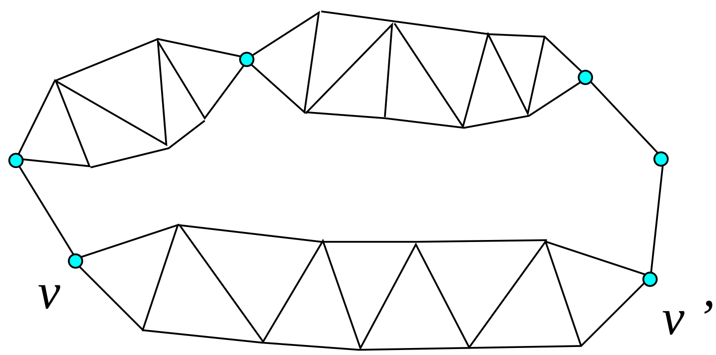
Constructing non-positively curved spaces and groups

Day 2: Algorithms



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Outline

I. Algorithm in dimension 3

II. 3-manifolds results

III. Higher dimensions: Positive curvature

IV. Higher dimensions: Moussong's lemma

I. Algorithm in dimension 3

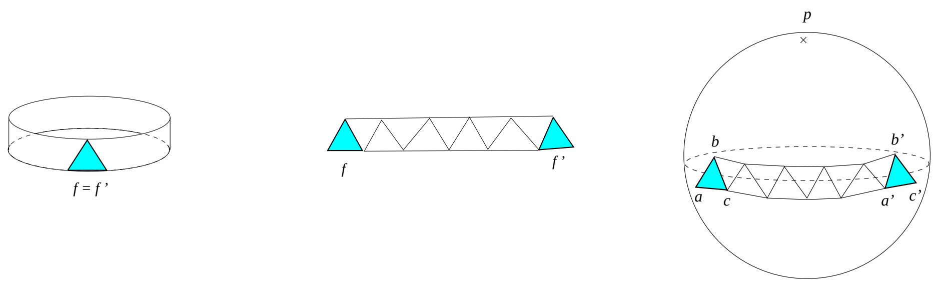
Thm(Elder-M): There is a “practical” algorithm which decides whether or not a 3-dimensional PE complex is nonpositively curved.

The argument uses elementary 3-dimensional geometry and has been implemented in **GAP** (Groups, Algorithms, Programming) and **Pari** (a number theory package).

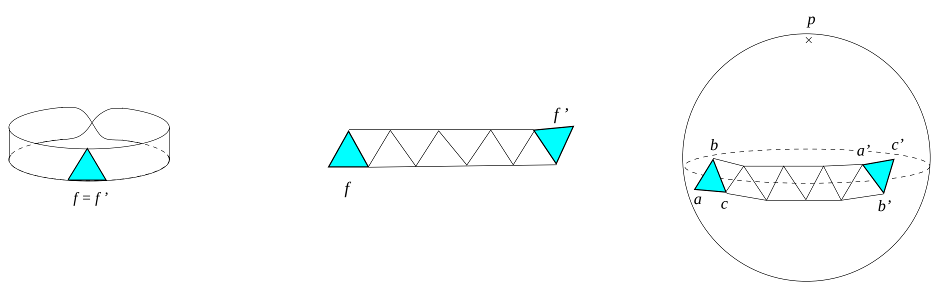
The packages are called `cat.g` and `cat.gp`, and are available from either of our webpages.

(switch to a demonstration)

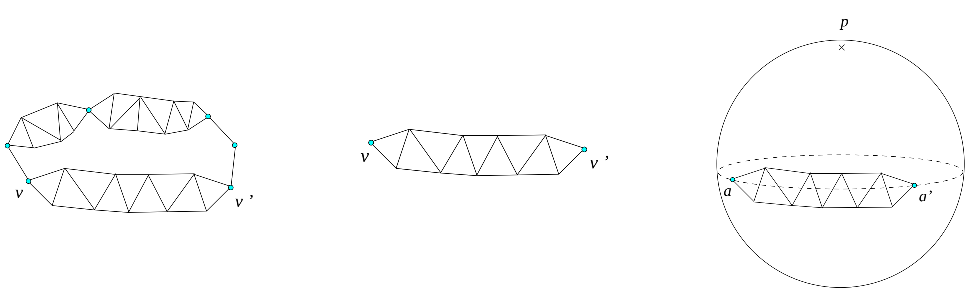
3-**dimensions**



An annular gallery, cut open and developed.



A Möbius gallery, cut open and developed.

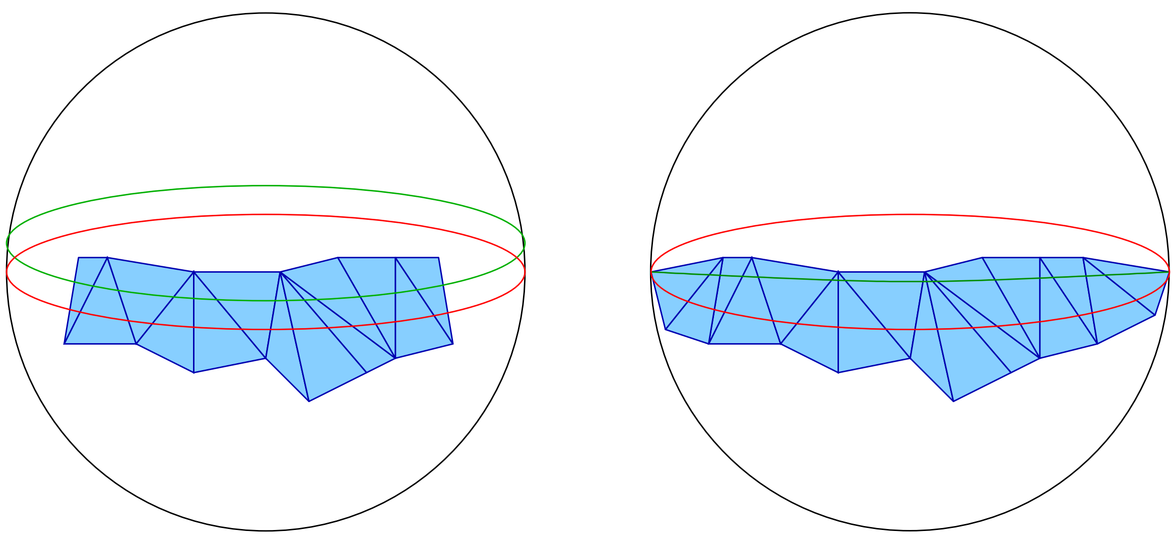


A bead developed.

Unshrinkable geodesics

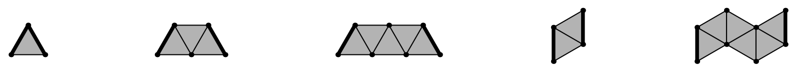
Lem(Bowditch) In a PS complex it is sufficient to search for a geodesic which can neither be shrunk nor homotoped (without increasing length) til it meets the boundary of its gallery.

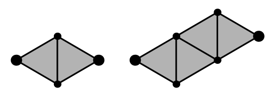
Lem: All annular galleries, Möbius galleries longer than π, and beads longer than π are shrinkable.



This simplifies the search for short geodesics immensely.

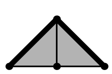
Regular tetrahedra

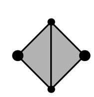
5 Möbius 

3 Bead 1 edge type and 

22 Necklaces A, A^2, A^3, A^4, A^5, B, BA, BA^2, BA^3, BA^4, B^2, B^2A, B^2A^2, $BABA$, B^3, C, CA, CA^2, CA^3, CB, CBA, C^2

Coxeter Shape

1 Möbius 

3 Beads 2 types of edges and 

19 Necklaces: A^2, A^4, A^6, A^2B, A^2B^2, A^2B^3, $ABAC$, A^2C, A^2C^2, A^4B, $CA^4\ B$, B^2, B^3, B^4, B^5, C, C^2, C^3

II. 3**-manifold results**

As an application of the 3-dimensional algorithm, we proved a new result about word-hyperbolicity for 3-manifolds.

$5/6^*$-**triangulations** Let M be a closed triangulated 3-manifold.

•The triangulation is a $5/6$-triangulation if every edge has degree 5 or 6.

•It is a $5/6^*$-triangulation if, in addition, each 2-cell contains at most one edge of degree 5.

A class of NPC 3-manifolds

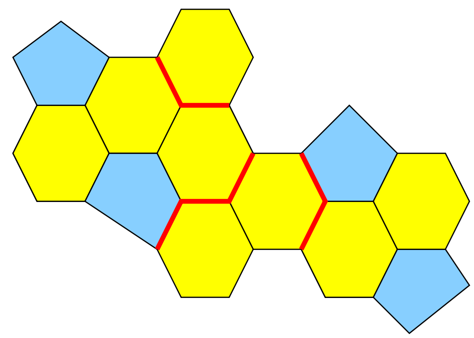
Thurston conjectured that a closed 3-manifold with a $5/6^*$-triangulation has a hyperbolic fundamental group.

Thm(Elder-M-Meier) Every $5/6^*$-triangulation of a closed 3-manifold admits a piecewise Euclidean metric of non-positive curvature with no isometrically embedded flat planes in its universal cover. Thus $\pi_1 M$ is hyperbolic.

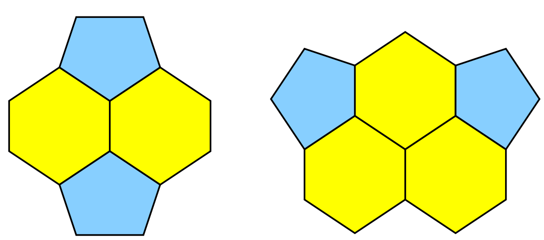
The proof involves a mixture of traditional combinatorial group theory, and computations carried out by `cat.g`, a collection of GAP routines developed by Murray Elder and myself.

Soccer diagrams

The dual tiling of a vertex link looks like a soccer ball, so we call these *soccer diagrams*.

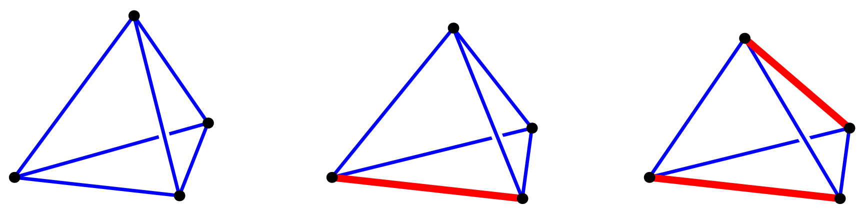


Key Lem: The only soccer diagrams with $\partial D \leq 14$, at most six pentagons, and no three consecutive right turns are the two diagrams shown below.



The metric

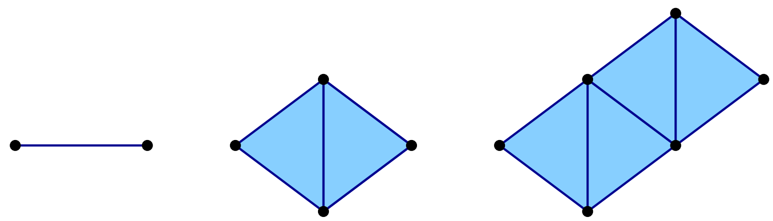
Def: If M is a $5/6^*$ triangulation, the assign a length of 2 to each edge of degree 5, and a length of $\sqrt{3}$ to each edge of degree 6.



It is easy to show that the edge links are CAT(1) by calculating dihedral angles.

Using the software `cat.g`, we've calculated the beads for these three shapes. Using the simplied search algorithm, it takes less than an hour to produce the list of 75 beads.

The output



The output is 4 types of edges, 26 beads with 2 triangles, and 45 beads with 4 triangles.

Minimum length

A	$.302\pi$
B	$.5\pi$
C	$.833\pi$

Necklaces

Using the explicit output, we can string together all possible necklaces.

The paths in the vertex links which determine these necklaces can be perturbed so the miss all the vertices.

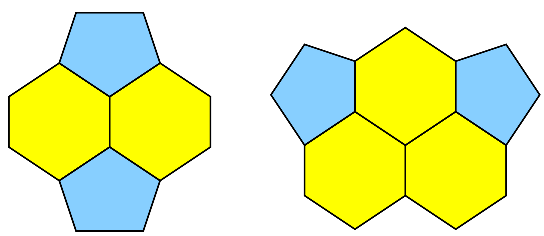
They now determine annular galleries with at most 14 triangles.

Because of the dihedral angles, there are never three right/left turns in a row.

Soccer diagrams revisited

The dual of this annular gallery is a simple path in a soccer tiling of $\mathbb{S}^2$ whose length is at most 14 and with no three consecutive left/right turns.

Thus if a short geodesic exists, it perturbs to bound one of these two diagrams.



But this is impossible by a careful look at the possible sequences of left and right turns coming from our three explicit (combinatorial) beads.

Edge degrees

The ubiquity of $5/6*$ triangulations is suggested by the following:

Thm(N.Brady-M-Meier) Every closed orientable 3-manifold has a triangulation in which each edge has degree 4, 5, or 6.

Idea of the proof: Use the universality of the figure eight knot and triangulate each piece carefully.

Rem: It is unlikely that these degrees can be further restricted because of the curvatures around the edges when the tetrahedra are regular.

There are also *foams*, 5/6-triangulations in which no 2-cell contain more than one edge of degree 6. See [J.Sullivan].

III. Positive curvature

Some recent results of Ezra Miller and Igor Pak imply:

Thm: If M is a PE n-manifold and every codimension 2 link is small, then $\pi_1 M$ is finite.

Cor: If M is a 3-manifold and every edge has degree at most 5, then $\pi_1 M$ is finite.

Positive vs. Negative curvature Positive curvature pushes geodesics away from the $n-2$ skeleton while negative curvature pushes geodesics towards the lower skeleta.

IV. Higher dimensions

There are some classes of higher dimensional complexes where easy algorithms are known for checking curvature conditons. Most follow from Moussong's lemma.

Def: A complex is *flag* if every 1-skeleton of a simplex is filled with a simplex.

Def: A metric complex is *metric flag* if every 1-skeleton of a metrically feasible simplex is filled with a simplex.

Lem(Moussong): If every edge in a PS complex has length at least $\pi/2$ then it is large $\Leftrightarrow$ it is a metric flag complex.

Thm(Moussong) All Coxeter groups are CAT(0) groups.

Why are higher dimensions hard?

Let T_n be a string of n PS tetrahedra strung together along edges. Testing whether such a configuration is shrinkable is encoded into a finite continued fraction expansion of a determinant.

In so far as continued fractions are notoriously sensitive to perturbations, this helps to explain the delicacy of the situation.