NEWTON'S LAW OF COOLING OR HEATING

Let
$T =$ temperature of an object,
$M =$ temperature of its surroundings, and
$t =$ time.

If the rate of change of the temperature T of the object is directly proportional to the difference in temperature between the object and its surroundings, then we get the following equation where k is a proportionality constant.

$$\frac{dT}{dt} = k(M - T), k > 0.$$

As the differential equation is separable, we can separate the equation to have one side solely dependent on T, and the other side solely dependent on t:

$$\frac{dT}{M - T} = kdt$$

Integrating both sides then gives the following:

$$\begin{aligned} \int \frac{dT}{M-T} &= \int kdt \\ -\ln|M-T| &= kt + C \\ \ln|M-T| &= -kt - C \\ e^{\ln|M-T|} &= e^{-kt-C} \\ |M-T| &= e^{-kt-C} \end{aligned}$$

Now here is where we need to be careful. We want to drop the absolute value signs to solve for T. To do so, we need to figure out whether $M - T$ is positive or negative. This depends on whether the object is cooling down to the surrounding temperature (in which case $T > M$ and $M - T$ is negative) or is warming up to the surrounding temperature ($T < M$ and $M - T$ is positive).

For *cooling*, as $M - T$ is negative, $|M - T| = -(M - T)$. So we get

$$\begin{aligned} |M-T| &= e^{-kt-C} \\ -(M-T) &= e^{-kt-C} \\ M-T &= -e^{-kt-C} \\ T &= M + e^{-kt-C} \\ T &= M + Ae^{-kt}, A = e^{-C} \end{aligned}$$

Since the object is cooling down to the surrounding temperature, T will always be greater than M so A will be a positive value. This agrees with the fact that $A = e^{-C}$ must be a positive value.

For *heating*, $M - T$ is positive, and so $|M - T| = (M - T)$ and we get

$$\begin{aligned} |M - T| &= e^{-kt-C} \\ M - T &= e^{-kt-C} \\ T &= M - e^{-kt-C} \\ T &= M - Ae^{-kt}, A = e^{-C} \end{aligned}$$

This time, as the object is warming up to the surrounding temperature, T is always less than M so A is again a positive value.