ON THE PROPAGATION OF REGULARITIES IN SOLUTIONS OF THE BENJAMIN-ONO EQUATION

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ABSTRACT. We shall deduce some special regularity properties of solutions to the IVP associated to the Benjamin-Ono equation. Mainly, for datum $u_0 \in H^{3/2}(\mathbb{R})$ whose restriction belongs to $H^m((b,\infty))$ for some $m \in \mathbb{Z}^+$, $m \geq 2$, and $b \in \mathbb{R}$ we shall prove that the restriction of the corresponding solution $u(\cdot,t)$ belongs to $H^m((\beta,\infty))$ for any $\beta \in \mathbb{R}$ and any $t > 0$. Therefore, this type of regularity of the datum travels with infinite speed to its left as time evolves.

1. INTRODUCTION

This work is mainly concerned with special properties of solutions to the initial value problem (IVP) associated to the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u - \partial_x^2 \mathsf{H} u + u\,\partial_x u = 0, & x,t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \qquad (1.1) \quad \boxed{\texttt{bo1}}$$

where H denotes the Hilbert transform,

$$\begin{aligned} \mathsf{H}f(x) &= \frac{1}{\pi}\mathrm{p.v.}\Big(\frac{1}{x}*f\Big)(x) \\ &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int\limits_{|y|\geq\varepsilon}\frac{f(x-y)}{y}\,dy = (-i\,\mathrm{sgn}(\xi)\widehat{f}(\xi))^{\vee}(x). \end{aligned} \qquad (1.2) \quad \boxed{\texttt{hita}}$$

The BO equation was first deduced by Benjamin [4] and Ono [31] as a model for long internal gravity waves in deep stratified fluids. Later, it was also shown to be a completely integrable system (see [2], [9] and references therein). Thus, solutions of the IVP (1.1) satisfy infinitely many conservation laws which provide an *a priori* estimate for the $H^{n/2}$-norm, $n \in \mathbb{Z}^+$, of the solution $u = u(x,t)$ for (1.1). Here we shall only consider real valued solutions of the IVP (1.1).

Following the definition in [18] it is said that the IVP (1.1) is locally well-posed (LWP) in the function space X if given any datum $u_0 \in X$ there exists

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$T > 0$ and a unique solution

$$u \in C([-T,T] : X) \cap ....$$

of the IVP (1.1) with the map data-solution, $u_0 \mapsto u$, being continuous. If T can be taken arbitrarily large, one says that the IVP (1.1) is globally well-posed (GWP) in X.

The problem of finding the minimal regularity property, measured in the Sobolev scale

$$H^s(\mathbb{R}) = \left(1 - \partial_x^2\right)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R},$$

required to guarantee that the IVP (1.1) is locally or globally well-posed in $H^s(\mathbb{R})$ has been extensively studied. Thus, one has the following list of works [1], [15], [32], [24], [20], [33], [7] and [14] where GWP was established in $H^0(\mathbb{R}) = L^2(\mathbb{R})$, (for further details and results regarding the well-posedness of the IVP (1.1) in $H^s(\mathbb{R})$ we refer to [27]). It should be pointed out that a result found in [30] (see also [25]) implies that none well-posedness in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$ for the IVP (1.1) can be established by a solely contraction principle argument.

Well-posedness of the IVP (1.1) has also been studied in weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad s, r \in \mathbb{R}, \qquad (1.3) \quad \boxed{\text{spaceZ}}$$

and

$$\dot{Z}_{s,r} = \{f \in H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx) : \widehat{f}(0) = 0\}, \quad s, r \in \mathbb{R}. \qquad (1.4) \quad \boxed{\text{spaceZdot}}$$

Notice that the conservation law for solutions of (1.1)

$$I_1(u_0) = \int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx,$$

guarantees that the property $\widehat{u}_0(0) = 0$ is preserved by the solution flow.

Motivated by the results in [15], [16] it was established in [11] that the IVP (1.1) is :

LWP (resp. GWP) in $Z_{s,r}$ for $s > 9/8$ (resp. $s \geq 3/2$), $s \geq r$ and $r \in (0,5/2)$, and

GWP in $\dot{Z}_{s,r}$ with $s \geq r$ and $r \in [5/2,7/2)$.

Moreover, it was also shown in [11] that the above results are optimal (for further details and results concerning the well-posedness of the IVP (1.1) in weighted Sobolev spaces we refer to [12]).

Although we shall be mainly engaged with the IVP (1.1) our results below will also apply to solutions of the IVP associated to the generalized Benjamin-Ono (g-BO) equation

$$\begin{cases} \partial_t u - \partial_x^2 \mathsf{H}u + u^k \partial_x u = 0, & x,t \in \mathbb{R}, \ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \qquad (1.5) \quad \boxed{\text{bok}}$$

In this case well-posedness of the IVP for $k \geq 2$ has been considered in [22], [28], [29], [23] where GWP for the IVP (1.5) with $k = 2$ was obtained in $H^s(\mathbb{R})$, $s \geq 1/2$ and [34] where LWP of the IVP (1.5) was proven in the critical Sobolev space $H^{s_k}(\mathbb{R})$ with $s_k = 1/2 - 1/k$ for $k \geq 4$ and with $s > 1/3$ for $k = 3$. Addressing the well-posedness of the IVP (1.5) in the weighted Sobolev spaces (1.3) and (1.4) one has that the positive results stated before for the IVP (1.1) also applies. However, the optimality of these results in the cases where the power k in (1.5) is an even integer has not been established (for further details see [26] and references therein).

To state our main result we need to describe the space of solutions where we shall be working on.

<div style="border:1px solid; display:inline-block; padding:2px">gwp</div> **Theorem A.** ([32]) *If $u_0 \in H^s(\mathbb{R})$ with $s \geq 3/2$, then there exists a unique solution $u = u(x,t)$ of the IVP (1.1) such that for any $T > 0$*

$$\text{(i)} \quad u \in C(\mathbb{R} : H^s(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^s(\mathbb{R})),$$

$$\text{(ii)} \quad \partial_x u \in L^4([-T,T] : L^\infty(\mathbb{R})), \quad (Strichartz),$$

$$\text{(iii)} \quad \int_{-T}^{T} \int_{-R}^{R} (|\partial_x D_x u|^2 + |\partial_x^2 u|^2)(x,t)|^2 dx dt \leq c_0, \qquad (1.6) \quad \boxed{\text{bo2}}$$

$$\text{(iv)} \quad \int_{-T}^{T} \int_{-R}^{R} |\partial_x D_x^{r-1/2} u(x,t)|^2 dx dt \leq c_1, \quad r \in [1,s],$$

with $c_0 = c_0(R,T,\|u_0\|_{3/2,2})$ and $c_1 = c_1(R,T,\|u_0\|_{s,2})$.

Remark 1.1. *From our previous comments it is clear that the results in Theorem A still holds for the IVP (1.5) with $k = 2$ (see [20]) and locally in time for $k \geq 3$. Indeed, one can lower the requirement $s \geq 3/2$ to a value between $[1,3/2]$, depending on the k considered, such that a well-posedness including the estimate (1.6) (ii) still holds. However, we will not consider this question here.*

To state our main result we introduce the two parameter family (ε, b) with $\varepsilon > 0$, $b \geq 5\varepsilon$ of functions $\chi_{\varepsilon,b} \in C^\infty(\mathbb{R})$ such that $\chi'_{\varepsilon,b}(x) \geq 0$ with

$$\chi_{\varepsilon,b}(x) = \begin{cases} 0, & x \leq \varepsilon, \\ 1, & x \geq b, \end{cases} \qquad (1.7) \quad \boxed{1.1}$$

satisfying (2.2)-(2.8) in section 2.

<div style="border:1px solid; display:inline-block; padding:2px">Th1</div> **Theorem 1.2.** *Let $u_0 \in H^{3/2}(\mathbb{R})$ and $u = u(x,t)$ be the corresponding solution of the IVP (1.1) provided by Theorem A. If for some $x_0 \in \mathbb{R}$ and for*

some $m \in \mathbb{Z}^+$, $m \geq 2$,

$$\int_{x_0}^{\infty} (\partial_x^m u_0)^2 (x) \, dx < \infty, \qquad\qquad (1.8) \quad \boxed{\texttt{bo3}}$$

then for any $v > 0$, $T > 0$, $\varepsilon > 0$, $b \geq 5\varepsilon$

$$\sup_{0 \leq t \leq T} \int (\partial_x^m u(x,t))^2 \, \chi_{\varepsilon,b}(x - x_0 + vt) \, dx$$

$$+ \int_0^T \int (D_x^{1/2}(\partial_x^m u(x,t)\eta_{\varepsilon,\beta}(x - x_0 + vt)))^2 \, dxdt < c = c(T,\varepsilon,b,v), \qquad (1.9) \quad \boxed{\texttt{bo4}}$$

where $\chi'_{\varepsilon,\beta} = \eta_{\varepsilon,\eta}^2$.
If in addition to (1.8) there exists $x_0 \in \mathbb{R}$ such that any $\varepsilon > 0$, $b > 5\varepsilon$

$$D_x^{1/2}(\partial_x^m u_0 \, \chi_{\varepsilon,b}(\cdot - x_0)) \in L^2(\mathbb{R}), \qquad\qquad (1.10) \quad \boxed{\texttt{bo5}}$$

then

$$\sup_{0 \leq t \leq T} \int (D_x^{1/2}(\partial_x^m u(x,t) \, \chi_{\varepsilon,b}(x - x_0 + vt)))^2 \, dx$$

$$+ \int_0^T \int (\partial_x^{m+1} u(x,t))^2 \, \chi'_{\varepsilon,b} \, \chi_{\varepsilon,b}(x - x_0 + vt) \, dxdt < c, \qquad (1.11) \quad \boxed{\texttt{bo6}}$$

with $c = c(T,\varepsilon,b,v)$.

$\boxed{\texttt{Th2}}$ **Theorem 1.3.** *With the same hypotheses the results in Theorem 1.2 apply to solutions of the IVP (1.5) globally in time if $k = 2$, and locally in time if $k \geq 3$.*

Remark 1.4.

(a) *From our comments above and our proof of Theorem 1.2 it will be clear that the requirement $u_0 \in H^{3/2}(\mathbb{R})$ in Theorem 1.2 can be lowered to $u_0 \in H^{9/8^+}(\mathbb{R})$ by considering the problem in a finite time interval (see [20]).*

(b) *It will be clear from our proofs of Theorem 1.2 and Theorem 1.3 that they still hold for solutions of the "defocussing" gBO equation*

$$\partial_t u - H\partial_x^2 u - u^k \, \partial_x u = 0, \quad x,t \in \mathbb{R}, \quad k \in \mathbb{Z}^+.$$

Therefore, our results apply to $u(-x,-t)$ if $u(x,t)$ is a solution of (1.5). In other words, Theorem 1.2 and Theorem 1.3 resp. remain valid, backwards in time, for datum satisfying the hypothesis (1.8) and (1.10) respectively on the left hand side of the real line.

(c) *It can be deduced from our proof of Theorem 1.2 that the first estimate in the inequality* (1.9) *can be made more precise, i.e. for any $t \in (0,1)$ and any $j = 1,..,m$*

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{2j+\delta}} (\partial_x^j u(x,t))^2 dx \leq \frac{c}{t},$$

with $c = c(\|u_0\|_{3/2,2}; \|\partial_x^j u_0\|_{L^2(x_0,\infty)}; x_0; \delta) > 0$.

The following are direct outcomes of the above comment, the group properties, and Theorems 1.2 and 1.3. In order to simplify the exposition we shall state them only for solutions of the IVP (1.1). First, as a direct consequence of Theorem 1.2 and the time reversible character of the equation in (1.1) one has :

cor.1 **Corollary 1.5.** *Let $u \in C(\mathbb{R} : H^{3/2}(\mathbb{R}))$ be a solution of the IVP* (1.1) *described in Theorem A. If there exist $m \in \mathbb{Z}^+, m \geq 2, \hat{t} \in \mathbb{R}, a \in \mathbb{R}$ such that*

$$\partial_x^m u(\cdot, \hat{t}) \notin L^2((a,\infty)),$$

then for any $t \in (-\infty, \hat{t})$ and any $\beta \in \mathbb{R}$

$$\partial_x^m u(\cdot, t) \notin L^2((\beta, \infty)).$$

cor.1a **Corollary 1.6.** *Let $u \in C(\mathbb{R} : H^{3/2}(\mathbb{R}))$ be a solution of the equation in* (1.1) *described in Theorem A. If there exist $m \in \mathbb{Z}^+, m \geq 2, t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$ and $a > 0$ such that*

$$\partial_x^m u(\cdot, t_1) \in L^2((a, \infty)) \quad and \quad \partial_x^m u(\cdot, t_2) \in L^2((-\infty, -a)),$$

then $u \in C(\mathbb{R} : H^m(\mathbb{R}))$

Next, one has that for appropriate class of data singularities of the corresponding solutions travel with infinite speed to the left as time evolves.

cor.2 **Corollary 1.7.** *Let $u \in C(\mathbb{R} : H^{3/2}(\mathbb{R}))$ be a solution of the equation in* (1.1) *described in Theorem A. If there exist $k, m \in \mathbb{Z}^+$ with $k \geq m$ and $a, b \in \mathbb{R}$ with $b < a$ such that*

$$\int_a^{\infty} |\partial_x^k u_0(x)|^2 dx < \infty \quad but \quad \partial_x^m u_0 \notin L^2((b, \infty)), \qquad (1.12) \quad \boxed{\text{A}}$$

then for any $t \in (0, \infty)$, $v > 0$ and $\varepsilon > 0$

$$\int_{a+\varepsilon-vt}^{\infty} |\partial_x^k u(x,t)|^2 dx < \infty,$$

and for any $t \in (-\infty, 0)$ and $\alpha \in \mathbb{R}$

$$\int_{\alpha}^{\infty} |\partial_x^m u(x,t)|^2 dx = \infty.$$

Remark 1.8.

(a) *If in Corollary 1.7 in addition to (1.12) one assumes that*

$$\int_{-\infty}^{b} |\partial_x^k u_0(x)|^2 dx < \infty,$$

then by combining the results in this corollary with the group properties it follows that

$$\int_{-\infty}^{\beta} |\partial_x^m u(x,t)|^2 dx = \infty, \quad \text{for any } \beta \in \mathbb{R} \text{ and } t > 0.$$

This shows that the regularity in the left hand side does not propagate forward in time. Also, one has that for all $t \neq 0$

$$u(\cdot,t) \in H_{loc}^m(\mathbb{R}).$$

Thus, the m-localized singularity at $t = 0$ in (a,b) can not reappear in the future $t > 0$ or in the past $t < 0$.

(b) *Notice that (1.9) implies: for any $\varepsilon > 0$, $v > 0$, $T > 0$*

$$\sup_{0 \leq t \leq T} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^k u(x,t))^2 dx \leq c = c(\varepsilon,v,T). \tag{1.13} \quad \boxed{\texttt{bo7}}$$

This tells us that the local regularity of the initial datum u_0 described in (1.8) propagates with infinite speed to its left as time evolves.

(c) *In [17] we proved the corresponding result concerning the IVP for the k-generalized Korteweg-de Vries equation*

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x,t \in \mathbb{R}, \; k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.14} \quad \boxed{\texttt{gkdv}}$$

More precisely, the following result was obtained in [17]:

Theorem B. *If $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $m \in \mathbb{Z}^+$, $m \geq 1$ and $x_0 \in \mathbb{R}$*

$$\| \partial_x^m u_0 \|_{L^2((x_0,\infty))}^2 = \int_{x_0}^{\infty} |\partial_x^m u_0(x)|^2 dx < \infty, \tag{1.15} \quad \boxed{\texttt{notes-3}}$$

then the solution of the IVP (1.14) satisfies that for any $v > 0$ and $\varepsilon > 0$

$$\sup_{0 \leq t \leq T} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^j u)^2(x,t) \, dx < c, \tag{1.16} \quad \boxed{\texttt{notes-4}}$$

for $j = 0,1,\ldots,m$ with $c = c(l,v,\varepsilon,T)$.

Moreover, for any $v \geq 0$, $\varepsilon > 0$ and $R > \varepsilon$

$$\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R-vt} (\partial_x^{j+1} u)^2(x,t) \, dxdt < c, \tag{1.17} \quad \boxed{\texttt{notes-5}}$$

with $c = c(l,v,\varepsilon,R,T)$.

However, the proof for the BO equation considered here is quite more involved. First, it includes a non-local operator, the Hilbert transform (1.2). Second, in the case of the k-gKdV the local smoothing effect yields a gain of one derivative which allows to pass to the next step in the inductive process. However, in the case of the BO equation the gain of the local smoothing is just 1/2-derivative so the iterative argument has to be carried out in two steps, one for positive integers m and another one for m + 1/2. Also the explicit identity obtained in [18] describing the local smoothing effect in solutions of the KdV equation is not available for the BO equation. In this case, to establish the local smoothing one has to rely on several commutator estimates. The main one is the extension of the Calderón first commutator estimate for the Hilbert transform [8] given by Bajvsank and Coifman in [3] (see Theorem 2.1 in section 2).

(d) *Without loss of generality from now on we shall assume that in Theorem 1.2 $x_0 = 0$.*

(e) *We recall that the above result still hold if one replaces $x, t > 0$ by $x, t < 0$.*

The rest of this paper is organized as follows: section 2 includes the description of the two parameter family of cut-off functions to be employed in the proof of Theorem 1.2. Also section 2 has the statements and some proofs of the commutator estimates and the interpolation inequalities needed in the proof of Theorem 1.2. The proof of Theorem 1.2 will be given in section 3.

2. PRELIMINARIES

For each $\varepsilon > 0$ and $b \geq 5\varepsilon$ we define a function $\chi_{\varepsilon,b} \in C^\infty(\mathbb{R})$ with $\chi'_{\varepsilon,b}(x) \geq 0$, and

$$\chi_{\varepsilon,b}(x) = \begin{cases} 0, & x \leq \varepsilon, \\ 1, & x \geq b, \end{cases} \tag{2.1}$$

which will be constructed as follows. Let $\rho \in C_0^\infty(\mathbb{R})$, $\rho(x) \geq 0$, even, with $\operatorname{supp}\rho \subseteq (-1,1)$ and $\int \rho(x)dx = 1$. Define

$$v_{\varepsilon,b}(x) = \begin{cases} 0, & x \leq 2\varepsilon, \\ \frac{1}{b-3\varepsilon}x - \frac{2\varepsilon}{b-3\varepsilon}, & x \in [2\varepsilon, b-\varepsilon], \\ 1, & x \geq b-\varepsilon, \end{cases} \tag{2.2}$$

and

$$\chi_{\varepsilon,b}(x) = \rho_\varepsilon * v_{\varepsilon,b}(x) \tag{2.3}$$

where $\rho_{\varepsilon}(x) = \varepsilon^{-1}\rho(x/\varepsilon)$. Thus

$$\begin{aligned}
\text{supp } \chi_{\varepsilon,b} &\subseteq [\varepsilon,\infty), \\
\text{supp } \chi'_{\varepsilon,b}(x) &\subseteq [\varepsilon,b].
\end{aligned} \tag{2.4}$$

If $x \in (3\varepsilon, b-2\varepsilon)$, then

$$\chi'_{\varepsilon,b}(x) \geq \frac{1}{b-3\varepsilon}. \tag{2.5}$$

If $x \in (3\varepsilon, \infty)$, then

$$\chi_{\varepsilon,b}(x) \geq \chi_{\varepsilon,b}(3\varepsilon) \geq \frac{1}{2}\frac{\varepsilon}{b-3\varepsilon}, \tag{2.6}$$

and for any $x \in \mathbb{R}$

$$\chi'_{\varepsilon,b}(x) \leq \frac{1}{b-3\varepsilon}. \tag{2.7}$$

Now we define $\eta_{\varepsilon,b}$ by the identities

$$\chi'_{\varepsilon,b}(x) = \eta^2_{\varepsilon,b}, \quad i.e. \quad \eta_{\varepsilon,b} = \sqrt{\chi'_{\varepsilon,b}(x)}. \tag{2.8}$$

<u>CLAIM</u> : For any $\varepsilon > 0$ and $b \geq 5\varepsilon$ $\eta_{\varepsilon,b} \in C_0^{\infty}(\mathbb{R})$ with supp $\eta_{\varepsilon,b} = $ supp $\chi'_{\varepsilon,b}$.

It suffices to show that if $f \in C_0^{\infty}(\mathbb{R})$ with supp $f = [0,1]$ and $f(x) > 0$, $x \in (0,1)$ then $\sqrt{f}$ is smooth at $x = 0$ and $x = 1$. Without loss of generality we only consider the case $x = 0$.

By hypothesis on f for every $n \in \mathbb{Z}^+$ there exist $M_n \geq 0$ and $\delta_n > 0$ such that

$$|f(x)| \leq M_n |x|^n, \quad |x| \leq \delta_n. \tag{2.9}$$

Thus

$$\frac{d}{dx}\left(\sqrt{f}\right)(0) = \lim_{h \to 0} \frac{\sqrt{f(h)} - 0}{h} = 0$$

by (2.9). Also

$$\frac{d^2}{dx^2}\left(\sqrt{f}\right)(0) = \lim_{h \to 0} \frac{\sqrt{f(h)} - 2\sqrt{f(0)} + \sqrt{f(-h)}}{h} = 0$$

by (2.9). Following this argument of writing the derivatives of $\sqrt{f}$ using finite differences, from (2.9) one obtains the desired result, i.e.

$$\frac{d^n}{dx^n}\left(\sqrt{f}\right)(0) = 0 \quad \text{for any } n,$$

so $\sqrt{f}$ is smooth.

We shall also use that given $\varepsilon > 0$, $b \geq 5\varepsilon$ there exists $c = c_{\varepsilon,b} > 0$ such that

$$\chi_{\varepsilon/5,\varepsilon}(x) = 1, \quad \text{on supp } \chi_{\varepsilon,b},$$
$$\chi'_{\varepsilon,b}(x) \leq c\,\chi'_{\varepsilon/3,b+2\varepsilon/3}(x)\,\chi_{\varepsilon/3,b+2\varepsilon/3}(x), \quad (2.10) \quad \boxed{\text{CL}}$$
$$\chi'_{\varepsilon,b}(x) \leq c\,\chi_{\varepsilon/5,\varepsilon}(x).$$

The following extension of the Calderón commutator theorem [8] established by Bajvsank and Coifman in [3] will be a crucial ingredient in our proof of Theorem 1.2.

$\boxed{\text{Cal}}$ **Theorem 2.1.** *Let H be the Hilbert transform. For any $p \in (1,\infty)$ and any $l,m \in \mathbb{Z}^+$, $l+m \geq 1$ there exists $c = c(p;l;m) > 0$ such that*

$$\|\partial_x^l[H;\psi]\,\partial_x^m f\|_p \leq c\,\|\partial_x^{m+l}\psi\|_\infty\|f\|_p. \quad (2.11) \quad \boxed{\text{CE}}$$

For a different proof of this estimate see Lemma 3.1 in [10]. Also we shall use the following commutator estimate:

Proposition 2.2.

$$\|[D_x^{1/2};h]\partial_x f\|_2 \leq c\,\|\widehat{\partial_x h}\|_1\|D_x^{1/2}f\|_2. \quad (2.12) \quad \boxed{\text{CE2}}$$

Proof. Taking Fourier transform it follows that

$$\widehat{[D^{1/2};h]\partial_x f}(\xi) = c\int (|\xi|^{1/2} - |\eta|^{1/2})\,\eta\,\widehat{h}(\xi-\eta)\,\widehat{f}(\eta)\,d\eta. \quad (2.13) \quad \boxed{11}$$

Using the mean value theorem it is easy to see that there exists $c > 0$ such that for any $\xi, \eta \in \mathbb{R}$

$$\big||\xi|^{1/2} - |\eta|^{1/2}\big||\eta| \leq c\,|\eta|^{1/2}|\xi - \eta|. \quad (2.14) \quad \boxed{12}$$

Therefore, inserting (2.14) into (2.13) and using Plancherel identity and Young's inequality we obtain the desired result (2.12).

$\square$

Next, we collect some inequalities concerning the Leibniz rule for fractional derivatives established in [19], [21], [13] to obtain :

Lemma 2.3. *For $\alpha \in (0,1)$, $p \in [1,\infty)$*

$$\|D_x^\alpha(fg)\|_p \leq c\left(\|f\|_{p_1}\|D_x^\alpha g\|_{p_2} + \|D_x^\alpha f\|_{p_3}\|g\|_{p_4}\right) \quad (2.15) \quad \boxed{\text{FD}}$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad p_j \in (1,\infty], \ \ j = 1,2,3,4.$$

Finally, we state some interpolation inequalities to be used in the proof of Theorem 1.2.

Lemma 2.4.

$$\|f\|_4 \le c\|D_x^{1/4} f\|_2 \le c\|D_x^{1/2} f\|_2^{1/2}\|f\|_2^{1/2},$$

$$\|D_x^{1/2} f\|_4 \le c\|\partial_x f\|_4^{1/2}\|f\|_4^{1/2}, \qquad (2.16) \quad \boxed{\text{INT}}$$

$$\|D_x^{1/2} f\|_4 \le c\|D_x^{3/4} f\|_2 \le c\|\partial_x f\|_2^{3/4}\|f\|_2^{1/4}.$$

For the proof of 2.16 we refer to [5].

3. PROOF OF THEOREM 1.2

We shall use induction in $m \in \mathbb{Z}^+$ with $m \ge 2$. First we consider the case $m = 2$.

Case $m = 2$ in (1.8) : Taking second derivative in the equation (1.1) multiplying the result by $\partial_x^2 u(x,t)\chi_{\varepsilon,b}(x+vt)$ and integrating in the x-variable one gets the identity (omitting the indices ε, b on χ)

$$\frac{1}{2}\frac{d}{dt}\int (\partial_x^2 u)^2 \chi\, dx - v\underbrace{\int (\partial_x^2 u)^2 \chi'\, dx}_{A_1}$$

$$\underbrace{-\int \mathsf{H}\partial_x^4 u\,\partial_x^2 u\,\chi\, dx}_{A_2} + \underbrace{\int \partial_x^2 (u\partial_x u)\,\partial_x^2 u\,\chi\, dx}_{A_3} = 0. \qquad (3.1) \quad \boxed{\text{bo8}}$$

Since given $T > 0$, $\varepsilon > 0$, $b > 5\varepsilon$, $v > 0$, there exist $c > 0$ and $R > 0$ such that

$$\chi'_{\varepsilon,b}(x+vt) \le c\,1_{[-R,R]}(x) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0,T], \qquad (3.2) \quad \boxed{\text{bo9}}$$

using (1.6) (iii) one has after integration in the time interval $[0,T]$ that

$$\int_0^T |A_1(t)|\, dt$$

$$\le c\,|v| \int_{-T}^T \int_{-R}^R (\partial_x^2 u)^2(x,t)\, dxdt \le c = c(v,R,T,\varepsilon,b,\|u_0\|_{3/2,2}). \qquad (3.3) \quad \boxed{\text{bo10}}$$

To control the contribution of A_2 in (3.1) we write after integration by parts that

$$A_2(t) = \int \mathsf{H}\partial_x^3 u\,\partial_x^3 u\,\chi\, dx + \int \mathsf{H}\partial_x^3 u\,\partial_x^2 u\,\chi'\, dx = A_{21}(t) + A_{22}(t). \qquad (3.4) \quad \boxed{\text{bo11}}$$

Now since the Hilbert transform is skew symmetric

$$A_{21} = -\int \partial_x^3 u \, \mathsf{H}(\partial_x^3 u \, \chi) \, dx$$

$$= -\int \partial_x^3 u \, \mathsf{H} \partial_x^3 u \, \chi \, dx - \int \partial_x^3 u \, [\mathsf{H}; \chi] \, \partial_x^3 u \, dx \qquad (3.5) \quad \boxed{\text{bo12}}$$

$$= -A_{21} - \int \partial_x^3 u \, [\mathsf{H}; \chi] \, \partial_x^3 u \, dx.$$

Therefore

$$A_{21} = -\frac{1}{2} \int \partial_x^3 u \, [\mathsf{H}; \chi] \, \partial_x^3 u \, dx = \frac{1}{2} \int u \, \partial_x^3 [\mathsf{H}; \chi] \, \partial_x^3 u \, dx. \qquad (3.6) \quad \boxed{\text{bo13}}$$

Consequently using the commutator estimate (2.11) it follows that

$$\sup_{0 \le t \le T} |A_{21}(t)| \le c \, \|u(t)\|_2^2 = c \, \|u_0\|_2^2. \qquad (3.7) \quad \boxed{\text{bo14}}$$

Next using that $\eta^2 = \chi'$ we rewrite A_{22} in (3.4) as

$$A_{22} = \int \mathsf{H} \partial_x^3 u \, \eta \, \partial_x^2 u \, \eta \, dx$$

$$= \int \mathsf{H}(\partial_x^3 u \, \eta) \, \partial_x^2 u \, \eta \, dx - \int [\mathsf{H}; \eta] \, \partial_x^3 u \, \partial_x^2 u \, \eta \, dx$$

$$= \int \mathsf{H} \partial_x (\partial_x^2 u \, \eta) \, \partial_x^2 u \, \eta \, dx - \int \mathsf{H}(\partial_x^2 u \, \eta') \, \partial_x^2 u \, \eta \, dx \qquad (3.8) \quad \boxed{\text{bo15}}$$

$$\quad - \int [\mathsf{H}; \eta] \partial_x^3 u \, \partial_x^2 u \, \eta \, dx$$

$$= \int D_x^{1/2}(\partial_x^2 u \, \eta) \, D_x^{1/2}(\partial_x^2 u \, \eta) \, dx - \int \mathsf{H}(\partial_x^2 u \, \eta') \, \partial_x^2 u \, \eta \, dx$$

$$\quad - \int [\mathsf{H}; \eta] \partial_x^3 u \, \partial_x^2 u \, \eta \, dx = A_{221} + A_{222} + A_{223}.$$

To estimate A_{222} in (3.8) we write

$$\int_0^T |A_{222}(t)| \, dt \le \int_0^T \|\partial_x^2 u \, \eta'\|_2 \|\partial_x^2 u \, \eta\|_2 \, dt$$

$$\le \int_0^T \int (\partial_x^2 u \, \eta')^2 \, dx dt + \int_0^T \int (\partial_x^2 u \, \eta)^2 \, dx dt \qquad (3.9) \quad \boxed{\text{bo16}}$$

$$\le c = c(v, R, T, \varepsilon, b, \|u_0\|_{3/2,2}),$$

by (1.6) (iii).

Now, after integration by parts we have that

$$A_{223}(t) = \int \partial_x [\mathsf{H}; \eta] \partial_x^3 u \, \partial_x u \eta \, dx + \int [\mathsf{H}; \eta] \partial_x^3 u \, \partial_x u \eta' \, dx. \qquad (3.10) \quad \boxed{\text{bo17a}}$$

Using the commutator estimate (2.11), (1.6) (iii) and (3.2), after integrating in the time interval $[0, T]$ one finds that

$$\int_0^T |A_{223}(t)|\, dt \le c \int_0^T \|u(t)\|_2 (\|\partial_x u\, \eta\|_2 + \|\partial_x u \eta'\|_2)\, dt$$

$$\le cT \sup_{0 \le t \le T} \|u(t)\|_2^2 + c \int_0^T \int (|\partial_x u \eta|^2 + |\partial_x u \eta'|^2)\, dx dt \qquad (3.11) \quad \boxed{\text{bo17}}$$

$$\le cT \|u_0\|_2^2 + c(v, R, T, \varepsilon, b, \|u_0\|_{3/2,2}).$$

Next, we observe that $A_{221}(t)$ in (3.8) is positive and represents the smoothing effect.

Finally we consider A_3 in (3.1). One has that

$$A_3(t) = \int u \partial_x^3 u \partial_x^2 u \chi\, dx + 3 \int \partial_x u \partial_x^2 u \partial_x^2 u \chi\, dx$$

$$= -\frac{1}{2} \int u \partial_x^2 u \partial_x^2 u \chi'\, dx + \frac{5}{2} \int \partial_x u \partial_x^2 u \partial_x^2 u \chi\, dx \qquad (3.12) \quad \boxed{\text{bo18}}$$

$$= A_{31}(t) + A_{32}(t).$$

We have

$$|A_{31}(t)| \le c\|u(t)\|_\infty \int (\partial_x^2 u)^2 \chi'\, dx, \qquad (3.13) \quad \boxed{\text{bo19}}$$

so after integration in the time interval $[0, T]$ a combination of the Sobolev embedding, (1.6) (i) and (1.6) (iii) yields

$$\int_0^T |A_{31}(t)|\, dt \le \sup_{0 \le t \le T} \|u(t)\|_{1,2} \int_0^T \int (\partial_x^2 u)^2 \chi'\, dx dt \qquad (3.14) \quad \boxed{\text{bo20}}$$

$$\le c(v, R, T, \varepsilon, b, \|u_0\|_{3/2,2}).$$

Also one gets that

$$|A_{32}(t)| \le \|\partial_x u(t)\|_\infty \int (\partial_x^2 u)^2 (x, t)\, \chi_{\varepsilon, b}(x + vt)\, dx. \qquad (3.15) \quad \boxed{\text{bo21}}$$

Inserting the above information in (3.1), using Gronwall's inequality and (1.6) (ii) we obtain (1.9) with $m = 2$ which is the desired result.

Case $m = 2$ in (1.10) : Assuming that (1.9) holds for $m = 2$ and that (1.10) holds for any $\varepsilon > 0$ and $b \ge 5\varepsilon$ with $x_0 = 0$ and $m = 2$ we shall prove (1.11).

From the previous step, $m = 2$, we know that the solution satisfies that for any $\varepsilon > 0$, $b > 5\varepsilon$, $v > 0$ and $T > 0$

$$\sup_{0 \le t \le T} \int (\partial_x^2 u(x,t))^2 \chi_{\varepsilon,b}(x+vt)\,dx$$

$$+ \int_0^T \int (D_x^{1/2}(\partial_x^2 u(x,t)\,\eta_{\varepsilon,b}(x+vt))^2\,dx dt \le c = c(T,\varepsilon,b,v). \tag{3.16} \quad \boxed{\texttt{bo22}}$$

where $\chi'_{\varepsilon,b}(x) = \eta^2_{\varepsilon,b}(x)$.

From the equation (1.1) one gets that

$$\partial_t(\partial_x^2 u \chi_{\varepsilon,b}) - v\partial_x^2 u\,\chi'_{\varepsilon,b} - \partial_x^2 \mathsf{H}\partial_x^2 u\,\chi_{\varepsilon,b} + \partial_x^2(u\partial_x u)\,\chi_{\varepsilon,b} = 0. \tag{3.17} \quad \boxed{\texttt{bo23}}$$

Therefore applying $D_x^{1/2}$ to (3.17), multiplying the result by $D_x^{1/2}(\partial_x^2 u \chi_{\varepsilon,b})$ and integrating in the space variable it leads to the identity (omitting the indices ε and b on χ)

$$\frac{1}{2}\frac{d}{dt}\int (D_x^{1/2}(\partial_x^2 u \chi))^2\,dx \underbrace{- v \int D_x^{1/2}(\partial_x^2 u \chi')\,D_x^{1/2}(\partial_x^2 u \chi)\,dx}_{A_1}$$

$$\underbrace{- \int D_x^{1/2}(\mathsf{H}\partial_x^4 u \chi)\,D_x^{1/2}(\partial_x^2 u \chi)\,dx}_{A_2} \tag{3.18} \quad \boxed{\texttt{bo24}}$$

$$\underbrace{+ \int D_x^{1/2}(\partial_x^2(u\partial_x u)\,\chi)\,D_x^{1/2}(\partial_x^2 u \chi)\,dx}_{A_3} = 0.$$

First we have

$$|A_1(t)| \le |v| \int (D_x^{1/2}(\partial_x^2 u \chi'))^2\,dx + |v| \int (D_x^{1/2}(\partial_x^2 u \chi))^2\,dx \tag{3.19} \quad \boxed{\texttt{bo25}}$$

$$= A_{11}(t) + A_{12}(t).$$

Using the previous step (3.16), (2.15), (2.16) and Young's inequality (and recalling the notation $\eta^2 = \chi'$) we obtain

$$A_{11}(t) = |v|\,\|D_x^{1/2}(\partial_x^2 u \eta^2)\|_2^2$$

$$\le |v|\,\|D_x^{1/2}(\partial_x^2 u \eta)\|_2^2\|\eta\|_\infty^2 + |v|\,\|\partial_x^2 u \eta\|_4^2\|D_x^{1/2}\eta\|_4^2$$

$$\le c\,|v|\,\|D_x^{1/2}(\partial_x^2 u \eta)\|_2^2 + c\,|v|\,\|D_x^{1/2}(\partial_x^2 u \eta)\|_2\|\partial_x^2 u \eta\|_2 \tag{3.20} \quad \boxed{\texttt{bo26}}$$

$$\le c\,|v|\,\|D_x^{1/2}(\partial_x^2 u \eta)\|_2^2 + c\,|v|\,\|\partial_x^2 u \eta\|_2^2.$$

So after integration in time interval the first term on the right hand side of (3.20) is bounded. To estimate the second term on the right hand side of

(3.20) we use that

$$\eta_{\varepsilon,b}^2 = \chi_{\varepsilon,b}' \leq c \chi_{\varepsilon/5,\varepsilon}.$$

Thus we have

$$\sup_{0 \leq t \leq T} \|\partial_x^2 u \, \eta\|_2^2 \leq \sup_{0 \leq t \leq T} \|\partial_x^2 u \chi_{\varepsilon/5,\varepsilon}\|_2^2 \leq c$$

by (3.16) with $(\varepsilon, b) = (\varepsilon/5, \varepsilon)$.

We observe that $A_{12}(t)$ is a multiple of the quantity we are estimating in (3.18).

Next we consider $A_2(t)$ in (3.18).

$$
\begin{aligned}
A_2(t) &= -\int D_x(\mathsf{H}\partial_x^4 u \chi) \, \partial_x^2 u \chi \, dx \\
&= -\int D_x \mathsf{H}(\partial_x^4 u \chi) \, \partial_x^2 u \chi, dx + \int D_x[\mathsf{H}; \chi] \, \partial_x^4 u \, \partial_x^2 u \chi \, dx \qquad (3.21) \quad \boxed{\text{bo27}} \\
&= A_{21}(t) + A_{22}(t).
\end{aligned}
$$

From the commutator estimate (2.11) and the previous step (3.16) it follows that

$$
\begin{aligned}
|A_{22}(t)| &= \Big| \int \mathsf{H}\partial_x[\mathsf{H}; \chi] \partial_x^4 u \, \partial_x^2 u \chi \, dx \Big| \\
&\leq c \|\chi^{(5)}\|_\infty \|u(t)\|_2 \|\partial_x^2 u(\cdot, t) \chi(\cdot + vt)\|_2 \qquad (3.22) \quad \boxed{\text{bo28}} \\
&\leq c \|u_0\|_2,
\end{aligned}
$$

with $c = c(T, \varepsilon, b, v)$ and $t \in [0, T]$.

We turn our attention to $A_{21}(t)$ in (3.21).

$$
\begin{aligned}
A_{21}(t) &= \int \partial_x(\partial_x^4 u \chi) \, \partial_x^2 u \chi \, dx \\
&= -\int \partial_x^4 u \chi \, \partial_x^3 u \chi \, dx - \int \partial_x^4 u \chi \, \partial_x^2 u \chi' \, dx \qquad (3.23) \quad \boxed{\text{bo29}} \\
&= 2 \int (\partial_x^3 u)^2 \chi \chi' \, dx - \frac{1}{2} \int (\partial_x^2 u)^2 (\chi \chi')'' \, dx \\
&= A_{211}(t) + A_{212}(t).
\end{aligned}
$$

From the previous step (3.16) with $(\varepsilon, b) = (\varepsilon/10, \varepsilon/2)$ one has that $A_{212}(t)$ is bounded and

$$A_{211}(t) \geq 0 \qquad (3.24) \quad \boxed{31}$$

which will provide the smoothing effect after being integrated in time.

So it remains to estimate $A_3 = A_3(t)$ in (3.18),

$$A_3(t) = \int D_x^{1/2}(u\partial_x^3 u\chi) D_x^{1/2}(\partial_x^2 u\chi)\,dx$$
$$+ 3\int D_x^{1/2}(\partial_x u\partial_x^2 u\chi) D_x^{1/2}(\partial_x^2 u\chi)\,dx \qquad (3.25) \quad \boxed{\text{bo32}}$$
$$= A_{31}(t) + A_{32}(t).$$

Thus using (2.10) and (2.15) we have

$$|A_{32}(t)| \leq c\|D_x^{1/2}(\partial_x u\chi_{\varepsilon/5,\varepsilon}\partial_x^2 u\chi)\|_2\|D_x^{1/2}(\partial_x^2 u\chi)\|_2$$
$$\leq c\|\partial_x u\|_\infty\|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2$$
$$+ c\|D_x^{1/2}(\partial_x u\chi_{\varepsilon/5,\varepsilon})\|_4\|\partial_x^2 u\chi\|_4\|D_x^{1/2}(\partial_x^2 u\chi)\|_2 \qquad (3.26) \quad \boxed{\text{bo33}}$$
$$= A_{321}(t) + A_{322}(t).$$

The term $A_{321}(t)$ will be handled by Gronwall's inequality and (1.6) (ii). Now using (2.16) and Young's inequality one gets

$$A_{322}(t) \leq c\|\partial_x(\partial_x u\,\chi_{\varepsilon/5,\varepsilon})\|_2^{3/4}\|\partial_x u\,\chi_{\varepsilon/5,\varepsilon}\|_2^{1/4}$$
$$\times \|D_x^{1/2}(\partial_x^2 u\chi)\|_2^{1/2}\|\partial_x^2 u\chi\|_2^{1/2}\|D_x^{1/2}(\partial_x^2 u\chi)\|_2 \qquad (3.27) \quad \boxed{\text{bo34}}$$
$$\leq c\|\partial_x(\partial_x u\,\chi_{\varepsilon/5,\varepsilon})\|_2^3\|\partial_x u\chi_{\varepsilon/5,\varepsilon}\|_2\|\partial_x^2 u\chi\|_2^2$$
$$+ c\,\|D^{1/2}(\partial_x^2 u\chi)\|_2^2,$$

where the first term on the right hand side of (3.27) is bounded in time, by our previous step (3.16), while the second term is the quantity to be estimated.

Now we consider $A_{31}(t)$ in (3.25). Thus, by (2.10)

$$A_{31}(t) = \int D_x^{1/2}(u\partial_x^3 u\chi) D_x^{1/2}(\partial_x^2 u\chi)\,dx$$
$$= c\int u\chi_{\varepsilon/5,\varepsilon} D_x^{1/2}(\partial_x(\partial_x^2 u\chi)) D_x^{1/2}(\partial_x^2 u\chi)\,dx$$
$$+ c\int [D_x^{1/2}; u\chi_{\varepsilon/5,\varepsilon}]\partial_x(\partial_x^2 u\chi) D_x^{1/2}(\partial_x^2 u\chi)\,dx \qquad (3.28) \quad \boxed{\text{bo35}}$$
$$- c\int D_x^{1/2}(u\partial_x^2 u\chi') D_x^{1/2}(\partial_x^2 u\chi)\,dx$$
$$= A_{311}(t) + A_{312}(t) + A_{313}(t).$$

Integration by parts yields

$$|A_{311}(t)| = \left|-\frac{1}{2}\int \partial_x(u\chi_{\varepsilon/5,\varepsilon})\,(D_x^{1/2}(\partial_x^2 u\chi))^2\,dx\right|$$
$$\leq c(\|\partial_x u\|_\infty + \|u\|_\infty)\int (D_x^{1/2}(\partial_x^2 u\chi))^2\,dx. \qquad (3.29) \quad \boxed{\text{bo36}}$$

After applying the Sobolev inequality this term will be handled by the Gronwall's inequality and (1.6) (ii).

From the commutator estimate (2.12) and the Sobolev inequality one gets

$$
\begin{aligned}
|A_{312}(t)| &\leq \|\partial_x\widehat{(u\chi_{\varepsilon/5,\varepsilon})}\|_1\|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2 \\
&\leq \|\partial_x(u\chi_{\varepsilon/5,\varepsilon})\|_{1,2}\|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2 \\
&\leq \big(\|\partial_x(u\chi_{\varepsilon/5,\varepsilon})\|_2 + \|\partial_x^2 u\chi_{\varepsilon/5,\varepsilon}\|_2 \\
&\quad + \|\partial_x u\chi'_{\varepsilon/5,\varepsilon}\|_2 + \|u\chi''_{\varepsilon/5,\varepsilon}\|_2\big)\|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2 \\
&= B(t)\,\|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2.
\end{aligned}
\tag{3.30}
$$ bo39

Since by (1.6) (i) and (3.16) $B(t)$ is bounded in the time interval $[0,T]$ and $\|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2$ is the quantity we are estimating, Gronwall's inequality provides the bound for $|A_{312}(t)|$.

Finally, we consider $A_{313}(t)$ in (3.28). Using (2.10) and (2.12) yields

$$
\begin{aligned}
|A_{313}(t)| &\leq c\|D_x^{1/2}(u\chi_{\varepsilon/5,\varepsilon}\partial_x^2 u\chi')\|_2\|D_x^{1/2}(\partial_x^2 u\chi)\|_2 \\
&\leq c\,\|u\|_\infty\|D_x^{1/2}(\partial_x^2 u\chi')\|_2\|D_x^{1/2}(\partial_x^2 u\chi)\|_2 \\
&\quad + \|D_x^{1/2}(u\chi_{\varepsilon/5,\varepsilon})\|_4\|\partial_x^2 u\chi'\|_4\|D_x^{1/2}(\partial_x^2 u\chi)\|_2 \\
&= A_{3131}(t) + A_{3132}(t).
\end{aligned}
\tag{3.31}
$$ 39b

From an argument similar to the one applied in (3.27) and (2.10) it follows that

$$
\begin{aligned}
A_{3132}(t) &\leq c\,\|\partial_x(u\chi_{\varepsilon/5,\varepsilon})\|_2^{3/4}\|u\chi_{\varepsilon/5,\varepsilon}\|_2^{1/4} \\
&\quad \times \|D_x^{1/2}(\partial_x^2 u\chi')\|_2^{1/2}\|\partial_x^2 u\chi'\|_2^{1/2}\|D_x^{1/2}(\partial_x^2 u\chi)\|_2 \\
&\leq c\|\partial_x(u\chi_{\varepsilon/5,\varepsilon})\|_2^3\|u\chi_{\varepsilon/5,\varepsilon}\|_2\|\partial_x^2 u\chi_{\varepsilon/5,\varepsilon}\|_2^2 \\
&\quad + \|D_x^{1/2}(\partial_x^2 u\chi')\|_2^2 + \|D_x^{1/2}(\partial_x^2 u\chi)\|_2^2.
\end{aligned}
\tag{3.32}
$$ 39c

The first term in the last inequality is bounded in time by (3.16) with $(\varepsilon,b) = (\varepsilon/5,\varepsilon)$. The last term is the quantity we want to estimate. It will be handled using Gronwall's inequality. The second term can be estimated employing (2.15), (2.16) and Young's inequality, that is,

$$
\begin{aligned}
\|D_x^{1/2}(\partial_x^2 u\chi')\|_2^2 &= \|D_x^{1/2}(\partial_x^2 u\eta\,\eta)\|_2^2 \\
&\leq c\|\eta\|_\infty^2\|D_x^{1/2}(\partial_x^2 u\eta)\|_2^2 + c\|D_x^{1/2}\eta\|_4^2\|\partial_x^2 u\eta\|_4^2 \\
&\leq c\|D_x^{1/2}(\partial_x^2 u\eta)\|_2^2 + c\|\partial_x^2 u\eta\|_2\|D_x^{1/2}(\partial_x^2 u\eta)\|_2 \\
&\leq c\|D_x^{1/2}(\partial_x^2 u\eta)\|_2^2 + c\|\partial_x^2 u\eta\|_2^2.
\end{aligned}
\tag{3.33}
$$ bo38

From (3.16) with $(\varepsilon/5, \varepsilon)$ instead of (ε, b) the last term is bounded in time $t \in [0, T]$ and the previous one is bounded after integrating in time (see (3.16)).

Next we deal with the term $A_{3131}(t)$. Young's inequality gives

$$A_{3131}(t) \le c\|u\|_\infty^2 \|D_x^{1/2}(\partial_x^2 u \chi)\|_2^2 + \|D_x^{1/2}(\partial_x^2 u \chi')\|_2^2. \qquad (3.34) \quad \boxed{\text{bo39d}}$$

The first term on the right hand side of (3.34) can be handled using Gronwall's inequality and (1.6) (i). The last term can be estimated using the argument in (3.33).

Collecting these results we obtain the desired estimate (1.11) with $m = 2$.

Following the induction argument we shall assume that (1.9) holds for $m \le j \in \mathbb{Z}^+$, $j \ge 2$, and prove that if (1.8) with $x_0 = 0$ and $m = j+1$ holds, then:

(a) (1.10) and (1.11) hold with $x_0 = 0$ and $m = j$, and

(b) (1.9) holds with $x_0 = 0$ for $m = j+1$.

Part (a) : From the hypothesis (1.8) with $m = j+1$ and since $u_0 \in H^{3/2}(\mathbb{R})$ one gets (1.10) with $m = j$ by interpolation.

Next, a familiar argument provides the identity

$$\frac{1}{2}\frac{d}{dt}\int (D_x^{1/2}(\partial_x^j u \chi))^2\,dx - v\underbrace{\int D_x^{1/2}(\partial_x^j u \chi')D_x^{1/2}(\partial_x^j u \chi)\,dx}_{A_1}$$

$$- \underbrace{\int D_x^{1/2}(\mathsf{H}\partial_x^{2+j}u \chi)D_x^{1/2}(\partial_x^j u \chi)\,dx}_{A_2} \qquad (3.35) \quad \boxed{\text{bo41}}$$

$$+ \underbrace{\int D_x^{1/2}(\partial_x^j(u\partial_x u)\chi)D_x^{1/2}(\partial_x^j u \chi)\,dx}_{A_3} = 0.$$

First we observe that

$$|A_1(t)| \le |v|\left(\int (D_x^{1/2}(\partial_x^j u \chi))^2\,dx + \int (D_x^{1/2}(\partial_x^j u \chi'))^2\,dx\right) \qquad (3.36) \quad \boxed{\text{bo42}}$$
$$= A_{11}(t) + A_{12}(t),$$

where $A_{11}(t)$ is multiple of the quantity we are estimating in (3.35) and from (2.15)–(2.16) we deduce

$$
\begin{aligned}
A_{12}(t) &= |v| \|D_x^{1/2}(\partial_x^j u\, \eta\, \eta)\|_2^2 \\
&\le |v| \Big(\|\eta\|_\infty^2 \|D_x^{1/2}(\partial_x^j u\, \eta)\|_2^2 + \|D_x^{1/2}\eta\|_4^2 \|\partial_x^j u\, \eta\|_4^2 \Big) \\
&\le c\|D_x^{1/2}(\partial_x^j u\, \eta)\|_2^2 + c\,\|D_x^{1/2}(\partial_x^j u\, \eta)\|_2 \|\partial_x^j u\, \eta\|_2 \\
&\le c\|D_x^{1/2}(\partial_x^j u\, \eta)\|_2^2 + c\|\partial_x^j u\, \eta\|_2^2.
\end{aligned}
\tag{3.37}\quad\boxed{\text{bo43}}
$$

By (1.9) with $m = j$ (induction hypothesis) the second term on the right hand side of (3.37) is bounded, while the first term is bounded after integration in time.

Next we turn our attention to $A_2(t)$ in (3.35)

$$
\begin{aligned}
A_2(t) &= - \int D_x(\mathsf{H}\partial_x^{j+2}u\,\chi)\,\partial_x^j u\,\chi\, dx \\
&= - \int D_x \mathsf{H}(\partial_x^{j+2}u\,\chi)\partial_x^j u\,\chi\, dx + \int D_x[\mathsf{H};\chi]\partial_x^{j+2}u\,\partial_x^j u\,\chi\, dx \\
&= A_{21}(t) + A_{22}(t).
\end{aligned}
\tag{3.38}\quad\boxed{\text{bo48}}
$$

From (2.11) and the conservation of the L^2-norm

$$
\begin{aligned}
|A_{22}(t)| &\le |\int \mathsf{H}\partial_x[\mathsf{H};\chi]\,\partial_x^{j+2}u\,\partial_x^j u\,\chi\, dx| \\
&\le c\|\chi^{(j+3)}\|_\infty \|u(t)\|_2 \|\partial_x^j u\,\chi\|_2 \\
&\le c\|u_0\|_2^2 + c\int (\partial_x^j u)^2\,\chi_{\varepsilon,b}(x+vt)\, dx \le c
\end{aligned}
\tag{3.39}\quad\boxed{\text{bo49}}
$$

where we used that $0 \le \chi \le 1$ and the induction hypothesis (1.9) with $m = j$. Now

$$
\begin{aligned}
A_{21}(t) &= \int \partial_x(\partial_x^{j+2}u\,\chi)\,\partial_x^j u\,\chi\, dx \\
&= - \int \partial_x^{j+2}u\,\chi\,\partial_x^{j+1}u\,\chi - \int \partial_x^{j+2}u\,\chi\,\partial_x^j u\,\chi'\, dx \\
&= 2\int (\partial_x^{j+1}u)^2\,\chi\,\chi'\, dx + \int \partial_x^{j+1}u\,\partial_x^j u\,(\chi\,\chi')'\, dx \\
&= 2\int (\partial_x^{j+1}u)^2\,\chi\,\chi'\, dx - \frac{1}{2}\int (\partial_x^j u)^2\,(\chi\,\chi')''\, dx \\
&= A_{211}(t) + A_{212}(t).
\end{aligned}
\tag{3.40}\quad\boxed{\text{bo50}}
$$

From the previous step (hypothesis of induction) $m = j$ with $(\varepsilon/4, \varepsilon)$ instead of (ε, b) one has that $A_{212}(t)$ is bounded and

$$
A_{211}(t) \ge 0,
\tag{3.41}\quad\boxed{\text{bo52}}
$$

which yields the smoothing effect after being integrated in time. So it remains only to consider A_3 in (3.35).

$$
\begin{aligned}
A_3(t) = & \int D_x^{1/2}(u\,\partial_x^{j+1}u\,\chi_{\varepsilon,b})\,D_x^{1/2}(\partial_x^j u\,\chi_{\varepsilon,b})\,dx \\
& + (j+1)\int D_x^{1/2}(\partial_x u\,\partial_x^j u\,\chi_{\varepsilon,b})D_x^{1/2}(\partial_x^j u\,\chi_{\varepsilon,b})\,dx \\
& + \sum_{l=2}^{j-1} c_l \int D_x^{1/2}(\partial_x^l u\,\partial_x^{j+1-l}u\,\chi_{\varepsilon,b})D_x^{1/2}(\partial_x^j u\,\chi_{\varepsilon,b})\,dx \\
= & A_{31}(t) + A_{32}(t) + \sum_{l=2}^{j-1} A_{3(l+1)}(t).
\end{aligned}
\tag{3.42}
$$ `bo53`

The estimates for A_{31} and A_{32} are similar to those described in (3.25)–(3.30) in the case $j=2$. So we restrict ourselves to consider $\{A_{3l}\}_{l=2}^{j-1}$.

We consider first the case $j=3$ where the sum in (3.42) reduces to the term $A_{33}(t)$.

$$
\begin{aligned}
|A_{33}(t)| = & \, c\,|\int D_x^{1/2}(\partial_x^2 u\,\partial_x^2 u\,\chi_{\varepsilon,b})D_x^{1/2}(\partial_x^3 u\,\chi_{\varepsilon,b})\,dx| \\
\le & \, c\|D_x^{1/2}(\partial_x^2 u\,\chi_{\varepsilon/5,\varepsilon}\,\partial_x^2 u\,\chi_{\varepsilon,b})\|_2^2 + \|D_x^{1/2}(\partial_x^3 u\,\chi_{\varepsilon,b})\|_2^2.
\end{aligned}
\tag{3.43}
$$ `bo54`

The last term above is the quantity to be estimated so we just need to concentrate in the first term on the right hand side of (3.43). Thus by (2.15) and (2.16) we deduce that

$$
\begin{aligned}
\|D_x^{1/2} & (\partial_x^2 u\chi_{\varepsilon/5,\varepsilon}\,\partial_x^2 u\chi_{\varepsilon,b})\|_2^2 \\
& \le c\|D_x^{1/2}(\partial_x^2 u\chi_{\varepsilon/5,\varepsilon})\|_4\|\partial_x^2 u\chi_{\varepsilon,b}\|_4 \\
& \quad + c\,\|\partial_x^2 u\chi_{\varepsilon/5,\varepsilon}\|_4\|D_x^{1/2}(\partial_x^2 u\chi_{\varepsilon,b})\|_4 \\
& \le c\|\partial_x(\partial_x^2 u\chi_{\varepsilon/5,\varepsilon})\|_2^{3/4}\|\partial_x^2 u\chi_{\varepsilon/5,\varepsilon}\|_2^{1/4} \\
& \quad \times \|D_x^{1/2}(\partial_x^2 u\chi_{\varepsilon,b})\|_2^{1/2}\|\partial_x^2 u\chi_{\varepsilon,b}\|_2^{1/2} \\
& \quad + c\,\|D_x^{1/2}(\partial_x^2 u\chi_{\varepsilon/5,\varepsilon})\|_2^{1/2}\|\partial_x^2 u\chi_{\varepsilon/5,\varepsilon}\|_2^{1/2} \\
& \quad \times \|\partial_x(\partial_x^2 u\chi_{\varepsilon,b})\|_2^{3/4}\|\partial_x^2 u\chi_{\varepsilon,b}\|_2^{1/4} \\
& \equiv A_{331},
\end{aligned}
\tag{3.44}
$$ `bo55`

where all the terms in $A_{331}(t)$ involve at most derivatives of order three which are bounded in $t \in [0,T]$ by hypothesis of induction (and the fact that $0 \le \chi \le 1$).

It is clear from the argument given in (3.43)–(3.44) for the case $j = 3$ that the general case $j \geq 4$ follows by using the same method which combines the inequalities (2.15), (2.16), Sobolev embedding theorem and induction hypothesis.

Inserting all the estimates above in (3.35) one gets that (1.11) holds for $m = j$.

This completes the proof of the first step, part (a), of our inductive argument.

Now we consider the next step in our induction argument:

<u>Part (b)</u> : We assume that (1.8) holds with $m = j+1$, our induction hypothesis, i.e., for $k = 2, ..., j$

$$\sup_{0 \leq t \leq T} \int (\partial_x^k u)^2 \chi_{\varepsilon,b}(x + vt) \, dx$$
$$+ \int_0^T \int (D_x^{1/2}(\partial_x^k u \eta_{\varepsilon,b}))^2 \, dxdt < c, \tag{3.45} \quad \boxed{\text{bo55b}}$$

and recall that in part (a) of our argument we have proven (1.11) with $m = j$. A familiar argument yields the identity

$$\frac{1}{2}\frac{d}{dt} \int (\partial_x^{j+1} u)^2 \chi_{\varepsilon,b}(x + vt) \, dx - \underbrace{v \int (\partial_x^{j+1} u)^2 \chi'_{\varepsilon,b}(x + vt) \, dx}_{A_1}$$

$$- \underbrace{\int \mathsf{H} \partial_x^{j+1} \partial_x^2 u \, \partial_x^{j+1} u \, \chi_{\varepsilon,b}(x + vt) \, dx}_{A_2} \tag{3.46} \quad \boxed{\text{bo55c}}$$

$$+ \underbrace{\int \partial_x^{j+1}(u \partial_x u) \, \partial_x^{j+1} u \, \chi_{\varepsilon,b}(x + vt) \, dxdx}_{A_3} = 0.$$

From our previous step, part (a), (1.11) holds with $m = j$. Therefore we have that for $\varepsilon' > 0$ and $b > 5\varepsilon'$

$$\int_0^T \int (\partial_x^{j+1} u)^2 \chi'_{\varepsilon',b} \chi_{\varepsilon',b}(x + vt) \, dxdt < c. \tag{3.47} \quad \boxed{56}$$

Using (2.10) after integrating in the time interval $[0, T]$ we have that

$$\int_0^T |A_1(t)| \, dt \leq c = c(\varepsilon, b, v, T). \tag{3.48} \quad \boxed{\text{bo57}}$$

Next we consider $A_2(t)$ in (3.46). We have

$$A_2 = \int \mathsf{H}\partial_x^{j+2}u\,\partial_x^{j+2}u\,\chi\,dx + \int \mathsf{H}\partial_x^{j+2}u\,\partial_x^{j+1}u\chi'\,dx$$
$$= A_{21} + A_{22},$$

(3.49) bo58

where

$$A_{21} = -\int \partial_x^{j+2}u\,\mathsf{H}(\partial_x^{j+2}u\,\chi)\,dx$$
$$= -\int \partial_x^{j+2}u\,\mathsf{H}\partial_x^{j+2}u\,\chi\,dx - \int \partial_x^{j+2}u\,[\mathsf{H};\chi]\partial_x^{j+2}u$$
$$= -A_{21} + (-1)^{j+3}\int u\partial_x^{j+2}[\mathsf{H};\chi]\partial_x^{j+2}u\,dx.$$

(3.50) bo59

Thus

$$A_{21}(t) = \frac{1}{2}(-1)^{j+3}\int u\partial_x^{j+2}[\mathsf{H};\chi]\partial_x^{j+2}u\,dx.$$

(3.51) bo60

So by the commutator estimate (2.11) and the conservation law

$$|A_{21}(t)| \le c\|\chi^{2j+4}\|_\infty\|u(t)\|_2^2 \le c_j\|u_0\|_2^2.$$

(3.52) bo61

To estimate A_{22}, recalling that $\eta^2 = \chi'$, we write

$$A_{22} = \int \mathsf{H}\partial_x^{j+2}u\,\partial_x^{j+1}u\,\eta^2(x+vt)\,dx$$
$$= \int \mathsf{H}(\partial_x^{j+2}u\,\eta)\,\partial_x^{j+1}u\,\eta\,dx - \int [\mathsf{H};\eta]\partial_x^{j+2}u\,\partial_x^{j+1}u\,\eta\,dx$$
$$= A_{221} + A_{222}.$$

(3.53) bo62

Now

$$A_{221} = \int \mathsf{H}\partial_x(\partial_x^{j+1}u\,\eta)\,\partial_x^{j+1}u\,\eta\,dx - \int \mathsf{H}(\partial_x^{j+1}u\,\eta')\partial_x^{j+1}u\,\eta\,dx$$
$$= \int D_x^{1/2}(\partial_x^{j+1}u\,\eta)\,D_x^{1/2}(\partial_x^{j+1}u\,\eta)$$
$$\quad - \int \mathsf{H}(\partial_x^{j+1}u\,\eta')\partial_x^{j+1}u\,\eta\,dx.$$

(3.54) bo63

Notice that the first term on the right hand side of (3.54) is positive (and will provide the smoothing effect) and the second one can be bounded by

$$\|\partial_x^{j+1}u\,\eta'\|_2^2 + \|\partial_x^{j+1}u\eta\|_2^2$$

(3.55) bo64

which after integration in time is bounded using part (a) in our argument, i.e. (1.11) with $m = j$, and the claim following (2.8) together with (2.5) and the estimates (2.10).

Similarly, A_{222} can be estimated with the aid of (2.11).

So it remains only to consider A_3. Now

$$A_3 = \int u \partial_x^{j+2} u \partial_x^{j+1} u \chi \, dx + (j+2) \int \partial_x u (\partial_x^{j+1} u)^2 \chi \, dx$$

$$+ \sum_{k=2}^{j} c_k \int \partial_x^k u \partial_x^{j+2-k} (\partial_x^{j+1} u) \chi \, dx \quad (j \geq 2) \qquad (3.56) \quad \boxed{\text{bo65}}$$

$$= A_{30} + A_{31} + \sum_{k=2}^{j} A_{3k}.$$

Thus

$$A_{30} = -\frac{1}{2} \int \partial_x u \partial_x^{j+1} u \partial_x^{j+1} u \chi \, dx - \int u \partial_x^{j+1} u \partial_x^{j+1} u \chi' \, dx$$

$$= A_{301} + A_{302} \qquad\qquad (3.57) \quad \boxed{\text{bo66}}$$

with

$$|A_{301}(t)| \leq \|\partial_x u(t)\|_\infty \int (\partial_x^{j+1} u)^2 \chi \, dx \qquad (3.58) \quad \boxed{\text{bo67}}$$

where the last term is the quantity to be estimated which will be handled in Gronwall's inequality using (1.6) (ii) and

$$|A_{302}(t)| \leq \|u(t)\|_\infty \int (\partial_x^{j+1} u)^2 \chi' \, dx \qquad (3.59) \quad \boxed{\text{bo68}}$$

which after integration in time is bounded using part (a) of our argument, i.e. (1.11) with $m = j$ and (2.10).

The estimate for $A_{31}(t)$ is similar to that one described in (3.58) for $A_{301}(t)$.

Next, we consider the case $j = 2$ where in (3.56) the term A_{32} appears, i.e.

$$\begin{aligned}
|A_{32}(t)| &\leq |c_2 \int \partial_x^2 u \partial_x^2 u (\partial_x^3 u) \chi \, dx| \\
&= |-\frac{c_2}{3} \int \partial_x^2 u \partial_x^2 u \partial_x^2 u \chi' \, dx| \\
&= |-\frac{c_2}{3} \int \partial_x^2 u \eta (\partial_x^2 u)^2 \eta \, dx| \\
&\leq c \|\partial_x^2 u \eta\|_\infty \int (\partial_x^2 u)^2 \eta \, dx.
\end{aligned} \qquad (3.60) \quad \boxed{\text{bo69}}$$

Since for the case $j = 2$ we have that

$$\int (\partial_x^2 u)^2 \eta_{\varepsilon,b}(x+vt) \, dx \leq \int (\partial_x^2 u)^2 \chi_{\varepsilon/5,\varepsilon}(x+vt) \, dx \qquad (3.61) \quad \boxed{\text{bo70}}$$

which is bounded in the time interval $[0,T]$. Now

$$\|\partial_x^2 u\,\eta\|_\infty \le c\|\partial_x(\partial_x^2 u\,\eta)\|_2^{1/2}\|\partial_x^2 u\,\eta\|_2^{1/2}$$
$$\le c\big(\|\partial_x^3 u\,\eta\|_2^{1/2} + \|\partial_x^2 u\,\eta'\|_2^{1/2}\big)\|\partial_x^2 u\,\eta\|_2^{1/2}. \qquad (3.62) \quad \boxed{\text{bo71}}$$

From the previous step we have that

$$\sup_{0\le t\le T}\big(\|\partial_x^2 u\,\eta\|_2 + \|\partial_x^2 u\,\eta'\|_2\big) \le c.$$

So

$$\|\partial_x^2 u\,\eta\|_\infty \le c + \|\partial_x^3 u\,\eta'\|_2^2 \qquad (3.63) \quad \boxed{\text{bo72}}$$

which is bounded after integrating in time from the part (a), (1.11) with $m = j$ (in this case $j = 3$), in our argument.

If $j \ge 2$ the new terms in (3.56) can be handled in a similar manner combining Sobolev theorem, the induction hypothesis and Gronwall's inequality. This completes the induction argument.

To justify the above formal computations we shall follow the following standard argument.

Consider data $u_0^\tau = \rho_\tau * u_0$ with $\rho \in C_0^\infty(\mathbb{R})$, $\mathrm{supp}\,\rho \in (-1,1)$, $\rho \ge 0$, $\int \rho(x)\,dx = 1$ and

$$\rho_\tau(x) = \frac{1}{\tau}\rho\Big(\frac{x}{\tau}\Big), \quad \tau > 0.$$

For $\tau > 0$ consider the solutions u^τ of the IVP (1.1) with data u_0^τ where $(u^\tau)_{\tau>0} \subseteq C([0,T] : H^\infty(\mathbb{R}))$.

Using the continuous dependence of the solution upon the data we have that

$$\sup_{t\in[0,T]}\|u^\tau(t) - u(t)\|_{3/2,2} \downarrow 0 \quad \text{as} \quad \tau \downarrow 0. \qquad (3.64) \quad \boxed{123}$$

Applying our argument to the smooth solutions $u^\tau(\cdot,t)$ one gets that

$$\sup_{[0,T]}\int (\partial_x^m u^\tau)^2\,\chi_{\varepsilon,b}(x+vt)\,dx \le c_0 \qquad (3.65) \quad \boxed{124}$$

for any $\varepsilon > 0$, $b \ge 5\varepsilon$, $v > 0$, $c_0 = c_0(\varepsilon;b;v) > 0$ but independent of $\tau > 0$ since for $0 < \tau < \varepsilon$

$$(\partial_x u_0^\tau)^2\chi_{\varepsilon,b}(x) = (\partial_x(\rho_\tau * u_0))^2\chi_{\varepsilon,b}(x) = (\rho_\tau * \partial_x u_0 1_{[0,\infty)})^2\chi_{\varepsilon,b}(x).$$

Combining (3.64) and (3.65) and a weak compactness argument one gets that

$$\sup_{[0,T]}\int (\partial_x u)^2\,\chi_{\varepsilon,b}(x+vt)\,dx \le c_0 \qquad (3.66) \quad \boxed{125}$$

which is the desired result. A similar argument provides the estimate for the second term in the left hand side of (1.9).

This completes the proof of Theorem 1.2.

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