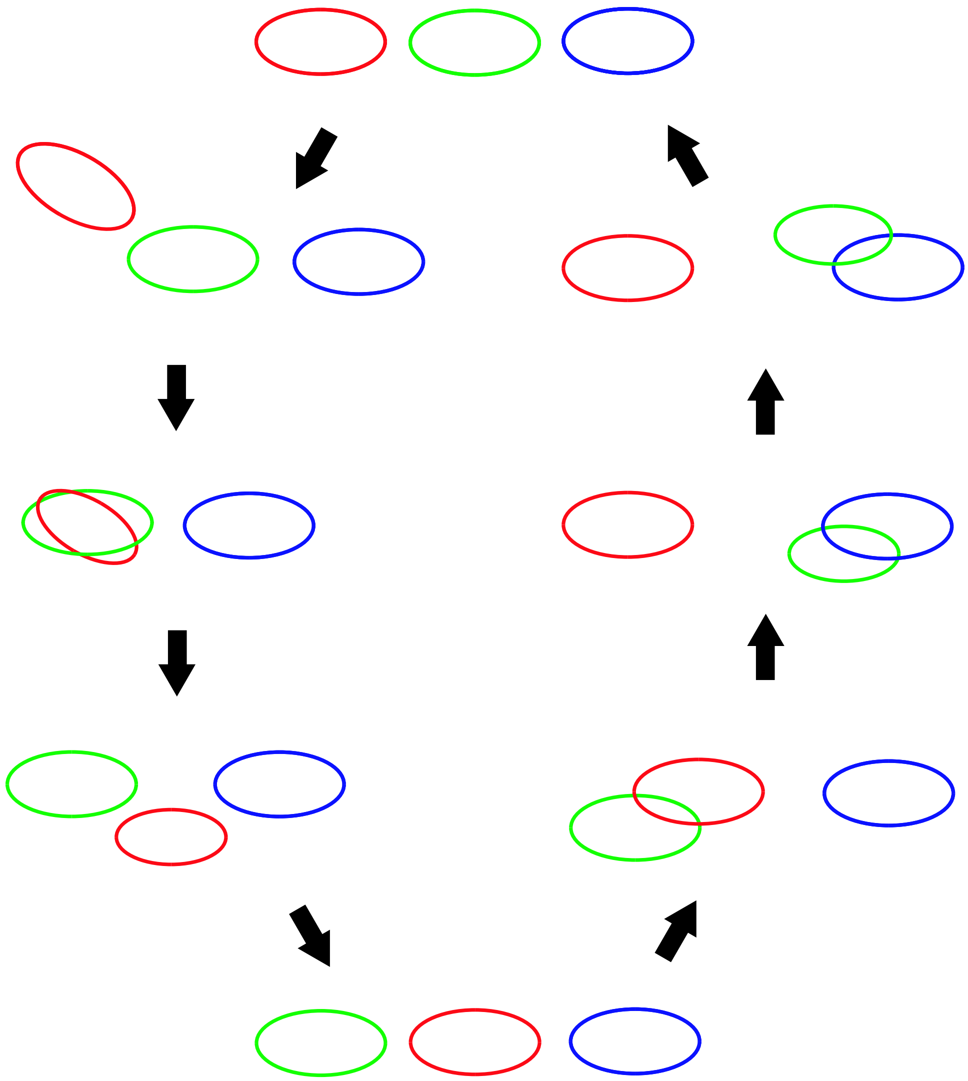
Hypertrees and the pure symmetric automorphism group



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Big Picture

Let X be a finite $K(G,1)$, so the cohomology of X is the cohomology of G.

The cohomology of $\widetilde{X}$ is rather dull, but

$\mathcal{H}^*(\widetilde{X}) = \ell^2$-cohomology and
$H_c^*(\widetilde{X}) =$ cohomology with compact supports

give interesting information about G.

Goal: Highlight how combinatorics and spectral sequences can be combined to help understand the asymptotic invariants of a group G. Our main example will be the group of motions of the trivial n-link.

"Motions"

$L_n =$ trivial n-link in S^3.

1. $H(S^3)$ is the space of self-homeomorphisms of the 3-sphere (compact-open topology).

2. $H(S^3, L_n)$ = the subspace of homeomorphisms with $\phi(L_n) = L_n$ — orientation preserved! — for a fixed embedding $L_n \hookrightarrow S^3$.

3. A *motion* of L_n is a path $\mu : [0, 1] \to H(S^3)$ such that $\mu(0) =$ the identity and $\mu(1) \in H(S^3, L_n)$.

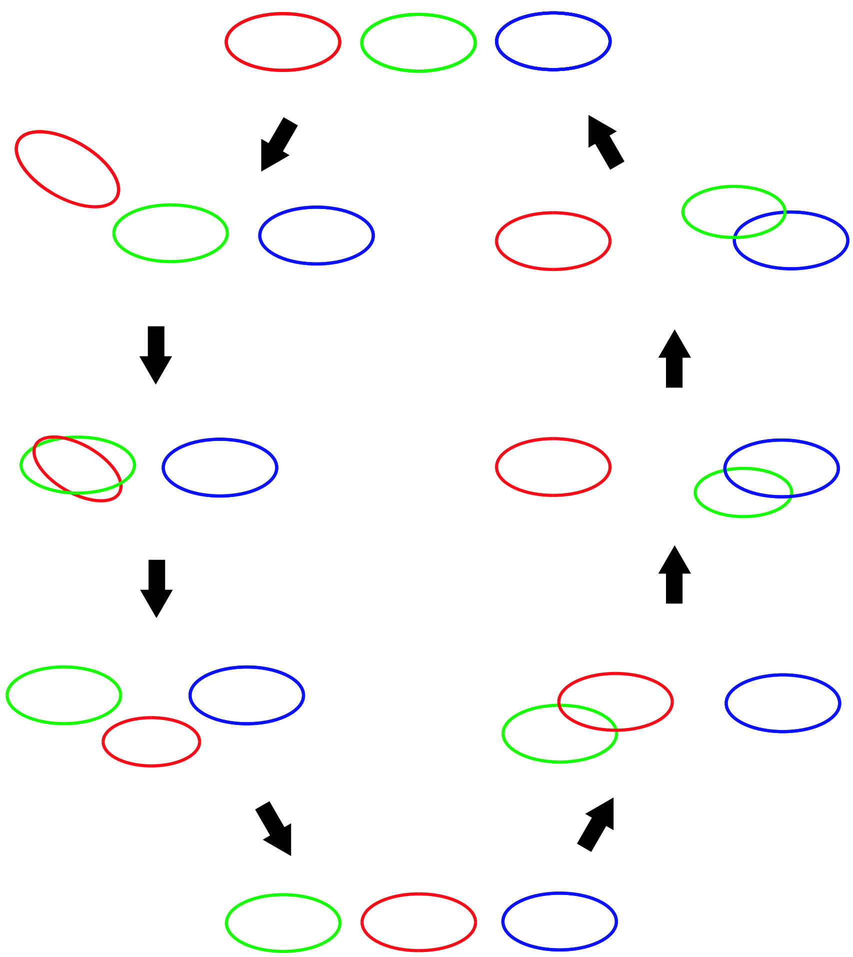
4. Two motions μ and ν are *equivalent* if $\mu^{-1}\nu$ is homotopic to a stationary motion, that is, a motion contained in $H(S^3, L_n)$.

Introduced by Fox $\Rightarrow$ Dahm $\Rightarrow$ Goldsmith $\cdots$

Σ_n and $P\Sigma_n$

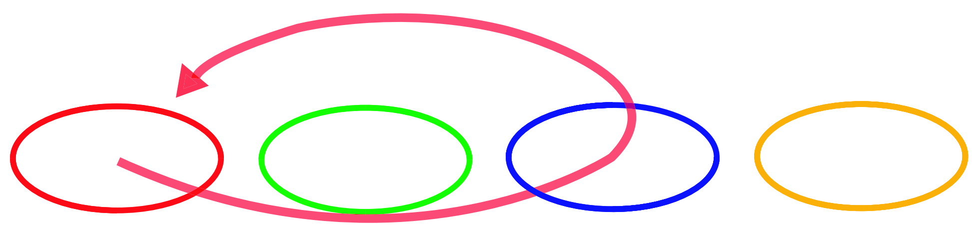
$\Sigma_n =$ the group of motions of L_n in S^3.

$P\Sigma_n =$ the index $n!$ subgroup of motions where the n components of L_n return to their original positions. (This is the *pure* motion group.)



Representing $P\Sigma_n$

Thm(Goldsmith, Mich. Math. J. '81)
There is a faithful representation of $P\Sigma_n$ into Aut $(F(x_1, \ldots, x_n))$ induced by sending the generators of $P\Sigma_n$



to automorphisms

$$\alpha_{ij}(x_k) = \begin{cases} x_k & k \neq i \\ x_j^{-1} x_i x_j & k = i \end{cases} .$$

The image in Aut(F_n) is referred to as the group of *pure symmetric automorphisms* since it is the subgroup of automorphisms where each generator is sent to a conjugate of itself.

Thinking of $P\Sigma_n$ as a subgroup of Aut(F_n) we can form the image of $P\Sigma_n$ in Out(F_n), denoted $OP\Sigma_n$.

Some of What's Known

- $P\Sigma_n$ contains PB_n.
- $P\Sigma_n$ has cohomological dimension $n-1$.
  (Collins, *CMH* '89)
- $P\Sigma_n$ has a regular language of normal forms.
  (Guttiérrez and Krstić, *IJAC* '98)

Our Results

Theorem A. *$P\Sigma_{n+1}$ is an n-dim'l duality group.*
(Brady-M-Meier-Miller, *J. Algebra*, '01)

Theorem B. *The ℓ^2-Betti numbers of $P\Sigma_{n+1}$ are all trivial except in top dimension, where*

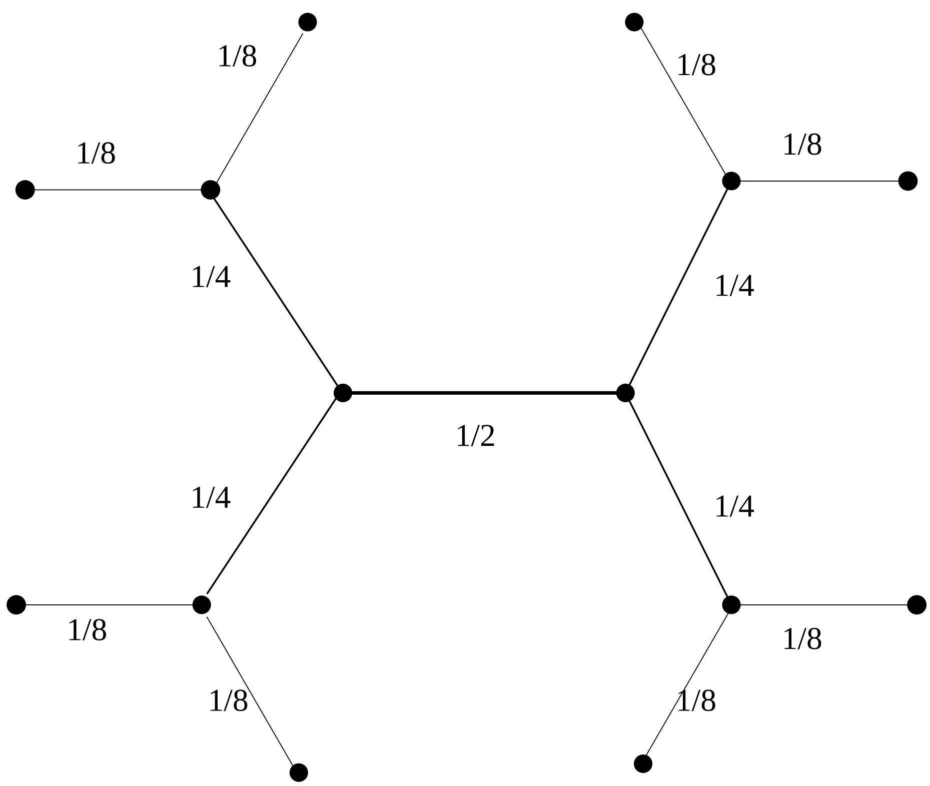
$$\chi(P\Sigma_{n+1}) = (-1)^n b_n^{(2)} = (-1)^n n^n \ .$$

(M-Meier, *New Stuff*)

Both are cohomology computations that occur in the universal cover of a $K(P\Sigma_{n+1}, 1)$. While both have to do with asymptotic properties of $P\Sigma_{n+1}$, the proofs ultimately boil down to interesting combinatorial arguments.

ℓ^2-Cohomology

For a group G (admitting a finite $K(G,1)$) let $\ell^2(G)$ be the Hilbert space of square-summable functions. The classic cocycle is:



In general, concrete computations are rare. One of the few is due to Davis and Leary who compute the ℓ^2-cohomology of arbitrary right-angled Artin groups (to appear, *Proc. LMS*).

Duality Groups

Def: (Bieri-Eckmann, *Invent. Math.* '73) A group G, with a finite $K(G, 1)$, X, is an n-*dimensional duality group* if ...

$H_c^*(\widetilde{X}) = H^*(G, \mathbb{Z}G)$ is torsion-free and concentrated in dimension n.

$$\Updownarrow$$

There is a G-module D such that

$$H^i(G, M) \simeq H_{n-i}(G, D \otimes M)$$

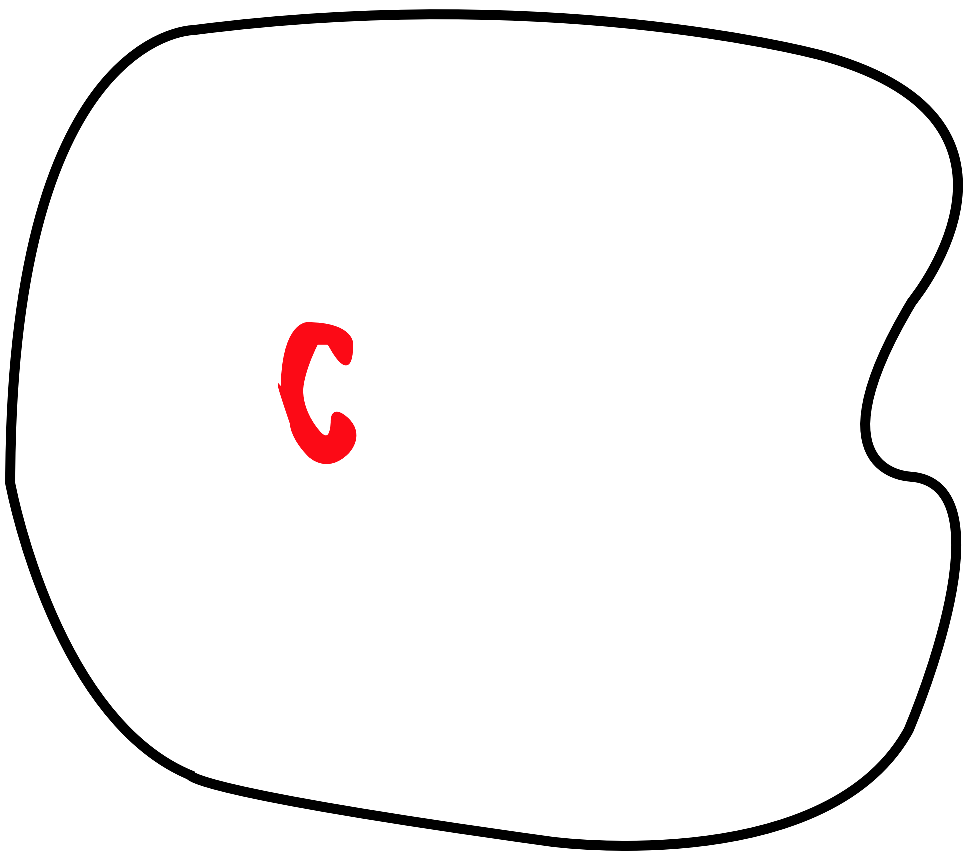
for all i and G-modules M.

$$\Updownarrow$$

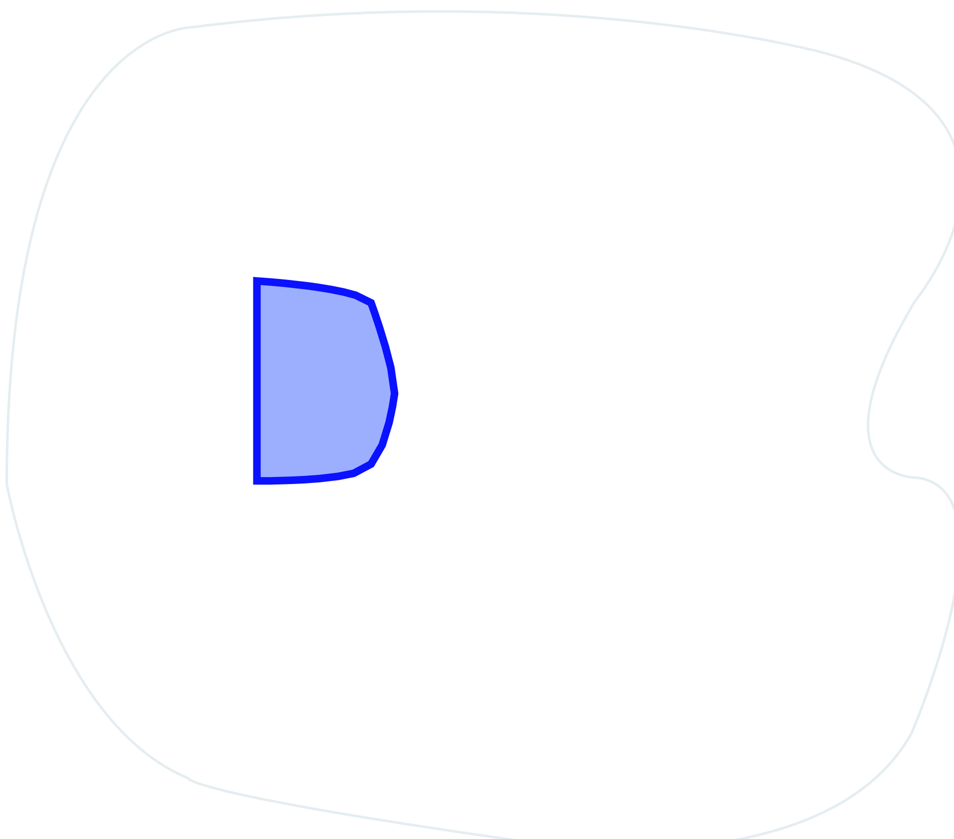
The universal cover $\widetilde{X}$ is $(n-2)$-acyclic at infinity. (Geoghegan-Mihalik, *JPAA* '85)

Acyclic at Infinity

Let X be a finite $K(\pi, 1)$. Then $\widetilde{X}$ is m-acyclic at infinity if given any compact $C \subset \widetilde{X}$,

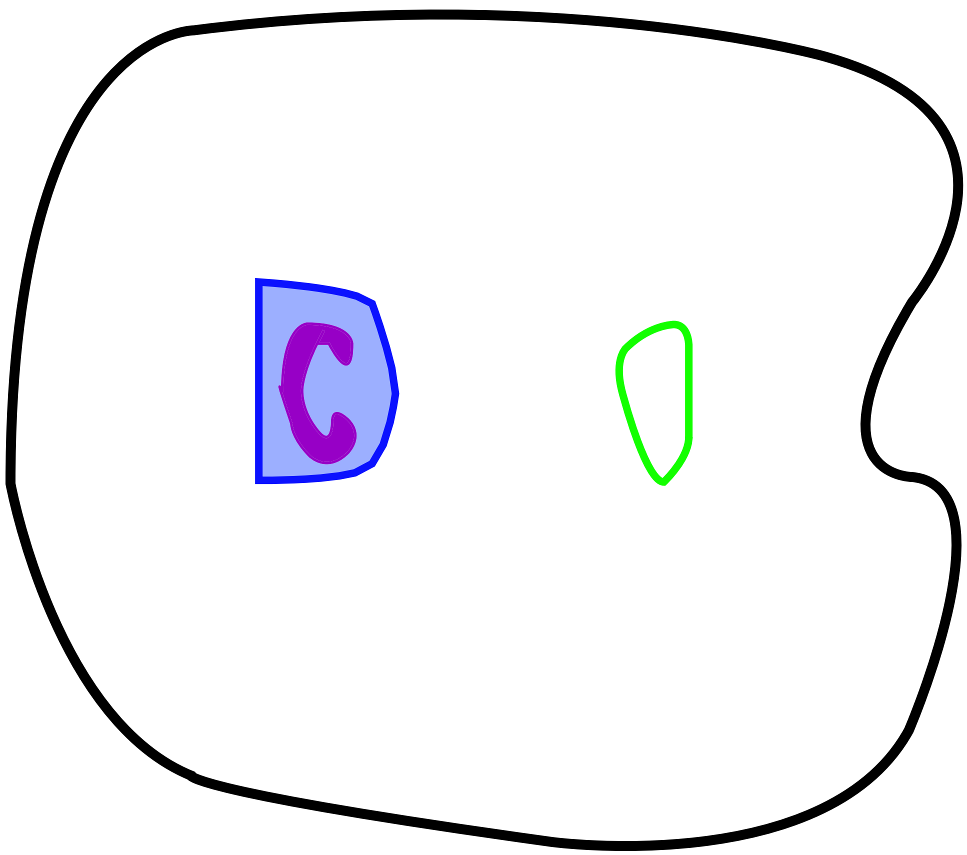


there is a compact $D \supset C$

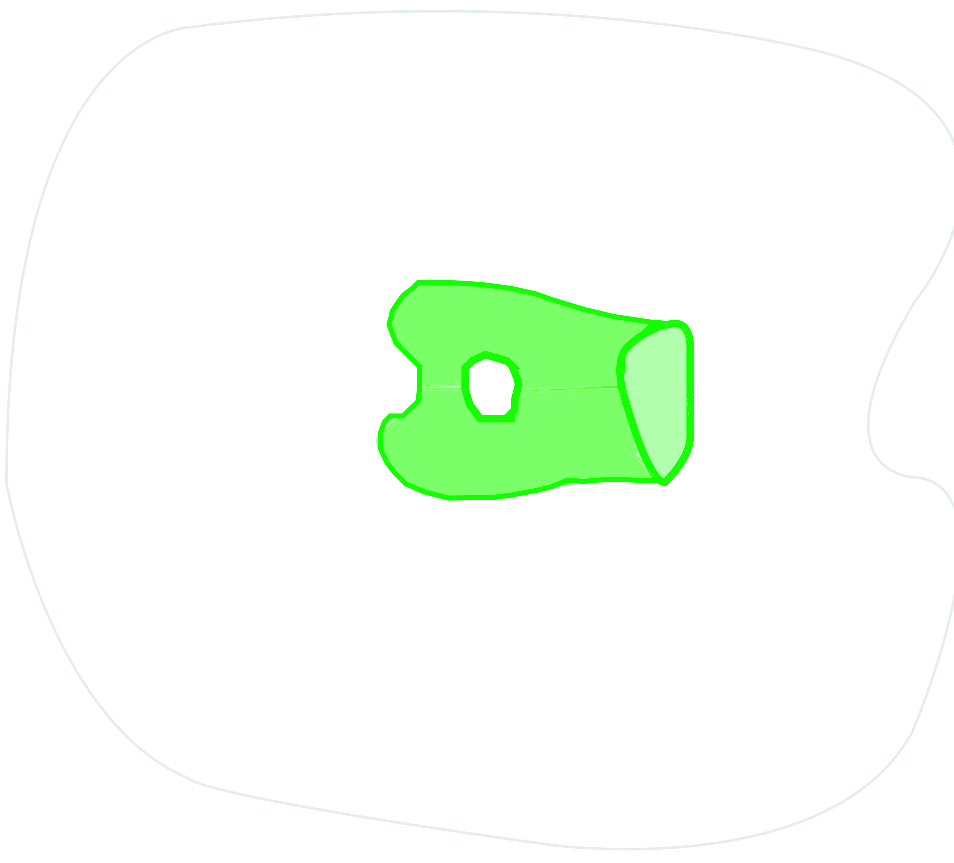


Acyclic at Infinity, II

Let X be a finite $K(\pi, 1)$. Then $\widetilde{X}$ is m-acyclic at infinity if given any compact $C \subset \widetilde{X}$, there is a compact $D \supset C$ such that every k-cycle supported in $\widetilde{X} - D$



is the boundary of a $(k+1)$-chain supported in $\widetilde{X}-C$. $(-1 \leq k \leq m)$



So duality groups are groups which are as acyclic at infinity as they can possibly be.

Examples of (Virtual) Duality Groups

- Braid groups as well as all Artin groups of finite type. (Squier, *Math. Scand.* 1995, or Bestvina, *Geom. & Top.* 1999)

- Mapping class groups of surfaces. (Harer, *Invent. Math.* 1986)

- Out(F_n) and Aut(F_n). (Bestvina and Feighn, *Invent. Math.* 2000)

- Groups like $SL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z}[1/p])$. (Borel and Serre, *CMH* 1974, *Topology* 1976)

McCullough-Miller Complex

The cohomology computations are done via an action of $OP\Sigma_n$ on a contractible simplicial complex MM_n, constructed by McCullough and Miller (*MAMS*, '96).

The complex MM_n is a space of F_n-actions on simplicial trees, where the actions all take the decomposition of F_n as a free product

$$F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ copies}}$$

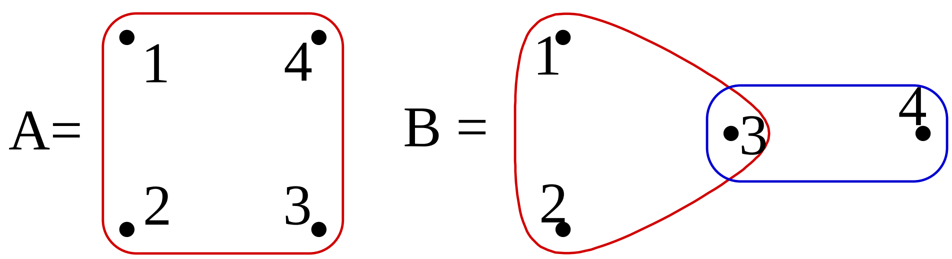
seriously.

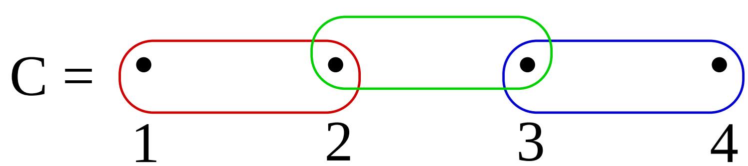
Each action in this space can be described by a marked hypertree ...

Hypertrees

Def: A *hypertree* is a connected hypergraph with no hypercycles.

In hypergraphs, the "edges" are subsets of the vertices, not just pairs of vertices.





The growth is quite dramatic: The number of hypertrees on $[n]$, for $n \geq 3$ is $=$

$\{4, 29, 311, 4447, 79745, 1722681, 43578820, \ldots\}$

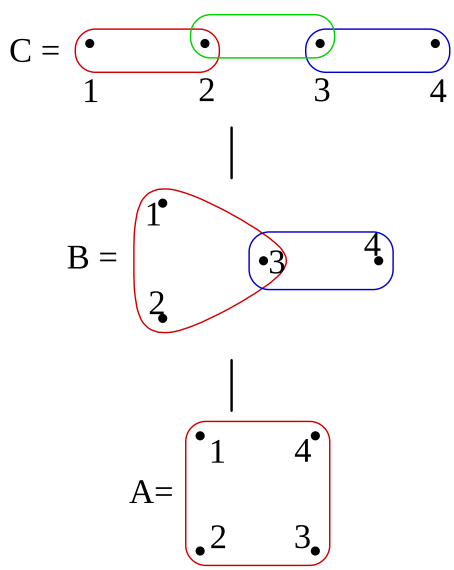
(Smith and Warme,Kalikow)

Hypertree Poset

The hypertrees on $[n]$ form a very nice poset, that is surprisingly unstudied in combinatorics.

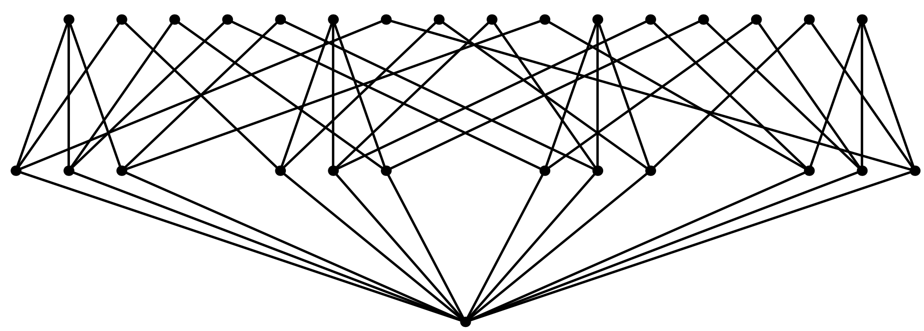
The elements of HT_n are n-vertex hypertrees with the vertices labelled by $[n] = \{1, \ldots, n\}$. The order relation is given by:
$\tau < \tau' \Leftrightarrow$ each hyperedge of τ' is contained in a hyperedge of τ.

The hypertree with only one edge is $\widehat{0}$, also called the *nuclear* element. If one adds a formal $\widehat{1}$ such that $\tau < \widehat{1}$ for all $\tau \in \mathsf{HT}_n$, the resulting poset is $\widehat{\mathsf{HT}}_n$.



Properties of $\mathbf{HT}_n$

The Hasse diagram of HT_4 is



Thm: $\widehat{\mathsf{HT}}_n$ is a finite lattice that is graded, bounded, and Cohen-Macaulay.

• Finite and Bounded are easy.

• Lattice is easy based on the similarities between HT_n and the partition lattice. (Lattice is the key element in the McCullough-Miller proof that MM_n is contractible.)

Cohen-Macaulay

A poset is *Cohen-Macaulay* if its geometric realization is Cohen-Macaulay, that is,

$$\widetilde{H}_i(\mathrm{lk}(\sigma), \mathbb{Z}) = 0$$

for all simplices σ (including the empty simplex) and all $i < \dim(\mathrm{lk}(\sigma))$.

(X Cohen-Macaulay $\Rightarrow$ X is h.e. to a bouquet of spheres.)

The Cohen-Macaulay property is actually the key step in BM^3's proof that $P\Sigma_n$ is a duality group.

We show HT_n is Cohen-Macaulay by showing that ...

$\widehat{\mathrm{HT}}_n$ is shellable, which we get by ...

Proving $\widehat{\mathrm{HT}}_n$ admits a recursive atom ordering.

Properties of MM_n

The McCullough-Miller space, MM_n, is the geometric realization of a poset of *marked* hypertrees. The marking is similar (and related) to the marked graph construction for outer space.

Some Useful Facts:

• MM_n admits $P\Sigma_n$ and $OP\Sigma_n$ actions.

• The fundamental domain for either action is the same, it's finite and isomorphic to the order complex of HT_n (also known as the *Whitehead* poset).

• The isotropy groups for the $OP\Sigma_n$ action are free abelian; the isotropy groups are free-by-(free abelian) for the action of $P\Sigma_n$.

Good News/Bad News

The asymptotic topology of a group G is the asymptotic topology of the universal cover of a $K(G,1)$.

Good News: We have a contractible, cocompact $P\Sigma_n$-complex.

Bad News: The action isn't free or even proper.

Good News: The stabilizers are well understood.

Punch Line: In order to understand the asymptotic topology of $P\Sigma_n$ we don't want to study the asymptotic topology of MM_n. We do want to understand the combinatorics of HT_n and the isotropy groups.

Proving Duality

You can prove that a group is a duality group by showing the cohomology with group ring coefficients is trivial, except in top dimension where it's torsion-free.

Idea: Use the equivariant spectral sequence with $\mathbb{Z}G$ coefficients

$$E_1^{pq} = \prod_{|\sigma|=p} H^q(G_\sigma, \mathbb{Z}G) \Rightarrow H^{p+q}(G, \mathbb{Z}G)$$

for the action of $OP\Sigma_n$ on MM_n.

Problem: The size of the isotropy groups for the action on the *poset* corresponds with the corank of the elements. But it does not correspond well with the dimension of simplices in the geometric realization.

The First Page

The standard equivariant spectral sequence applied to the action of $OP\Sigma_5$ on MM_5 has its first page something like:

$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	0	0	0
$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	0	0
$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	0
$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	$\bigoplus_{i=0}^{\infty} \mathbb{Z}$	$\bigoplus_{i=0}^{\infty} \mathbb{Z}$

A Better Idea

Filter MM_n by the poset rank not dimension. Then the first page becomes

$$
\begin{array}{cccc}
\bigoplus_{i=0}^{\infty} \mathbb{Z} & 0 & 0 & 0 \\
0 & \bigoplus_{i=0}^{\infty} \mathbb{Z} & 0 & 0 \\
0 & 0 & \bigoplus_{i=0}^{\infty} \mathbb{Z} & 0 \\
0 & 0 & 0 & \bigoplus_{i=0}^{\infty} \mathbb{Z}
\end{array}
$$

It is this simple because the fundamental domain is Cohen-Macaulay.
(Brown-Meier, *CMH* ‘00)

ℓ^2-Betti Numbers

We compute the ℓ^2-Betti numbers of $OP\Sigma_{n+1}$ via its action on MM_{n+1}. In order to do this we have to switch to an algebraic standpoint, using group cohomology with coefficients in the group von Neumann algebra $\mathcal{N}(G)$.

We also are really computing the equivariant ℓ^2-Betti numbers of the action of $OP\Sigma_{n+1}$ on MM_{n+1}. We can get away with this because

Lemma. *The ℓ^2-cohomology of $\mathbb{Z}^n$ is trivial.*

Lemma. *Let X be a contractible G-complex. Suppose that each isotropy group G_σ is finite or satisfies $b_p^{(2)}(G_\sigma) = 0$ for $p \geq 0$. Then $b_p^{(2)}(X, \mathcal{N}(G)) = b_p^{(2)}(G)$ for $p \geq 0$.*

(cf. Lück's L^2*-Invariants: Theory and Applications ...*)

Reduction to Euler characteristics

In looking at the resulting equivariant spectral sequence we find we are really looking at the homology of

$$\mathsf{HT}^{\circ}_{n+1} = \mathsf{HT}_{n+1} - \{\text{the nuclear vertex}\}$$

(this is the singular set for the $OP\Sigma_{n+1}$ action.)

Since this poset is Cohen-Macaulay, all we really care about is

$$\text{rank}\left(H_{n-2}(\mathsf{HT}^{\circ}_{n+1})\right) = |\widetilde{\chi}(\mathsf{HT}^{\circ}_{n+1})|$$

and so computing the ℓ^2-Betti numbers of the group $OP\Sigma_{n+1}$ has boiled down to computing the Euler characteristic of the poset $\mathsf{HT}^{\circ}_{n+1}$.

Reduction to Möbius functions

Realizing we need to compute $\widetilde{\chi}(\mathsf{HT}^\circ_{n+1})$ we start filling up chalk boards with Hasse diagrams and compute ...

$$\chi(\mathsf{HT}^\circ_4) = 28 - 36 = -8$$
$$\chi(\mathsf{HT}^\circ_5) = 310 - 855 + 610 = 65$$

etc.

Luckily, Euler characteristics are well studied in enumerative combinatorics. In particular we can get to the Euler characteristic of HT°_{n+1} by studying the Möbius function μ of $\widehat{\mathsf{HT}}_{n+1}$.

Fact: If μ is the Möbius function of $\widehat{\mathsf{HT}}_{n+1}$ then $\mu(\widehat{0}, \widehat{1}) = \widetilde{\chi}(\mathsf{HT}^\circ_{n+1})$

$$\widetilde{\chi}(\mathsf{HT}^\circ_4) = -9$$
$$\widetilde{\chi}(\mathsf{HT}^\circ_5) = 64$$

Exponential generating functions

The *weight* of a hypertree on $[n]$ is $u_2^{\lambda_2} \cdots u_n^{\lambda_n}$ where λ_i counts the number of i-edges.

Let T_n be the sum of all the weights of hypertrees on $[n]$, and let R_n be the sum of all the weights of rooted hypertrees on $[n]$.

$T_3 = u_3 + 3u_2^2$
$T_4 = u_4 + 12u_2u_3 + 16u_2^3$
$R_n = n \cdot T_n$

Let $T = \sum_n T_n \frac{t^n}{n!}$ and let $R = \sum_n R_n \frac{t^n}{n!}$

Thm(Kalikow) $R = te^y$ where

$$y = \sum_{j \geq 1} u_{j+1} \frac{R^j}{j!}$$

The Calculation and Its Corollaries

Using various recursion formulas for Möbius functions, and Kalikow's functional equation, it only takes 3 or 4 pages of work to show:

Thm: $\widetilde{\chi}(\mathsf{HT}^\circ_{n+1}) = (-1)^n n^{n-1}$.

Cor 1: *The* ℓ^2*-Betti numbers of* $OP\Sigma_{n+1}$ *are trivial, except* $b^{(2)}_{n-1} = n^{n-1}$*. It follows that*

$$b^{(2)}_{n-1}(O\Sigma_{n+1}) = \frac{n^{n-1}}{(n+1)!} \ .$$

Cor 2: *The* ℓ^2*-Betti numbers of* $P\Sigma_{n+1}$ *are trivial, except* $b^{(2)}_{n} = n^{n}$*. It follows that*

$$b^{(2)}_{n}(\Sigma_{n+1}) = \frac{n^{n}}{(n+1)!} \ .$$

Open Questions

- Is $P\Sigma_n$ (bi)automatic?
  (Guttiérrez and Krstić)

- Is $P\Sigma_n$ CAT(0)?
  (The complex MM_n is not CAT(0))

- Is $P\Sigma_n$ linear?

- What about the motion groups of other links?

- What about the motion groups of higher dimensional links? (2-spheres in $\mathbb{R}^4$, for example.)

- What's the asymptotic topology of Aut($G_1 * \cdots * G_n$) where $|G_i| < \infty$?