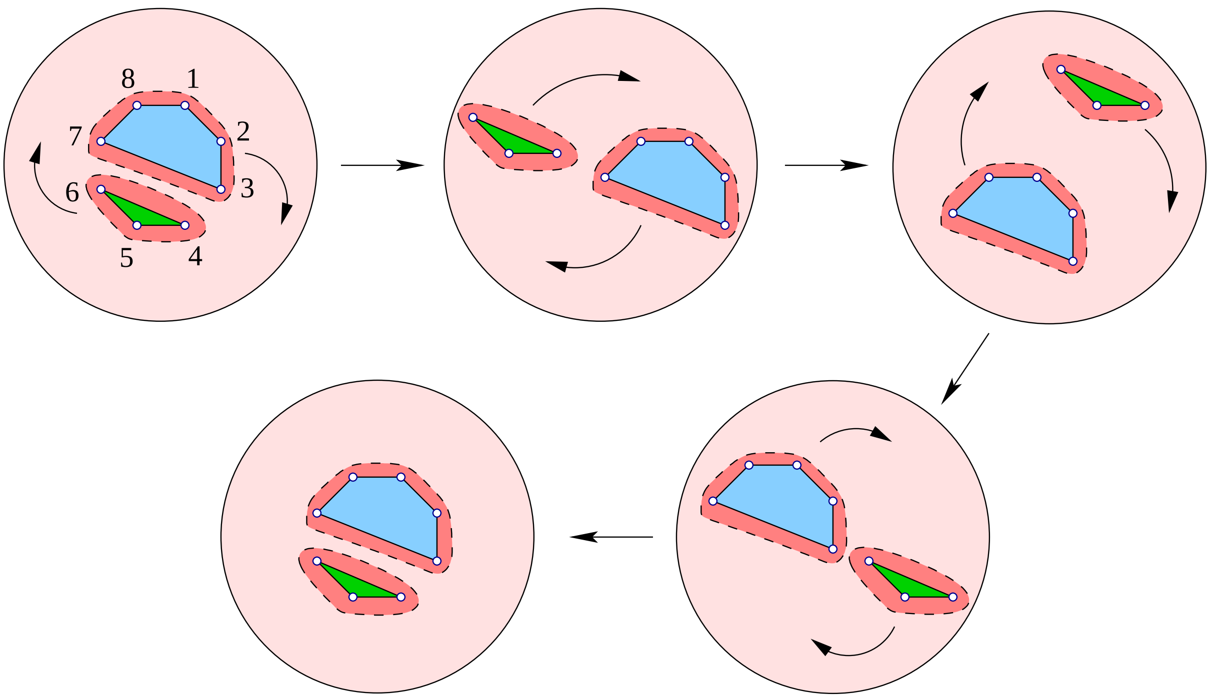
Square Dancing in the Pure Braid Group



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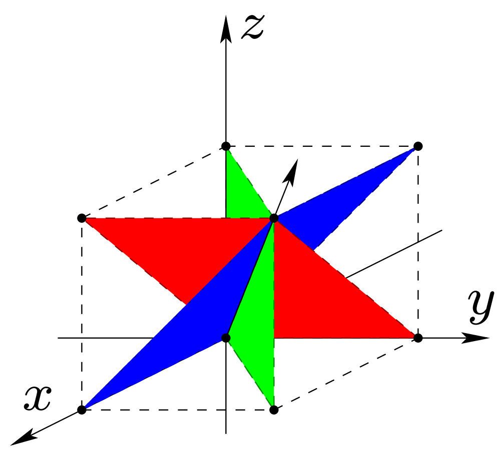
The braid arrangement

The (real) *braid arrangement* is the space of all n-tuples of distinct real numbers

$$(x_1, x_2, \ldots, x_n)$$

Ex: In $\mathbb{R}^2$ this consists of the stuff above the line $y = x$ and the stuff below the line $y = x$.

Ex: In $\mathbb{R}^3$ it consists of 6 connected pieces separated by the planes $x = y$, $x = z$ and $y = z$.



Ex: In $\mathbb{R}^n$ it has $n!$ connected pieces separated by the hyperplanes $\{x_i = x_j\}$.

The complexified braid arrangement

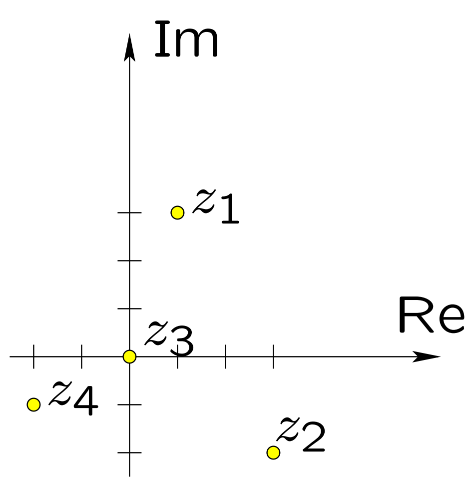
The *complexified braid arrangement* is the space of all n-tuples of distinct complex numbers

$$(z_1, z_2, \ldots, z_n)$$

There's a trick to visualizing this space.

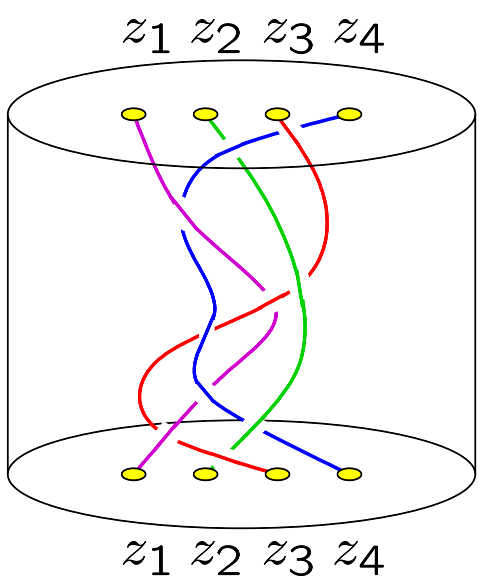
Ex: The figure below encodes the point

$$(z_1, z_2, z_3, z_4) = (1 + 3i, 3 - 2i, 0, -2 - i)$$



The braid group

Moving around in the complexified braid arrangement corresponds to moving the labeled points in the complex plane without letting them collide.



If we keep track of this movement by tracing out what happens over time we see actual braided strings—hence the name.

Note: The path I've drawn is a non-trivial closed loop.

Artin's presentation

Artin's presentation of the pure braid group is generated by elements S_{ij}, $(1 \leq i < j \leq n)$ and subject to the following four types of relations of the form

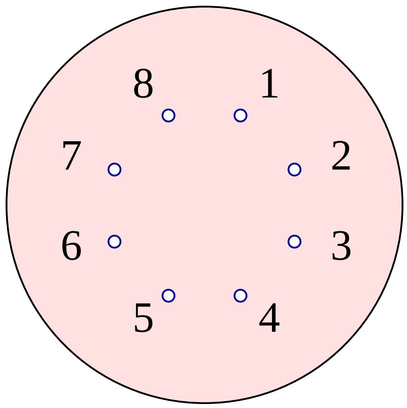
$$S_{rs}^{-1} S_{ij} S_{rs} = U S_{ij} U^{-1}$$

where

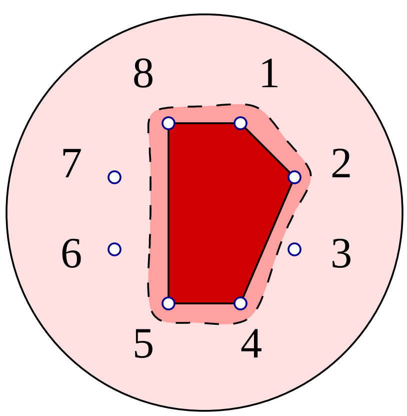
$U = 1$ when $r < s < i < j$ or $i < r < s < j$,
$U = S_{rj}$ when $r < s = i < j$,
$U = S_{ij} S_{sj}$ when $r = i < j < s$, and
$U = S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1}$ when $r < i < s < j$.

Convexly punctured discs

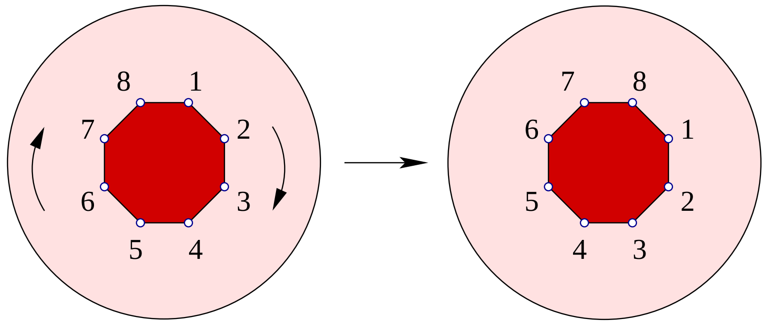
A convexly punctured disc with 8 punctures and a standard labeling.



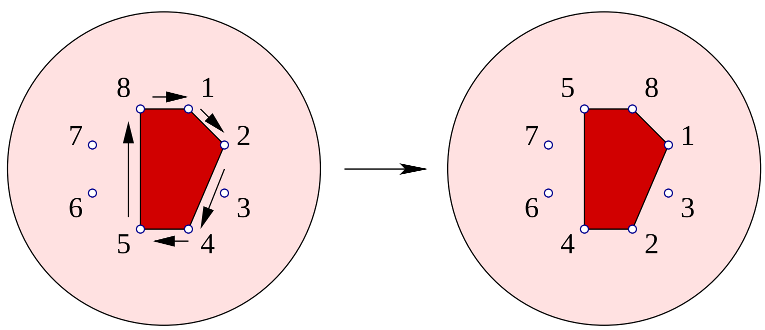
The convexly punctured subdisc D_B when $B = \{1, 2, 4, 5, 8\}$.



Rotations

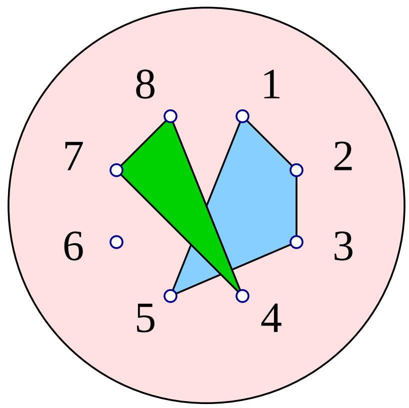


Rotating all the punctures.

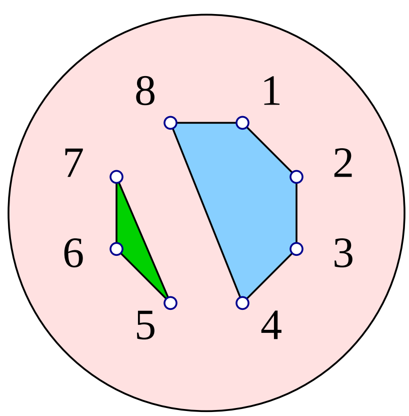


The rotation R_B inside $_{[8]}$ when $B = \{1, 2, 4, 5, 8\}$.

Noncrossing sets

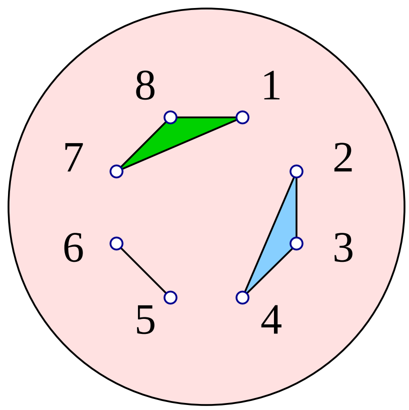


The subsets $\{1, 2, 3, 5\}$ and $\{4, 7, 8\}$ are crossing subsets of $[8]$ and the subsets $\{1, 2, 3, 4, 8\}$ and $\{5, 6, 7\}$ are noncrossing.

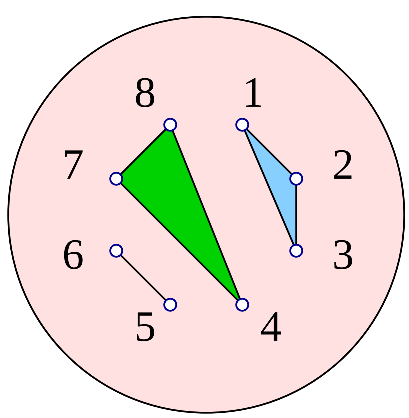


Admissible partitions

If $B = \{2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{7, 8, 1\}$ then (B, C, D) is an admissible partition, as is (C, D, B), but (C, B, D) is not.

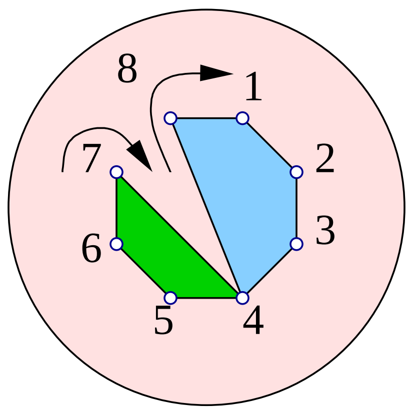


On the other hand, although the subsets $\{1, 2, 3\}$, $\{4, 7, 8\}$ and $\{5, 6\}$ are non-crossing, no ordering of these three subsets is admissible.



Factoring rotations

An illustration that $R_{45678123}$ is the same as $R_{4567} \cdot R_{48123}$.



In general $R_{iBC} = R_{iB} \cdot R_{iC}$ when $(\{i\}, B, C)$ is admissible.

We are using iBC as an abbreviation of $\{i\} \cup B \cup C$.

Presenting the braid group

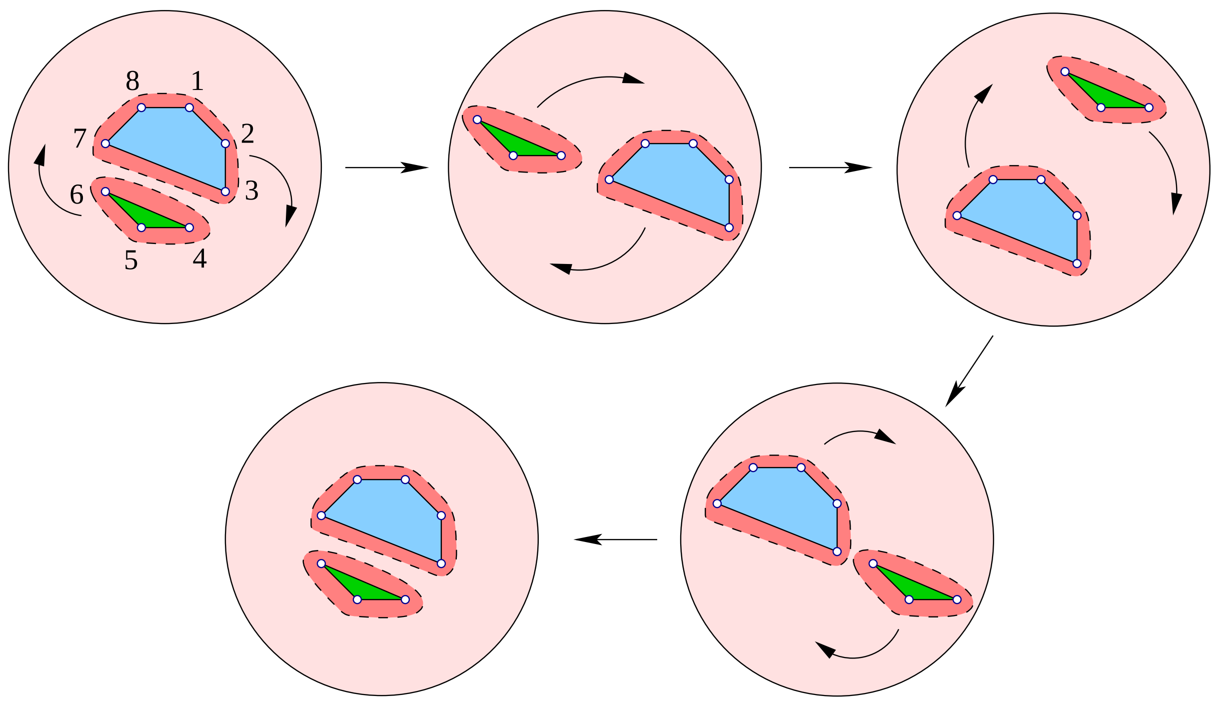
Thm (Margalit-M) The braid group has a finite presentation generated by the rotations R_B where B is a subset of the punctures and the only relations we need are those which assert that

1. Noncrossing rotations commute
2. Rotations decompose

Pf: These relations hold geometrically, and they include all of the Birman-Ko-Lee generators and relations.

Square dancing

We call the following motion a do-si-do.

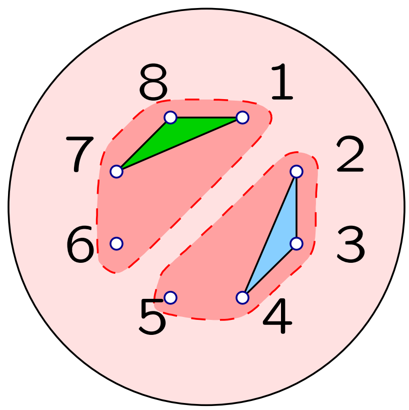


If B and C are the two sets of punctures doing the dancing, we denote this $T_{B,C}$ (since we are twisting B and C).

If $B \cup C$ isn't all the punctures we do the twisting inside the convexly punctured subdisc D_{BC}.

Noncrossing do-si-dos commute

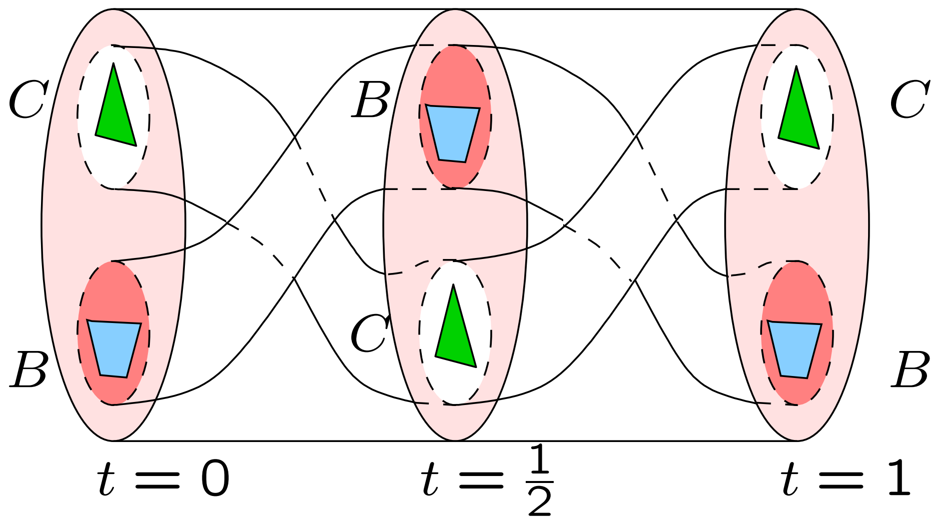
It should be clear that noncrossing do-si-dos commute since they take place inside disjoint subdiscs.



For example, the do-si-dos $T_{234,5}$ and $T_{6,781}$ commute.

Nested do-si-dos commute

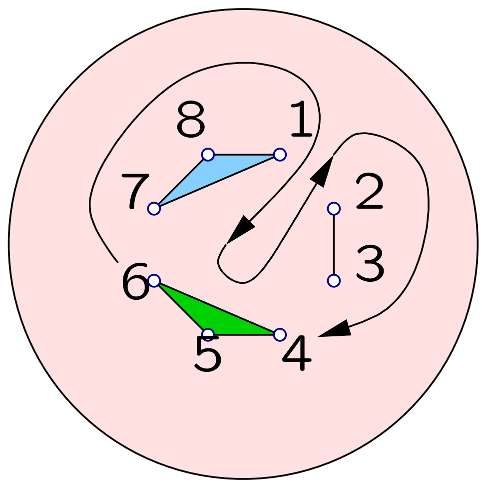
Do-si-dos $T_{B,C}$ and $T_{D,E}$ are called *nested* when $(D \cup E)$ is a subset of B (or a subset of C).



The 3-dimensional trace of a convex twist over time. Because the tube containing B is internally untwisted, any action taking place inside the subdisc D_B commutes with the do-si-do $T_{B,C}$.

Do-si-dos decompose

The third type of relation describes how do-si-dos decompose into smaller dance moves.



For example, $T_{B,CD} = T_{B,C} \cdot T_{B,D}$ when $\{B,C,D\}$ is an admissible partition of $B \cup C \cup D$.

A new presentation

Thm (Margalit-M) The pure braid group has a finite presentation generated by the do-si-dos $T_{B,C}$ (where B and C are noncrossing subsets of punctures) and the only relations we need are those which assert that

1. Noncrossing do-si-dos commute
2. Nested do-si-dos commute
3. Do-si-dos decompose

Pf: The relations hold and they imply Artin's relations.

Open questions

1. Are there similarly nice presentations for the other pure Artin groups of finite type?

2. What are the combinatorics of the finite posets of factorizations of the generator of the center? They look like interesting yet new finite posets.

3. Are these finite posets Garside structures for the pure Artin groups? They are in the low cases we've checked.