ETS® GRE®

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GRADUATE RECORD EXAMINATIONS®

Mathematics Test Practice Book

This practice book contains

- one actual, full-length GRE® Mathematics Test
- test-taking strategies

Become familiar with

- test structure and content
- test instructions and answering procedures

Compare your practice test results with the performance of those who took the test at a GRE administration.

This book is provided **FREE** with test registration by the Graduate Record Examinations Board.

www.ets.org/gre

Note to Test Takers: Keep this practice book until you receive your score report.
This book contains important information about scoring.

Table of Contents

Purpose of the GRE Subject Tests

The GRE Subject Tests are designed to help graduate school admission committees and fellowship sponsors assess the qualifications of applicants in specific fields of study. The tests also provide you with an assessment of your own qualifications.

Scores on the tests are intended to indicate knowledge of the subject matter emphasized in many undergraduate programs as preparation for graduate study. Because past achievement is usually a good indicator of future performance, the scores are helpful in predicting success in graduate study. Because the tests are standardized, the test scores permit comparison of students from different institutions with different undergraduate programs. For some Subject Tests, subscores are provided in addition to the total score; these subscores indicate the strengths and weaknesses of your preparation, and they may help you plan future studies.

The GRE Board recommends that scores on the Subject Tests be considered in conjunction with other relevant information about applicants. Because numerous factors influence success in graduate school, reliance on a single measure to predict success is not advisable. Other indicators of competence typically include undergraduate transcripts showing courses taken and grades earned, letters of recommendation, and GRE General Test scores. For information about the appropriate use of GRE scores, see the *GRE Guide to the Use of Scores* at **ets.org/gre/stupubs**.

Development of the Subject Tests

Each new edition of a Subject Test is developed by a committee of examiners composed of professors in the subject who are on undergraduate and graduate faculties in different types of institutions and in different regions of the United States and Canada. In selecting members for each committee, the GRE Program seeks the advice of the appropriate professional associations in the subject.

The content and scope of each test are specified and reviewed periodically by the committee of examiners. Test questions are written by committee members and by other university faculty members who are subject-matter specialists. All questions proposed for the test are reviewed and revised by the committee and subject-matter specialists at ETS. The tests are assembled in accordance with the content specifications developed by the committee to ensure adequate coverage of the various aspects of the field and, at the same time, to prevent overemphasis on any single topic. The entire test is then reviewed and approved by the committee.

Subject-matter and measurement specialists on the ETS staff assist the committee, providing information and advice about methods of test construction and helping to prepare the questions and assemble the test. In addition, each test question is reviewed to eliminate language, symbols, or content considered potentially offensive, inappropriate for major subgroups of the test-taking population, or likely to perpetuate any negative attitude that may be conveyed to these subgroups.

Because of the diversity of undergraduate curricula, it is not possible for a single test to cover all the material you may have studied. The examiners, therefore, select questions that test the basic knowledge and skills most important for successful graduate study in the particular field. The committee keeps the test up-to-date by regularly developing new editions and revising existing editions. In this way, the test content remains current. In addition, curriculum surveys are conducted periodically to ensure that the content of a test reflects what is currently being taught in the undergraduate curriculum.

After a new edition of a Subject Test is first administered, examinees' responses to each test question are analyzed in a variety of ways to determine whether each question functioned as expected. These analyses may reveal that a question is ambiguous, requires knowledge beyond the scope of the test, or is inappropriate for the total group or a particular subgroup of examinees taking the test. Such questions are not used in computing scores.

Following this analysis, the new test edition is equated to an existing test edition. In the equating process, statistical methods are used to assess the difficulty of the new test. Then scores are adjusted so that examinees who took a more difficult edition of the test are not penalized, and examinees who took an easier edition of the test do not have an advantage. Variations in the number of questions in the different editions of the test are also taken into account in this process.

Scores on the Subject Tests are reported as three-digit scaled scores with the third digit always zero. The maximum possible range for all Subject Test total scores is from 200 to 990. The actual range of scores for a particular Subject Test, however, may be smaller. For Subject Tests that report subscores, the maximum possible range is 20 to 99; however, the actual range of subscores for any test or test edition may be smaller. Subject Test score interpretive information is provided in *Interpreting Your GRE Scores*, which you will receive with your GRE score report. This publication is also available at **ets.org/gre/stupubs.**

Content of the Mathematics Test

The test consists of approximately 66 multiple-choice questions drawn from courses commonly offered at the undergraduate level. Approximately 50 percent of the questions involve calculus and its applications—subject matter that can be assumed to be common to the backgrounds of almost all mathematics majors. About 25 percent of the questions in the test are in elementary algebra, linear algebra, abstract algebra, and number theory. The remaining questions deal with other areas of mathematics currently studied by undergraduates in many institutions.

The following content descriptions may assist students in preparing for the test. The percents given are estimates; actual percents will vary somewhat from one edition of the test to another.

Calculus—50%

- Material learned in the usual sequence of elementary calculus courses—differential and integral calculus of one and of several variables—includes calculus-based applications and connections with coordinate geometry, trigonometry, differential equations, and other branches of mathematics

Algebra—25%

- Elementary algebra: basic algebraic techniques and manipulations acquired in high school and used throughout mathematics
- Linear algebra: matrix algebra, systems of linear equations, vector spaces, linear transformations, characteristic polynomials, and eigenvalues and eigenvectors
- Abstract algebra and number theory: elementary topics from group theory, theory of rings and modules, field theory, and number theory

Additional Topics—25%

- Introductory real analysis: sequences and series of numbers and functions, continuity, differentiability and integrability, and elementary topology of $\mathbb{R}$ and $\mathbb{R}^n$
- Discrete mathematics: logic, set theory, combinatorics, graph theory, and algorithms
- Other topics: general topology, geometry, complex variables, probability and statistics, and numerical analysis

The above descriptions of topics covered in the test **should not be considered exhaustive**; it is necessary to understand many other related concepts. Prospective test takers should be aware that questions requiring no more than a good precalculus background may be quite challenging; such questions can be among the most difficult questions on the test. In general, the questions are intended not only to test recall of information but also to assess test takers' understanding of fundamental concepts and the ability to apply those concepts in various situations.

Preparing for a Subject Test

GRE Subject Test questions are designed to measure skills and knowledge gained over a long period of time. Although you might increase your scores to some extent through preparation a few weeks or months before you take the test, last minute cramming is unlikely to be of further help. The following information may be helpful.

- A general review of your college courses is probably the best preparation for the test. However, the test covers a broad range of subject matter, and no one is expected to be familiar with the content of every question.
- Use this practice book to become familiar with the types of questions in the GRE Mathematics Test, taking note of the directions. If you understand the directions before you take the test, you will have more time during the test to focus on the questions themselves.

Test-Taking Strategies

The questions in the practice test in this book illustrate the types of multiple-choice questions in the test. When you take the actual test, you will mark your answers on a separate machine-scorable answer sheet. Total testing time is two hours and fifty minutes; there are no separately timed sections. Following are some general test-taking strategies you may want to consider.

- Read the test directions carefully, and work as rapidly as you can without being careless. For each question, choose the best answer from the available options.
- All questions are of equal value; do not waste time pondering individual questions you find extremely difficult or unfamiliar.
- You may want to work through the test quite rapidly, first answering only the questions about which you feel confident, then going back and answering questions that require more thought, and concluding with the most difficult questions if there is time.
- If you decide to change an answer, make sure you completely erase it and fill in the oval corresponding to your desired answer.
- Questions for which you mark no answer or more than one answer are not counted in scoring.
- Your score will be determined by subtracting one-fourth the number of incorrect answers from the number of correct answers. If you have some knowledge of a question and are able to rule out one or more of the answer choices as incorrect, your chances of selecting the correct answer are improved, and answering such questions will likely improve your score. It is unlikely that pure guessing will raise your score; it may lower your score.
- Record all answers on your answer sheet. Answers recorded in your test book will not be counted.
- Do not wait until the last five minutes of a testing session to record answers on your answer sheet.

What Your Scores Mean

Your raw score — that is, the number of questions you answered correctly minus one-fourth of the number you answered incorrectly — is converted to the scaled score that is reported. This conversion ensures that a scaled score reported for any edition of a Subject Test is comparable to the same scaled score earned on any other edition of the same test. Thus, equal scaled scores on a particular Subject Test indicate essentially equal levels of performance regardless of the test edition taken. Test scores should be compared only with other scores on the same Subject Test. (For example, a 680 on the Computer Science Test is not equivalent to a 680 on the Mathematics Test.)

Before taking the test, you may find it useful to know approximately what raw scores would be required to obtain a certain scaled score. Several factors influence the conversion of your raw score to your scaled score, such as the difficulty of the test edition and the number of test questions included in the computation of your raw score. Based on recent editions of the Mathematics Test, the following table gives the range of raw scores associated with selected scaled scores for three different test editions. (Note that when the number of scored questions for a given test is greater than the number of actual scaled score points, it is likely that two or more raw scores will convert to the same scaled score.) The three test editions in the table that follows were selected to reflect varying degrees of difficulty. Examinees should note that future test editions may be somewhat more or less difficult than the test editions illustrated in the table.

Range of Raw Scores* Needed to Earn Selected Scaled Score on Three Mathematics Test Editions that Differ in Difficulty

	Raw Scores		
Scaled Score	Form A	Form B	Form C
800	49	47	45
700	39	36	35
600	28	25	25
500	18	14	16
Number of Questions Used to Compute Raw Score			
	66	66	66

*Raw Score = Number of correct answers minus one-fourth the number of incorrect answers, rounded to the nearest integer.

For a particular test edition, there are many ways to earn the same raw score. For example, on the edition listed above as "Form A," a raw score of 28 would earn a scaled score of 600. Below are a few of the possible ways in which a scaled score of 600 could be earned on the edition:

Examples of Ways to Earn a Scaled Score of 600 on the Edition Labeled as "Form A"

Raw Score	Questions Answered Correctly	Questions Answered Incorrectly	Questions Not Answered	Number of Questions Used to Compute Raw Score
28	28	0	38	66
28	32	15	19	66
28	36	30	0	66

PRACTICE TEST

To become familiar with how the administration will be conducted at the test center, first remove the answer sheet (pages 69 and 70). Then go to the back cover of the test book (page 64) and follow the instructions for completing the identification areas of the answer sheet. When you are ready to begin the test, note the time and begin marking your answers on the answer sheet.

GRE®

FORM GR0568

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GRADUATE RECORD EXAMINATIONS®

MATHEMATICS TEST

Do not break the seal
until you are told to do so.

The contents of this test are confidential.
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THIS TEST BOOK MUST NOT BE TAKEN FROM THE ROOM.

MATHEMATICS TEST

Time—170 minutes

66 Questions

Directions: Each of the questions or incomplete statements below is followed by five suggested answers or completions. In each case, select the one that is the best of the choices offered and then mark the corresponding space on the answer sheet.

Computation and scratch work may be done in this examination book.

Note: In this examination:

(1) All logarithms with an unspecified base are natural logarithms, that is, with base e.
(2) The set of all real numbers x such that $a \le x \le b$ is denoted by $[a, b]$.
(3) The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

---

1. In the *xy*-plane, the curve with parametric equations $x = \cos t$ and $y = \sin t$, $0 \le t \le \pi$, has length

(A) 3 (B) π (C) 3π (D) $\frac{3}{2}$ (E) $\frac{\pi}{2}$

---

2. Which of the following is an equation of the line tangent to the graph of $y = x + e^x$ at $x = 0$?

(A) $y = x$

(B) $y = x + 1$

(C) $y = x + 2$

(D) $y = 2x$

(E) $y = 2x + 1$

SCRATCH WORK

3. If V and W are 2-dimensional subspaces of $\mathbb{R}^4$, what are the possible dimensions of the subspace $V \cap W$?

(A) 1 only (B) 2 only (C) 0 and 1 only (D) 0, 1, and 2 only (E) 0, 1, 2, 3, and 4

---

4. Let k be the number of real solutions of the equation $e^x + x - 2 = 0$ in the interval $[0, 1]$, and let n be the number of real solutions that are <u>not</u> in $[0, 1]$. Which of the following is true?

(A) $k = 0$ and $n = 1$ (B) $k = 1$ and $n = 0$ (C) $k = n = 1$ (D) $k > 1$ (E) $n > 1$

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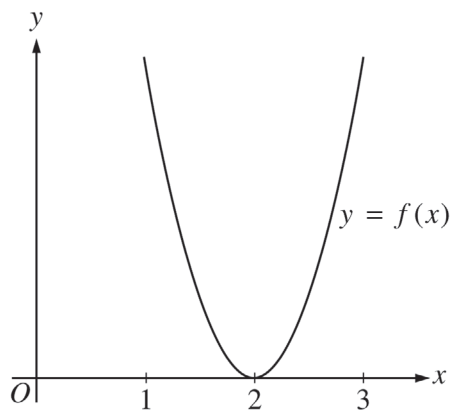
SCRATCH WORK



5. Suppose b is a real number and $f(x) = 3x^2 + bx + 12$ defines a function on the real line, part of which is graphed above. Then $f(5) =$

(A) 15 (B) 27 (C) 67 (D) 72 (E) 87

---

6. Which of the following circles has the greatest number of points of intersection with the parabola $x^2 = y + 4$?

(A) $x^2 + y^2 = 1$

(B) $x^2 + y^2 = 2$

(C) $x^2 + y^2 = 9$

(D) $x^2 + y^2 = 16$

(E) $x^2 + y^2 = 25$

SCRATCH WORK

7. $\int_{-3}^{3} |x + 1|\, dx =$

(A) 0 (B) 5 (C) 10 (D) 15 (E) 20

---

8. What is the greatest possible area of a triangular region with one vertex at the center of a circle of radius 1 and the other two vertices on the circle?

(A) $\frac{1}{2}$ (B) 1 (C) $\sqrt{2}$ (D) π (E) $\frac{1+\sqrt{2}}{4}$

$$J = \int_0^1 \sqrt{1 - x^4}\, dx$$
$$K = \int_0^1 \sqrt{1 + x^4}\, dx$$
$$L = \int_0^1 \sqrt{1 - x^8}\, dx$$

9. Which of the following is true for the definite integrals shown above?

(A) $J < L < 1 < K$

(B) $J < L < K < 1$

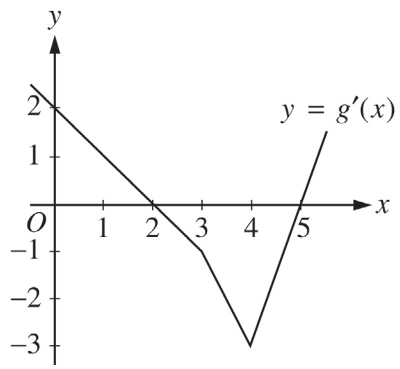
(C) $L < J < 1 < K$

(D) $L < J < K < 1$

(E) $L < 1 < J < K$

GO ON TO THE NEXT PAGE.

SCRATCH WORK



10. Let g be a function whose derivative g' is continuous and has the graph shown above. Which of the following values of g is largest?

(A) $g(1)$ (B) $g(2)$ (C) $g(3)$ (D) $g(4)$ (E) $g(5)$

---

11. Of the following, which is the best approximation of $\sqrt{1.5}\,(266)^{3/2}$?

(A) 1,000 (B) 2,700 (C) 3,200 (D) 4,100 (E) 5,300

---

12. Let A be a 2×2 matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Which of the following must be an eigenvector of A ?

I. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

II. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

III. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

SCRATCH WORK

13. A total of x feet of fencing is to form three sides of a level rectangular yard. What is the maximum possible area of the yard, in terms of x ?

(A) $\frac{x^2}{9}$ (B) $\frac{x^2}{8}$ (C) $\frac{x^2}{4}$ (D) x^2 (E) $2x^2$

---

14. What is the units digit in the standard decimal expansion of the number 7^{25} ?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

---

15. Let f be a continuous real-valued function defined on the closed interval $[-2, 3]$. Which of the following is NOT necessarily true?

(A) f is bounded.

(B) $\int_{-2}^{3} f(t)\,dt$ exists.

(C) For each c between $f(-2)$ and $f(3)$, there is an $x \in [-2, 3]$ such that $f(x) = c$.

(D) There is an M in $f([-2, 3])$ such that $\int_{-2}^{3} f(t)\,dt = 5M$.

(E) $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$ exists.

GO ON TO THE NEXT PAGE.

SCRATCH WORK

16. What is the volume of the solid formed by revolving about the x-axis the region in the first quadrant of the xy-plane bounded by the coordinate axes and the graph of the equation $y = \dfrac{1}{\sqrt{1+x^2}}$?

(A) $\dfrac{\pi}{2}$ (B) π (C) $\dfrac{\pi^2}{4}$ (D) $\dfrac{\pi^2}{2}$ (E) ∞

---

17. How many real roots does the polynomial $2x^5 + 8x - 7$ have?

(A) None (B) One (C) Two (D) Three (E) Five

---

18. Let V be the real vector space of all real 2×3 matrices, and let W be the real vector space of all real 4×1 column vectors. If T is a linear transformation from V onto W, what is the dimension of the subspace $\{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

SCRATCH WORK

19. Let f and g be twice-differentiable real-valued functions defined on $\mathbb{R}$. If $f'(x) > g'(x)$ for all $x > 0$, which of the following inequalities must be true for all $x > 0$?

(A) $f(x) > g(x)$

(B) $f''(x) > g''(x)$

(C) $f(x) - f(0) > g(x) - g(0)$

(D) $f'(x) - f'(0) > g'(x) - g'(0)$

(E) $f''(x) - f''(0) > g''(x) - g''(0)$

---

20. Let f be the function defined on the real line by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is rational} \\ \dfrac{x}{3} & \text{if } x \text{ is irrational.} \end{cases}$$

If D is the set of points of discontinuity of f, then D is the

(A) empty set

(B) set of rational numbers

(C) set of irrational numbers

(D) set of nonzero real numbers

(E) set of real numbers

---

21. Let P_1 be the set of all primes, $\{2, 3, 5, 7, \ldots\}$, and for each integer n, let P_n be the set of all prime multiples of n, $\{2n, 3n, 5n, 7n, \ldots\}$. Which of the following intersections is nonempty?

(A) $P_1 \cap P_{23}$ (B) $P_7 \cap P_{21}$ (C) $P_{12} \cap P_{20}$ (D) $P_{20} \cap P_{24}$ (E) $P_5 \cap P_{25}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

22. Let $C(\mathbb{R})$ be the collection of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Then $C(\mathbb{R})$ is a real vector space with pointwise addition and scalar multiplication defined by

$$(f+g)(x) = f(x) + g(x) \text{ and } (rf)(x) = rf(x)$$

for all $f, g \in C(\mathbb{R})$ and all $r, x \in \mathbb{R}$. Which of the following are subspaces of $C(\mathbb{R})$?

I. $\{f : f$ is twice differentiable and $f''(x) - 2f'(x) + 3f(x) = 0$ for all $x\}$

II. $\{g : g$ is twice differentiable and $g''(x) = 3g'(x)$ for all $x\}$

III. $\{h : h$ is twice differentiable and $h''(x) = h(x) + 1$ for all $x\}$

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

---

23. For what value of b is the line $y = 10x$ tangent to the curve $y = e^{bx}$ at some point in the xy-plane?

(A) $\frac{10}{e}$ (B) 10 (C) $10e$ (D) e^{10} (E) e

---

24. Let h be the function defined by $h(x) = \int_0^{x^2} e^{x+t}\,dt$ for all real numbers x. Then $h'(1) =$

(A) $e - 1$ (B) e^2 (C) $e^2 - e$ (D) $2e^2$ (E) $3e^2 - e$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

25. Let $\{a_n\}_{n=1}^{\infty}$ be defined recursively by $a_1 = 1$ and $a_{n+1} = \left(\frac{n+2}{n}\right)a_n$ for $n \geq 1$. Then a_{30} is equal to

(A) $(15)(31)$ (B) $(30)(31)$ (C) $\frac{31}{29}$ (D) $\frac{32}{30}$ (E) $\frac{32!}{30!\,2!}$

---

26. Let $f(x, y) = x^2 - 2xy + y^3$ for all real x and y. Which of the following is true?

(A) f has all of its relative extrema on the line $x = y$.

(B) f has all of its relative extrema on the parabola $x = y^2$.

(C) f has a relative minimum at $(0, 0)$.

(D) f has an absolute minimum at $\left(\frac{2}{3}, \frac{2}{3}\right)$.

(E) f has an absolute minimum at $(1, 1)$.

SCRATCH WORK

27. Consider the two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in $\mathbb{R}^3$. Which of the following sets is the intersection of these planes?

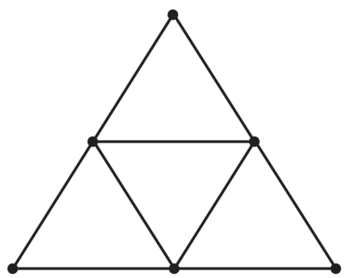
(A) $\varnothing$

(B) $\{(0, 3, 1)\}$

(C) $\{(x, y, z) : x = t,\ y = 3t,\ z = 7 - 2t,\ t \in \mathbb{R}\}$

(D) $\{(x, y, z) : x = 7t,\ y = 3 + t,\ z = 1 + 5t,\ t \in \mathbb{R}\}$

(E) $\{(x, y, z) : x - 2y - z = -7\}$



28. The figure above shows an undirected graph with six vertices. Enough edges are to be deleted from the graph in order to leave a spanning tree, which is a connected subgraph having the same six vertices and no cycles. How many edges must be deleted?

(A) One (B) Two (C) Three (D) Four (E) Five

SCRATCH WORK

29. For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

$$f \sim g \text{ if and only if } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Which of the following is NOT a consequence of $f \sim g$?

(A) $f^2 \sim g^2$ (B) $\sqrt{f} \sim \sqrt{g}$ (C) $e^f \sim e^g$ (D) $f + g \sim 2g$ (E) $g \sim f$

---

30. Let f be a function from a set X to a set Y. Consider the following statements.

P: For each $x \in X$, there exists $y \in Y$ such that $f(x) = y$.
Q: For each $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
R: There exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

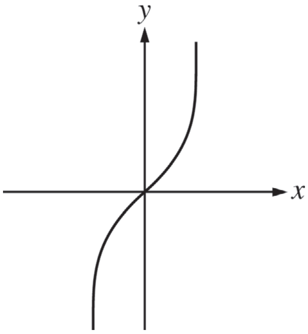
The negation of the statement "f is one-to-one and onto Y" is

(A) P or not R
(B) R or not P
(C) R or not Q
(D) P and not R
(E) R and not Q

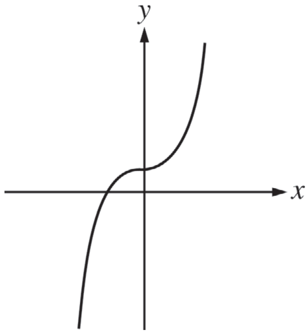
SCRATCH WORK

31. Which of the following most closely represents the graph of a solution to the differential equation $\frac{dy}{dx} = 1 + y^4$?

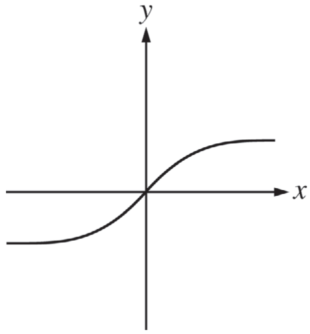
(A)



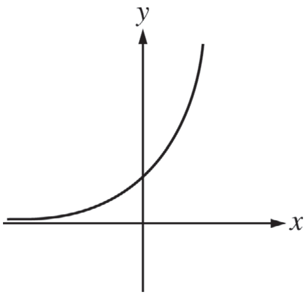
(B)



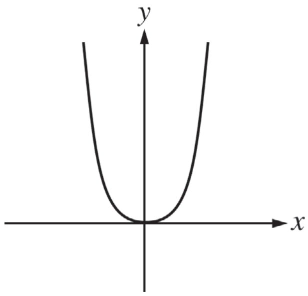
(C)



(D)



(E)



GO ON TO THE NEXT PAGE.

SCRATCH WORK

32. Suppose that two binary operations, denoted by $\oplus$ and $\odot$, are defined on a nonempty set S, and that the following conditions are satisfied for all x, y, and z in S:

(1) $x \oplus y$ and $x \odot y$ are in S.
(2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and $x \odot (y \odot z) = (x \odot y) \odot z$.
(3) $x \oplus y = y \oplus x$

Also, for each x in S and for each positive integer n, the elements nx and x^n are defined recursively as follows:

$1x = x^1 = x$ and
if kx and x^k have been defined, then $(k+1)x = kx \oplus x$ and $x^{k+1} = x^k \odot x$.

Which of the following must be true?

I. $(x \odot y)^n = x^n \odot y^n$ for all x and y in S and for each positive integer n.

II. $n(x \oplus y) = nx \oplus ny$ for all x and y in S and for each positive integer n.

III. $x^m \odot x^n = x^{m+n}$ for each x in S and for all positive integers m and n.

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

GO ON TO THE NEXT PAGE.

SCRATCH WORK

33. The Euclidean algorithm is used to find the greatest common divisor (`gcd`) of two positive integers `a` and `b`.

```
input(a)
input(b)
while b > 0
  begin
    r := a mod b
    a := b
    b := r
  end
gcd := a
output(gcd)
```

When the algorithm is used to find the greatest common divisor of `a = 273` and `b = 110`, which of the following is the sequence of computed values for `r` ?

(A) `2, 26, 1, 0`

(B) `2, 53, 1, 0`

(C) `53, 2, 1, 0`

(D) `53, 4, 1, 0`

(E) `53, 5, 1, 0`

---

34. The minimal distance between any point on the sphere $(x-2)^2+(y-1)^2+(z-3)^2=1$ and any point on the sphere $(x+3)^2+(y-2)^2+(z-4)^2=4$ is

(A) 0 (B) 4 (C) $\sqrt{27}$ (D) $2(\sqrt{2}+1)$ (E) $3(\sqrt{3}-1)$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

35. At a banquet, 9 women and 6 men are to be seated in a row of 15 chairs. If the entire seating arrangement is to be chosen at random, what is the probability that all of the men will be seated next to each other in 6 consecutive positions?

(A) $\dfrac{1}{\binom{15}{6}}$ (B) $\dfrac{6!}{\binom{15}{6}}$ (C) $\dfrac{10!}{15!}$ (D) $\dfrac{6!\,9!}{14!}$ (E) $\dfrac{6!\,10!}{15!}$

---

36. Let M be a 5×5 real matrix. Exactly four of the following five conditions on M are equivalent to each other. Which of the five conditions is equivalent to NONE of the other four?

(A) For any two distinct column vectors $\mathbf{u}$ and $\mathbf{v}$ of M, the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

(B) The homogeneous system $M\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(C) The system of equations $M\mathbf{x} = \mathbf{b}$ has a unique solution for each real 5×1 column vector $\mathbf{b}$.

(D) The determinant of M is nonzero.

(E) There exists a 5×5 real matrix N such that NM is the 5×5 identity matrix.

---

37. In the complex z-plane, the set of points satisfying the equation $z^2 = |z|^2$ is a

(A) pair of points

(B) circle

(C) half-line

(D) line

(E) union of infinitely many different lines

GO ON TO THE NEXT PAGE.

SCRATCH WORK

38. Let A and B be nonempty subsets of $\mathbb{R}$ and let $f : A \to B$ be a function. If $C \subseteq A$ and $D \subseteq B$, which of the following must be true?

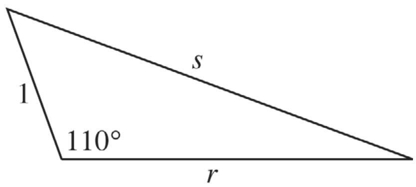
(A) $C \subseteq f^{-1}\big(f(C)\big)$

(B) $D \subseteq f\big(f^{-1}(D)\big)$

(C) $f^{-1}\big(f(C)\big) \subseteq C$

(D) $f^{-1}\big(f(C)\big) = f\big(f^{-1}(D)\big)$

(E) $f\big(f^{-1}(D)\big) = f^{-1}(D)$



39. In the figure above, as r and s increase, the length of the third side of the triangle remains 1 and the measure of the obtuse angle remains 110°. What is $\lim\limits_{\substack{s\to\infty \\ r\to\infty}} (s - r)$?

(A) 0

(B) A positive number less than 1

(C) 1

(D) A finite number greater than 1

(E) ∞

SCRATCH WORK

40. For which of the following rings is it possible for the product of two nonzero elements to be zero?

(A) The ring of complex numbers

(B) The ring of integers modulo 11

(C) The ring of continuous real-valued functions on $[0, 1]$

(D) The ring $\{a + b\sqrt{2} : a \text{ and } b \text{ are rational numbers}\}$

(E) The ring of polynomials in x with real coefficients

---

41. Let C be the circle $x^2 + y^2 = 1$ oriented counterclockwise in the xy-plane. What is the value of the line integral $\oint_C (2x - y)\,dx + (x + 3y)\,dy$?

(A) 0 (B) 1 (C) $\frac{\pi}{2}$ (D) π (E) 2π

---

42. Suppose X is a discrete random variable on the set of positive integers such that for each positive integer n, the probability that $X = n$ is $\frac{1}{2^n}$. If Y is a random variable with the same probability distribution and X and Y are independent, what is the probability that the value of at least one of the variables X and Y is greater than 3 ?

(A) $\frac{1}{64}$ (B) $\frac{15}{64}$ (C) $\frac{1}{4}$ (D) $\frac{3}{8}$ (E) $\frac{4}{9}$

SCRATCH WORK

43. If $z = e^{2\pi i/5}$, then $1 + z + z^2 + z^3 + 5z^4 + 4z^5 + 4z^6 + 4z^7 + 4z^8 + 5z^9 =$

(A) 0 (B) $4e^{3\pi i/5}$ (C) $5e^{4\pi i/5}$ (D) $-4e^{2\pi i/5}$ (E) $-5e^{3\pi i/5}$

---

44. A fair coin is to be tossed 100 times, with each toss resulting in a head or a tail. If H is the total number of heads and T is the total number of tails, which of the following events has the greatest probability?

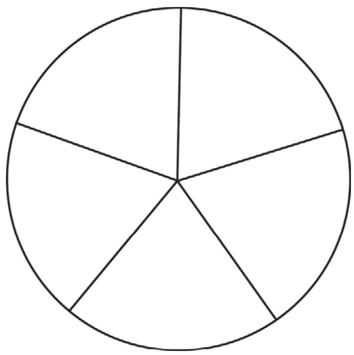
(A) $H = 50$

(B) $T \geq 60$

(C) $51 \leq H \leq 55$

(D) $H \geq 48$ and $T \geq 48$

(E) $H \leq 5$ or $H \geq 95$



45. A circular region is divided by 5 radii into sectors as shown above. Twenty-one points are chosen in the circular region, none of which is on any of the 5 radii. Which of the following statements must be true?

I. Some sector contains at least 5 of the points.

II. Some sector contains at most 3 of the points.

III. Some pair of adjacent sectors contains a total of at least 9 of the points.

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

GO ON TO THE NEXT PAGE.

SCRATCH WORK

46. Let G be the group of complex numbers $\{1, i, -1, -i\}$ under multiplication. Which of the following statements are true about the homomorphisms of G into itself?

I. $z \mapsto \overline{z}$ defines one such homomorphism, where $\overline{z}$ denotes the complex conjugate of z.

II. $z \mapsto z^2$ defines one such homomorphism.

III. For every such homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.

(A) None (B) II only (C) I and II only (D) II and III only (E) I, II, and III

---

47. Let $\mathbf{F}$ be a constant unit force that is parallel to the vector $(-1, 0, 1)$ in xyz-space. What is the work done by $\mathbf{F}$ on a particle that moves along the path given by $\left(t, t^2, t^3\right)$ between time $t = 0$ and time $t = 1$?

(A) $-\frac{1}{4}$ (B) $-\frac{1}{4\sqrt{2}}$ (C) 0 (D) $\sqrt{2}$ (E) $3\sqrt{2}$

---

48. Consider the theorem: If f and f' are both strictly increasing real-valued functions on the interval $(0, \infty)$, then $\lim_{x\to\infty} f(x) = \infty$. The following argument is suggested as a proof of this theorem.

(1) By the Mean Value Theorem, there is a c_1 in the interval $(1, 2)$ such that

$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) > 0.$$

(2) For each $x > 2$, there is a c_x in $(2, x)$ such that $\dfrac{f(x) - f(2)}{x - 2} = f'(c_x)$.

(3) For each $x > 2$, $\dfrac{f(x) - f(2)}{x - 2} = f'(c_x) > f'(c_1)$ since f' is strictly increasing.

(4) For each $x > 2$, $f(x) > f(2) + (x - 2)f'(c_1)$.

(5) $\lim_{x\to\infty} f(x) = \infty$

Which of the following statements is true?

(A) The argument is valid.

(B) The argument is not valid since the hypotheses of the Mean Value Theorem are not satisfied in (1) and (2).

(C) The argument is not valid since (3) is not valid.

(D) The argument is not valid since (4) cannot be deduced from the previous steps.

(E) The argument is not valid since (4) does not imply (5).

GO ON TO THE NEXT PAGE.

SCRATCH WORK

49. Up to isomorphism, how many additive abelian groups G of order 16 have the property that $x + x + x + x = 0$ for each x in G ?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

---

50. Let A be a real 2×2 matrix. Which of the following statements must be true?

I. All of the entries of A^2 are nonnegative.

II. The determinant of A^2 is nonnegative.

III. If A has two distinct eigenvalues, then A^2 has two distinct eigenvalues.

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

---

51. If $\lfloor x \rfloor$ denotes the greatest integer not exceeding x, then $\int_0^{\infty} \lfloor x \rfloor e^{-x}\, dx =$

(A) $\dfrac{e}{e^2 - 1}$ (B) $\dfrac{1}{e - 1}$ (C) $\dfrac{e - 1}{e}$ (D) 1 (E) $+\infty$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

52. If A is a subset of the real line $\mathbb{R}$ and A contains each rational number, which of the following must be true?

(A) If A is open, then $A = \mathbb{R}$.

(B) If A is closed, then $A = \mathbb{R}$.

(C) If A is uncountable, then $A = \mathbb{R}$.

(D) If A is uncountable, then A is open.

(E) If A is countable, then A is closed.

---

53. What is the minimum value of the expression $x + 4z$ as a function defined on $\mathbb{R}^3$, subject to the constraint $x^2 + y^2 + z^2 \le 2$?

(A) 0 (B) -2 (C) $-\sqrt{34}$ (D) $-\sqrt{35}$ (E) $-5\sqrt{2}$

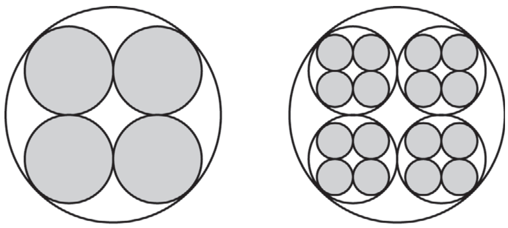


Figure 1 Figure 2

54. The four shaded circles in Figure 1 above are congruent and each is tangent to the large circle and to two of the other shaded circles. Figure 2 is the result of replacing each of the shaded circles in Figure 1 by a figure that is geometrically similar to Figure 1. What is the ratio of the area of the shaded portion of Figure 2 to the area of the shaded portion of Figure 1 ?

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{1+\sqrt{2}}$ (C) $\frac{4}{1+\sqrt{2}}$ (D) $\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^2$ (E) $\left(\frac{2}{1+\sqrt{2}}\right)^2$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

55. For how many positive integers k does the ordinary decimal representation of the integer $k!$ end in exactly 99 zeros?

(A) None (B) One (C) Four (D) Five (E) Twenty-four

---

56. Which of the following does NOT define a metric on the set of all real numbers?

(A) $\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y \end{cases}$

(B) $\rho(x, y) = \min\{|x - y|, 1\}$

(C) $\sigma(x, y) = \dfrac{|x - y|}{3}$

(D) $\tau(x, y) = \dfrac{|x - y|}{|x - y| + 1}$

(E) $\omega(x, y) = (x - y)^2$

---

57. The set of real numbers x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^{2n}}{n^n\left(1 + x^{2n}\right)}$ converges is

(A) $\{0\}$

(B) $\{x : -1 < x < 1\}$

(C) $\{x : -1 \leq x \leq 1\}$

(D) $\{x : -\sqrt{e} \leq x \leq \sqrt{e}\}$

(E) $\mathbb{R}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

58. Suppose A and B are $n \times n$ invertible matrices, where $n > 1$, and I is the $n \times n$ identity matrix. If A and B are similar matrices, which of the following statements must be true?

I. $A - 2I$ and $B - 2I$ are similar matrices.

II. A and B have the same trace.

III. A^{-1} and B^{-1} are similar matrices.

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

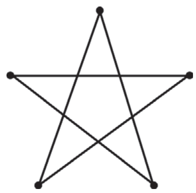
---

59. Suppose f is an analytic function of the complex variable $z = x + iy$ given by

$$f(z) = (2x + 3y) + ig(x, y),$$

where $g(x, y)$ is a real-valued function of the real variables x and y. If $g(2, 3) = 1$, then $g(7, 3) =$

(A) -14 (B) -9 (C) 0 (D) 11 (E) 18



60. The group of symmetries of the regular pentagram shown above is isomorphic to the

(A) symmetric group S_5

(B) alternating group A_5

(C) cyclic group of order 5

(D) cyclic group of order 10

(E) dihedral group of order 10

GO ON TO THE NEXT PAGE.

SCRATCH WORK

61. Which of the following sets has the greatest cardinality?

(A) $\mathbb{R}$

(B) The set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$

(C) The set of all functions from $\mathbb{R}$ to $\{0, 1\}$

(D) The set of all finite subsets of $\mathbb{R}$

(E) The set of all polynomials with coefficients in $\mathbb{R}$

---

62. Let K be a nonempty subset of $\mathbb{R}^n$, where $n > 1$. Which of the following statements must be true?

I. If K is compact, then every continuous real-valued function defined on K is bounded.

II. If every continuous real-valued function defined on K is bounded, then K is compact.

III. If K is compact, then K is connected.

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

---

63. If f is the function defined by

$$f(x) = \begin{cases} xe^{-x^2 - x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

at how many values of x does the graph of f have a horizontal tangent line?

(A) None (B) One (C) Two (D) Three (E) Four

GO ON TO THE NEXT PAGE.

SCRATCH WORK

64. For each positive integer n, let f_n be the function defined on the interval $[0, 1]$ by $f_n(x) = \dfrac{x^n}{1 + x^n}$. Which of the following statements are true?

I. The sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to a limit function f.

II. The sequence $\{f_n\}$ converges uniformly on $[0, 1]$ to a limit function f.

III. $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \left(\lim\limits_{n\to\infty} f_n(x)\right) dx$

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

---

65. Which of the following statements are true about the open interval $(0, 1)$ and the closed interval $[0, 1]$?

I. There is a continuous function from $(0, 1)$ onto $[0, 1]$.

II. There is a continuous function from $[0, 1]$ onto $(0, 1)$.

III. There is a continuous one-to-one function from $(0, 1)$ onto $[0, 1]$.

(A) None (B) I only (C) II only (D) I and III only (E) I, II, and III

SCRATCH WORK

66. Let R be a ring with a multiplicative identity. If U is an additive subgroup of R such that $ur \in U$ for all $u \in U$ and for all $r \in R$, then U is said to be a right ideal of R. If R has exactly two right ideals, which of the following must be true?

I. R is commutative.

II. R is a division ring (that is, all elements except the additive identity have multiplicative inverses).

III. R is infinite.

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

S T O P

If you finish before time is called, you may check your work on this test.

DO NOT USE INK

Use only a pencil with soft, black lead (No. 2 or HB) to complete this answer sheet.
Be sure to fill in completely the space that corresponds to your answer choice.
Completely erase any errors or stray marks.

GRADUATE RECORD EXAMINATIONS® - ETS® GRE® - SUBJECT TEST

SIDE 1

1. NAME Enter your last name, first name initial (given name), and middle initial if you have one. Omit spaces, apostrophes, Jr., II., etc.

Last Name only (Family Name or Surname) - First 15 Letters	First Name Initial	Middle Initial
(A)–(Z) ×15	(A)–(Z)	(A)–(Z)

2.

YOUR NAME: (Print) ____ Last Name (Family or Surname) First Name (Given) M.I.

MAILING ADDRESS: (Print) ____ P.O. Box or Street Address

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BE SURE EACH MARK IS DARK AND COMPLETELY FILLS THE INTENDED SPACE AS ILLUSTRATED HERE: ⬤ .

YOU MAY FIND MORE RESPONSE SPACES THAN YOU NEED. IF SO, PLEASE LEAVE THEM BLANK.

1	(A) (B) (C) (D) (E)	39	(A) (B) (C) (D) (E)	77	(A) (B) (C) (D) (E)
2	(A) (B) (C) (D) (E)	40	(A) (B) (C) (D) (E)	78	(A) (B) (C) (D) (E)
3	(A) (B) (C) (D) (E)	41	(A) (B) (C) (D) (E)	79	(A) (B) (C) (D) (E)
4	(A) (B) (C) (D) (E)	42	(A) (B) (C) (D) (E)	80	(A) (B) (C) (D) (E)
5	(A) (B) (C) (D) (E)	43	(A) (B) (C) (D) (E)	81	(A) (B) (C) (D) (E)
6	(A) (B) (C) (D) (E)	44	(A) (B) (C) (D) (E)	82	(A) (B) (C) (D) (E)
7	(A) (B) (C) (D) (E)	45	(A) (B) (C) (D) (E)	83	(A) (B) (C) (D) (E)
8	(A) (B) (C) (D) (E)	46	(A) (B) (C) (D) (E)	84	(A) (B) (C) (D) (E)
9	(A) (B) (C) (D) (E)	47	(A) (B) (C) (D) (E)	85	(A) (B) (C) (D) (E)
10	(A) (B) (C) (D) (E)	48	(A) (B) (C) (D) (E)	86	(A) (B) (C) (D) (E)
11	(A) (B) (C) (D) (E)	49	(A) (B) (C) (D) (E)	87	(A) (B) (C) (D) (E)
12	(A) (B) (C) (D) (E)	50	(A) (B) (C) (D) (E)	88	(A) (B) (C) (D) (E)
13	(A) (B) (C) (D) (E)	51	(A) (B) (C) (D) (E)	89	(A) (B) (C) (D) (E)
14	(A) (B) (C) (D) (E)	52	(A) (B) (C) (D) (E)	90	(A) (B) (C) (D) (E)
15	(A) (B) (C) (D) (E)	53	(A) (B) (C) (D) (E)	91	(A) (B) (C) (D) (E)
16	(A) (B) (C) (D) (E)	54	(A) (B) (C) (D) (E)	92	(A) (B) (C) (D) (E)
17	(A) (B) (C) (D) (E)	55	(A) (B) (C) (D) (E)	93	(A) (B) (C) (D) (E)
18	(A) (B) (C) (D) (E)	56	(A) (B) (C) (D) (E)	94	(A) (B) (C) (D) (E)
19	(A) (B) (C) (D) (E)	57	(A) (B) (C) (D) (E)	95	(A) (B) (C) (D) (E)
20	(A) (B) (C) (D) (E)	58	(A) (B) (C) (D) (E)	96	(A) (B) (C) (D) (E)
21	(A) (B) (C) (D) (E)	59	(A) (B) (C) (D) (E)	97	(A) (B) (C) (D) (E)
22	(A) (B) (C) (D) (E)	60	(A) (B) (C) (D) (E)	98	(A) (B) (C) (D) (E)
23	(A) (B) (C) (D) (E)	61	(A) (B) (C) (D) (E)	99	(A) (B) (C) (D) (E)
24	(A) (B) (C) (D) (E)	62	(A) (B) (C) (D) (E)	100	(A) (B) (C) (D) (E)
25	(A) (B) (C) (D) (E)	63	(A) (B) (C) (D) (E)	101	(A) (B) (C) (D) (E)
26	(A) (B) (C) (D) (E)	64	(A) (B) (C) (D) (E)	102	(A) (B) (C) (D) (E)
27	(A) (B) (C) (D) (E)	65	(A) (B) (C) (D) (E)	103	(A) (B) (C) (D) (E)
28	(A) (B) (C) (D) (E)	66	(A) (B) (C) (D) (E)	104	(A) (B) (C) (D) (E)
29	(A) (B) (C) (D) (E)	67	(A) (B) (C) (D) (E)	105	(A) (B) (C) (D) (E)
30	(A) (B) (C) (D) (E)	68	(A) (B) (C) (D) (E)	106	(A) (B) (C) (D) (E)
31	(A) (B) (C) (D) (E)	69	(A) (B) (C) (D) (E)	107	(A) (B) (C) (D) (E)
32	(A) (B) (C) (D) (E)	70	(A) (B) (C) (D) (E)	108	(A) (B) (C) (D) (E)
33	(A) (B) (C) (D) (E)	71	(A) (B) (C) (D) (E)	109	(A) (B) (C) (D) (E)
34	(A) (B) (C) (D) (E)	72	(A) (B) (C) (D) (E)	110	(A) (B) (C) (D) (E)
35	(A) (B) (C) (D) (E)	73	(A) (B) (C) (D) (E)	111	(A) (B) (C) (D) (E)
36	(A) (B) (C) (D) (E)	74	(A) (B) (C) (D) (E)	112	(A) (B) (C) (D) (E)
37	(A) (B) (C) (D) (E)	75	(A) (B) (C) (D) (E)	113	(A) (B) (C) (D) (E)
38	(A) (B) (C) (D) (E)	76	(A) (B) (C) (D) (E)	114	(A) (B) (C) (D) (E)

Item responses continued on reverse side.

3. DATE OF BIRTH

Month	Day	Year
◯ Jan. ◯ Feb. ◯ Mar. ◯ April ◯ May ◯ June ◯ July ◯ Aug. ◯ Sept. ◯ Oct. ◯ Nov. ◯ Dec.	(0)–(3), (0)–(9)	(0)–(9), (0)–(9)

4. SOCIAL SECURITY NUMBER (U.S.A. only)

(0)–(9) ×9

5. REGISTRATION NUMBER (from your admission ticket)

(0)–(9) ×7

6. TITLE CODE (on back cover of your test book)

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