees on $[n]$ form a very nice poset, that is surprisingly understudied in combinatorics. The elements of HT_n are n-vertex hypertrees with the vertices labelled by $[n] = \{1, \ldots, n\}$. The order relation is given by: $\tau < \tau' \Leftrightarrow$ each hyperedge of τ' is contained in a hyperedge of τ. The hypertree with only one edge is $\widehat{0}$, also called the *nuclear* element. If one adds a formal $\widehat{1}$ such that $\tau < \widehat{1}$ for all $\tau \in \mathsf{HT}_n$, the resulting poset is $\widehat{\mathsf{HT}}_n$.



First properties of $\mathbf{HT}_n$

The Hasse diagram of HT_4 is



Thm: $\widehat{\mathsf{HT}}_n$ is a finite, graded, bounded lattice.

Pf: Finite, graded, and bounded are easy. Lattice is easy based on the similarities between HT_n and the partition lattice (and is the key element in the McCullough-Miller proof that MM_n is contractible.)

What we do

My co-authors and I:

- prove that $\widehat{\mathsf{HT}}_n$ is Cohen-Macaulay, and use this to prove that $P\Sigma_n$ is a duality group.

- calculate the Möbius function of $\widehat{\mathsf{HT}}_n$ and use this to calculate the ℓ^2-betti numbers of $P\Sigma_n$.

- calculate Euler characteristics for large classes of groups by deriving various hypertree identities, and we

- calculate the full integral cohomology of $P\Sigma_n$ (including the ring structure) using the hypertree poset structure to separate the relevant spectral sequences into pieces we can analyze.

Cohen-Macaulay

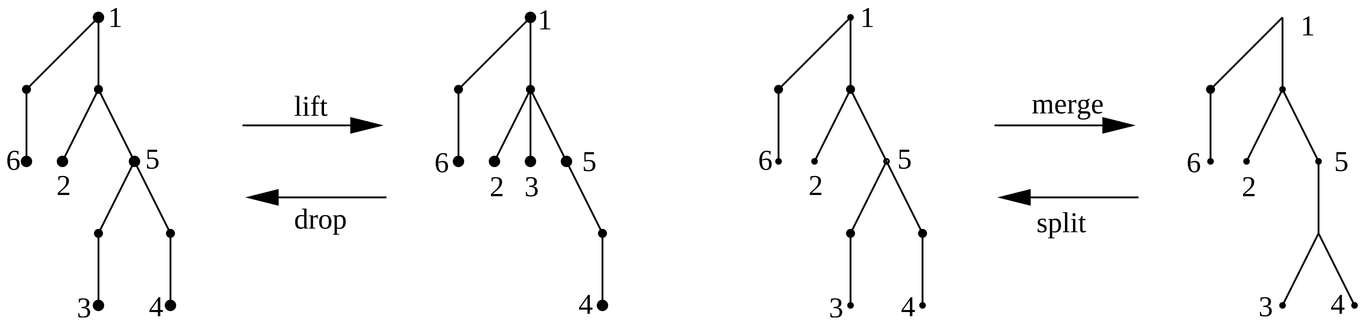
A poset is *Cohen-Macaulay* if its geometric realization is Cohen-Macaulay in the sense that $\widetilde{H}_i(\mathsf{lk}(\sigma), \mathbb{Z}) = 0$ for all simplices σ (including the empty simplex) and all $i < \dim(\mathsf{lk}(\sigma))$.

(When X is Cohen-Macaulay, this implies that X is homotopy equivalent to a bouquet of spheres.)

We show HT_n is Cohen-Macaulay by showing that $\widehat{\mathsf{HT}}_n$ is shellable, which we get by proving $\widehat{\mathsf{HT}}_n$ admits a recursive atom ordering.

The recursive atom ordering

Our recursive atom ordering roots the hypertrees at the vertex 1 and then orders by the depth of the vertices (details omitted). Moving around the ordering involves dropping and lifting, and splitting and merging.



The Möbius function

Let μ be the Möbius function of $\widehat{\mathsf{HT}}_{n+1}$ and recall that $\mu(\hat{0},\hat{1}) = \tilde{\chi}(\mathsf{HT}^{\circ}_{n+1})$ where the circle indicates that $\hat{0}$ and $\hat{1}$ are removed.

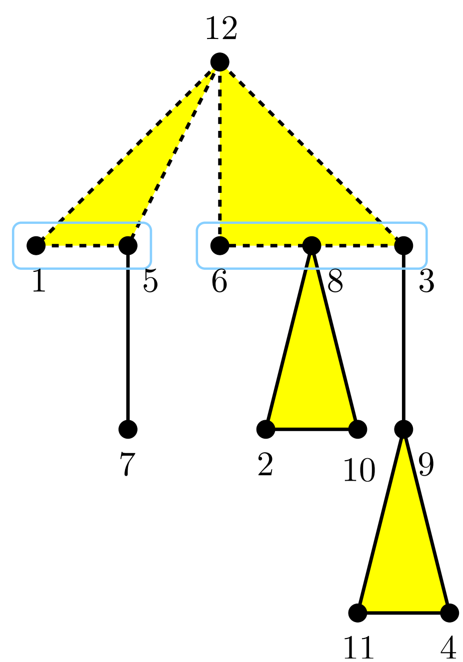
Using recursion formulas for Möbius functions, and Kalikow's functional equation, we show

Thm(M-Meier) $\mu(\hat{0},\hat{1}) = \tilde{\chi}(\mathsf{HT}^{\circ}_{n+1}) = (-1)^n n^{n-1}$.

For example, $\tilde{\chi}(\mathsf{HT}^{\circ}_{3}) = 2$, $\tilde{\chi}(\mathsf{HT}^{\circ}_{4}) = -9$, $\tilde{\chi}(\mathsf{HT}^{\circ}_{5}) = 64$.

Rooted hypertree and planted hyperforests

A rooted hypertree on [12] is shown with its associated planted hyperforest on [11]. One can pass from the hyperforest back to the hypertree using the partition of $\{1, 3, 5, 6, 8\}$ indicated by the lightly colored boxes.



A sample hypertree identity

Let the weight of a hyperedge be $(e-1)^{e-2}$ where e is its size. Let the weight of vertex i be $x_i^{\mathrm{val}(i)-1}$ where val(i) is its valence. Let the weight of a hypertree be the product of its vertex and edge weights.

Thm (Jensen-M-Meier)

$$\sum_{\tau \in HT_n} \mathrm{Weight}(\tau) = (x_1 + x_2 + \cdots + x_n + n)^{n-1}$$

This is proved starting with Abel's identities and a tree result from Stanley. We then prove several partition identities and rooted tree and planted forest identities before reaching this one.

III. Geometric Group Theory

(only if time permitting)

Duality groups

Def: (Bieri-Eckmann, *Invent. Math.* '73) A group G, with a finite $K(G,1)$, X, is an *n-dimensional duality group* if ...

$H_c^*(\widetilde{X}) = H^*(G, \mathbb{Z}G)$ is torsion-free and concentrated in dim n.

$$\Updownarrow$$

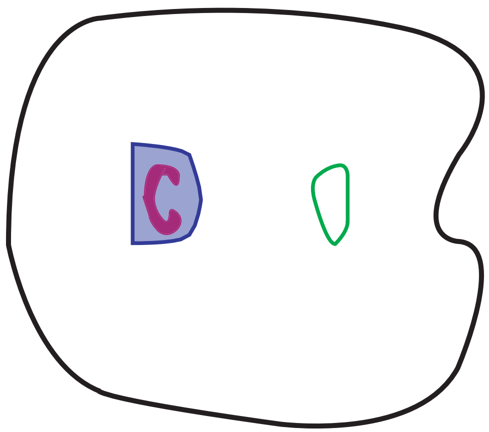
There is a G-module D such that $H^i(G, M) \simeq H_{n-i}(G, D \otimes M)$ for all i and G-modules M.

$$\Updownarrow$$

The universal cover $\widetilde{X}$ is $(n-2)$-acyclic at infinity. (Geoghegan-Mihalik, *JPAA* '85)

Acyclic at infinity

Let X be a finite $K(\pi, 1)$. Then $\widetilde{X}$ is m-acyclic at infinity if given any compact $C \subset \widetilde{X}$, there is a compact $D \supset C$ such that every k-cycle supported in $\widetilde{X} - D$ is the boundary of a $(k+1)$-chain supported in $\widetilde{X} - C$. $(-1 \leq k \leq m)$



Duality groups are groups which are as acyclic at infinity as they can possibly be.

Examples of (virtual) duality groups

• Groups like $SL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z}[1/p])$. (Borel, *CMH* 1974, Serre, *Topology* 1976)

• Mapping class groups of surfaces. (Harer, *Invent. Math.* 1986)

• Braid groups as well as all Artin groups of finite type. (Squier, *Math. Scand.* 1995, or Bestvina, *Geom. & Top.* 1999)

• Out($\mathbb{F}_n$) and Aut($\mathbb{F}_n$). (Bestvina-Feighn, *Invent. Math.* 2000)

• $P\Sigma_n$ and $OP\Sigma_n$ (Brady-M-Meier-Miller, *J. Algebra*, 2001)

Proving duality

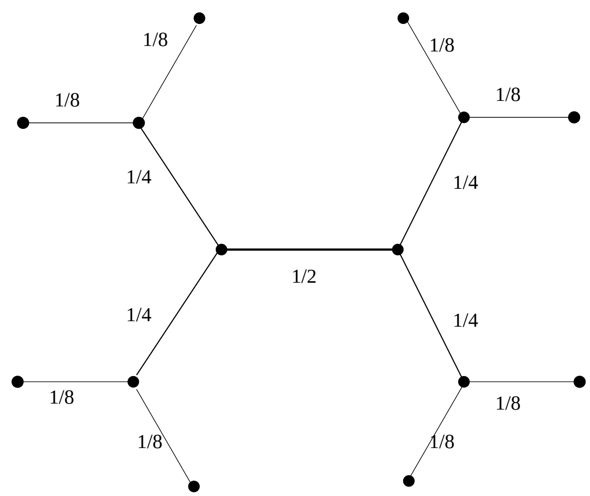
You can prove that a group is a duality group by showing the cohomology with group ring coefficients is trivial, except in top dimension where it's torsion-free.

The standard equivariant spectral sequence with $\mathbb{Z}G$ coefficients for the action of $OP\Sigma_n$ on MM_n has a complicated first page because the size of the isotropy groups for the action on the poset corresponds with the corank of the elements. It does not correspond well with the dimension of simplices in the geometric realization.

On the other hand, the Brown-Meier spectral sequence filters by the poset rank not dimension and collapses immediately when the poset is Cohen-Macaulay. (Brown-Meier, *CMH* '00)

ℓ^2-cohomology

For a group G (admitting a finite $K(G, 1)$), let $\ell^2(G)$ be the Hilbert space of square-summable functions. The classic cocycle is:



In general, concrete computations are rare. One of the few is due to Davis and Leary who compute the ℓ^2-cohomology of arbitrary right-angled Artin groups (*Proc. LMS*).

ℓ^2-betti numbers

We compute the ℓ^2-betti numbers of $OP\Sigma_{n+1}$ via its action on MM_{n+1}. In order to do this we have to switch to an algebraic standpoint, using group cohomology with coefficients in the group von Neumann algebra $\mathcal{N}(G)$. We also are really computing the equivariant ℓ^2-betti numbers of the action of $OP\Sigma_{n+1}$ on MM_{n+1}. We can get away with this because

Lemma. *The ℓ^2-cohomology of $\mathbb{Z}^n$ is trivial.*

Lemma. *Let X be a contractible G-complex. Suppose that each isotropy group G_σ is finite or satisfies $b_p^{(2)}(G_\sigma) = 0$ for $p \geq 0$. Then $b_p^{(2)}(X, \mathcal{N}(G)) = b_p^{(2)}(G)$ for $p \geq 0$.*

(cf. Lück's L^2-*Invariants: Theory and Applications ...*)

Reduction to Euler characteristics

In looking at the resulting equivariant spectral sequence we find we are really looking at the homology of

$$\mathsf{HT}^{\circ}_{n+1} = \mathsf{HT}_{n+1} - \{\text{the nuclear vertex}\}$$

(this is the singular set for the $OP\Sigma_{n+1}$ action.)

Since this poset is Cohen-Macaulay, all we really care about is

$$\text{rank}\left(H_{n-2}(\mathsf{HT}^{\circ}_{n+1})\right) = |\widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1})|$$

and so computing the ℓ^2-betti numbers of the group $OP\Sigma_{n+1}$ has boiled down to computing the Euler characteristic of the poset $\mathsf{HT}^{\circ}_{n+1}$.

Reduction to Möbius functions

To compute $\widetilde{\chi}(\mathsf{HT}^\circ_{n+1})$ we fill up chalk boards with Hasse diagrams and compute ...

$$\chi(\mathsf{HT}^\circ_3) = 3 = 3$$
$$\chi(\mathsf{HT}^\circ_4) = 28 - 36 = -8$$
$$\chi(\mathsf{HT}^\circ_5) = 310 - 855 + 610 = 65, \text{ etc.}$$

Luckily, Euler characteristics are well studied in enumerative combinatorics. In particular we can get to the Euler characteristic of HT°_{n+1} by studying the Möbius function μ of $\widehat{\mathsf{HT}}_{n+1}$.

Fact: If μ is the Möbius function of $\widehat{\mathsf{HT}}_{n+1}$ then $\mu(\hat{0}, \hat{1}) = \widetilde{\chi}(\mathsf{HT}^\circ_{n+1})$

In our case $\widetilde{\chi}(\mathsf{HT}^\circ_3) = 2$, $\widetilde{\chi}(\mathsf{HT}^\circ_4) = -9$, $\widetilde{\chi}(\mathsf{HT}^\circ_5) = 64$.

The Calculation and Its Corollaries

Using recursion formulas for Möbius functions, and Kalikow's functional equation, it only takes 3 or 4 pages to show:

Thm: $\widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1}) = (-1)^n n^{n-1}$.

Cor 1: *The ℓ^2-Betti numbers of $OP\Sigma_{n+1}$ are trivial, except $b^{(2)}_{n-1} = n^{n-1}$. It follows that $b^{(2)}_{n-1}(O\Sigma_{n+1}) = \frac{n^{n-1}}{(n+1)!}$.*

Cor 2: *The ℓ^2-Betti numbers of $P\Sigma_{n+1}$ are trivial, except $b^{(2)}_n = n^n$. It follows that $b^{(2)}_n(\Sigma_{n+1}) = \frac{n^n}{(n+1)!}$.*