ON THE PROPAGATION OF REGULARITY AND DECAY OF SOLUTIONS TO THE k-GENERALIZED KORTEWEG-DE VRIES EQUATION

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ABSTRACT. We study special regularity and decay properties of solutions to the IVP associated to the k-generalized KdV equations. In particular, for datum $u_0 \in H^{3/4^+}(\mathbb{R})$ whose restriction belongs to $H^l((b,\infty))$ for some $l \in \mathbb{Z}^+$ and $b \in \mathbb{R}$ we prove that the restriction of the corresponding solution $u(\cdot,t)$ belongs to $H^l((\beta,\infty))$ for any $\beta \in \mathbb{R}$ and any $t \in (0,T)$. Thus, this type of regularity propagates with infinite speed to its left as time evolves.

1. INTRODUCTION

In this work we shall deduce some special properties of solutions to the k-generalized Korteweg-de Vries (k-gKdV) equations. These properties are concerned with the propagation of regularity and decay of their solutions on the line. Thus, we shall consider the initial value problem (IVP)

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \, \partial_x u = 0, & x,t \in \mathbb{R}, \ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \quad (1.1) \quad \boxed{\texttt{gkdv}}$$

The cases $k = 1$ and $k = 2$ in (1.1) correspond to the KdV and modified KdV (mKdV) equations, respectively. Initially they arise as a model in nonlinear wave propagation in a shallow channel and later as models in several other physical phenomena (see [21] and references therein). Also they have been shown to be completely integrable. In particular, they possess infinitely many conservation laws. In the case of powers $k = 3, 4, \ldots$ the solutions of the equation in (1.1) satisfy just three conservation laws.

Following [13] it is said that the IVP (1.1) is locally well-posed (LWP) in the function space X if given any datum $u_0 \in X$ there exist $T > 0$ and a unique solution

$$u \in C([-T,T] : X) \cap \ldots.$$

of the IVP (1.1) with the map data-solution, $u_0 \to u$, being continuous. In the cases where T can be taken arbitrarily large, one says that the problem

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is globally well-posed (GWP). In both cases, the solution flow defines a dynamical system on X.

For the IVP (1.1) the natural function space is the classical Sobolev family

$$H^s(\mathbb{R}) = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R}. \tag{1.2} \quad \boxed{\texttt{sob}}$$

The problem of finding the minimal regularity on the data u_0 in the scale in (1.2) which guarantees that the IVP (1.1) is well posed has been studied extensively. After the works of [24], [1] and [13] gathering the local and the global well-posedness results, in [15], [2], [16], [3], [4], [26], [11], [12] and [17] one has:

The IVP (1.1) with $k = 1$ is globally well-posed in $H^s(\mathbb{R})$ for $s \geq -3/4$ (see [2] (GWP for $s = 0$), [16] (LWP for $s > -3/4$), [3] (LWP for $s \geq -3/4$), [4] (GWP for $s > -3/4$) , and [12], [17] (GWP for $s \geq -3/4$)).

The IVP (1.1) with $k = 2$ is globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1/4$ (see [15] (LWP for $s \geq 1/4$), [4] (GWP for $s > 1/4$) and [12], [17] (GWP for $s \geq 1/4$)).

The IVP (1.1) with $k = 3$ is locally well-posed in $H^s(\mathbb{R})$ for $s \geq -1/6$ (see [10] for LWP for $s > -1/6$ and [26] for LWP for $s = -1/6$) and GWP in $H^s(\mathbb{R})$ for $s > -1/42$ ([11]).

The IVP (1.1) with $k \geq 4$ is locally well-posed in $H^s(\mathbb{R})$ for $s \geq (k-4)/2k$ (see [15]).

In [20] it was shown that a local solution of the IVP (1.1) with $k = 4$ corresponding to smooth initial data can develop singularities in finite time.

The well-posedness of the IVP (1.1) has been also studied in the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx), \quad s, r \in \mathbb{R}. \tag{1.3} \quad \boxed{\texttt{wei-sob}}$$

In [13] it was established the LWP of the IVP (1.1) in $Z_{s,r}$ with r an even integer and $s \geq r$. In particular, this implies the well-posedness of the IVP (1.1) in the Schwartz space $\mathscr{S}(\mathbb{R})$. The proof of this result is based in the commutative relation of the operators

$$\mathscr{L} = \partial_t + \partial_x^3 \quad \text{and} \quad \Gamma = x - 3t\partial_x^2.$$

Roughly, in [22] and [7] the above result was extended to the case $Z_{s,r}$ with $r > 0$, $s \geq r$ and $s \geq \max\{s_k; 0\}$ with $s_1 = -3/4$, $s_2 = 1/4$ and $s_k = (k-4)/2k$ for $k \geq 3$ (for further details see [19] and references therein).

In [8] it was shown that the hypothesis $s \geq r$ is necessary. More precisely, it was established in [8] that if $u \in C([-T,T] : H^s(\mathbb{R})) \cap \ldots$ is a solution of the IVP (1.1) with $s \geq \max\{s_k; 0\}$ (with s_k as above) and there exist two different times $t_1, t_2 \in [-T,T]$ such that $|x|^\alpha u(\cdot, t_j) \in L^2(\mathbb{R})$ for $j = 1, 2$ with $2\alpha > s$, then $u \in C([-T,T] : H^{2\alpha}(\mathbb{R}))$.

Thus, if $u_0 \in Z_{s,r}$ with $r > s$, then at time $t \neq 0$ the solution $u(\cdot, t)$ stays only in $Z_{s,s}$, i.e. the extra decay $r - s$ is not preserved by the solution flow. As a consequence of our results obtained here we shall see that this extra-decay, no preserved by the flow, is transformed into extra regularity in a precise manner (see Corollary 1.7).

The above results are concerned with regularity and decay properties of solutions of the IVP (1.1) in symmetric spaces.

For asymmetric weighted spaces, for which the outcome has to be restricted to forward times $t > 0$, one has the following result found in [13] for the KdV equation, $k = 1$ in (1.1), in the space $L^2(e^{\beta x} dx)$, $\beta > 0$. In [13] it was shown that the persistence property holds for L^2-solutions in $L^2(e^{\beta x} dx)$, $\beta > 0$, for $t > 0$. Moreover, formally in this space the operator $\partial_t + \partial_x^3$ becomes $\partial_t + (\partial_x - \beta)^3$ so the solutions of the equation exhibit a parabolic behavior. More precisely, the following result for the KdV equation was proven in [13] (Theorem 11.1 and Theorem 12.1).

$\boxed{\text{Ka}}$ **Theorem A.** *Let $u \in C([0,\infty) : H^2(\mathbb{R}))$ be a solution of the IVP (1.1) with $k = 1$. If*

$$u_0 \in H^2(\mathbb{R}) \cap L^2(e^{\beta x} dx), \quad \text{for some } \beta > 0, \qquad (1.4) \quad \boxed{1.3}$$

then

$$e^{2\beta x} u \in C([0,\infty) : L^2(\mathbb{R})) \cap C((0,\infty) : H^\infty(\mathbb{R})), \qquad (1.5) \quad \boxed{1.4}$$

with

$$\|u(t)\|_2 = \|u_0\|_2, \qquad \|u(t) - u_0\|_{-3,2} \leq Kt, \quad t > 0, \qquad (1.6) \quad \boxed{1.5}$$

and

$$\|e^{\beta x} u(t)\|_2 \leq e^{Kt} \|e^{\beta x} u_0\|_2, \quad t > 0, \qquad (1.7) \quad \boxed{1.6}$$

where $K = K(\beta, \|u_0\|_2)$. Moreover, the map data-solution $u_0 \to u$ is continuous from $L^2(\mathbb{R}) \cap L^2(e^{\beta x} dx)$ to $C([0,T] : L^2(e^{\beta x} dx))$, for any $T > 0$.

It is easy to see that the result of Theorem A extends to solutions of the IVP (1.1) for any $k \in \mathbb{Z}^+$ in their positive time interval of existence $[0, T]$.

The problem of the uniqueness of solutions in polynomial weighted $L^2(\mathbb{R})$-spaces was considered in [18] and [9]. In particular, in [9] uniqueness and *a priori* estimates, for positive times, were deduced for solutions of the KdV equation ($k = 1$ in (1.1)) with data $u_0 \in L^2((1 + x_+)^{3/4} dx)$ and for the mKdV equation ($k = 2$ in (1.1)) for data $u_0 \in L^2((1 + x_+)^{1/4} dx)$.

In view of the results in Theorem A is natural to ask what is the strongest weighted space where persistence of the solutions of (1.1) holds. The following uniqueness result obtained in [6] gives an upper bound of this weight.

$\boxed{\text{EKPV06}}$ **Theorem B.** *There exists $c_0 = c_0(k) > 0$ such that for any pair*

$$u_1, u_2 \in C([0,1] : H^4(\mathbb{R}) \cap L^2(|x|^2 dx))$$

of solutions of (1.1), *if*

$$u_1(\cdot,0) - u_2(\cdot,0), \ u_1(\cdot,1) - u_2(\cdot,1) \in L^2(e^{c_0 x_+^{3/2}} dx), \qquad (1.8) \quad \boxed{\texttt{3:2}}$$

then $u_1 \equiv u_2$.

Thus, by taking $u_2 \equiv 0$ one gets a restriction on the possible decay of a non-trivial solution of (1.1) at two different times. It was established in [8] that this result is optimal. More precisely, the following was proven in [8]:

$\boxed{\texttt{C}}$ **Theorem C.** *Let a_0 be a positive constant. For any given data*

$$u_0 \in H^s(\mathbb{R}) \cap L^2(e^{a_0 x_+^{3/2}} dx), \quad s > s_k \ \text{defined above} \qquad (1.9) \quad \boxed{\texttt{0.1}}$$

the unique solution of the IVP (1.1) *satisfies that for any $T > 0$*

$$\sup_{t \in [0,T]} \int_{-\infty}^{\infty} e^{a(t)x_+^{3/2}} |u(x,t)|^2 dx \le c^* \qquad (1.10) \quad \boxed{\texttt{0.2}}$$

with $= c^*(a_0; \|u_0\|_2; \|e^{a_0 x_+^{3/2}/2} u_0\|_2; T; k)$ *and*

$$a(t) = \frac{a_0}{(1 + 27a_0^2 t/4)^{1/2}}. \qquad (1.11) \quad \boxed{\texttt{0.3}}$$

Before stating our results we need to define the class of solutions to the IVP (1.1) to which it applies. Thus, we shall rely on the following well-posedness result which is a consequence of the arguments deduced in [15]:

$\boxed{\texttt{TD}}$ **Theorem D.** *If $u_0 \in H^{3/4^+}(\mathbb{R})$, then there exist $T = T(\|u_0\|_{3/4^+,2}; k) > 0$ and a unique solution of the IVP* (1.1) *such that*

(i) $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$,

(ii) $\partial_x u \in L^4([-T,T] : L^\infty(\mathbb{R}))$, *(Strichartz)*,

(iii) $\displaystyle\sup_x \int_{-T}^T |J^r \partial_x u(x,t)|^2 dt < \infty \ \text{ for } \ r \in [0, 3/4^+]$, $\quad$ (1.12) $\boxed{\texttt{notes-2}}$

(iv) $\displaystyle\int_{-\infty}^{\infty} \sup_{-T \le t \le T} |u(x,t)|^2 dx < \infty$,

with $J = (1 - \partial_x^2)^{1/2}$. Moreover, the map data-solution, $u_0 \to u(x,t)$ is locally continuos (smooth) from $H^{3/4+}(\mathbb{R})$ into the class defined in (1.12).

As it was already commented, in the cases $k = 1, 2, 3$ the value of T in Theorem D can be taken arbitrarily large. Also we shall remark that for powers $k = 2, 3, ..$ it suffices to assume that $u_0 \in H^{3/4}(\mathbb{R})$ to obtain the crucial estimates (1.12) (ii) (see remark (b) below).

Our first result is concerned with the propagation of regularity in the right hand side of the data for positive times. It affirms that this regularity moves with infinite speed to its left as time evolves.

t1   **Theorem 1.1.** *If $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $l \in \mathbb{Z}^+$, $l \geq 1$ and $x_0 \in \mathbb{R}$*

$$\| \partial_x^l u_0 \|_{L^2((x_0,\infty))}^2 = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2 dx < \infty, \tag{1.13}$$ notes-3

then the solution of the IVP (1.1) provided by Theorem D satisfies that for any $v > 0$ and $\varepsilon > 0$

$$\sup_{0 \leq t \leq T} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\, dx < c, \tag{1.14}$$ notes-4

for $j = 0, 1, \ldots, l$ with $c = c(l; \|u_0\|_{3/4^+,2}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \varepsilon; T)$.

  In particular, for all $t \in (0,T]$, the restriction of $u(\cdot,t)$ to any interval (x_0,∞) belongs to $H^l((x_0,\infty))$.

  Moreover, for any $v \geq 0$, $\varepsilon > 0$ and $R > 0$

$$\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R-vt} (\partial_x^{l+1} u)^2(x,t)\, dxdt < c, \tag{1.15}$$ notes-5

with $c = c(l; \|u_0\|_{3/4^+,2}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \varepsilon; R; T)$.

  Our second result describes the persistence properties and regularity effects, for positive times, in solutions associated with data having polynomial decay in the positive real line.

t2   **Theorem 1.2.** *If $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $n \in \mathbb{Z}^+$, $n \geq 1$,*

$$\| x^{n/2} u_0 \|_{L^2((0,\infty))}^2 = \int_0^{\infty} |x^n| |u_0(x)|^2 dx < \infty, \tag{1.16}$$ b1

then the solution u of the IVP (1.1) provided by Theorem D satisfies that

$$\sup_{0 \leq t \leq T} \int_0^{\infty} |x^n| |u(x,t)|^2\, dx \leq c \tag{1.17}$$ b2

with $c = c(n; \|u_0\|_{3/4^+,2}; \|x^{n/2} u_0\|_{L^2((0,\infty))}; T)$.

  Moreover, for any $\varepsilon, \delta, R > 0, v \geq 0, m, j \in \mathbb{Z}^+, m+j \leq n, m \geq 1$,

$$\sup_{\delta \leq t \leq T} \int_{\varepsilon-vt}^{\infty} (\partial_x^m u)^2(x,t)\, x_+^j\, dx$$
$$+ \int_\delta^T \int_{\varepsilon-vt}^{\infty} (\partial_x^{m+1} u)^2(x,t)\, x_+^{j-1}\, dxdt \leq c, \tag{1.18}$$ b3

with $c = c(n; \|u_0\|_{3/4^+,2}; \|x^{n/2} u_0\|_{L^2((0,\infty))}; T; \delta; \varepsilon; R; v)$.

  <u>Remark:</u> Answering a question of K. Nakanishi it can be deduced from our proof of Theorem 1.1 that the inequality (1.14) can be more precise i.e. for $\delta > 0$ and $t \in (0,1)$ and $j = 1, \ldots, l$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{j+\delta}} (\partial_x^j u)^2(x,t)\, dx \leq \frac{c}{t},$$

with $c = c(\|u_0\|_{3/4^+,2}; \|\partial_x^j u_0\|_{L^2((x_0,\infty))}; x_0; \delta)$, (the details will given somewhere else).

It will be clear from our proofs of Theorem 1.1 and Theorem 1.2 that they still hold for solutions of the "defocussing" k-gKdV

$$\partial_t u + \partial_x^3 u - u^k \partial_x u = 0, \quad x, t \in \mathbb{R}, \ k \in \mathbb{Z}^+.$$

Therefore, our results apply to $u(-x, -t)$ if $u(x,t)$ is a solution of (1.1). In other words, Theorem 1.1 and Theorem 1.2 resp. remain valid, backward in time, for datum satisfying the hypothesis (1.13) and (1.16) resp. on the left hand side of the real line.

As a direct consequence of Theorem 1.1 and Theorem 1.2, the above comments and the time reversible character of the equation in (1.1) one has:

cor1 **Corollary 1.3.** *Let* $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ *be a solution of the equation in* (1.1) *described in Theorem D. If there exist* $m \in \mathbb{Z}^+$, $\hat{t} \in (-T,T)$, $a \in \mathbb{R}$ *such that*

$$\partial_x^m u(\cdot, \hat{t}) \notin L^2((a, \infty)),$$

then for any $t \in [-T, \hat{t})$ *and any* $\beta \in \mathbb{R}$

$$\partial_x^m u(\cdot, t) \notin L^2((\beta, \infty)), \quad \text{and} \quad x^{m/2} u(\cdot, t) \notin L^2((0, \infty)).$$

Next, as a consequence of Theorem 1.1 and Theorem 1.2 one has that for an appropriate class of data the singularity of the solution travels with infinite speed to the left as time evolves. In the integrable cases $k = 1, 2$ this is expected as part of the so called *resolution conjecture*, (see [5]). Also as a consequence of the time reversibility property one has that the solution cannot have had some regularity in the past.

cor2 **Corollary 1.4.** *Let* $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ *be a solution of the equation in* (1.1) *described in Theorem D. If there exist* $n, m \in \mathbb{Z}^+$ *with* $m \leq n$ *such that for some* $a, b \in \mathbb{R}$ *with* $a < b$

$$\int_b^\infty |\partial_x^n u_0(x)|^2 dx < \infty \quad \text{but} \quad \partial_x^m u_0 \notin L^2((a, \infty)), \qquad (1.19) \quad \boxed{A}$$

then for any $t \in (0, T)$ *and any* $v > 0$ *and* $\varepsilon > 0$

$$\int_{b+\varepsilon-vt}^\infty |\partial_x^n u(x,t)|^2 dx < \infty,$$

and for any $t \in (-T, 0)$ *and any* $\alpha \in \mathbb{R}$

$$\int_\alpha^\infty |\partial_x^m u(x,t)|^2 dx = \infty.$$

Remark: If in Corollary 1.4 in addition to (1.19) one assumes that

$$\int_{-\infty}^{a} |\partial_x^n u_0(x)|^2 dx < \infty, \qquad (1.20) \quad \boxed{\text{Ab}}$$

then by combining the results in this corollary with the group properties it follows that

$$\int_{-\infty}^{\beta} |\partial_x^n u(x,t)|^2 dx = \infty, \quad \text{for any } \beta \in \mathbb{R} \text{ and } t > 0.$$

This tell us that in general the regularity in the left hand side of the real line does not propagate forward in time. Also it follows from Theorem 1.1 that under the hypotheses (1.19)-(1.20) the solution $u(\cdot,t)$ satisfies

$$u(\cdot,t) \in H_{\text{loc}}^n(\mathbb{R}) \quad \text{for all } t \in [-T,T]\backslash\{0\},$$

hence the local (m) singularity $\partial_x^m u_0 \notin L^2((a,b))$ does not reappear in the future or in the past.

The same argument shows that for initial datum

$$u_0 \in L^2(|x|^n dx) \cap H^{3/4^+}(\mathbb{R})\backslash H^m(\mathbb{R})$$

with $m,n \in \mathbb{Z}^+$, $m < n$, the corresponding solution $u(\cdot,t)$ of (1.1) satisfies

$$x_+^{n/2} u(\cdot,t) \in L^2((0,\infty)) \quad \text{and} \quad x_-^{m/2} u(\cdot,t) \notin L^2((-\infty,0)), \quad t \in (0,T].$$

Thus, one has that in general the decay in the left hand side of the real line does not propagate forward in time.

In a similar manner we also have,

$\boxed{\text{cor3}}$ **Corollary 1.5.** *Let $u \in C([-T,T]:H^{3/4^+}(\mathbb{R}))$ be a solution of the equation in (1.1) described in Theorem D. If for $j,m \in \mathbb{Z}^+$, $j < m$,*

$$x_+^{m/2} u_0 \in L^2((0,\infty)), \text{ and } \partial_x^j u_0(x) \notin L^2((\beta,\infty)), \text{ for some } \beta \in \mathbb{R},$$

then for any $t \in (0,T]$

$$x_+^{m/2} u(\cdot,t) \in L^2((0,\infty)), \text{ and } \partial_x^m u(\cdot,t) \in L^2((\alpha,\infty)), \ \forall \alpha \in \mathbb{R},$$

and for any $t \in [-T,0)$

$$x_+^{j/2} u(\cdot,t) \notin L^2((0,\infty)), \text{ and } \partial_x^j u(\cdot,t) \notin L^2((\alpha,\infty)), \ \forall \alpha \in \mathbb{R}.$$

As a consequence of Theorem 1.2 we can improve the result in Theorem 1.4 in [8] in the case of a positive integer. More precisely,

$\boxed{\text{cor4}}$ **Corollary 1.6.** *Let $u \in C([-T,T]:H^{3/4^+}(\mathbb{R}))$ be a solution of the equation (1.1) described in Theorem D. If there exist $n_j \in \mathbb{Z}^+ \cup \{0\}$, $j = 1,2,3,4$, $t_0,t_1 \in [-T,T]$ with $t_0 < t_1$ and $a,b \in \mathbb{R}$ such that*

$$\int_0^\infty |x|^{n_1} |u(x,t_0)|^2 dx < \infty \text{ and } \int_a^\infty |\partial_x^{n_2} u(x,t_0)|^2 dx < \infty,$$

and

$$\int_{-\infty}^{0} |x|^{n_3} |u(x,t_1)|^2 dx < \infty \quad and \quad \int_{-\infty}^{b} |\partial_x^{n_4} u(x,t_1)|^2 dx < \infty,$$

then

$$u \in C([-T,T] : H^s(\mathbb{R}) \cap L^2(|x|^r dx))$$

where

$$s = \min\{\max\{n_1;n_2\};\max\{n_3;n_4\}\} \quad and \quad r = \min\{n_1;n_3\}.$$

Remarks: (a) The improvement of Theorem 1.4 in [8] in the case of a positive follows by taking $n_1 = n_3 \in \mathbb{Z}^+$ and $n_2 = n_4 = 0$.

(b) Although for the sake of the simplicity we shall not pursue this issue here, we remark that the results in Theorem 1.2 and Corollary 1.6 can be extended to non-integer values of the parameter n. In this case, one needs to combine the argument given below with those found in [23].

As it was mentioned above the solution flow of the IVP (1.1) does not preserve the class $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx)$ when $r > s$. Our next result describes how the decay $(r-s)$, not preserved by the flow, is transformed into extra regularity of the solution.

$\boxed{\text{cor7}}$ **Corollary 1.7.** *If $u_0 \in Z_{s,r}$ with $s > 3/4^+$, $r \in \mathbb{Z}^+$ and $r > s$. Then the solution of the IVP (1.1) $u \in C([-T,T] : Z_{s,s})$ also satisfies that for any $b \in \mathbb{R}$*

$$\int_{b}^{\infty} (\partial_x^r u(x,t))^2 dx < \infty, \quad for \ any \ t \in (0,T],$$

$$\int_{0}^{\infty} |x|^r |u(x,t)|^2 dx < \infty, \quad for \ any \ t \in [0,T],$$

and

$$\int_{-\infty}^{-b} (\partial_x^r u(x,t))^2 dx < \infty, \quad for \ any \ t \in [-T,0),$$

$$\int_{-\infty}^{0} |x|^r |u(x,t)|^2 dx < \infty, \quad for \ any \ t \in [-T,0].$$

Remarks: (a) Combining the results in Theorems 1.1 and 1.2 with those in [8] previously described one can deduce further properties of the solution $u(x,t)$ as a consequence of the regularity and decay assumptions on the data u_0. For example, the solution of the IVP (1.1) with data $u_0 \in H^{3/4^+}(\mathbb{R}) \backslash H^1(\mathbb{R})$ for which there exists $n \in \mathbb{Z}^+$

$$\int_{0}^{\infty} |x^n| |u_0(x)|^2 dx < \infty$$

one has, in addition to (1.17)–(1.18), that for any $M > 0$ and any $t > 0$

$$\int_{-\infty}^{-M} |xu(x,t)|^2 dx = \infty \quad and \quad \int_{-\infty}^{-M} |\partial_x u(x,t)|^2 dx = \infty.$$

(b) The proof of Theorems 1.1 and 1.2 relies on weighted energy estimates and an iterative process for which the estimate (1.12) (ii) is essential. This is a consequence of the following version of the Strichartz estimates [25] obtained in [14]. The solution of the linear IVP

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{1.21} \quad \boxed{\texttt{linearIVP}}$$

is given by the group $\{U(t) : t \in \mathbb{R}\}$

$$U(t)v_0(x) = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{\cdot}{\sqrt[3]{3t}}\right) * v_0(x).$$

where $Ai(\cdot)$ denotes the Airy function

$$Ai(x) = c \int_{-\infty}^{\infty} e^{ix\xi + i\xi^3/3} d\xi. \tag{1.22} \quad \boxed{\texttt{airy}}$$

The following inequality was established in [14] : for any $(\theta, \alpha) \in [0,1] \times [0,1/2]$

$$\|D^{\theta\alpha/2}U(t)u_0\|_{L^q(\mathbb{R}:L^p(\mathbb{R}))} \leq c\|u_0\|_2, \tag{1.23} \quad \boxed{\texttt{stric}}$$

with $(q,p) = (6/\theta(1+\alpha), 2/(1-\theta))$. In particular, by taking $(\theta, \alpha) = (1, 1/2)$ one has

$$\left(\int_{-\infty}^{\infty} \|D^{1/4}U(t)u_0\|_{\infty}^4 dt\right)^{1/4} \leq c\|u_0\|_2, \tag{1.24} \quad \boxed{\texttt{stric1}}$$

which explains the hypothesis $s = 3/4^+$ in Theorem D and the conclusion (1.12) (ii) on it.

(c) Even for the associated linear IVP (1.21) the results in Theorems 1.1 and 1.2 do not seem to have been appreciated before, even that in this (linear) case a different proof should follow based on estimates of the Airy function $Ai(\cdot)$ (see (1.22)) and its derivatives.

(d) Although we will not pursue this issue here, it should be remarked that the results in Theorems 1.1 and 1.2 can be extended to include some continuity property in time. For example, in Theorem 1.1 in addition to (1.17) one can show that for any $t_0 \in (0,T)$ and $\varepsilon > 0$ and any $v > 0$

$$\lim_{t \to t_0} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)dx = \int_{x_0+\varepsilon-vt_0}^{\infty} (\partial_x^j u)^2(x,t_0)dx.$$

In this case the proof follows by using an argument similar to that given in [1].

(e) Without loss of generality from now on we shall assume that in Theorem 1.1 $x_0 = 0$.

(f) We recall that the above results still hold if one replaces $x,t > 0$ by $x,t < 0$.

(g) In a forthcoming work we shall extend some of the results obtained here to other dispersive models.

The rest of this paper is organized in the following manner: in section 2 we introduce the family of cut-off functions to be used in the proof of Theorem 1.1 and Theorem 1.2. This is an important part in our iterative argument used in the proof. The proof of Theorem 1.1 will be given in section 3 and the proof of Theorem 1.2 in section 4.

2. PRELIMINARIES

We shall construct a class of real functions $\chi_{0,\varepsilon,b}(x)$ for $\varepsilon > 0$ and $b \geq 5\varepsilon$ such that

$$\chi_{0,\varepsilon,b} \in C^{\infty}(\mathbb{R}), \ \chi'_{0,\varepsilon,b} \geq 0,$$

$$\chi_{0,\varepsilon,b}(x) = \begin{cases} 0, & x \leq \varepsilon, \\ 1, & x \geq b, \end{cases}$$

therefore

$$\operatorname{supp} \chi_{0,\varepsilon,b} \subseteq [\varepsilon, \infty),$$

$$\chi'_{0,\varepsilon,b}(x) \geq \frac{1}{b-3\varepsilon} \, 1_{[3\varepsilon, b-2\varepsilon]}(x),$$

and

$$\operatorname{supp} \chi'_{0,\varepsilon,b}(x) \subseteq [\varepsilon, b].$$

Thus

$$\chi'_{0,\varepsilon/3,b+\varepsilon}(x) \geq c_j |\chi^{(j)}_{0,\varepsilon,b}(x)|, \quad \forall x \in \mathbb{R}, \ \forall j \geq 1.$$

Also, if $x \in (3\varepsilon, \infty)$, then

$$\chi_{0,\varepsilon,b}(x) \geq \chi_{0,\varepsilon,b}(3\varepsilon) \geq \frac{1}{2} \frac{\varepsilon}{b-3\varepsilon},$$

and for any $x \in \mathbb{R}$

$$\chi'_{0,\varepsilon/3,b+\varepsilon}(x) \leq \frac{1}{b-3\varepsilon}.$$

We also have that given $\varepsilon > 0, \ b \geq 5\varepsilon$ there exist $c_1, c_2 > 0$ such that

$$\chi'_{0,\varepsilon,b}(x) \leq c_1 \chi'_{0,\varepsilon/3,b+\varepsilon}(x)\chi_{0,\varepsilon/3,b+\varepsilon}(x),$$

$$\chi'_{0,\varepsilon,b}(x) \leq c_2 \chi_{0,\varepsilon/5,\varepsilon}(x).$$

We shall obtain this family $\{\chi_{0,\varepsilon,b} : \varepsilon > 0, \ b \geq 5\varepsilon\}$ by first considering $\rho \in C_0^\infty(\mathbb{R})$, $\rho(x) \geq 0$, even, with $\operatorname{supp}\rho \subseteq (-1,1)$ and $\int \rho(x)dx = 1$. Then defining

$$
v_{\varepsilon,b}(x) = \begin{cases} 0, & x \leq 2\varepsilon, \\[2mm] \frac{1}{b-3\varepsilon}x - \frac{2\varepsilon}{b-3\varepsilon}, & x \in [2\varepsilon, b-\varepsilon], \\[2mm] 1, & x \geq b-\varepsilon, \end{cases}
$$

and

$$
\chi_{0,\varepsilon,b}(x) = \rho_\varepsilon * v_{\varepsilon,b}(x),
$$

where $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(x/\varepsilon)$.

For any $n \in \mathbb{Z}^+$ define

$$
\chi_{n,\varepsilon,b}(x) = x^n \chi_{0,\varepsilon,b}(x).
$$

Thus, for any $n \in \mathbb{Z}^+$

$$
\chi'_{n+1,\varepsilon,b}(x) \geq (n+1)\chi_{n,\varepsilon,b}(x).
$$

3. PROOF OF THEOREM 1.1

We shall use an induction argument. First, we shall prove (1.14) for $l = 1$ and $l = 2$ (to illustrate our method).

Case $l = 1$

Without loss of generality we restrict ourselves to the case $k = 1$ in (1.1), (KdV case).

Formally, take partial derivative with respect to x of the equation in (1.1) and multiply by $\partial_x u \chi_{0,\varepsilon,b}(x+vt)$ to obtain after integration by parts the identity

$$
\frac{1}{2}\frac{d}{dt}\int (\partial_x u)^2(x,t)\chi_0(x+vt)\,dx - \underbrace{v\int (\partial_x u)^2(x,t)\chi_0'(x+vt)\,dx}_{A_1}
$$

$$
+ \frac{3}{2}\int (\partial_x^2 u)^2(x,t)\chi_0'(x+vt)\,dx - \underbrace{\frac{1}{2}\int (\partial_x u)^2(x,t)\chi_0'''(x+vt)\,dx}_{A_2} \quad (3.1) \quad \boxed{\texttt{notes-10}}
$$

$$
+ \underbrace{\int \partial_x(u\partial_x u)\partial_x u(x,t)\chi_0(x+vt)\,dx}_{A_3} = 0
$$

where in χ_0 we omit the index ε,b (fixed).

Then after integrating in the time interval $[0, T]$ using the estimate (1.12)(iii) with $r = 0$ we have

$$\int_0^T |A_1(t)|\, dt \le v \int_0^T \int (\partial_x u)^2 \chi_0'(x+vt)\, dxdt < c, \qquad (3.2) \quad \boxed{\text{notes-11}}$$

since given v, ε, b, T as above there exist $c_0 > 0$ and $R > 0$ such that

$$\chi_0'(x+vt) \le c_0 1_{[-R,R]}(x), \quad \forall (x,t) \in \mathbb{R} \times [0,T].$$

The same argument shows that

$$\int_0^T |A_2(t)|\, dt \le c_0. \qquad (3.3) \quad \boxed{\text{notes-12}}$$

Finally, since

$$\begin{aligned} A_3 &= \int \partial_x u\, \partial_x u\, \partial_x u\, \chi_0\, dx + \int u \partial_x^2 u\, \partial_x u\, \chi_0\, dx \\ &= \frac{1}{2} \int \partial_x u\, \partial_x u\, \partial_x u\, \chi_0(x+vt)\, dx - \frac{1}{2} \int u \partial_x u\, \partial_x u\, \chi_0'(x+vt)\, dx \qquad (3.4) \quad \boxed{\text{notes-13}} \\ &= A_{31} + A_{32} \end{aligned}$$

one has

$$|A_{31}| \le \|\partial_x u\|_\infty \int (\partial_x u)^2 \chi_0(x+vt)\, dx \qquad (3.5) \quad \boxed{\text{notes-14}}$$

and

$$|A_{32}| \le \|u(t)\|_\infty \int (\partial_x u)^2 \chi_0'(x+vt)\, dx. \qquad (3.6) \quad \boxed{\text{notes-15}}$$

Hence by Sobolev embedding and (3.2) after integrating in the time interval $[0, T]$ one gets

$$\int_0^T |A_{32}|\, dt \le \sup_{[0,T]} \|u(t)\|_{3/4^+} \int_0^T \int (\partial_x u)^2 \chi_0'(x+vt)\, dxdt \le c_0. \qquad (3.7) \quad \boxed{\text{notes-16}}$$

Inserting the above information in (3.1), Gronwall's inequality and (1.12) (ii) yield the estimate

$$\sup_{[0,T]} \int (\partial_x u)^2 \chi_{0,\varepsilon,b}(x+vt)\, dx + \int_0^T \int (\partial_x^2 u)^2 \chi_{0,\varepsilon,b}'(x+vt)\, dxdt \le c_0 \qquad (3.8) \quad \boxed{\text{notes-17}}$$

with $c_0 = c_0(\varepsilon; b; v) > 0$ for any $\varepsilon > 0$, $b \ge 5\varepsilon$, $v > 0$, which proves the case $l = 1$.

Case $l = 2$

We can assume that the solution $u(\cdot)$ satisfies (1.12) and (3.8) (case $l = 1$). Then formally one has the following identity

$$\frac{1}{2}\frac{d}{dt}\int (\partial_x^2 u)^2(x,t)\chi_0(x+vt)\,dx - \underbrace{v\int (\partial_x^2 u)^2(x,t)\chi_0'(x+vt)\,dx}_{A_1}$$

$$+ \frac{3}{2}\int (\partial_x^3 u)^2(x,t)\chi_0'(x+vt)\,dx - \underbrace{\frac{1}{2}\int (\partial_x^2 u)^2(x,t)\chi_0'''(x+vt)\,dx}_{A_2} \qquad (3.9) \quad \boxed{\texttt{notes-18}}$$

$$+ \underbrace{\int \partial_x^2(u\partial_x u)\partial_x^2 u(x,t)\,\chi_0(x+vt)\,dx}_{A_3} = 0.$$

After integration in time we have from (3.8)

$$\int_0^T |A_1(t)|\,dt \le |v| \int_0^T \int (\partial_x^2 u)^2 \chi_0'(x+vt)\,dxdt \le |v|\,c_0. \qquad (3.10) \quad \boxed{\texttt{notes-19}}$$

Next by the construction of the $\chi_{0,\varepsilon,b}$'s one has that for $\varepsilon > 0$, $b \ge 5\varepsilon$, there exists $c > 0$ such that

$$\left|\chi_{0,\varepsilon,b}'''(x)\right| \le c_{\varepsilon,b}\,\chi_{0,\varepsilon/3,b+\varepsilon}'(x) \quad \forall x \in \mathbb{R}. \qquad (3.11) \quad \boxed{\texttt{notes-20}}$$

Therefore, after integration in time using (3.8) with $(\varepsilon/2, b+\varepsilon)$ instead of (ε, b), it follows that

$$\int_0^T |A_2(t)|\,dt \le \int_0^T \int (\partial_x^2 u)^2\,|\chi_{0,\varepsilon,b}'''(x+vt)|\,dxdt$$

$$\le c \int_0^T \int (\partial_x^2 u)^2\,|\chi_{0,\varepsilon/3,b+\varepsilon}'(x+vt)|\,dxdt \le c. \qquad (3.12) \quad \boxed{\texttt{notes-21}}$$

Finally we have to consider A_3 in (3.9). Thus

$$\int \partial_x^2(u\partial_x u)\,\partial_x^2 u\,\chi_0(x+vt)\,dx$$

$$= \int u\partial_x^3 u\,\partial_x^2 u\chi_0(x+vt)\,dx + 3\int \partial_x u\,\partial_x^2 u\,\partial_x^2 u\,\chi_0(x+vt)\,dx \qquad (3.13) \quad \boxed{\texttt{notes-22}}$$

$$= \frac{5}{2}\int \partial_x u\,\partial_x^2 u\,\partial_x^2 u\,\chi_0(x+vt)\,dx - \frac{1}{2}\int u\partial_x^2 u\,\partial_x^2 u\,\chi_0'(x+vt)\,dx$$

$$= A_{31} + A_{32}.$$

Then

$$|A_{31}| \le c\|\partial_x u(t)\|_\infty \int (\partial_x^2 u)^2\,\chi_0(x+vt)\,dx \qquad (3.14) \quad \boxed{\texttt{notes-23}}$$

where the last integral is the quantity to be estimated. After integration in time, (3.8) and Sobolev embedding we obtain that

$$\int_0^T |A_{32}(t)| \, dt \le \sup_{0 \le t \le T} \|u(t)\|_\infty \int_0^T \int (\partial_x^2 u)^2 \chi_0'(x+vt) \, dxdt \le c. \quad (3.15) \quad \boxed{\text{notes-24}}$$

Inserting the above information in (3.9) and using Gronwall's inequality it follows that

$$\sup_{0 \le t \le T} \int (\partial_x^2 u)^2 \chi_{0,\varepsilon,b}(x+vt) \, dx$$

$$+ \int_0^T \int (\partial_x^3 u)^2 \chi_{0,\varepsilon,b}'(x+vt) \, dxdt \le c_0 \quad (3.16) \quad \boxed{\text{notes-25}}$$

with $c_0 = c_0(\varepsilon; b; v) > 0$ for any $\varepsilon > 0$, $b \ge 5\varepsilon$, $v > 0$.

We shall prove the case $l+1$ assuming the case $l \ge 2$. More precisely, we assume:

If u_0 satisfies (1.13) then (1.14) holds, i.e.

$$\sup_{0 \le t \le T} \int (\partial_x^j u)^2 \chi_{0,\varepsilon,b}(x+vt) \, dx$$

$$+ \int_0^T \int (\partial_x^{j+1} u)^2 \chi_{0,\varepsilon,b}'(x+vt) \, dxdt \le c \quad (3.17) \quad \boxed{\text{notes-26}}$$

for $j = 1, 2, \ldots, l$, $l \ge 2$, for any $\varepsilon > 0$, $b \ge 5\varepsilon$, $v > 0$.

Now we have that

$$u_0\big|_{(0,\infty)} \in H^{l+1}([0,\infty)). \quad (3.18) \quad \boxed{\text{notes-27}}$$

Thus from the previous step (3.17) holds. Also formally we have for $\varepsilon > 0$, $b \ge 5\varepsilon$, the identity

$$\frac{1}{2}\frac{d}{dt} \int (\partial_x^{l+1} u)^2 \chi_0(x+vt)dx - \underbrace{v \int (\partial_x^{l+1} u)^2 \chi_0'(x+vt)dx}_{A_1}$$

$$+ \frac{3}{2} \int (\partial_x^{l+2} u)^2 \chi_0'(x+vt)dx - \underbrace{\frac{1}{2} \int (\partial_x^{l+1} u)^2 \chi_0'''(x+vt)dx}_{A_2} \quad (3.19) \quad \boxed{\text{notes-28}}$$

$$+ \underbrace{\int \partial_x^{l+1}(u\partial_x u)\partial_x^{l+1} u(x,t) \chi_0(x+vt)dx}_{A_3} = 0.$$

Using (3.17) with $j = l$, after integration in time one has

$$\int_0^T |A_1(t)| \, dt \le |v| \int_0^T \int (\partial_x^{l+1} u)^2(x,t)\chi_0'(x+vt) \, dx \le c_0. \quad (3.20) \quad \boxed{\text{notes-29}}$$

We recall (3.11) i.e. given $\varepsilon > 0$, $b \geq 5\varepsilon$, there exists $c >$ such that

$$\left|\chi'''_{0,\varepsilon,b}(x)\right| \leq c_{\varepsilon,b}\, \chi'_{0,\varepsilon/3,b+\varepsilon}(x), \quad \forall x \in \mathbb{R}. \tag{3.21} \quad \boxed{\texttt{notes-30}}$$

Hence integrating in the time interval $[0,T]$ and using (3.17) with $j = l$ and $(\varepsilon, b) = (\varepsilon/3, b+\varepsilon)$ one has

$$\int_0^T |A_2(t)|\, dt \leq \int_0^T \int (\partial_x^{l+1} u)^2 (x,t) \left|\chi'''_{0,\varepsilon,b}(x+vt)\right| dx dt$$

$$\leq c\,|v| \int_0^T \int (\partial_x^{l+1} u)^2 (x,t)\, \chi'_{0,\varepsilon/3,b+\varepsilon}(x+vt)\, dx dt \leq c_0. \tag{3.22} \quad \boxed{\texttt{notes-31}}$$

So it remains to consider $A_3(t)$ in (3.19). Then we have to distinguish two cases:

Case $l + 1 = 3$

Thus

$$A_3 = \int \partial_x^3 (u\partial_x u)\, \partial_x^3 u\, \chi_{0,\varepsilon,b}(x+vt)\, dx$$

$$= 4 \int \partial_x u (\partial_x^3 u)^2\, \chi_{0,\varepsilon,b}(x+vt)\, dx + \int u \partial_x^4 u\, \partial_x^3 u\, \chi_{0,\varepsilon,b}(x+vt)\, dx$$

$$+ 3 \int \partial_x^2 u\, \partial_x^2 u\, \partial_x^3 u\, \chi_{0,\varepsilon,b}(x+vt)\, dx$$

$$= \frac{7}{2} \int \partial_x u (\partial_x^3 u)^2\, \chi_{0,\varepsilon,b}(x+vt)\, dx - \frac{1}{2} \int u (\partial_x^3 u)^2\, \chi'_{0,\varepsilon,b}(x+vt)\, dx$$

$$+ 3 \int \partial_x^2 u\, \partial_x^2 u\, \partial_x^3 u\, \chi_{0,\varepsilon,b}(x+vt)\, dx$$

$$= A_{31} + A_{32} + A_{33}. \tag{3.23} \quad \boxed{\texttt{notes-32}}$$

Then

$$|A_{31}(t)|\, dt \leq c \|\partial_x u(t)\|_\infty \int (\partial_x^3 u)^2\, \chi_{0,\varepsilon,b}(x+vt)\, dx \tag{3.24} \quad \boxed{\texttt{notes-33}}$$

where the last integral is the quantity to be estimated. After integration in time and using (3.17) with $j = l = 2$ and Sobolev embedding one gets

$$\int_0^T |A_{32}(t)|\, dt \leq c \sup_{0 \leq t \leq T} \|u(t)\|_\infty \int_0^T \int (\partial_x^3 u)^2\, \chi'_{0,\varepsilon,b}(x+vt)\, dx dt$$

$$\leq \sup_{0 \leq t \leq T} \|u(t)\|_{3/4^+,2} \int_0^T \int (\partial_x^3 u)^2\, \chi'_{0,\varepsilon,b}(x+vt)\, dx \leq c_0. \tag{3.25} \quad \boxed{\texttt{notes-34}}$$

So it remains consider $A_{33}(t)$. First we observe that

$$\chi_{0,\varepsilon/5,\varepsilon}(x) = 1 \quad \text{on} \ \operatorname{supp} \chi_{0,\varepsilon,b} \subseteq [\varepsilon, \infty) \tag{3.26} \quad \boxed{\texttt{notes-35}}$$

and since

$$\chi'_{0,\varepsilon/3,b+\varepsilon}(x) \geq c_{\varepsilon,b} 1_{[\varepsilon,b]}(x) \quad \text{and} \quad \operatorname{supp} \chi''_{0,\varepsilon,b} \subseteq [\varepsilon, b] \tag{3.27} \quad \boxed{\texttt{notes-36}}$$

one has

$$\chi'_{0,\varepsilon/3,b+\varepsilon}(x) \geq c|\chi''_{0,\varepsilon,b}(x)| \quad \forall x \in \mathbb{R}. \tag{3.28}$$ `notes-37`

Thus

$$\begin{aligned}
A_{33}(t) &= \int \partial_x^2 u \, \partial_x^2 u \, \partial_x^3 u \, \chi_{0,\varepsilon,b}(x+vt) \, dx \\
&= -\frac{1}{3} \int (\partial_x^2 u)^3 \, \chi'_{0,\varepsilon,b}(x+vt) \, dx
\end{aligned} \tag{3.29}$$ `notes-38`

and so by (3.26),

$$|A_{33}(t)| \leq \|\partial_x^2 u \, \chi'_{0,\varepsilon,b}(\cdot + vt)\|_\infty \int (\partial_x^2 u)^2 \, \chi_{0,\varepsilon/5,\varepsilon}(x+vt) \, dx \tag{3.30}$$ `notes-39a`

but the last term is bounded in t for $t \in [0,T]$ by a constant $c_{\varepsilon,b,v}$ according to (3.17) ($j = 2$).

Hence

$$\begin{aligned}
|A_{33}(t)| &\leq c \|\partial_x^2 u \, \chi'_{0,\varepsilon,b}(\cdot + vt)\|_\infty^2 + c_{\varepsilon,b,v} \\
&\leq c^* \|(\partial_x^2 u)^2 \, \chi'_{0,\varepsilon,b}(\cdot + vt)\|_\infty + c_{\varepsilon,b,v}, \quad (c^* = c^*(\|\chi'_{0,\varepsilon,b}(\cdot)\|_{L_x^\infty}) \\
&\leq c_{\varepsilon,b} \int |\partial_x((\partial_x^2 u)^2 \, \chi'_{0,\varepsilon,b}(x+vt))| \, dx + c_{\varepsilon,b,v} \\
&\leq c \int |\partial_x^2 u \, \partial_x^3 u \, \chi'_{0,\varepsilon,b}(x+vt)| \, dx \\
&\quad + \int |(\partial_x^2 u)^2 \, \chi''_{0,\varepsilon,b}(x+vt)| \, dx + c_{\varepsilon,b,v} \\
&\leq c \int (\partial_x^2 u)^2 \, \chi'_{0,\varepsilon,b}(x+vt) \, dx + c \int (\partial_x^3 u)^2 \, \chi'_{0,\varepsilon,b}(x+vt) \, dx \\
&\quad + c \int (\partial_x^2 u)^2 \, \chi'_{0,\varepsilon/3,b+\varepsilon}(x+vt) \, dx + c_{\varepsilon,b,v}.
\end{aligned} \tag{3.31}$$ `notes-39b`

By (3.17) with $j = 1,2$ after integration in time we have that

$$\int_0^T |A_{33}(t)| \, dt \leq c = c_{\varepsilon,b,v,T}. \tag{3.32}$$ `notes-40`

Remark 3.1. *In the case $k \geq 2$ in (1.1) when one applies the argument above one basically needs to consider another term of the form ($k = 2$).*

$$A_4 = \int \partial_x u \, \partial_x^2 u \, \partial_x^2 u \, \partial_x^2 u \chi_{0,\varepsilon,b}(x+vt) \, dx. \tag{3.33}$$ `notes-41`

Thus in this case we write (using (3.27))

$$
\begin{aligned}
|A_4(t)| \\
&\leq \int |\partial_x u\, \partial_x^2 u| \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx\, \|(\partial_x^2 u)^2 \chi_{0,\varepsilon,b}\|_\infty \\
&\leq \left(\int (\partial_x u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x+vt)dx\right)^{1/2} \left(\int (\partial_x^2 u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x+vt)dx\right)^{1/2} \\
&\quad \times \|\partial_x((\partial_x^2 u)^2 \chi_{0,\varepsilon,b})\|_1 \\
&\leq \left(\int (\partial_x u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx + \int (\partial_x^2 u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx\right) \\
&\quad \times \left(\int \partial_x^2 u\, \partial_x^3 u\, \chi_{0,\varepsilon,b}(x+vt)\,dx + \int (\partial_x^2 u)^2\, \chi'_{0,\varepsilon,b}(x+vt)\,dx\right).
\end{aligned}
$$

(3.34) `notes-42`

Next we observe that

$$
\sum_{j=1}^{2} \int (\partial_x^j u)^2 \, \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx
$$

is bounded for t in $[0,T]$ by a constant $c_{\varepsilon,b,v,T} = c$.

Coming back to (3.34) we have

$$
\begin{aligned}
|A_4(t)| &\leq c\left(\int (\partial_x^2 u)^2\, \chi_{0,\varepsilon,b}(x+vt)\,dx + \int (\partial_x^2 u)^2 \chi'_{0,\varepsilon,b}(x+vt)\,dx \right. \\
&\qquad \left. + \int (\partial_x^3 u)^2\, \chi_{0,\varepsilon,b}(x+vt)\,dx\right) \\
&\leq c\left(1 + \int (\partial_x^2 u)^2 \chi'_{0,\varepsilon,b}(x+vt)\,dx \right. \\
&\qquad \left. + \int (\partial_x^3 u)^2 \chi_{0,\varepsilon,b}(x+vt)\,dx\right)
\end{aligned}
$$

(3.35) `notes-42b`

where we used that the first integral in the first inequality is bounded in $[0,T]$. After integration in time the first integral in the second inequality is bounded by (3.17) ($j = 1$). Finally, the last integral in the second inequality is the quantity to be estimated. Gathering the above information yields the desired result.

Case: $l+1 \geq 4$ ($\iff l \geq 3$).

Returning to (3.19) we obtain after integration by parts that

$$
\begin{aligned}
A_3 &= \int \partial_x^{l+1}(u\partial_x u)\, \partial_x^{l+1} u\, \chi_0(x+vt)\, dx \\
&= c_0 \int u\, \partial_x^{l+1} u\, \partial_x^{l+1} u\, \chi_0'(x+vt)\, dx \\
&\quad + c_1 \int \partial_x u\, (\partial_x^{l+1} u)^2\, \chi_0(x+vt)\, dx \\
&\quad + c_2 \int \partial_x^2 u\, \partial_x^l u\, \partial_x^{l+1} u\, \chi_0(x+vt)\, dx \\
&\quad + \sum_{j=3}^{l-1} \int \partial_x^j u\, \partial_x^{l+2-j} u\, \partial_x^{l+1} u\, \chi_0(x+vt)\, dx \\
&= A_{3,0} + A_{3,1} + A_{3,2} + \sum_{j=3}^{l-1} A_{3,j}.
\end{aligned}
\tag{3.36}
$$

notes-43

Next we have

$$
|A_{3,0}(t)| \le \|u(t)\|_\infty \int (\partial_x^{l+1} u)^2\, \chi_0'(x+vt)\, dx
\tag{3.37}
$$

notes-44

which after integration in time is bounded by Sobolev embedding and (3.17) $j = l$ yield the appropriate bound.

$$
|A_{3,1}(t)| \le \|\partial_x u(t)\|_\infty \int (\partial_x^{l+1} u)^2\, \chi_0(x+vt)\, dx.
\tag{3.38}
$$

notes-45

We use the estimate (1.12) (ii) to bound the first term on the right hand side of (3.38) when we later apply Gronwall's Lemma. The last term is the quantity to be estimated.

To estimate $A_{3,2}$ we follow the argument in the previous case (3.29)–(3.34).

So we need to consider $\sum_{j=3}^{l-1} A_{3,j}(t)$ which only appears if $l - 1 \ge 3$. Using an elementary inequality it follows that

$$
\begin{aligned}
|A_{3,j}(t)| &= \Big| \int \partial_x^j u\, \partial_x^{l+2-j} u\, \partial_x^{l+1} u\, \chi_{0,\varepsilon,b}(x+vt)\, dx \Big| \\
&\le \frac{1}{2} \int (\partial_x^j u\, \partial_x^{l+2-j} u)^2\, \chi_{0,\varepsilon,b}(x+vt)\, dx \\
&\quad + \frac{1}{2} \int (\partial_x^{l+1} u)^2\, \chi_{0,\varepsilon,b}(x+vt)\, dx \\
&= A_{3,j,1} + \int (\partial_x^{l+1} u)^2\, \chi_{0,\varepsilon,b}(x+vt)\, dx
\end{aligned}
\tag{3.39}
$$

notes-47

where the last integral is the quantity to be estimated. To treat $A_{3,j,1}$ observe that $j, l+2-j \le l-1$ and therefore

$$|A_{3,j,1}(t)|$$
$$\le \|(\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon}(\cdot + vt)\|_\infty \int (\partial_x^{l+2-j} u)^2 \chi_{0,\varepsilon,b}(x+vt)\,dx \qquad (3.40) \quad \boxed{\text{notes-48}}$$

Observe that the last integral is bounded by previous step (induction) (3.17). Also, by Sobolev

$$\|(\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon}(\cdot + vt)\|_\infty \le \|\partial_x \left((\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon} \right)\|_1$$
$$\le \|\partial_x^j u\, \partial_x^{j+1} u\, \chi_{0,\varepsilon/5,\varepsilon}\|_1 + \|(\partial_x^j u)^2 \chi'_{0,\varepsilon/5,\varepsilon}\|_1$$
$$\le c \left(\int (\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx + \int (\partial_x^{j+1} u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx \right. \qquad (3.41) \quad \boxed{\text{notes-49}}$$
$$\left. + \int (\partial_x^j u)^2 \chi'_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx \right)$$

where the first integrals are bounded in the time interval $[0,T]$ (see (3.17)) and the last one is bounded after integration in time (see (3.17)), which provides the desired result.

To justify the previous formal computations we shall follow the following argument. Without loss of generality we assume $l = 1$.

Consider the family of data $u_0^\mu = \rho_\mu * u_0$ with $\rho \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\rho \in (-1,1)$, $\rho \ge 0$, $\int \rho(x)\,dx = 1$ and

$$\rho_\mu(x) = \frac{1}{\mu}\rho\left(\frac{x}{\mu}\right), \ \ \mu > 0.$$

For $\mu > 0$ consider u^μ the solutions of the IVP (1.1) with data u_0^μ where $(u^\mu)_{\mu>0} \subseteq C([0,T] : H^\infty(\mathbb{R}))$.

Using the continuous dependence of the solution upon the data we have that

$$\sup_{t \in [0,T]} \|u^\mu(t) - u(t)\|_{3/4^+,2} \downarrow 0 \ \ \text{as} \ \ \mu \downarrow 0. \qquad (3.42) \quad \boxed{123}$$

Applying the argument in (3.1)-(3.8) to the smooth solutions $u^\mu(\cdot,t)$ one gets that

$$\sup_{[0,T]} \int (\partial_x u^\mu)^2 \chi_{0,\varepsilon,b}(x+vt)\,dx$$
$$+ \int_0^T \int (\partial_x^2 u^\mu)^2 \chi'_{0,\varepsilon,b}(x+vt)\,dxdt \le c_0, \qquad (3.43) \quad \boxed{124}$$

for any $\varepsilon > 0$, $b \ge 5\varepsilon$, $v > 0$, $c_0 = c_0(\varepsilon;b;v) > 0$ but independent of $\mu > 0$ since for $0 < \mu < \varepsilon$

$$(\partial_x u_0^\mu)^2 \chi_{0,\varepsilon,b}(x) = (\partial_x(\rho_\mu * u_0))^2 \chi_{0,\varepsilon,b}(x) = (\rho_\mu * \partial_x u_0\, 1_{[0,\infty)})^2 \chi_{0,\varepsilon,b}(x).$$

Combining (3.42) and (3.43), and Theorem D (the continuity on the initial data of the class defined in (1.12)), and weak compactness and Fatou's lemma argument one gets that

$$\sup_{[0,T]} \int (\partial_x u)^2 \, \chi_{0,\varepsilon,b}(x+vt) \, dx + \int_0^T \int (\partial_x^2 u)^2 \, \chi'_{0,\varepsilon,b}(x+vt) \, dxdt \le c_0 \quad (3.44) \quad \boxed{125}$$

which is the desired result.

4. PROOF OF THEOREM 1.2

Proof of (1.17) for any $n \in \mathbb{Z}^+$. First, we observe that

$x_+^n \, u_0 \in L^2(\mathbb{R})$, $x_+ = \max\{x; 0\}$, implies that for any $\varepsilon > 0$ and any $b \ge 5\varepsilon$,

$$\chi_{n,\varepsilon,b}(\cdot) \, u_0 \in L^2(\mathbb{R}).$$

$$(4.1) \quad \boxed{b4}$$

Now using the identity

$$\frac{1}{2} \frac{d}{dt} \int u^2 \, \chi_{n,\varepsilon,b}(x+vt) \, dx + \frac{3}{2} \int (\partial_x u)^2 \, \chi'_{n,\varepsilon,b}(x+vt) \, dx$$

$$+ v \underbrace{\int u^2 \, \chi'_{n,\varepsilon,b}(x+vt) \, dx}_{A_1}$$

$$(4.2) \quad \boxed{b5}$$

$$- \frac{1}{2} \underbrace{\int u^2 \chi'''_{n,\varepsilon,b}(x+vt) \, dx}_{A_2} + \underbrace{\int u \partial_x u \, u \, \chi_{n,\varepsilon,b}(x+vt) \, dx}_{A_3} = 0.$$

We observe that (in general)

$$\chi'_{n,\varepsilon,b}(x) = (x^n \, \chi_{0,\varepsilon,b})' = n x^{n-1} \, \chi_{0,\varepsilon,b}(\cdot) + x^n \, \chi'_{0,\varepsilon,b}(\cdot)$$

$$\chi'''_{n,\varepsilon,b}(x) = (x^n \, \chi_{0,\varepsilon,b})''' = n(n-1)(n-2) x^{n-3} \, \chi_{0,\varepsilon,b}(\cdot) \quad (4.3) \quad \boxed{b6}$$

$$+ 3n(n-1)x^{n-2} \, \chi'_{0,\varepsilon,b}(\cdot) + 3n x^{n-1} \, \chi''_{0,\varepsilon,b}(\cdot) + x^n \, \chi'''_{0,\varepsilon,b}(\cdot).$$

Thus for any $b > 5\varepsilon$, there exists $c_{n,b} > 0$ such that

$$\left| \chi^{(j)}_{n,\varepsilon,b}(x) \right| \le c_{n,b,j} + \chi_{n,\varepsilon,b}(x), \quad j = 1,2,3. \quad (4.4) \quad \boxed{b7}$$

Hence

$$|A_2(t)| \le c_{n,b} \int u^2(x,t) \, dx + \int u^2 \chi_{n,\varepsilon,b}(x+vt) \, dx$$

$$\le c_{n,b} \|u_0\|_2^2 + \int u^2 \chi_{n,\varepsilon,b}(x+vt) \, dx. \quad (4.5) \quad \boxed{b8}$$

Similarly,

$$|A_1(t)| \leq |v| \int u^2(x,t) \, \chi_{n,\varepsilon,b}(x+vt) \, dx + |v| c_b \int u^2(x,t) \, dx$$
$$\leq |v| \int u^2 \chi_{n,\varepsilon,b}(x+vt) \, dx + |v| c_b \|u_0\|_2^2 \tag{4.6}$$

and

$$|A_3(t)| \leq \|\partial_x u(t)\|_\infty \int u^2(x,t) \, \chi_{n,\varepsilon,b}(x+vt) \, dx. \tag{4.7}$$

This together with Gronwall's inequality shows that for any $\varepsilon > 0$, $b \geq 5\varepsilon$

$$\sup_{0 \leq t \leq T} \int u^2(x,t) \, \chi_{n,\varepsilon,b}(x+vt) \, dx$$
$$+ \int_0^T \int (\partial_x u)^2(x,t) \, \chi'_{n,\varepsilon,b}(x+vt) \, dx < c. \tag{4.8}$$

Next we prove (1.18) in Theorem 1.2. We divide the proof in several cases.

Case $n = 1$

From (4.8) and the fact that

$$\chi_{1,\varepsilon,b}{}'(x) = (x\chi_{0,\varepsilon,b})' = \chi_{0,\varepsilon,b}(x) + x\chi'_{0,\varepsilon,b}(x) \geq \chi_{0,\varepsilon,b}(x) \tag{4.9}$$

it follows that for any $\delta > 0$ there exists $\hat{t} \in (0,\delta)$ such that

$$\int (\partial_x u)^2(x,\hat{t}) \, \chi_{0,\varepsilon,b}(x) \, dx < \infty. \tag{4.10}$$

Now using the identity

$$\frac{1}{2}\frac{d}{dt} \int (\partial_x u)^2 \, \chi_{0,\varepsilon,b}(x+vt) \, dx + \frac{3}{2} \int (\partial_x^2 u)^2 \, \chi'_{0,\varepsilon,b}(x+vt) \, dx$$
$$\underbrace{- v \int (\partial_x u)^2 \, \chi'_{0,\varepsilon,b}(x+vt) \, dx}_{A_1} \underbrace{- \frac{1}{2} \int (\partial_x u)^2 \chi'''_{0,\varepsilon,b}(x+vt) \, dx}_{A_2}$$
$$+ \underbrace{\int \partial_x(u\partial u) \, \partial_x u \, \chi_{0,\varepsilon,b}(x+vt) \, dx}_{A_3} = 0. \tag{4.11}$$

By (4.8) with $n = 1$, after time integration we have that

$$\int_{\hat{t}}^T |A_1(t)| \, dt < c. \tag{4.12}$$

Also using that for any $\varepsilon > 0$ and $b \geq 5\varepsilon$ there exists $c_{\varepsilon,b}$ such that

$$\left|\chi'''_{0,\varepsilon,b}(y)\right| \leq c_{\varepsilon,b} \, \chi'_{0,\varepsilon/3,b+\varepsilon}(y), \quad \text{for all} \quad y \in \mathbb{R}, \tag{4.13}$$

from (4.8) with $n = 1$ and $(\varepsilon/3, b + \varepsilon)$ instead of (ε, b) and after integrating in time one has

$$\int_{\hat{t}}^{T} |A_2(t)| \, dt < c_{\varepsilon,b}. \qquad (4.14) \quad \boxed{\text{b15}}$$

Finally, we have (after integration by parts)

$$\int \partial_x(u \partial_x u) \, \partial_x u \, \chi_{0,\varepsilon,b} \, dx$$
$$= \frac{1}{2} \int \partial_x u \, \partial_x u \, \partial_x u \, \chi_{0,\varepsilon,b} \, dx - \frac{1}{2} \int u \partial_x u \, \partial_x u \, \chi'_{0,\varepsilon,b} \, dx \qquad (4.15) \quad \boxed{\text{b16}}$$
$$= A_{3,1} + A_{3,2},$$

where

$$|A_{3,1}(t)| \le \|\partial_x u(t)\|_\infty \int (\partial_x u)^2 \, \chi_{0,\varepsilon,b}(x + vt) \, dx \qquad (4.16) \quad \boxed{\text{b17}}$$

and (after integration in time)

$$\int_{\hat{t}}^{T} |A_{3,2}(t)| \, dt \le \int_{\hat{t}}^{T} \|u(t)\|_\infty \int (\partial_x u)^2 \, \chi'_{0,\varepsilon,b}(x + vt) \, dx dt$$
$$\le \sup_{0 \le t \le T} \|u(t)\|_{3/4^+} \int_{\hat{t}}^{T} \int (\partial_x u)^2 \, \chi'_{0,\varepsilon,b}(x + vt) \, dx dt < c. \qquad (4.17) \quad \boxed{\text{b18}}$$

by (4.8) and Sobolev embedding. Hence from (4.11) and Gronwall's inequality we have that

$$\sup_{\hat{t} \le t \le T} \int (\partial_x u)^2(x,t) \, \chi_{0,\varepsilon,b}(x + vt) \, dx$$
$$+ \int_{\hat{t}}^{T} \int (\partial_x^2 u)^2(x,t) \, \chi'_{0,\varepsilon,b}(x + vt) \, dx < c. \qquad (4.18) \quad \boxed{\text{b19}}$$

Before proving the general induction case we shall prove the case $n = 2$.

<u>Case $n = 2$</u>

By hypotheses since $x_+ u_0 \in L^2(\mathbb{R})$ by step 0 we have

$$\sup_{0 \le t \le T} \int u^2(x,t) \, \chi_{2,\varepsilon,b}(x + vt) \, dx$$
$$+ \int_0^T \int (\partial_x u)^2(x,t) \, \chi'_{2,\varepsilon,b}(x + vt) \, dx < c. \qquad (4.19) \quad \boxed{\text{b20}}$$

Since

$$\chi'_{n,\varepsilon,b}(x) = (x^n \chi_{0,\varepsilon,b})' = nx^{n-1} \chi_{0,\varepsilon,b} + x^n \chi'_{0,\varepsilon,b} \ge n \chi_{n-1,\varepsilon,b}(x) \qquad (4.20) \quad \boxed{\text{b21}}$$

from (4.19) and (4.20) ($n = 2$) for any $\delta > 0$ there exists $\hat{t} \in (0, \delta)$ such that ($v \equiv 0$)

$$\int (\partial_x u)^2 (x, \hat{t}) \, \chi_{1,\varepsilon,b}(x) \, dx < \infty. \tag{4.21} \quad \boxed{\text{b22}}$$

Now consider the identity

$$\frac{1}{2} \frac{d}{dt} \int (\partial_x u)^2 \chi_1(x+vt) \, dx + \frac{3}{2} \int (\partial_x^2 u)^2 \chi_1'(x+vt) \, dx$$

$$- v \underbrace{\int (\partial_x u)^2 \chi_1'(x+vt) \, dx}_{A_1} - \frac{1}{2} \underbrace{\int (\partial_x u)^2 \chi_1'''(x+vt) \, dx}_{A_2} \tag{4.22} \quad \boxed{\text{b23}}$$

$$+ \underbrace{\int \partial_x (u \partial_x u) \, \partial_x u \, \chi_1(x+vt) \, dx}_{A_3} = 0.$$

By the case $n = 1$ using (4.8) after integrating in time we have

$$\int_{\hat{t}}^{T} |A_1(t)| \, dt \leq |v| \int_{\hat{t}}^{T} \int (\partial_x u)^2 (x,t) \, \chi_{1,\varepsilon,b}'(x+vt) \, dxdt \leq c_v. \tag{4.23} \quad \boxed{\text{b24}}$$

Now since

$$(\chi_1)''' = (x \chi_0)''' = 3\chi_0'' + x\chi_0''' \tag{4.24} \quad \boxed{\text{b25}}$$

we have that given ε, b, v, T there exist $c > 0$ and $R > 0$ such that

$$(\chi_1(x+vt))''' \leq 3\chi_0''(x+vt) + x\chi_0'''(x+vt)$$
$$\leq c \, 1_{[-R,R]}(x) \text{ for all } (x,t) \in \mathbb{R} \times [0,T]. \tag{4.25} \quad \boxed{\text{b26}}$$

Thus by (1.12) (iii)

$$\int_{\hat{t}}^{T} |A_2(t)| \, dt \leq c \int_{\hat{t}}^{T} \int_{-R}^{R} (\partial_x u)^2 (x,t) \, dxdt < c. \tag{4.26} \quad \boxed{\text{b27}}$$

To treat A_3 we proceed as we did to obtain (4.15) to (4.17), with χ_1 instead of χ_0. Thus collecting the above information we get that

$$\sup_{\hat{t} \leq t \leq T} \int (\partial_x u)^2 (x,t) \, \chi_{1,\varepsilon,b}(x+vt) \, dx$$

$$+ \int_{\hat{t}}^{T} \int (\partial_x^2 u)^2 (x,t) \, \chi_{1,\varepsilon,b}'(x+vt) \, dxdt < \infty \tag{4.27} \quad \boxed{\text{b29}}$$

for any $\varepsilon > 0$ and $b \geq 5\varepsilon$.

Now by (4.27) for any $\delta > 0$ there exists $\hat{\hat{t}} \in (\hat{t}, \delta)$ such that

$$\int (\partial_x^2 u)^2 (x, \hat{\hat{t}}) \, \chi_{1,\varepsilon,b}'(x) \, dx < \infty \quad v \equiv 0, \tag{4.28} \quad \boxed{\text{b30}}$$

which implies

$$\int (\partial_x^2 u)^2 (x, \hat{\hat{t}}) \, \chi_{0,\varepsilon,b}(x) \, dx < \infty. \tag{4.29} \quad \boxed{\text{b31}}$$

Thus

$$\int (\partial_x^2 u)^2(x,\hat{t})\chi_{0,\varepsilon,b}(x)\,dx < \infty, \qquad \forall \varepsilon > 0, \ \forall b \geq 5\varepsilon. \tag{4.30}$$ b32

Hence using our result in propagation of regularity (see (3.16)) one has: for any $\varepsilon > 0$, $b \geq 5\varepsilon$, $v > 0$

$$\sup_{\delta \leq t \leq T} \int (\partial_x^2 u)^2(x,t)\,\chi_{0,\varepsilon,b}(x+vt)\,dx$$
$$+ \int_\delta^T \int (\partial_x^3 u)^2(x,t)\,\chi'_{0,\varepsilon,b}(x+vt)\,dx < c. \tag{4.31}$$ b33

This completes the proof of the case $n = 2$.

So from now on we shall assume that $n \geq 3$.

To prove Theorem 1.2 for the general case $n \geq 3$ we shall use induction.

Given $(m,l) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ we say that

$$(m,l) > (\hat{m},\hat{l}) \iff \begin{cases} \text{(i)} & m > \hat{m} \\ \text{or} \\ \text{(ii)} & m = \hat{m} \text{ and } l > \hat{l}. \end{cases} \tag{4.32}$$ b34

Similarly, we say that $(m,l) \geq (\hat{m},\hat{l})$ if (ii) in the right hand side of (4.32) holds with $\geq$ insted of $>$.

Our statement (m,l)

For any $\delta > 0$, $v > 0$, $\varepsilon > 0$, $b \geq 5\varepsilon$

$$\sup_{\delta \leq t \leq T} \int (\partial_x^l u)^2(x,t)\,\chi_{m,\varepsilon,b}(x+vt)\,dx$$
$$+ \int_\delta^T \int (\partial_x^{l+1} u)^2(x,t)\,\chi'_{m,\varepsilon,b}(x+vt)\,dx \leq c_{\delta,v,\varepsilon,b}(u_0). \tag{4.33}$$ b35

Notice that we have already proved the following cases:

(1) $(0,1)$ and $(1,0)$.
(2) $(0,2)$, $(1,1)$ and $(2,0)$.
(3) Under the hypothesis $x_+^{n/2} u_0 \in L^2(\mathbb{R})$ we establish (4.8) i.e. $(n,0)$ for all $n \in \mathbb{Z}^+$.
(4) By Theorem A propagation of regularity: If (4.33) holds with $(m,l) = (1,l)$ ($\delta/2$ instead of δ), then there exists $\hat{t} \in (\delta/2,\delta)$ such that

$$\int (\partial_x^{l+1} u)^2(x,\hat{t})\,\chi'_{1,\varepsilon,b}(x+v\hat{t})\,dx < \infty, \quad v \equiv 0$$

which implies that

$$\int (\partial_x^{l+1} u)^2(x,t)\, \chi_{0,\varepsilon,b}(x+vt)\, dx < \infty \quad v \equiv 0.$$

By the propagation of regularity (Theorem 1.1) one has the result (4.33) with $(m,l) = (0, l+1)$, i.e. $(1,l)$ implies $(0, l+1)$ for any $l \in \mathbb{Z}^+$.

Hence to complete our induction we assume (4.33) for (m,k) such that

$$\begin{cases} \text{(a) } (m,k) \leq (n-j,j) \text{ for some } j = 0,1,\ldots,n \\ \quad \text{and} \\ \text{(b) } (m,k) = (n+1,0),(n,1),\ldots,(n+1-l,l) \text{ for some } l \leq n. \end{cases} \tag{4.34} \quad \boxed{\text{b36}}$$

We need to prove the case $(n+1-(l+1), l+1) = (n-l, l+1)$. Observe that from (4) above, since $(1,l)$ implies $(0, l+1)$, this case is ready for $l = n$. Thus it suffices to consider $l \leq n-1$.

From the previous step $((n-l+1, l))$ we have that for any $\delta' > 0$, $v > 0$, ε, and $b \geq 5\varepsilon$

$$\sup_{\delta' \leq t \leq T} \int (\partial_x^l u)^2(x,t)\, \chi_{n+1-l,\varepsilon,b}(x+vt)\, dx$$
$$+ \int_{\delta'}^{T} \int (\partial_x^{l+1} u)^2(x,t)\, \chi'_{n+1-l,\varepsilon,b}(x+vt)\, dx < \infty. \tag{4.35} \quad \boxed{\text{b37}}$$

Since

$$\chi'_{n+1-l,\varepsilon,b}(x) \geq c\, \chi_{n-l,\varepsilon,b}(x) \tag{4.36} \quad \boxed{\text{b38}}$$

one has (by (4.35)) that there exists $\hat{t} \in (\delta', 2\delta')$ such that

$$\int (\partial_x^{l+1} u)^2(x,\hat{t})\, \chi_{n-l,\varepsilon,b}(x)\, dx < \infty, \quad (v \equiv 0). \tag{4.37} \quad \boxed{\text{b39}}$$

Consider the identity (for $t > \hat{t}$)

$$
\frac{1}{2}\frac{d}{dt}\int (\partial_x^{l+1}u)^2(x,t)\,\chi_{n-l,\varepsilon,b}(x+vt)\,dx
$$
$$
+\frac{3}{2}\int (\partial_x^{l+2}u)^2(x,t)\,\chi'_{n-l,\varepsilon,b}(x+vt)\,dx
$$
$$
\underbrace{-v\int (\partial_x^{l+1}u)^2(x,t)\,\chi'_{n-l,\varepsilon,b}(x+vt)\,dx}_{A_1}
$$
$$
\underbrace{-\frac{1}{2}\int (\partial_x^{l+1}u)^2(x,t)\,\chi'''_{n-l,\varepsilon,b}(x+vt)\,dx}_{A_2}
$$
$$
\underbrace{+\int \partial_x^{l+1}(u\partial_x u)\,\partial_x^{l+1}u\,\chi_{n-l,\varepsilon,b}(x+vt)\,dx = 0.}_{A_3}
$$

$$(4.38)\quad \boxed{\text{b40}}$$

From the previous step $(n-l,l)$ after integration in time $[\hat{t},T]$ one gets

$$
\int_{\hat{t}}^{T}|A_1(t)|\,dt \le |v|\int_{\hat{t}}^{T}\int (\partial_x^{l+1}u)^2\,\chi'_{n-l,\varepsilon,b}(x+vt)\,dxdt \le |v|\,c. \qquad (4.39)\quad \boxed{\text{b41}}
$$

Next using that

$$
|\chi'''_{n-l,\varepsilon,b}(y)| = |(y^{n-l}\chi_{0,\varepsilon,b})'''(y)|
$$
$$
\le c_{nl}\chi_{n-l-3,\varepsilon,b}(y) + \big|c_{nl}^1 y^{n-l-2}\chi'_{0,\varepsilon,b}(y)
$$
$$
+ c_{nl}^2 y^{n-l-1}\chi''_{0,\varepsilon,b}(y) + c_{nl}^3 y^{n-l}\chi'''_{0,\varepsilon,b}(y)\big|
$$
$$
\le c_{nl}\,\chi_{n-l-3,\varepsilon,b}(y) + c_{n,l,b}\,\chi_{0,\varepsilon/5,\varepsilon}(y).
$$

$$(4.40)\quad \boxed{\text{b42}}$$

Hence

$$
|A_2(t)| \le c_{ln}\int (\partial_x^{l+1}u)^2\,\chi_{n-l-3,\varepsilon,b}(x+vt)\,dx
$$
$$
+ c_{l,n,b}\int (\partial_x^{l+1}u)^2\,\chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx
$$

$$(4.41)\quad \boxed{\text{b43}}$$

which are bounded in $[\hat{t},T]$ by the step $(n-l-3,l+1)$ and $(0,l+1)$ which is implied by the step $(1,l) = (l+1-l,l) \le (n-l,l)$, a previous case since $l \le n-1$.

Finally, we need to consider A_3 in (4.38). Since

$$
\partial_x^{l+1}(u\partial_x u) = u\partial_x^{l+2}u + c_1\partial_x u\,\partial_x^{l+1}u + c_2\,\partial_x^2 u\,\partial_x^l u
$$
$$
+ \sum_{j=3}^{l-1}c_j\,\partial_x^j u\partial_x^{l+2-j}u.
$$

$$(4.42)\quad \boxed{\text{b44}}$$

we have the following terms in A_3

$$
\begin{aligned}
A_3(t) = &\int u \partial_x^{l+2} u \partial_x^{l+1} u \chi_{n-l,\varepsilon,b} \, dx \\
&+ c_1 \int \partial_x u \, \partial_x^{l+1} u \, \partial_x^{l+1} u \chi_{n-l,\varepsilon,b} \, dx \\
&+ c_2 \int \partial_x^2 u \, \partial_x^l u \, \partial_x^{l+1} u \chi_{n-l,\varepsilon,b} \, dx \\
&+ \sum_{j=3}^{l-1} c_j \int \partial_x^j u \, \partial_x^{l+2-j} u \, \partial_x^{l+1} u \chi_{n-l,\varepsilon,b} \, dx \\
= &\, A_{30} + A_{31} + A_{32} + \sum_{j=3}^{l-1} A_{3j}.
\end{aligned}
$$
(4.43) b45

Thus after integration by parts

$$
\begin{aligned}
A_{30} = &-\frac{1}{2} \int \partial_x u \, (\partial_x^{l+1} u)^2 \chi_{n-l,\varepsilon,b} \, dx - \frac{1}{2} \int u \, (\partial_x^{l+1} u)^2 \chi'_{n-l,\varepsilon,b} \, dx \\
= &\, A_{301} + A_{302}
\end{aligned}
$$
(4.44) b46

with $A_{301} = -\frac{1}{2} A_{31}$ and after integrating in time

$$
\int_{\hat{t}}^T |A_{302}| \, dt \leq \sup_{\hat{t} \leq t \leq T} \|u(t)\|_\infty \int_{\hat{t}}^T \int (\partial_x^{l+1} u)^2 \chi'_{n-l,\varepsilon,b}(x+vt) \, dx.
$$
(4.45) b47

Notice that the last integral on the right hand side corresponds to the case $(n-l,l)$ (see (4.34)) which is part of our hypothesis of induction and the first term can be bounded as usual using Sobolev embedding.

Next

$$
|A_{31}(t)| \leq c \, \|\partial_x u(t)\|_\infty \int (\partial_x^{l+1} u)^2 \chi_{n-l,\varepsilon,b}(x+vt) \, dx.
$$
(4.46) b48

The last term in the inequality above is the quantity to be estimated.

Consider now A_{32} which appears only if $l \geq 2$ (we recall that $n \geq 3$ (to be proved $(n-l,l+1)$))

$$
A_{32}(t) = c \int \partial_x^2 u \, \partial_x^l u \, \partial_x^{l+1} u \chi'_{n-l,\varepsilon,b}(x+vt) \, dx.
$$
(4.47) 49

We study two cases:

Case $l = 2$

By integration by parts (similar to (3.30)–(3.32) in the proof of Theorem 1.1) we have

$$
\begin{aligned}
|A_{32}(t)| &= \left| -\frac{c}{3} \int\int \partial_x^2 u\, \partial_x^2 u\, \partial_x^2 u\, \chi'_{n-l,\varepsilon,b}(x+vt)\,dx \right| \\
&\leq c\|(\partial_x^2 u)\chi'_{n-l,\varepsilon,b}(\cdot+vt)\|_\infty \int (\partial_x^2 u)^2\, \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\,dx \\
&\leq c\|(\partial_x^2 u)\chi'_{n-l,\varepsilon,b}(\cdot+vt)\|_\infty^2 + c \\
&\leq cc^*\,\|(\partial_x^2 u)^2\, \chi'_{n-l,\varepsilon,b}(\cdot+vt)\|_\infty + c \quad (c^* = \|\chi'_{n-l,\varepsilon,b}(\cdot)\|_{L_x^\infty}) \\
&\leq c\int \left|\partial_x((\partial_x^2 u)^2\, \chi'_{n-l,\varepsilon,b}(x+vt))\right|dx + c \qquad\qquad (4.48)\quad \boxed{\text{b50}} \\
&\leq c\int (\partial_x^2 u)^2\, \chi'_{n-l,\varepsilon,b}(x+vt)\,dx \\
&\quad + c\int (\partial_x^3 u)^2\, \chi'_{n-l,\varepsilon,b}(x+vt)\,dx \\
&\quad + c\int (\partial_x^2 u)^2\, \chi''_{n-l,\varepsilon,b}(x+vt)\,dx + c \\
&= A_{321} + A_{322} + A_{323} + c.
\end{aligned}
$$

We have used in the second inequality that the second term on the right hand side is bounded in $t \in [\hat{t}, T]$ step $(0, l) = (0, 2)$. After integrating in time A_{321} corresponds to the case $(n-l, 1) = (n-2, 1)$, A_{322} corresponds to the case $(n-l, 2) = (n-2, 2)$ since $l = 2$. So by the hypothesis of induction they are bounded (the previous step to $(n-2, 3)$ which we want to prove).

Finally,

$$
\begin{aligned}
\chi''_{n-l,\varepsilon,b}(y) &= (y^{n-l}\,\chi_{0,\varepsilon,b})'' \\
&= c_{nl}\,y^{n-l-2}\,\chi_{0,\varepsilon,b} + c_{nl}^1\,y^{n-l-1}\,\chi'_{0,\varepsilon,b} + y^{n-l}\,\chi''_{0,\varepsilon,b}.
\end{aligned}
\qquad (4.49)\quad \boxed{\text{b51}}
$$

Hence there exists $c^* = c^*(b, v, T)$ such that

$$
|\chi''_{n-l,\varepsilon,b}(x+vt)| \leq c\,\chi_{n-l-2,\varepsilon,b}(x+vt) + c_b^*\,\chi'_{0,\varepsilon/5,\varepsilon}(x+vt).
$$

Thus

$$
\begin{aligned}
|A_{323}(t)| &\leq c\int (\partial_x^2 u)^2(x,t)\, \chi_{n-l-2,\varepsilon,b}(x+vt)\,dx \\
&\quad + c_b^*\int (\partial_x^2 u)^2(x,t)\, \chi_{0,\varepsilon/3,b+\varepsilon'}(x+vt)\,dx \qquad (4.50)\quad \boxed{\text{b52}} \\
&= A_{3231} + A_{3232}.
\end{aligned}
$$

The term A_{3231} is bounded in time $t \in [\hat{t}, T]$ (case $(n-l-2, 2)$) and A_{3232} is also bounded in time (case $(0, 2)$).

This handles all the terms in (4.38) for the case $l = 2$. Thus after applying Gronwall's inequality one gets

$$\sup_{\hat{t} \leq t \leq T} \int (\partial_x^{l+1} u)^2(x,t)\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx$$
$$+ \int_{\hat{t}}^{T} \int (\partial_x^{l+2} u)^2(x,t)\, \chi'_{n-l,\varepsilon,b}(x+vt)\, dx dt < c. \tag{4.51}$$

It remains to consider the case $l \geq 3$ in (4.43).

Case $l \geq 3$

Observe that in this case after integration by parts

$$A_{32} = c_2 \int \partial_x^2 u\, \partial_x^l u\, \partial_x^{l+1} u\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx$$
$$= -\frac{c_2}{2} \int \partial_x^3 u\, (\partial_x^l u)^2\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx \tag{4.52}$$
$$+ \frac{c_2}{2} \int \partial_x^2 u\, (\partial_x^l u)^2\, \chi'_{n-l,\varepsilon,b}(x+vt)\, dx$$

whose bound is similar to that given above for the case $l = 2$. Thus it suffices to consider the reminder terms in (4.43), i.e.

$$A_{3j}(t) = c_j \int \partial_x^j u\, \partial_x^{l+2-j} u\, \partial_x^{l+1} u\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx \tag{4.53}$$

without loss of generality (since $l \geq 3$ and $l+2 = j + (l+2-j)$ we can assume $3 \leq j \leq l/2 + 1$.

Thus

$$|A_{3j}(t)| \leq c_j \int (\partial_x^j u\, \partial_x^{l+2-j} u)^2\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx$$
$$+ c_j \int (\partial_x^{l+1} u)^2\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx \tag{4.54}$$

where the second term is the quantity to be estimated.

So we need to bound

$$\int (\partial_x^j u)^2\, (\partial_x^{l+2-j} u)^2\, \chi_{n-l,\varepsilon,b}(x+vt)\, \chi_{0,\varepsilon/5,\varepsilon}(x+vt)\, dx$$
$$\leq \|(\partial_x^j u)^2\, \chi_{0,\varepsilon/5,\varepsilon}(\cdot+vt)\|_\infty \int (\partial_x^{l+2-j} u)^2\, \chi_{n-l,\varepsilon,b}(x+vt)\, dx. \tag{4.55}$$

Noticing that the second term is bounded in time (case $(n-l, l+2-j)$, $j \geq 3$) by the induction hypothesis.

Thus it remains to bound

$$
\begin{aligned}
\|(\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon}(\cdot + vt)\|_\infty &\le \int |\partial_x((\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon}(\cdot + vt))|\, dx \\
&\le c \int |\partial_x^j u\, \partial_x^{j+1} u\, \chi_{0,\varepsilon/5,\varepsilon}(\cdot + vt)|\, dx + \int |(\partial_x^j u)^2 \chi'_{0,\varepsilon/5,\varepsilon}(\cdot + vt)|\, dx \\
&\le \int (\partial_x^j u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x + vt)\, dx + \int (\partial_x^{j+1} u)^2 \chi_{0,\varepsilon/5,\varepsilon}(x + vt)\, dx \\
&\quad + \int (\partial_x^j u)^2 \chi'_{0,\varepsilon/5,\varepsilon}(x + vt)\, dx.
\end{aligned}
\tag{4.56}
$$

b58

The first two terms are bounded in time, cases $(0, j)$, $(0, j+1)$ since $j \le l - 1$ implies $j + 1 \le l$ and $l \le n$.

For the third one after integration in time one gets

$$
\int_{\hat{t}}^{T} \int (\partial_x^j u)^2 \chi'_{0,\varepsilon/5,\varepsilon}(x + vt)\, dx dt
\tag{4.57}
$$

b59

corresponding to the case $(0, j-1)$ previous case.

This basically completes the proof of Theorem 1.2.

The previous formal computation can be justified by approximating the initial data u_0 by elements in the Schwartz space $\mathscr{S}(\mathbb{R})$ say u_0^μ, $\mu > 0$. This is done by an appropriate truncation and smoothing proccess. Using the well-posedness in this class one gets a family of solutions $u^\mu(\cdot, t)$ for which each step of the above argument can be justified. Due to our construction the estimates are uniform in the parameter $\mu > 0$ which yields the desired estimate by passing to the limit.

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