DISPERSIVE BLOW UP FOR NONLINEAR SCHRÖDINGER EQUATIONS REVISITED

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Abstract. The possibility of finite-time, dispersive blow up for nonlinear equations of Schrödinger type is revisited. This mathematical phenomena is one of the possible explanations for oceanic and optical rogue waves. In dimension one, the possibility of dispersive blow up for nonlinear Schrödinger equations already appears in [9]. In the present work, the existing results are extended in several ways. In one direction, the theory is broadened to include the Davey-Stewartson and Gross-Pitaevskii equations. In another, dispersive blow up is shown to obtain for nonlinear Schrödinger equations in spatial dimensions larger than one and for more general power-law nonlinearities. As a by-product of our analysis, a sharp global smoothing estimate for the integral term appearing in Duhamel's formula is obtained.

Résumé. Nous revisitons la possibilité d'apparition de singularités dispersives (dispersive blow-up) pour des solutions d'équations de Schrödinger non linéaires. Ce phénomène mathématique pourrait être une explication pour l'apparition des "vagues scélérates" (rogue waves) en océanographie et optique non linéaire. La possibilité de singularités dispersives pour des équations de Schrödinger non linéaires en dimension spatiale un a été prouvée dans [9]. Ces résultats sont étendus ici dans plusieurs directions. D'une part la théorie est étendue à des équations de Schrödinger en dimension spatiale quelconque, avec des non-linéarités de type puissance générales. D'autre part nous traitons également le cas des systèmes de Davey-Stewartson et de l'équation de Gross-Pitaevskii. Un sous-produit de notre analyse est un effet de lissage global précis pour le terme intégral de la représentation de Duhamel.

1. Introduction

This paper continues the theory of dispersive blow up which was initiated and developed in [8] and [9]. The present contribution is especially relevant to nonlinear Schrödinger-type equations, and includes theory for the Davey-Stewartson and the Gross-Pitaevskii equations. The work on Schrödinger equations substantially extends the results already available in [9].

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Dispersive blow up of wave equations is a phenomenon of focusing of smooth initial disturbances with finite-mass (or, finite-energy, depending on the physical context) that relies upon the dispersion relation guaranteeing that, in the linear regime, different wavelengths propagate at different speeds. This is especially the case for models wherein the linear dispersion is unbounded, so that energy can be moved around at arbitrarily high speeds, but even bounded dispersion can exhibit this type of blow up.

To be more concrete, consider the Cauchy problem for the linear (free) Schrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u\big|_{t=0} = u_0(x), \tag{1.1}$$

where $x \in \mathbb{R}^n$ for some $n \in \mathbb{N}$. For $u_0 \in L^2(\mathbb{R}^n)$, elementary Fourier analysis shows the solution to this initial-value problem is

$$u(x,t) = e^{it\Delta} u_0(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-it|\xi|^2} \widehat{u}_0(\xi) e^{i\xi\cdot x} d\xi. \tag{1.2}$$

Here, $\widehat{u}_0$ denotes the Fourier transformed initial data, *viz.*

$$\mathcal{F}u_0(\xi) \equiv \widehat{u}_0(\xi) = \int_{\mathbb{R}^n} u_0(x) e^{-i\xi\cdot x}\, dx.$$

The corresponding inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$. From (1.2), it is immediately inferred that for any $s \in \mathbb{R}$, solutions lie in $C(\mathbb{R}; H^s)$ whenever u_0 lies in the L^2-based Sobolev space H^s. Moreover, the evolution preserves all these Sobolev-space norms, which is to say

$$\|u(\cdot,t)\|_{H^s(\mathbb{R}^n)} = \|u_0\|_{H^s(R^n)}$$

for $t \in \mathbb{R}$. In certain applications of this model, the case $s = 0$ in the last formula corresponds to conservation of total mass in the underlying physical system.

However, in Theorem 2.1 of [9], it was shown that for any given point $(x_*, t_*) \in \mathbb{R}^n \times \mathbb{R}_+$, there exists initial data $u_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that the solution $u(x,t)$ of the corresponding initial-value problem (1.1) for the free Schrödinger equation is continuous on $\mathbb{R}^n \times \mathbb{R}_+ \setminus \{(x_*, t_*)\}$, but

$$\lim_{(x,t)\in\mathbb{R}^n\times\mathbb{R}_+\to(x_*,t_*)} |u(x,t)| = +\infty.$$

This fact is referred to as (finite-time) dispersive blow up and will sometimes be abbreviated DBU in the following. The analogous phenomena also appears in other linear dispersive equations, such as the linear Korteweg-de Vries equation [5] and the free surface water waves system linearized around the rest state [9].

At first sight, one would expect that nonlinear terms would destroy dispersive blow up. What is a little surprising is that even the inclusion of physically relevant nonlinearities in various models of wave propagation does not prevent dispersive blow up. Indeed, theory shows in some important cases that initial data leading to this focusing singularity under the linear evolution continues to blow up in exactly the same way when nonlinear terms are included. In [8], this was shown to be true for the Korteweg-de Vries equation, a model for shallow water waves and other simple wave phenomena. This result and analogous dispersive blow up theory for solutions of the one-dimensional nonlinear Schrödinger equations [9],

$$i\partial_t u + \partial_x^2 u \pm |u|^p u = 0, \quad u\big|_{t=0} = u_0(x), \tag{1.3}$$

where $x \in \mathbb{R}$ and $p \in (0,3)$, lead to the speculation that such focusing might be one road to the formation of rogue waves in shallow and deep water and in nonlinear optics. (see [17, 18, 28, 37]).

The analysis of [9] revolves around providing bounds on the nonlinear terms in a Duhamel representation of the evolution. Because the phenomenon is due to the linear terms in the equation, data of arbitrarily small size will still exhibit dispersive blow up, and indeed it can be organized to happen arbitrarily quickly. This emphasizes the linear aspect of these singularities and differentiates it from the blow up that occurs for some of the same models when the nonlinear term is focusing and sufficiently strong (see [39] for a general overview of this aspect of Schrödinger equations). Moreover, even though the theory begins by showing that there are specific initial data that lead to dispersive blow up, the result is in fact self-improving. Dispersive blow up continues to hold if this special initial data is subjected to a smooth perturbation. The theory further implies that there is C^∞ initial data with compact support which can be taken as small as we like that will, in finite time, become large in a neighborhood of a prescribed spatial point (see Remark 3.5 for more details). Dispersive blow up thereby also serves to demonstrate ill-posedness of the considered models in L^∞–spaces.

The aim of the present work is to generalize the results mentioned above in several respects. Most importantly, the dispersive blow up that in [9] was obtained for (1.3) will be shown to hold true of nonlinear Schrödinger equations in *all dimensions* $n \geq 1$ and for the whole range of nonlinearities $p \geq [\frac{n}{2}]$, with or without a (possibly unbounded) real-valued potential. Here and in the following, for $\mu \in \mathbb{R}$, the quantity $[\mu]$ is the greatest integer less than or equal to μ. Higher-order Schrödinger equations are also countenanced. Our theory relies especially on the results of Cazenave and Weissler established in [15]. In addition to Schrödinger equations, dispersive blow up is proved for Gross-Pitaevskii equations *with non-trivial* boundary conditions at infinity and for the Davey-Stewartson systems.

As a by-product of our analysis, a sharp global smoothing effect is obtained for the nonlinear integral term in the equation derived from (1.3) by use of Duhamel's formula.

The paper proceeds as follows: Section 2 is concerned with some preliminaries which are mostly linear in nature. Dispersive blow up for nonlinear Schrödinger equations is tackled in Section 3. Section 4 deals with the just mentioned sharp global smoothing property. Dispersive blow up for the Davey-Stewartson systems then follows more or less as a corollary to the results in Sections 2 and 3. The Gross-Pitaevskii equation takes center stage in Section 5 whilst higher-order Schrödinger equations are studied in Section 6.

2. Mathematical preliminaries

In this section, a review of the basic idea behind dispersive blow up is provided in the context of nonlinear Schrödinger equations. Parts of the currently available theory for the linear Schrödinger group are also recalled in preparation for the analysis in Section 3.

2.1. **Dispersive blow up in linear Schrödinger equations.**

To understand the appearance of dispersive blow up in the solution of (1.1), start by explicitly computing the inverse Fourier transformation in (1.2) to see that the free Schrödinger group admits the representation

$$u(x,t) = \frac{1}{(4i\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)\,dy, \quad \text{for } t \neq 0. \tag{2.1}$$

This representation formula is the starting point of the following lemma.

Lemma 2.1. *Let $\alpha \in \mathbb{R}$, $q \in \mathbb{R}^n$ and*

$$u_0(x) = \frac{e^{-i\alpha|x-q|^2}}{(1+|x|^2)^m}\ , \quad \textit{with } \frac{n}{4} < m \leq \frac{n}{2}.$$

Then, $u_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and the associated global in-time solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ of (1.1) has the following properties.

(1) *At the point $(x_*, t_*) = (q, \frac{1}{4\alpha})$, the solution u in (2.1) blows up, which is to say,*

$$\lim_{(x,t)\in\mathbb{R}^{n+1}\to(x_*,t_*)} |u(x,t)| = +\infty,$$

(2) *it is a continuous function of (x,t) on $\mathbb{R}^n \times \mathbb{R} \setminus \{t_*\}$ and*

(3) *$u(x,t_0)$ is a continuous function of $x \in \mathbb{R}^n \setminus \{x_*\}$.*

Proof. First note that $u_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, that $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ and that the L^2–norm of u is constant in view of mass conservation. On the other hand, evaluating (2.1) at $t = \frac{1}{4\alpha}$ for this particular u_0 gives

$$u\left(x, \frac{1}{4\alpha}\right) = \left(\frac{\alpha}{i\pi}\right)^{n/2} e^{i\alpha(|x|^2-|q|^2)} \int_{\mathbb{R}^n} e^{-2i\alpha y\cdot(x-q)} \frac{dy}{(1+|y|^2)^m}.$$

Thus at $x = q$, it transpires that

$$\left|u\left(q, \frac{1}{4\alpha}\right)\right| = \left(\frac{\alpha}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^m} = +\infty$$

provided $m \leq \frac{n}{2}$. Assertions (ii) and (iii) can then be proved by the same arguments as in the proof of [9, Theorem 2.1]. In this endeavor, it is useful to note that $(1+x^2)^{-m}$ is closely related to the inverse Fourier transform of the modified Bessel functions $K_\nu(|x|)$, where $\nu = \frac{n}{2} - m$. □

In other words, for any given $q \in \mathbb{R}^n, \alpha \in \mathbb{R}$, we have constructed an explicit family of bounded smooth initial data (with finite mass) for which the solution of the free Schrödinger equation (1.1) exhibits dispersive blow up at the point $(x_*, t_*) = (q, \frac{1}{4\alpha})$ in space and time. This result can be immediately generalized in various ways. The following sequence of remarks indicates some of them.

Remark 2.2. The same argument shows that any initial data of the form

$$u_0(x) = e^{-i\alpha|x-q|^2} a(x),$$

with an amplitude $a \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ but $a \notin L^1(\mathbb{R}^n)$ will exhibit dispersive blow up. Using the superposition principle, one can construct initial data which yield dispersive blow up at any countably many isolated points in space-time $\mathbb{R}^n \times \mathbb{R}$. In addition, multiplying u_0 by δ with $0 < \delta \ll 1$, allows for initial data which are arbitrarily small, but which nevertheless blow up at (x_*, t_*). By a suitable spatial truncation of u_0, one can also construct small, smooth, bounded initial data with finite mass, all of whose derivatives also have finite L^2–norm, such that the corresponding solution u remains smooth but achieves arbitrarily large values at a given point in space-time (see [9] for more details).

Remark 2.3. It was also proven in [9] that dispersive blow up holds true for the class

$$i\partial_t u + (-\Delta)^{\frac{a}{2}} u = 0, \quad 0 < a < 1,$$

of fractional Schrödinger equations in $\mathbb{R}^n \times \mathbb{R}^+$. However, observe that, in contrast to the classical Schrödinger equation, the phase velocity for (2.3) becomes arbitrarily large in the long wave limit but is bounded (and actually tends to zero) in the short wave limit.

A natural extension of DBU for linear Schrödinger equations is the initial-value problem in the presence of external potentials $V(x,t) \in \mathbb{R}$. While we are not going to deal here with this issue in full generality, we note that an immediate consequence of Remark 2.2 is the appearance of dispersive blow up for Schrödinger equations with a Stark potential, *i.e.*

$$i\partial_t v + \Delta v - (E \cdot x) v = 0, \quad v\big|_{t=0} = v_0(x), \tag{2.2}$$

where $E \in \mathbb{R}^n$. This equation models electromagnetic wave propagation in a constant electric field. Solutions of (2.2) are connected to solutions of the free Schrödinger equation through the Avron-Herbst formula [1]. Indeed, it is easy to check that if v solves (2.2), then

$$u(x,t) = v\left(x + t^2 E, t\right) e^{-itE\cdot x - i\frac{t^3}{3}|E|^2},$$

solves the linear problem (1.1) with the same initial data. Thus, if the u that solves the free Schrödinger equation exhibits dispersive blow up at a given (x_*, t_*), then so does the solution v of (2.2) at the point $(x_* + 2t_*^2 E, t_*)$.

An analogous result is also true in the case of linear Schrödinger equations with isotropic quadratic potentials

$$i\partial_t u + \Delta u \pm \omega^2 |x|^2 u = 0, \quad u\big|_{t=0} = u_0(x), \tag{2.3}$$

where $\omega \in \mathbb{R}$. The two signs correspond, respectively, to attractive $(-)$ and repulsive $(+)$ potentials. Following [12], we find that if v solves (2.3) with attractive harmonic potential, then

$$u(x,t) = \frac{1}{(1+(2\omega t)^2)^{n/4}}\, v\left(\frac{\arctan(2\omega t)}{\omega}, \frac{x}{\sqrt{1+(2\omega t)^2}}\right) e^{i\frac{2\omega^2 x^2 t}{(1+(2\omega t)^2)}},$$

solves (1.1). Thus, dispersive blow up for u again implies dispersive blow up for v, although at a shifted point in space-time. A similar formula can be derived in the repulsive case (see [13]).

Remark 2.4. Alternatively, one can prove dispersive blow up for linear Schrödinger equations with quadratic potentials using (generalized) Mehler formulas for the associated Schrödinger group. The Mehler formulas are

$$u(x,t) = e^{-in\frac{\pi}{4}\operatorname{sgn} t} \left|\frac{\omega}{2\pi \sin \omega t}\right|^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i\omega}{\sin \omega t}\left(\frac{|x|^2+|y|^2}{2}\cos \omega t - x\cdot y\right)} u_0(y)\, dy,$$

for (2.3) in the attractive situation, while in the repulsive case, one has

$$u(x,t) = e^{-in\frac{\pi}{4}\operatorname{sgn} t} \left|\frac{\omega}{2\pi \sinh \omega t}\right|^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i\omega}{\sinh \omega t}\left(\frac{|x|^2+|y|^2}{2}\cosh \omega t - x\cdot y\right)} u_0(y)\, dy$$

(see [13] again). In the attractive case, Mehler's formula is valid for $|t| < \frac{\pi}{2\omega}$, while in the repulsive case it makes sense for all $t > 0$. Generalizations of these formulas are available and allow one to infer that dispersive blow up can occur in the presence of anisotropic quadratic potentials (see [13]).

We close this subsection by noting, that at least in $n = 1$, it is easy to show that dispersive blow up is stable under the influence of a rather general class of external (real-valued) potentials $V \in C(\mathbb{R}_t; L^2(\mathbb{R}_x))$. To this end, consider the linear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - V(x,t)u = 0, \quad u\big|_{t=0} = u_0(x), \tag{2.4}$$

which we can rewrite, using Duhamel's formula, as

$$u(x,t) = e^{it\partial_x^2}u_0(x) - i\int_0^t e^{i(t-s)\partial_x^2}V(s,x)u(x,s)\,ds =: e^{it\Delta}u_0(x) - iI_V(x,t). \tag{2.5}$$

In view of (2.1), we formally have

$$I_V(x,t) = \frac{1}{(4i\pi t)^{1/2}}\int_0^t\int_{\mathbb{R}^n}\frac{1}{(t-s)^{\frac12}}\exp\left(i\frac{|x-y|^2}{4(t-s)}\right)V(s,y)u(y,s)\,dy\,ds. \tag{2.6}$$

Now, assume that the first term on the right-hand side of (2.5) exhibits dispersive blow up at some (x_*, t_*). Then, it suffices to show that $I_V(x,t)$ is continuous for $(x,t) \in \mathbb{R}^n \times [0,T]$, for any $T > t_*$, in order to conclude that the solution of the (2.4) exhibits dispersive blow up at the same point (x_*, t_*). Having in mind Lebesgue's dominated convergence theorem, we only need to prove that $I_V(x,t)$ is locally bounded as a function of x and t. To do so, we first apply the Cauchy-Schwartz inequality to find

$$|I_V(x,t)| \leq \frac{1}{(4\pi|t|)^{1/2}}\int_0^t\frac{1}{|t-s|^{\frac12}}\,\|V(\cdot,s)\|_{L^2}\|\|u_0\|_{L^2}\,ds, \tag{2.7}$$

where we have also used mass conservation, i.e. $\|u(\cdot,t)\|_{L^2} = \|u_0\|_{L^2}$. Due to our assumption on V and the fact that $t \mapsto t^{-1/2}$ is locally integrable, the right hand side of (2.7) is finite and we are done.

Indeed, a similar argument will be used in the study of DBU for nonlinear Schrödinger equations in $n = 1$, see below. It is clear, however, that in general dimensions $n > 1$ a more refined analysis is needed.

2.2. **Smoothing properties of the free Schrödinger group.** In this subsection, some results on the smoothing properties of the free Schrödinger group $S(t) = e^{it\Delta}$ are reviewed. They will find use in Section 4.

First, recall the notion of admissible index-pairs.

Definition 2.5. The pair (p,q) is called *admissible* if

$$\frac{2}{q} = \frac{n}{2} - \frac{n}{p}, \quad \text{and} \quad \begin{cases} 2 \leq p < \frac{2n}{n-2}, & \text{for } n \geq 3, \\ 2 \leq p < +\infty, & \text{if } n = 2, \\ 2 \leq p \leq +\infty, & \text{if } n = 1. \end{cases}$$

From now on, for any index $r > 0$, its Hölder dual is denoted r', *i.e.* $\frac{1}{r} + \frac{1}{r'} = 1$. The well known Strichartz estimates for the Schrödinger group $S(t) = e^{it\Delta}$ are recounted in the next lemma (see [14, 29] for more details).

Lemma 2.6. *If (p,q) is admissible, then the group $\{e^{it\Delta}\}_{t\in\mathbb{R}}$ satisfies*

$$\left(\int_{-\infty}^{\infty}\left\|e^{it\Delta}f\right\|_{L^p(\mathbb{R}^n)}^q\,dt\right)^{\frac1q} \leq C\left\|f\right\|_{L^2(\mathbb{R}^n)}$$

and

$$\left(\int_{-\infty}^{\infty}\Big\|\int_0^t e^{i(t-s)\Delta}g(\cdot,s)\,ds\Big\|_{L^p(\mathbb{R}^n)}^q dt\right)^{\frac{1}{q}} \le C\left(\int_{-\infty}^{\infty}\big\|g(\cdot,t)\big\|_{L^{p'}(\mathbb{R}^n)}^{q'} dt\right)^{\frac{1}{q'}},$$

where $C = C(p,n)$*.*

The estimates stated above can be interpreted as global smoothing properties of the free Schrödinger group $S(t)$. In addition to that, $S(t)$ is known to also induce local smoothing effects, some of which are collected in the following lemma (for proofs, see [29, Chapter 4]). For $1 \le j \le n$, denote the so-called homogenous derivatives of order $s > 0$ by

$$\begin{aligned} &D_{x_j}^s f(x) := \mathcal{F}^{-1}\big(|\xi_j|^s \widehat{f}(\xi)\big)(x) \quad \text{and, for } n=1,\\ &D^s f(x) := \mathcal{F}^{-1}\big(|\xi|^s \widehat{f}(\xi)\big)(x). \end{aligned} \tag{2.8}$$

Lemma 2.7. *If* $n = 1$ *and for* $f \in L^2(\mathbb{R})$*,*

$$\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}\big|D_x^{1/2}e^{it\partial_x^2}f(x)\big|^2\,dt \le C\|f\|_{L^2(\mathbb{R})}^2.$$

Let $n \ge 2$*. Then, for all* $j \in \{1,\dots,n\}$ *and* $f \in L^2(\mathbb{R}^n)$*,*

$$\sup_{x_j\in\mathbb{R}}\int_{\mathbb{R}^n}\big|D_{x_j}^{1/2}e^{it\Delta}f(x)\big|^2 dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n dt \le C\|f\|_{L^2(\mathbb{R}^n)}^2.$$

Helpful inequalities involving the Schrödinger maximal function

$$S_T^* f(x) := \sup_{0\le t\le T}|e^{it\Delta}f(x)|$$

are derived in [36] and [41]. They are reported in the next lemma.

Lemma 2.8. *The inequality*

$$\|S_T^* f\|_{L^q(\mathbb{R}^n)} \le C_T\|f\|_{H^\sigma(\mathbb{R}^n)} \tag{2.9}$$

holds if either

$$n=1 \quad \textit{and} \quad \begin{cases} q>2 \quad \textit{and} \quad \sigma \ge \max\{\frac{1}{q},\frac{1}{2}-\frac{1}{q}\},\\ q=2 \quad \textit{and} \quad \sigma > \frac{1}{2}, \end{cases} \tag{2.10}$$

or

$$n>1 \quad \textit{and} \quad \begin{cases} q\in(2+\frac{4}{(n+1)},\infty) \quad \textit{and} \quad \sigma > n(\frac{1}{2}-\frac{1}{q}),\\ q\in[2,2+\frac{4}{(n+1)}] \quad \textit{and} \quad \sigma > \frac{3}{q}-\frac{1}{2}. \end{cases} \tag{2.11}$$

With these results at hand, attention is turned to establishing dispersive blow up for nonlinear Schrödinger equations.

3. Dispersive blow up for nonlinear Schrödinger equations

In this section, the initial-value problem

$$i\partial_t u + \Delta u \pm |u|^p u = 0, \quad u\Big|_{t=0} = u_0(x), \tag{3.1}$$

for the nonlinear Schrödinger equation is considered. Here, $x \in \mathbb{R}^n$ and $p > 0$ is not necessarily an integer. Finite-time dispersive blow up was established for $n = 1$ and $p \in (0,3)$ in [9]. Our strategy to improve upon this result relies upon the theory developed in [15], where the Cauchy problem (3.1) was studied for $u_0 \in H^s(\mathbb{R}^n)$ for various values of s.

3.1. **Local well-posedness in H^s.** For $1 \le r < \infty$ and $s > 0$, define

$$H^{s,r}(\mathbb{R}^n) = \{f \in L^r(\mathbb{R}^n) : \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\widehat{f} \in L^r(\mathbb{R}^n)\}.$$

These are the standard Bessel potential spaces. According to [38], which uses the notation $L^{s,r}$ instead of $H^{s,r}$, these spaces may be characterized in the following manner. Let $s \in (0,1)$ and $\frac{2n}{(n+2s)} < r < \infty$. Then $f \in H^{s,r}(\mathbb{R}^n)$ if and only if $f \in L^r(\mathbb{R}^n)$ and

$$\mathcal{D}^s f(x) = \left(\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\, dy \right)^{1/2} \in L^r(\mathbb{R}^n). \tag{3.2}$$

The space $H^{s,r}(\mathbb{R}^n) \equiv (I-\Delta)^{-s/2}L^r(\mathbb{R}^n)$ is equipped with the norm

$$\begin{aligned}\|f\|_{L^{s,r}(\mathbb{R}^n)} = \|(I-\Delta)^{s/2}f\|_{L^r(\mathbb{R}^n)} &= \|f\|_{L^r(\mathbb{R}^n)} + \|\mathcal{D}^s f\|_{L^r(\mathbb{R}^n)} \\ &\simeq \|f\|_{L^r(\mathbb{R}^n)} + \|D^s f\|_{L^r(\mathbb{R}^n)},\end{aligned}$$

where D^s is the homogenous derivative defined in (2.8). Observe that, for $r = 2$, $H^{s,2}(\mathbb{R}^n) \equiv H^s(\mathbb{R}^n)$, the usual L^2–based Hilbert space. In addition, a straightforward calculation reveals that

$$\|\mathcal{D}^s f\|_{L^2(\mathbb{R}^n)} = c_n \||\xi|^s \widehat{f}\,\|_{L^2(\mathbb{R}^n)} \equiv c_n \|D^s f\|_{L^2(\mathbb{R}^n)}. \tag{3.3}$$

If (q,r) is an admissible pair as defined in Section 2, then the space

$$W^T_{s,n} \,=\, C\big([0,T]; H^s(\mathbb{R}^n)\big) \cap L^q\big([0,T|; H^{s,r}(\mathbb{R}^n)\big) \tag{3.4}$$

will also appear. Of course, these spaces depend on the admissible pair (q,r), but this dependence is suppressed in the notation. The following local well-posedness theorem established in [15] makes use of both the Bessel-potential spaces and the latter, spatial-temporal spaces.

Proposition 3.1. *Let $s > s_{p,n} = \frac{n}{2} - \frac{2}{p}$, $s > 0$, with s otherwise arbitrary if p is an even integer, $s < p+1$ if p is an odd integer and $[s] < p$ if p is not an integer. For given initial data $u_0 \in H^s(\mathbb{R}^n)$,*

1. *there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution*
$$u \in C\big([0,T]; H^s(\mathbb{R}^n)\big) \cap L^q\big([0,T|; H^{s,r}(\mathbb{R}^n)\big) \equiv W^T_{s,n}$$
for all pairs (q,r) admissible in the sense of Definition 2.5, and
2. *the local existence time $T = T(\|u_0\|_{H^s}) \to +\infty$, as $\|u_0\|_{H^s(\mathbb{R}^n)} \to 0$.*

Remark 3.2. The proof of this result follows from a fixed point argument based on the Strichartz estimates displayed in Lemma 2.6. Recall that the notation $[s]$ connotes the largest integer less than or equal to s.

To apply Proposition 3.1 in our context, we need to show that the class of initial data constructed in Section 2.1 (yielding dispersive blow up for the free Schrödinger evolution) admits Sobolev class regularity that will allow the use of Lemma 3.1. Via scaling and translation, dispersive blow up can be achieved at any point (x_*, t_*) in space-time, so without loss of generality, fix $(x_*, t_*) = (0,1)$ and focus upon the initial data

$$u_0(x) = \frac{e^{-4i|x|^2}}{(1+|x|^2)^m}\ , \quad \text{with } \frac{n}{4} < m \le \frac{n}{2}. \tag{3.5}$$

This initial value u_0 has Sobolev regularity explained in the next lemma.

Lemma 3.3. *Let u_0 be as depicted in* (3.5)*. Then $u_0 \in H^s(\mathbb{R}^n)$ if $2m > s + \frac{n}{2}$. In particular, if $m = \frac{n}{2}$, $s \in (0, \frac{n}{2})$ whereas if $m = \frac{n}{4}^+$, then $s = 0^+$.*

Proof. Notice first that it suffices to consider the case $0 < s < 1$. For s in this range, Propositions 1 and 2 in [32] provide the inequalities

$$|\mathcal{D}^s e^{i|x|^2}| \le c_n(1 + |x|^s)$$

and

$$\|\mathcal{D}^s(fg)\|_{L^2(\mathbb{R}^n)} \le \|f\mathcal{D}^s g\|_{L^2(\mathbb{R}^n)} + \|g\mathcal{D}^s f\|_{L^2(\mathbb{R}^n)},$$

where $\mathcal{D}^s$ is defined in (3.2). Combining these estimates with identity (3.3) and using interpolation, one arrives at the inequality

$$\begin{aligned}
&\left\|\mathcal{D}^s\left(\frac{e^{i|x|^2}}{(1+|x|^2)^m}\right)\right\|_{L^2(\mathbb{R}^n)} \\
&\le \left\|\frac{1}{(1+|x|^2)^m}\mathcal{D}^s(e^{i|x|^2})\right\|_{L^2(\mathbb{R}^n)} + \left\|\mathcal{D}^s\left(\frac{1}{(1+|x|^2)^m}\right)\right\|_{L^2(\mathbb{R}^n)} \\
&\le c_n\left\|\frac{1}{(1+|x|^2)^m}\right\|_{L^2(\mathbb{R}^n)} + c_n\left\|\frac{|x|^s}{(1+|x|^2)^m}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad + \left\|\frac{1}{(1+|x|^2)^m}\right\|_{L^2(\mathbb{R}^n)}^{1-s}\left\|D\left(\frac{1}{(1+|x|^2)^m}\right)\right\|_{L^2(\mathbb{R}^n)}^{s}
\end{aligned}$$

which is finite if and only if $2m - s > \frac{n}{2}$. The result follows □

Notice that if $m = \frac{n}{2}$, we can certainly choose the value s in the interval $(\frac{n}{2} - \frac{2}{p}, \frac{n}{2}]$ where Proposition 3.1 applies.

3.2. **Proof of dispersive blow up for nonlinear Schrödinger equations.** Here is the detailed statement of dispersive blow up for the initial-value problem (3.1), with $p \ge [\frac{n}{2}]$ if p is not an even integer.

Theorem 3.4. *Given $t_* > 0$ and $x_* \in \mathbb{R}^n$, there are initial data $u_0 \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, with $s \in (\frac{n}{2} - \frac{1}{2p}, \frac{n}{2}]$, such that*

1. *the initial-value problem* (3.1) *has a unique solution u defined in the time interval $[0, T]$ belonging to the class described in Proposition 3.1 with $T = T(\|u_0\|_{H^s}) > t_*$,*
2. *u exhibits dispersive blow up, which is to say,*

$$\lim_{(x,t)\in\mathbb{R}^n\times[0,T]\to(x_*,t_*)} |u(x,t)| = +\infty.$$

3. *Moreover, u is a continuous function of (x, t) on $\mathbb{R}^n \times ([0, T]) \setminus \{t_*\})$ and*
4. *$u(\cdot, t_*)$ is a continuous function of x on $\mathbb{R}^n \setminus \{x_*\}$.*

This theorem extends the results of [9] to the cases where $n \ge 2$ and $p \ge 3$. Notice that the nonlinearity $y \mapsto |y|^p y$ is smooth when p is an even integer. Otherwise, it has finite regularity and hence the restriction on p in those cases.

Proof. The proof is provided in detail for $p > 0$ in the case $n = 1$ and, when $n \ge 2$, for the case $p = 2k$, k a positive integer. It will be clear from the argument that the result extends to the case of $p \ge [\frac{n}{2}]$ if p is not an even integer.

As already mentioned, we may assume that the dispersive blow up for the free Schrödinger group $S(t)$ occurs at $x_* = 0$ and $t_* = 1$. Note that the same is true for initial data of the form δu_0, where u_0 is as in (3.5) with $m = \frac{n}{2}$, say, $\delta > 0$ arbitrary and s satisfying the conditions in Proposition 3.1. In view of part (3) of the latter proposition, the local existence time $T^* = T(\|\delta u_0\|_{H^s}) > 0$ can be made arbitrarily large by choosing δ sufficiently small and hence we can always achieve $T^* > 1 = t_*$.

Step 1. Take f as in (3.5) with $m = \frac{n}{2}$ and take as initial data $u_0 = \delta f$. Let $s \in (\frac{n}{2} - \frac{2}{p}]$ with $p \geq [\frac{n}{2}]$. Then s satisfies the conditions of Proposition 3.1. As noted above, by choosing δ small enough, we can be sure that the solution u of (3.1) emanating from u_0 exists and is unique in $C([0,T] : H^s)$ where $T > t_* = 1$.

Duhamel's formula allows us to represent u in the form

$$u(x,t) = e^{it\Delta}u_0(x) \pm i\int_0^t e^{i(t-s)\Delta}|u(x,s)|^p u(x,s)\,ds =: e^{it\Delta}u_0(x) \pm iI(x,t), \tag{3.6}$$

where, at least formally, $I(x,t)$ can be written as the double integral

$$I(x,t) = \frac{1}{(4i\pi t)^{n/2}}\int_0^t \int_{\mathbb{R}^n} \frac{1}{(t-s)^{\frac{n}{2}}}\exp\left(i\frac{|x-y|^2}{4(t-s)}\right)|u(y,s)|^p u(y,s)\,dy\,ds. \tag{3.7}$$

The first term on the right-hand side of (3.6) exhibits dispersive blow up at $(x_*, t_*) = (0,1)$ on account of the choice of u_0. If it turns out that I is continuous for $(x,t) \in \mathbb{R}^n \times [0,T]$, then it is immediately concluded that (3.6) (and thus (3.1)) exhibits dispersive blow up at the same point $(x_*, t_*) = (0,1)$. To show that $I(x,t)$ is continuous, it suffices to prove that it is locally bounded as a function of x and t, since then Lebesgue's dominated convergence theorem will imply I is continuous on $\mathbb{R}^n \times [0,T]$.

Step 2. Consider $n = 1$ first, since a more direct proof can be made in this case. The initial data is

$$u_0(x) = \frac{\delta e^{-i4x^2}}{(1+x^2)^{\frac{1}{2}}}, \quad x \in \mathbb{R}.$$

The function u_0 lies in $H^s(\mathbb{R})$ for any s in the range $0 \leq s < \frac{1}{2}$. Proposition 3.1 then provides a local in time solution $u \in C([0,T]; H^s(\mathbb{R}))$ to the Cauchy problem (3.1) provided that

$$0 < s < \frac{1}{2} \quad \text{and} \quad 0 < p \leq \frac{4}{1-2s}.$$

As mentioned already, T may be taken larger than 1 by choosing δ small. By Sobolev imbedding, $H^s(\mathbb{R}) \subset L^{r+1}(\mathbb{R})$ if $\frac{1}{r+1} \geq \frac{1}{2} - s$. Hence, for $s = \frac{1}{2} - \varepsilon$, where $\varepsilon > 0$ is fixed and small, it is inferred that $u \in C([0,T]; L^{r+1}(\mathbb{R}))$ for all r in the range

$$0 < r \leq \min\left\{\frac{2}{\varepsilon}, \frac{1-\varepsilon}{\varepsilon}\right\} = \frac{1-\varepsilon}{\varepsilon}.$$

As $\epsilon > 0$ was arbitrary, it follows that $u(\cdot,t) \in L^{r+1}(\mathbb{R})$, for $r \geq 1$ arbitrarily large. In consequence, $I(x,t)$ is locally bounded. Indeed, using Hölder's inequality, it is seen that

$$|I(x,t)| \leq \int_0^t \frac{1}{(t-s)^{1/2}} \left\|u(\cdot,s)\right\|_{L^{p+1}}^{p+1} ds$$
$$\leq \left(\int_0^t \frac{1}{(t-s)^{\gamma/2}} ds\right)^{\frac{1}{\gamma}} \left(\int_0^t \left\|u(\cdot,s)\right\|_{L^{p+1}}^{\gamma'(p+1)} ds\right)^{\frac{1}{\gamma'}}$$

where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ and $\gamma \in (1,2)$. Having in mind that $u \in C([0,T]; L^{p+1}(\mathbb{R}))$, it transpires that

$$|I(x,t)| \leq CT^{\frac{1}{\gamma'}} \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^{p+1}}^{\gamma'(p+1)},$$

for all $x \in \mathbb{R}$, which concludes the proof in the case $n = 1$.

Step 3. In case $n \geq 2$, the strategy employed above no longer works beause the factor $t \mapsto t^{-n/2}$ appearing in the representation formula (3.7) is no longer locally integrable. However, it will be shown in Proposition 4.1 below that the double integral I is in fact half a derivative smoother than one would naively expect. To make use of this result, choose initial data u_0 of the form (3.5) with $m \in (\frac{n}{2} - \frac{1}{4p}, \frac{n}{2}]$, where $p \geq [\frac{n}{2}]$. It is immediately inferred from Lemma 3.3 that

$$u_0 \in H^s(\mathbb{R}^n) \quad \text{for any } s \in \Big(\frac{n}{2} - \frac{1}{2p}, \frac{n}{2}\Big].$$

Notice that

$$\frac{n}{2} - \frac{1}{2p} > \frac{n}{2} - \frac{2}{p} = s_{p,n}$$

so that Proposition 3.1 applies and therefore $u \in W_{s,n}^T$ (see (3.4)) satisfies the integral form (3.6) of the nonlinear Schrödinger equation. Proposition 4.1 below then shows that

$$I \in C([0,T]; H^{s+1/2}(\mathbb{R}^n)),$$

and hence, for $s \in (\frac{n}{2} - \frac{1}{2p}, \frac{n}{2}]$, one has

$$I \in C(\mathbb{R}^n \times [0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T]).$$

The proof is complete. □

Remark 3.5. It was shown in [9, Theorem 2.1] that dispersive blow up results are stable under smooth and localized perturbations of the data. That is to say, if they hold for data u_0, then they also hold for data $u_0 + w$ where, for instance, $w \in H^\infty(\mathbb{R}^n)$. In particular the data leading to DBU do *not* need to be radially symmetric. The same is true for Theorem 3.4. The proofs of these results consists of writing the equation satisfied by w and showing that it has bounded, continuous solutions. The details follow exactly the argument given already in [9].

In addition, there is a kind of density of initial data leading to dispersive blow up. More precisely, given $u_0 \in H^s(\mathbb{R}^n)$ with $s > n/2$ and $\epsilon > 0$, there exists $\phi \in H^r(\mathbb{R}^n)$, $r \in \big(\frac{n}{2} - \frac{1}{2p}, \frac{n}{2}\big]$ with

$$\|u_0 - \phi\|_{H^r(\mathbb{R}^n)} < \epsilon, \tag{3.8}$$

such that the initial data ϕ leads to dispersive blow up in the sense of Theorem 3.4. Indeed, it suffices to take $\phi = u_0 + \delta v_0$, where v_0 leads to dispersive blow up and $\delta > 0$ is small enough that (3.8) holds. A combination of Theorem 3.4 and Proposition 3.1 then implies the above assertion.

4. Global smoothing of the Duhamel term and applications

In this section, the proof of Theorem 3.4 is completed by showing the Duhamel term in the integral representation (3.6) of the solution of the initial-value problem (3.1) is smoother than is the linear term involving only the initial data. In fact, several different results of smoothing by the Duhamel term will be developed, though the first one is enough for the dispersive blow up result in Section 3.

4.1. **Smoothing by half a derivative.** The following proposition suffices to complete the proof of Theorem 3.4.

Proposition 4.1. *Let $u_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} - \frac{1}{2p}$ with $p \geq 1$ and $[p+1] \geq s + \frac{1}{2}$ if p is not an even integer. Let $u \in W^T_{s,n}$ be the solution of (3.1) satisfying*

$$u(x,t) = e^{it\Delta}u_0(x) \pm i\int_0^t e^{i(t-s)\Delta}|u(x,s)|^p u(x,s)ds =: e^{it\Delta}u_0(x) \pm iI(x,t).$$

Then $I \in C([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^n))$.

In other words, the integral term I is "smoother" than the free propagator $e^{it\Delta}u_0$ by half a derivative. This is the key point needed in the proof of Theorem 3.4 for $n \geq 2$. In the special case wherein the nonlinearity $|u|^p u$ is smooth, so when $p = 2k, k$ an integer, Proposition 4.3 will show that the Duhamel term is almost one derivative smoother than one would expect.

Remark 4.2. The fact that the nonlinear integral term in Duhamel's formula is smoother than the linear one in certain circumstances has been used in other works on nonlinear dispersive equations. For example, in [30] it was employed to give a different proof of some of the results obtained in [8]. In [11], this smoothing effect was applied to deduce global well-posedness below the regularity index provided by the conservation laws of mass and energy. So far as we are aware, however, the result stated in Proposition 4.1 has not previously been explicitly written down.

Proof. The details are provided for the case $p = 2k, k \in \mathbb{N}$. It will be clear that the arguments extend to the case where $p \geq 1$ is not an even integer, but $p \geq [\frac{n}{2}]$ and $n \geq 2$.

Step 1. In the first step, useful estimates on $e^{it\Delta}u_0$ are derived. Start by fixing a $j \in \{1, \dots, n\}$ and noticing that

$$\sup_{0\leq t\leq T}\left\{\sup_{x_1\dots x_{j-1}x_{j+1}\dots x_n}\left\{|e^{it\Delta}u_0(x)|\right\}\right\} \lesssim \sup_{x_1\dots x_{j-1}x_{j+1}\dots x_n}\left\{|e^{it(x_j)\Delta}u_0(x)|\right\} \tag{4.1}$$

for some $t(x_j) \in [0,T]$. Thus, for $q \geq 2$ and $s > \frac{(n-1)}{q}$, it follows by Sobolev embedding that

$$\begin{aligned}\sup_{x_1\dots x_{j-1}x_{j+1}\dots x_n}|e^{it(x_j)\Delta}u_0(x)| &\lesssim \left\|e^{it(x_j)\Delta}u_0\right\|_{H^{s,q}\left(\mathbb{R}^{n-1}_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n}\right)} \\ &= c\left\|e^{it(x_j)\Delta}J^s_{\hat{j}}u_0\right\|_{L^q\left(\mathbb{R}^{n-1}_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n}\right)} \\ &\lesssim \left\|\sup_t|e^{it\Delta}J^s_{\hat{j}}u_0|\;\right\|_{L^q\left(\mathbb{R}^{n-1}_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n}\right)}\end{aligned} \tag{4.2}$$

where here and in the following,

$$J^s_{\hat{j}} = (1 - (\partial^2_{x_1} + \dots + \partial^2_{x_{j-1}} + \partial^2_{x_{j+1}} + \dots + \partial^2_{x_n}))^{s/2}$$

is defined via the associated Fourier symbol. Using (4.1) and (4.2) together with the estimates on the Schrödinger maximal function given in Lemma 2.8, there appears the inequality

$$
\begin{aligned}
&\big\|e^{it\Delta}u_0\big\|_{L^q_{x_j}(\mathbb{R};L^\infty_{x_1\dots x_{j-1}x_{j+1}\dots x_n t}(\mathbb{R}^{n-1}\times[0,T]))}\\
&\quad = \Big\| \sup_{0\le t\le T}\ \sup_{x_1\dots x_{j-1}x_{j+1}\dots x_n} |e^{it\Delta}u_0| \ \Big\|_{L^q(\mathbb{R}_{x_j})}\\
&\quad \le C_T\Big\| \sup_{0\le t\le T} |e^{it\Delta}J^s_j u_0| \ \Big\|_{L^q(\mathbb{R}^n)}\\
&\quad \lesssim \big\|u_0\big\|_{H^{\sigma+s}(\mathbb{R}^n)},
\end{aligned}
\tag{4.3}
$$

with $q\ge 2$, $s>\frac{n-1}{q}$ and $\sigma>0$ as specified in Lemma 2.8. The inequality (4.3), together with the local smoothing estimates stated in Lemma 2.7, reveal that

$$
\big\|D^{\theta/2}_{x_j}e^{it\Delta}u_0\big\|_{L^{q/(1-\theta)}_{x_j}(\mathbb{R};L^{2/\theta}_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n t}(\mathbb{R}^{n-1}\times[0,T]))} \le c_T\big\|u_0\big\|_{H^{(1-\theta)(\sigma+s)}(\mathbb{R}^n)},
\tag{4.4}
$$

where $\theta\in[0,1]$, $q\ge 2$, $s>\frac{n-1}{q}$ and $\sigma>0$ as before.

Step 2. To bound $\|D^{s+1/2}_{x_j}I\|_{L^\infty_t([0,T]:L^2(\mathbb{R}^n))}$ above, for $j\in\{1,\dots,n\}$ and $p=2k$, write

$$
\begin{aligned}
&\Big\|D^{s+1/2}_{x_j}\int_0^t e^{i(t-s)\Delta}(|u|^{2k}u)(s)\,ds\Big\|_{L^\infty([0,T];L^2(\mathbb{R}^n))}\\
&\quad \le\ c\big\|D^{s+1/2}_{x_j}(|u|^{2k}u)\big\|_{L^1([0,T];L^2(\mathbb{R}^n))}\\
&\quad \le\ cT^{1/2}\big\|D^{s+1/2}_{x_j}(|u|^{2k}u)\big\|_{L^2([0,T]\times\mathbb{R}^n)}.
\end{aligned}
\tag{4.5}
$$

To estimate the right-hand side of (4.5), the calculus of inequalities involving fractional derivatives derived in [27] is helpful. More precisely, the following inequality, which is a particular case of those proved in [27, Theorem A.8], will be used. Let $\alpha\in(0,1)$, $\alpha_1,\alpha_2\in[0,\alpha]$ with $\alpha=\alpha_1+\alpha_2$ and let $p_1,p_2,q_1,q_2\in[2,\infty)$ be such that

$$
\frac{1}{2}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}.
$$

Then

$$
\begin{aligned}
&\big\|D^\alpha_{x_j}(fg)-fD^\alpha_{x_j}f-gD^\alpha_{x_j}f\big\|_{L^2_{x_j}(\mathbb{R};L^2(Q))}\\
&\qquad \le c\big\|D^{\alpha_1}_{x_j}f\big\|_{L^{p_1}_{x_j}(\mathbb{R};L^{q_1}(Q))}\big\|D^{\alpha_2}_{x_j}g\big\|_{L^{p_2}_{x_j}(\mathbb{R};L^{q_2}(Q))},
\end{aligned}
\tag{4.6}
$$

where $Q=\mathbb{R}^{n-1}\times[0,T]$.

To illustrate the use of the inequality (4.6) in estimating the right-hand side of (4.5), assume without loss of generality that $s+\frac{1}{2}=1+\alpha$ with $\alpha\in(0,1)$. Thus, omitting the domains of integration $\mathbb{R}$ and Q,

$$
\begin{aligned}
\big\|D^{s+1/2}_{x_j}(fg)\big\|_{L^2_{x_j}L^2} &= \big\|D^{1+\alpha}_{x_j}(fg)\big\|_{L^2_{x_j}L^2} \simeq \big\|D^\alpha_{x_j}\partial_{x_j}(fg)\big\|_{L^2_{x_j}L^2}\\
&\qquad \le c\Big(\big\|D^\alpha_{x_j}(f\partial_{x_j}g)\big\|_{L^2_{x_j}L^2}\big\|D^\alpha_{x_j}(\partial_{x_j}fg)\big\|_{L^2_{x_j}L^2}\Big).
\end{aligned}
\tag{4.7}
$$

By symmetry it suffices to consider only one of the terms on the right-hand side of (4.7). From (4.6), with $\alpha_1 = 0$, there obtains

$$\begin{aligned}\big\|D^{\alpha}_{x_j}(g\partial_{x_j} f)\big\|_{L^2_{x_j}L^2} &\le \big\|D^{\alpha}_{x_j}(g\partial_{x_j} f) - gD^{\alpha}_{x_j}\partial_{x_j} f - \partial_{x_j} f D^{\alpha}_{x_j} g\big\|_{L^2_{x_j}L^2}\\ &\quad + \big\|gD^{\alpha}_{x_j}\partial_{x_j} f\big\|_{L^2_{x_j}L^2} + \big\|\partial_{x_j} f D^{\alpha}_{x_j} g\big\|_{L^2_{x_j}L^2}\\ &\le \big\|gD^{\alpha}_{x_j}\partial_{x_j} f\big\|_{L^2_{x_j}L^2} + c\big\|\partial_{x_j} f\big\|_{L^{p_1}_{x_j}L^{q_1}}\big\|D^{\alpha_2}_{x_j} g\big\|_{L^{p_2}_{x_j}L^{q_2}}\end{aligned}$$

with p_1, p_2, q_1, q_2 restricted as above.

Using the latter inequality to continue the inequality (4.5) yields

$$\begin{aligned}&\big\|D^{s+1/2}_{x_j} I\big\|_{L^\infty_t([0,T]:L^2(\mathbb{R}^n))}\\ &\qquad\le\ cT^{1/2}\big\|D^{s+1/2}_{x_j}(|u|^{2k}u)\big\|_{L^2([0,T]\times\mathbb{R}^n)}\\ &\qquad\le\ cT^{1/2}\Big(\big\|u\big\|^{2k}_{L^{4k}_{x_j}(\mathbb{R};L^\infty_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n,t}(\mathbb{R}^{n-1}\times[0,T]))}\\ &\qquad\qquad\times\big\|D^{s+1/2}_{x_j} u\big\|_{L^\infty_{x_j}(\mathbb{R};L^2_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n,t}(\mathbb{R}^{n-1}\times[0,T]))} + R\Big),\end{aligned}\tag{4.8}$$

where the remainder R includes only estimates for terms involving powers of u, $\partial_{x_j} u$ and $D^{\alpha}_{x_j} u$. These are straightforwardly bounded above by use of (4.4). In fact, a bound for them is an interpolation between the first two terms on the right-hand side of (4.8). It therefore remains to bound only the terms

$$\big\|u\big\|^{2k}_{L^{4k}_{x_j}(\mathbb{R};L^\infty_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n,t}(\mathbb{R}^{n-1}\times[0,T]))}\tag{4.9}$$

and

$$\big\|D^{s+1/2}_{x_j} u\big\|_{L^\infty_{x_j}(\mathbb{R};L^2_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n,t}(\mathbb{R}^{n-1}\times[0,T]))},\tag{4.10}$$

$j = 1, \cdots, n$, appearing in (4.8).

Step 3. To bound the quantity appearing in (4.9), first note that (4.3) implies

$$\Big\|\sup_{0\le t\le T}\ \sup_{x_1\dots x_{j-1}x_{j+1}\dots x_n}|e^{it\Delta}u_0|\ \Big\|_{L^{4k}_{x_j}(\mathbb{R})} \lesssim \Big\|\sup_{0\le t\le T}|e^{it\Delta}J^s_{\hat{j}}u_0|\ \Big\|_{L^{4k}(\mathbb{R}^n)},$$

with $s > \frac{(n-1)}{4k}$. This estimate can be extended using Lemma 2.8 by observing that

$$\Big\|\sup_{0\le t\le T}|e^{it\Delta}J^s_{\hat{j}}u_0|\ \Big\|_{L^{4k}(\mathbb{R}^n)} \le \big\|J^s_{\hat{j}}u_0\big\|_{H^\sigma(\mathbb{R}^n)} \lesssim \big\|u_0\big\|_{H^{\sigma+s}(\mathbb{R}^n)} = \big\|u_0\big\|_{H^{s^*}(\mathbb{R}^n)},$$

where

$$s^* = s+\sigma > \frac{n-1}{4k} + n\Big(\frac12 - \frac{1}{4k}\Big) = \frac{n}{2} - \frac{1}{4k}.$$

Inserting this inequality in the Duhamel representation (3.6) with

$$s \in \Big(\frac{n}{2} - \frac{1}{4k}, \frac{n}{2}\Big] = \Big(\frac{n}{2} - \frac{1}{2p}, \frac{n}{2}\Big],$$

it follows that

$$\big\|u\big\|_{L^{4k}_{x_j}(\mathbb{R};L^\infty_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n t}(\mathbb{R}^{n-1}\times[0,T]))} \le C\big\|u_0\big\|_{s,2} + \big\|J^s(|u|^p u)\big\|_{L^1_t([0,T];L^2(\mathbb{R}^n))}.$$

Since $p = 2k$, use of a fractional Leibniz rule (see [26]) implies that

$$\big\|J^s(|u|^{2k}u)\big\|_{L^1_t([0,T];L^2(\mathbb{R}^n))} \le c\big\|u\big\|^{2k}_{L^{2k}_t([0,T];L^\infty(\mathbb{R}^n))}\big\|J^s u\big\|_{L^\infty_t([0,T];L^2(\mathbb{R}^n))}.\tag{4.11}$$

If it was known that

$$\|u\|_{L_t^{2k}([0,T];L^\infty(\mathbb{R}^n)))} \le cT^\theta \|J^s u\|_{L_t^q([0,T];L^r(\mathbb{R}^n))} \tag{4.12}$$

for some $\theta > 0$ and for some admissible Strichartz pair (r, q), then the sequence of inequalities could be closed. To obtain (4.12), recall that

$$s > \frac{n}{2} - \frac{1}{4k} = \frac{2kn-1}{4k} = \frac{n}{4kn/(2kn-1)},$$

so we can take

$$r = \frac{4kn}{(2kn-1)} < \frac{2n}{(n-2)}, \quad \text{if } n \ge 3,$$

(the case $n = 1, 2$ is immediate) and $q = 8k$. By Sobolev embedding, the inequality (4.12) is seen to hold with $\theta = \frac{3}{8k}$. In summary, all the terms in (4.9) are shown to be bounded.

Step 4. Finally, attention is turned to terms of the form appearing in (4.10). The local smoothing estimate enunciated in Lemma 2.7 together with Duhamel's formula imply that

$$\begin{aligned}
&\left\|D_{x_j}^{s+1/2} u\right\|_{L_{x_j}^\infty(\mathbb{R}; L^2_{x_1\cdots x_{j-1}x_{j+1}\cdots x_n,t}(\mathbb{R}^{n-1}\times[0,T]))} \\
&\qquad \le \left\|u_0\right\|_{H^s(\mathbb{R}^n)} + \left\|J^s(|u|^{2k}u)\right\|_{L_t^1([0,T];L^2(\mathbb{R}^n))}.
\end{aligned}$$

The right-hand side was already estimated in (4.11)–(4.12). Because of (4.8), this shows that there exists a $C = C(T, n) > 0$ such that

$$\left\|D_{x_j}^{s+1/2} I\right\|_{L_t^\infty([0,T]:L^2(\mathbb{R}^n))} \le C.$$

Summing these estimates over j for $j = 1 \cdots, n$ yields the result advertised in the proposition.

Finally, we remark that in the case where p is not an even integer, one needs to supplement the Leibnitz-type inequality (4.6) with the chain rule for fractional derivatives adduced in the Appendix of [27] together with the restriction on p. □

4.2. **An even stronger smoothing property.** For large s and higher values of p, a stronger smoothing result than that established in Proposition 4.1 holds.

Proposition 4.3. *Let $u_0 \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2} - \frac{1}{2p}$ with $p \ge 2$ and $[p+1] \ge s + \frac{1}{2}$ if p is not an even integer. Under these hypothses, it follows that for any $\varepsilon > 0$,*

$$I \in C([0,T]; H^{s+1-\varepsilon}(\mathbb{R}^n)),$$

where the notation is taken from Proposition 4.1

Remark 4.4. The loss of ε in the regularity of I is needed to obtain a factor $T^{\delta(\varepsilon)}, \delta(\varepsilon) > 0$, on the right-hand side of the inequalities below which allows them to be closed. It can be recovered by assuming that the data u_0 is small enough in $H^s(\mathbb{R}^n)$.

The proof of Proposition 4.3 uses the following smoothing estimate, which is a direct consequence of Lemma 2.7, a duality argument and the Christ-Kiselev lemma [16]. For a proof, see [29, Chapter 4].

Lemma 4.5. *For any $n \in \mathbb{N}$, the inequality*

$$\left\| D_{x_j}^{1/2} \int_0^t e^{i(t-s)\Delta} f(\cdot, s)\, ds \right\|_{L_t^\infty(\mathbb{R}; L_x^2(\mathbb{R}^n))} \leq C \left\| \mathcal{H}_j f \right\|_{L^1_{x_j}(\mathbb{R}; L^2_{x_1 \cdots x_{j-1} x_{j+1} \cdots x_n t}(\mathbb{R}^n))}$$

holds, where $\mathcal{H}_j$ denotes the Hilbert transform in the j-th variable, which is to say,

$$\mathcal{H}_j f(x) := -i\, \mathcal{F}^{-1}\Big(\operatorname{sign}(\xi_j) \widehat{f}(\xi) \Big)(x).$$

Proof of Proposition 4.3. The proof is similar to that of Proposition 4.1 and hence, we only sketch the main differences.

First consider the case of data $u_0 \in H^s(\mathbb{R}^n)$ which is small, so that all the norms involved are indeed "small". We want to show that the integral term I in Duhamel's formula is one order smoother in the Sobolev scale $C([0,T]; H^s(\mathbb{R}^n))$ than the the free propagation $e^{it\Delta}u_0$. To this end, apply Lemma 4.5 together with the commutator estimate in [27, Theorem A.13] to write

$$\begin{aligned} &\left\| \mathcal{H}_j D_{x_j}^{s+1} I \right\|_{L^\infty([0,T]; L^2(\mathbb{R}^n))} \\ &\quad\lesssim \left\| D_{x_j}^{s+1/2}(|u|^{2k}u) \right\|_{L^1_{x_j}(\mathbb{R}; L^2_{x_1 \cdots x_{j-1} x_{j+1} \cdots x_n t}(\mathbb{R}^{n-1} \times [0,T]))} \\ &\quad\lesssim \Big(\|u\|^{2k}_{L^{2k}_{x_j}(\mathbb{R}; L^2_{x_1 \cdots x_{j-1} x_{j+1} \cdots x_n t}(\mathbb{R}^{n-1} \times [0,T]))} \\ &\qquad \times \| D_{x_j}^{s+1/2} u \|_{L^\infty_{x_j}(\mathbb{R}; L^2_{x_1 \cdots x_{j-1} x_{j+1} \cdots x_n t}(\mathbb{R}^{n-1} \times [0,T]))} + R \Big). \end{aligned} \tag{4.13}$$

To estimate the two explicit quantities on the right-hand side of the last inequality, one uses arguments similar to those given in the proof of Proposition 4.1. The estimates for the remainder terms represented by R then follow by interpolation of the previous estimates. Since the terms on the right-hand side of (4.13) are quadratic and each factor is small, one can close the estimate and get the desired result, but only provided that u_0 is sufficiently small.

For data $u_0 \in H^s(\mathbb{R}^n)$ of arbitrary size one gives up ε-amount of spatial smoothing for a little temporal smoothing, thereby obtaining the factor $T^{\delta(\epsilon)}$, $\delta(\varepsilon) > 0$ on the right-hand side. The right-hand side of the estimate then has lower homogeneity than the left side and the prrof proceeds. $\square$

4.3. **Extension to the case of non-elliptic Schrödinger equations.** The results above extend to the case of non-elliptic, non-degenerate nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta_{\mathrm{H}} u \pm |u|^p u = 0, \quad u\big|_{t=0} = u_0(x), \tag{4.14}$$

where

$$\Delta_{\mathrm{H}} := \partial^2_{x_1} + \dots \partial^2_{x_j} - \partial^2_{x_{j+1}} \dots - \partial^2_{x_n}.$$

Proposition 4.6. *The result of Theorem 3.4 also holds for the initial-value problem delineated in* (4.14) .

Proof. Remark first that for initial data of the form

$$\widetilde{u}_0(x) = \frac{e^{-i\alpha\big((x_1-q_1)^2 + \dots + (x_j-q_j)^2 - (x_{j+1}-q_{j+1})^2 - \dots - (x_n-q_n)^2\big)}}{(1+|x|^2)^m}, \tag{4.15}$$

with $n/4 < m \leq n/2$, the solution of the initial-value problem associated to the linearization of (4.14) around the rest state, i.e.,

$$u(x,t) = e^{it\Delta_H}\widetilde{u}_0(x),$$

satisfies the conclusions of Lemma 2.1, and in particular, blows up at the point $\widetilde{q} = (q_1, \dots, q_j, -q_{j+1}, \dots, -q_n)$.

Next, notice that the the local well-posedness result stated in Proposition 3.1, is solely based on Strichartz estimates, which are exactly the same for the two groups $\{e^{it\Delta}\}_{t\in\mathbb{R}}$ and $\{e^{it\Delta_{\rm H}}\}_{t\in\mathbb{R}}$ (*cf.* [22]). In other words, Lemma 2.6 also holds in the non-elliptic case. Together with Sobolev embeddings, this yields a unique solution $u \in W^T_{s,n}$ to (4.14), by the same arguments as in [15]. Furthermore, the local smoothing estimates stated in Lemma 2.7 carry over to the non-elliptic situation.

Finally, for the boundedness of the associated maximal function (see Lemma 2.8), we point to the estimate

$$\Big\| \sup_{0\leq t\leq T} \big| e^{it(\partial^2_{x_1} - \partial^2_{x_2})} f \big| \Big\|_{L^4(\mathbb{R}^n)} \leq C_T \Big\| D^{1/2}_x f \Big\|_{L^2(\mathbb{R}^2)}, \tag{4.16}$$

proved in [35, Theorem 2.6] and observe that the same argument used there to establish (4.16) shows that

$$\Big\| \sup_{0\leq t\leq T} \big| e^{it\Delta_{\rm H}} f \big| \Big\|_{L^4(\mathbb{R}^n)} \leq C_T \Big\| D^{n/4}_x f \Big\|_{L^2(\mathbb{R}^n)},$$

which is the desired estimate. The result then follows along the same lines as given in the proof of Theorem 3.4. □

An important consequence of this is the possibility of dispersive blow up for the Davey-Stewartson system in the "elliptic/elliptic" or "hyperbolic/elliptic" cases (see [22] for more details about these systems, in particular for theory of local well-posedness). This system arose originally as an approximate description of surface gravity-capillary waves in shallow water, but has other applications as well.

Corollary 4.7. *For $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ consider the Davey-Stewartson system*

$$\begin{cases} i\partial_t u \pm \partial^2_{x_1} u + \partial^2_{x_2} u = \alpha |u|^2 u + \beta u \partial_{x_1}\phi, & x = (x_1, x_2) \in \mathbb{R}^2, \\ \Delta\phi = \partial_{x_1} |u|^2. \end{cases} \tag{4.17}$$

Then there exist initial values $u_0 \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$, with $s \in (\frac{1}{2}, 1]$ such that the solution u of (4.17) *with the initial condition $u(0) = u_0$ exhibits dispersive blow up.*

Proof. Rewrite the system (4.17) as a single equation

$$i\partial_t u \pm \partial^2_{x_1} u + \partial^2_{x_2} u = \alpha |u|^2 u - \beta u \partial^2_{x_1} (-\Delta)^{-1}(|u|^2),$$

in the usual way. The latter equation has the equivalent form

$$i\partial_t u \pm \partial^2_{x_1} u + \partial^2_{x_2} u = \alpha |u|^2 u - \beta u R_1 R_1(|u|^2),$$

where

$$R_1 f(x_1, x_2) := \mathcal{F}^{-1} \left(i \frac{\xi_1}{|\xi|} \widehat{f}(\xi_1, \xi_2) \right) (x_1, x_2)$$

denotes the 1-Riesz transform in $\mathbb{R}^2$

From the L^p-continuity of the Riesz transform, it is clear that the result in Proposition 3.1 and the argument entailed in the proof of Proposition 4.1 still hold. Hence, Theorem 3.4 extends to solutions of the system (4.17). □

5. Dispersive blow up in the Gross-Pitaevskii equation

In this section, the discussion is moved to the initial-value problem for the Gross-Pitaevskii equation,

$$i\partial_t\psi + \Delta\psi + (1-|\psi|^2)\psi = 0, \quad \psi\big|_{t=0} = \psi_0(x). \tag{5.1}$$

Here $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and ψ is subject to the boundary condition

$$\lim_{|x|\to\infty} \psi(x,t) = 1, \quad \text{for all } t \in \mathbb{R}. \tag{5.2}$$

The Gross-Pitaevskii equation arises, for example, in the description of Bose-Einstein condensates, superfluid helium He^2 and, in one spatial dimension, as a model for light propagation in a fiber optics cable (see for instance the survey article [7] and other articles in the same volume).

Remark 5.1. An important conserved quantity of (5.1) is the Ginzburg-Landau energy, defined by

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^n} (1-|\psi|^2)^2\,dx.$$

This invariant indicates a natural energy space

$$\mathcal{E}(\mathbb{R}^n) = \{\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^n), E(\psi) < +\infty\}$$

for the Gross-Pitaevskii equation (see [20, 21] for more details).

5.1. **A reformulation of the Gross-Pitaevskii Equation.** Due to the non-zero boundary condition (5.2) at infinity, the ideas that worked in the earlier sections do not apply directly to show dispersive blow up. To prove dispersive blow up for the Gross-Pitaevskii equation, make the change

$$\psi(t,x) = 1 + u(t,x),$$

of the dependent variable so that $u \in L^2(\mathbb{R}^n)$ describes the deviation from the steady state. In terms of u, the Gross-Pitaevskii equation (5.1) becomes

$$i\partial_t u + \Delta u - 2\mathrm{Re}\, u = F(u), \quad u\big|_{t=0} = \psi_0(x) - 1, \tag{5.3}$$

where

$$F(u) = u^2 + 2|u|^2 + |u|^2u.$$

Even the transformed initial-value problem (5.3) does not fall directly to the general lines of argument proposed earlier, due to the appearance of the slightly odd, $\mathbb{R}$-linear term $-2\mathrm{Re}\, u$, which at least in principle might cancel out the effect of dispersive blow up stemming from the Laplacian.

This obstacle will be surmounted by using the further reformulation developed by Gustafson, Nakanishi and Tsai in [23, 24]. Following them, we introduce the Fourier multipliers

$$A := \sqrt{-\Delta(2-\Delta)} \quad \text{and} \quad B := \sqrt{-\Delta(2-\Delta)^{-1}}\,. \tag{5.4}$$

These operators satisfy $A = -\Delta B^{-1} = (2-\Delta)B = \sqrt{-\Delta(2-\Delta)}$. Next, define the $\mathbb{R}$-linear operator

$$\Upsilon u := B\,\mathrm{Re}\, u + i\,\mathrm{Im}\, u \equiv Bu_1 + iu_2,$$

where $u = u_1 + iu_2$. One checks (see [23, 24]) that the left-hand side of the equation in (5.3) can be written in the form

$$i\partial_t u + \Delta u - 2\mathrm{Re}\, u = \Upsilon(i\partial_t - A)\Upsilon^{-1}u. \tag{5.5}$$

Denote by v the function

$$v := \Upsilon^{-1}u \equiv \Upsilon^{-1}(u_1 + iu_2) = B^{-1}u_1 + iu_2$$

and rewrite (5.3) as

$$i\partial_t v - Av = \Upsilon^{-1}F(u), \quad v\big|_{t=0} = \Upsilon^{-1}(\psi_0 - 1). \tag{5.6}$$

The associated free evolution is

$$w(x,t) = e^{-itA}v_0(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{it\sqrt{|\xi|^2(|\xi|^2+2)}}\widehat{u}_0(\xi)e^{i\xi\cdot x}\,d\xi,$$

which can be represented as follows.

Lemma 5.2. *For any $f \in L^2(\mathbb{R}^n)$ and any $t \neq 0$, the group $\{e^{-itA}\}_{t\in\mathbb{R}}$ has the representation*

$$e^{-itA}f(x) = (G(\cdot,t) * f)(x),$$

where the kernel is $G = G_1 + G_2$ with

$$G_1(x,t) = \frac{e^{it}}{(4\pi it)^{n/2}}\, e^{\frac{i|x|^2}{4t}} \quad \text{and} \quad G_2(x,t) := \frac{e^{it}}{(2\pi)^n}\int_{\mathbb{R}^n} \kappa(\xi,t)e^{it|\xi|^2}e^{ix\cdot\xi}d\xi.$$

Here, the convolution is with respect to the spatial variable over $\mathbb{R}^n$ and the kernel κ is

$$\kappa(\xi,t) = 2i\sin\Big(\frac{t}{2}r(\xi)\Big)e^{i\frac{t}{2}r(\xi)}$$

with

$$r(\xi) = \frac{-2|\xi|^2}{\Big(\sqrt{|\xi|^2(|\xi|^2+2)} + |\xi|^2\Big)^2} \sim O(|\xi|^{-2}), \ \text{as } |\xi| \to \infty.$$

Moreover, r lies in $C_{\mathrm{b}}(\mathbb{R}^n)$ and is smooth away from the origin.

Note that G_1 is the usual Schrödinger group multiplied by e^{it}. Remark also that, as $|\xi| \to \infty$, $\kappa(\cdot,t)$ decays to zero like $|\xi|^{-2}$, uniformly on compact time-intervals.

Proof. The convolution kernel G is

$$G(x,t) = \mathcal{F}^{-1}\Big(e^{it\sqrt{(|\xi|^2(|\xi|^2+2)}}\Big)(x,t).$$

Observe that

$$\sqrt{(|\xi|^2(|\xi|^2+2)} = |\xi|^2 + a(\xi),$$

where

$$a(\xi) = \frac{2|\xi|^2}{\sqrt{(|\xi|^2(|\xi|^2+2)} + |\xi|^2} = 1 - \frac{2|\xi|^2}{\Big(\sqrt{(|\xi|^2(|\xi|^2+2)} + |\xi|^2\Big)^2} = 1 + r(\xi).$$

Consequently, it follows that

$$e^{it\sqrt{(|\xi|^2(|\xi|^2+2)}} = e^{it|\xi|^2}e^{it}e^{itr(\xi)} = e^{it}e^{it|\xi|^2}(1 + f_t(|\xi|)),$$

where

$$f_t(|\xi|) = 2i\sin\Big(\frac{t}{2}r(\xi)\Big)e^{it\frac{r(\xi)}{2}}$$

is continuous, smooth on $\mathbb{R}^n$ and decays to zero like $|\xi|^{-2}$, as $|\xi| \to \infty$, uniformly on compact time intervals in $(0, \infty)$, since $r(|\xi|)$ does so. This allows the propagator in Fourier space to be written as

$$\widehat{G}(\xi, t) = e^{it\sqrt{|\xi|^2(|\xi|^2+2)}} = e^{it} e^{it|\xi|^2}(1 + \kappa(\xi, t))$$

with κ as above. □

With this representation in hand, dispersive blow up for the evolutionary group $\{e^{-itA}\}_{t\in\mathbb{R}}$ can be established for at least spatial dimensions less than or equal to three.

Lemma 5.3. *Let $n \leq 3$. Given $x_* \in \mathbb{R}^n$ and $t_* > 0$, there exist initial data $v_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that*

$$w(\cdot, t) = e^{-itA} v_0 \in L^2(\mathbb{R}^n),$$

exhibits dispersive blow up at (x_*, t_*).

Proof. Choose $(x_*, t_*) = (0, \frac{1}{4})$ without loss of generality. As before, let

$$v_0(x) = \frac{e^{-i|x|^2}}{(1 + |x|^2)^m}, \quad \text{with } \frac{n}{4} < m \leq \frac{n}{2}.$$

Using Lemma 5.2, it is found that

$$e^{-itA} v_0(x) = (G_1(\cdot, t) * v_0)(x) + (G_2(\cdot, t) * v_0)(x).$$

In view of the calculations in Section 2.1 it is inferred that the first term on the right-hand side will exhibit dispersive blow up at $(0, \frac{1}{4})$. It thus suffices to show that $G_2(\cdot, t) * v_0 \in C_{\rm b}(\mathbb{R}^n)$, uniformly on compact time-intervals.

The kernel $G_2(\cdot, t) \in L^2(\mathbb{R}^n)$ for $n \leq 3$, due to the decay properties of $r(\xi)$. Since $v_0 \in L^2(\mathbb{R}^n)$ by construction, the product $\widehat{G}_2(\cdot, t)\widehat{v}_0 \in L^1(\mathbb{R}^n)$, uniformly on compact time-intervals. The Riemann-Lebesgue lemma then implies the desired result. □

Note that even though e^{-itA} represents the linear evolution operator associated to (5.6), it includes part of the nonlinearity of the original Gross-Pitaevskii equation (5.1), as can be seen from (5.5).

5.2. **Proof of dispersive blow up for Gross-Pitaevskii.** Lemma 5.3 together with the results of Section 3 now allow us to prove dispersive blow up for equation (5.6) and, consequently, also for the original Gross-Pitaevskii equation (in physically relevant dimensions $n = 1, 2, 3$). To effect a proof, the following lemma on the mapping properties of B and its inverse will be used.

Lemma 5.4. *The operator B in* (5.4) *may be written in the form*

$$B = \mathrm{Id} + B_1, \quad B^{-1} = \mathrm{Id} + B_2,$$

where B_1, $B_2 \in \mathcal{L}(H^s(\mathbb{R}^n), H^{s+2}(\mathbb{R}^n))$, for all $s \in \mathbb{R}$.

Proof. The proof is a simple computation on the level of the Fourier symbols b_1 and b_2 associated to B_1 and B_2. For example, the symbol b_1 is

$$b_1(\xi) = -\frac{2}{(2 + |\xi|^2)(\sqrt{|\xi|^2(2 + |\xi|^2)^{-1}} + 1)}.$$

From this formula, one sees that b_1 is in fact uniformly bounded and is $O(|\xi|^{-2})$ as $|\xi| \to \infty$. Similarly, one finds

$$b_2(\xi) = \frac{2}{(2+|\xi|^2)\sqrt{|\xi|^2(2+|\xi|^2)^{-1}}[1+(|\xi|^2(2+|\xi|^2)^{-1}]^{1/2}},$$

from which one infers the assertion about B_2. □

In particular, since $\Upsilon^{-1}u = B^{-1}u_1 + iu_2$, this implies that, for all $s \in \mathbb{R}$, $\Upsilon^{-1} \in \mathcal{L}(H^s(\mathbb{R}^n), H^s(\mathbb{R}^n))$.

Here is the main result of this section.

Theorem 5.5. *Let $n \le 3$. Given $x_* \in \mathbb{R}^n$, $t_* > 0$, there exist initial data $v_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that the corresponding solution of* (5.3) *exhibits dispersive blow up at (x_*, t_*).*

Proof. We follow the same strategy as for the nonlinear Schrödinger equation and rewrite the solution of (5.6) as

$$v(x,t) = e^{-itA}v_0(x) + i\int_0^t e^{-i(t-s)A}\Upsilon^{-1}F(u(x,s))\,ds \tag{5.7}$$

using Duhamel's formula, where the nonlinearity is explicitly given by

$$\Upsilon^{-1}F(u) = B^{-1}(3u_1^2 + u_2^2 + |u|^2u_1) + iu_2(2u_1 + |u|^2)$$

since $u = u_1 + iu_2$. In view of Lemma 5.3, the first term on the right-hand side of (5.7) exhibits dispersive blow up at $(x_*, t_*) = (0, \frac{1}{4})$ provided $n < 4$. The advertised result will be in hand when the second term is known to be uniformly bounded.

The Cazenave-Weissler theory implies that corresponding to initial data $u_0 \in H^s(\mathbb{R}^n)$, the initial-value problem (5.3) has a local solution $u \in C([0,T], H^s(\mathbb{R}^n))$ for some values of $s < \frac{n}{2}$. The integral term in (5.7) splits into $I_1 + I_2$ where

$$I_j(x,t) = \int_0^t G_j(\cdot, t-s) * \Upsilon^{-1}F(u(\cdot,s))\,ds, \quad j = 1,2,$$

corresponding to the decomposition $G = G_1 + G_2$. The fact that I_2 is a bounded continuous function can be concluded by the same argument as in the proof of Lemma 5.3. For $n < 4$, $G_2(\cdot,t) \in L^2(\mathbb{R}^n)$, and so is $\Upsilon^{-1}F(u)$ because $u \in H^s(\mathbb{R}^n)$ and $\Upsilon^{-1} \in \mathcal{L}(H^s(\mathbb{R}^n), H^s(\mathbb{R}^n))$.

On the other hand, I_1 is given by

$$I_1(x,t) = \int_0^t G_1(\cdot, t-s) * \big(B^{-1}[3u_1^2 + u_2^2 + |u|^2u_1] + u_2(2u_1 + |u|^2)\big)(\cdot, s)\,ds.$$

Inasmuch as G_1 is, up to a multiplicative constant, the fundamental solution of the usual linear Schrödinger equation (see Lemma 5.2), the double integral I_1 is of the same form (up the the appearance of B^{-1}) as the usual nonlinearity in the Duhamel representation of the nonlinear Schrödinger equation. In view of the mapping properties of B^{-1}, the desired result of boundedness and continuity then follows from the corresponding proof for the usual nonlinear Schrödinger equation given in Sections 3.2 and 4. □

As a corollary, we infer the appearance of dispersive blow up for the original Gross-Pitaevskii equation (5.1) in physically relevant dimensions.

Corollary 5.6. *Let $n \leq 3$. Given $x_* \in \mathbb{R}^n$, $t_* > 0$, there exist smooth and bounded initial data $\psi_0 \in C_{\mathrm{b}}(\mathbb{R}^n)$ with $\psi_0 - 1 \in L^2(\mathbb{R}^n)$, such that the solution ψ exhibits dispersive blow up at (x_*, t_*).*

Proof. To establish dispersive blow up for (5.1), the results for v need to be transferred back to the variable ψ. In search of such a conclusion, note that the initial data in (5.6) is given by

$$v_0(x) = \Upsilon^{-1}(\psi_0 - 1) = B^{-1}(\operatorname{Re}\psi_0 - 1) + i \operatorname{Im}\psi_0.$$

The specific choice $v_0(x) = \frac{e^{-i|x|^2}}{(1+|x|^2)^m}$ then corresponds to the initial data

$$\psi_0(x) = 1 + B \operatorname{Re} v_0(x) + i \operatorname{Im} v_0(x) = 1 + B\left(\frac{\cos|x|^2}{(1+|x|^2)^m}\right) + \frac{i \sin|x|^2}{(1+|x|^2)^m}$$

for (5.1). Since $B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, it is clear from the last formula that $\psi_0 - 1 \in L^2(\mathbb{R}^n)$. By Lemma 5.4, $B = \mathrm{Id} + B_1$ with $B_1 \in \mathcal{L}(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n))$ and $H^2(\mathbb{R}^n) \subset C_{\mathrm{b}}(\mathbb{R}^n)$ for $n \leq 3$, so it is concluded that $\psi_0 \in C_{\mathrm{b}}(\mathbb{R}^n)$.

Now, if $v = v_1 + iv_2$ is a solution of (5.6) with the dispersive blow up property at a point (x_*, t_*) provided by Theorem 5.5, the corresponding $u = u_1 + iu_2$ is given by

$$u_1 = B^{-1}v_1 = (I + B_2)v_1, \quad u_2 = v_2.$$

Since B_2 is a smoothing operator, u, and thus $\psi = 1 + u$ satisfies the DBU property at the same point (x_*, t_*). $\square$

6. Higher-order nonlinear Schrödinger equations

In this final section, we indicate how results of dispersive blow up can be extended to higher-order nonlinear Schrödinger equations. It is mathematically natural to inquire whether or not higher-order terms destroy dispersive blow up, but the practical motivation for considering such an extension is perhaps even more telling. In nonlinear optics, third and fourth-order Schrödinger-type equations frequently appear in the description of various wave phenomena. In particular, the analysis of optical rogue-wave formation has been based on higher-order nonlinear Schrödinger equations (see, for example, [17, 18, 31, 40]).

As the ideas and even much of the technical detail mirror closely what has gone before, we content ourselves with admittedly sketchy indications of how the theory is developed. The one point which would require serious new effort has to do with an appropriate generalization of the Cazenave–Weissler theory in [15] to a higher-order setting. This is not attempted here, but is deserving of further investigation at a later stage.

6.1. Fourth-order nonlinear Schrödinger equation.

In this subsection, initial-value problems for fourth-order nonlinear Schrödinger equations of the form

$$i\partial_t u + \alpha\Delta u + \beta\Delta^2 u + \lambda|u|^p u = 0, \quad u\big|_{t=0} = u_0(x), \tag{6.1}$$

are considered. Here, the parameters α, β, λ are real constants, with $\beta \neq 0$. Theory for this initial-value problem can be found, for example, in [19, 34] and in the references cited in these works. If $\alpha = 0$, the partial differential equation is often referred to as the bi-harmonic NLS equation (see, e.g., [2]). A simple scaling allows us to assume $\beta = 1$ and to consider only the values $\alpha \in \{0, -1, +1\}$, though time may need to be reversed.

To establish dispersive blow up for (6.1), the dispersive properties of the associated linear equation

$$i\partial_t u - \alpha\Delta u + \Delta^2 u = 0, \quad \alpha \in \{0, -1, +1\} \tag{6.2}$$

are helpful, just as for the lower-order cases. As should be clear from the preceding theory, the possibility of dispersive blow up for (6.2) is linked to the dispersive properties of the fundamental solution

$$\Sigma_\alpha(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{it(|\xi|^4+\alpha|\xi|^2)+ix\cdot\xi}\, dx \tag{6.3}$$

of (6.2), which have in fact been established already in [3].

Lemma 6.1. *Let Σ_α be as in (6.3) and $\mu \in \mathbb{N}^n$ a multi-index.*

(1) *If $\alpha = 0$, there exists a $C > 0$ such that for $x \in \mathbb{R}^n$ and $t > 0$,*

$$|\partial^\mu \Sigma_0(x,t)| \le C t^{-(n+|\mu|)/4} \left(1 + \frac{|x|}{t^{1/4}}\right)^{-(n-|\mu|)/3}.$$

(2) *For $t > 0$ and either $t \le 1$ or $|x| \ge t$, there exists a $C > 0$ such that*

$$|\partial^\mu \Sigma_\alpha(x,t)| \le C t^{-(n+|\mu|)/4} \left(1 + \frac{|x|}{t^{1/4}}\right)^{-(n-|\mu|)/3}$$

for $\alpha = \pm 1$.

Strichartz estimates then follow pretty much directly from Proposition 6.1. In some detail, we say that the pair (q,r) is admissible for the fourth-order Schrödinger group $\{e^{it(\Delta^2-\alpha\Delta)}\}_{t\in\mathbb{R}}$ if

$$\frac{1}{q} = \frac{n}{4}\left(\frac{1}{2} - \frac{1}{r}\right), \tag{6.4}$$

for $2 \le r \le \frac{2n}{n-2}$ if $n \ge 3$, respectively, $2 \le r \le \infty$ if $n = 1$ and $2 \le r < \infty$ if $n = 2$. Using this, one has the following estimates, which are the fourth-order counterpart to the ones established in Lemma 2.6. In what follows $T = +\infty$ when $\alpha = 0$ and T is any nonnegative number when $\alpha = \pm 1$.

Lemma 6.2 ([3])**.** *Let (q,r) be admissible in the sense of (6.4). Then there exists $c = c(n,r,T)$ such that*

$$\left\| e^{it(\Delta^2-\alpha\Delta)} f \right\|_{L^q((-T,T);L^r(\mathbb{R}^n))} \le c\|f\|_{L^2(\mathbb{R}^n)}.$$

The linear operator

$$\Phi f = \int_0^t e^{i(t-s)(\Delta^2-\alpha\Delta)} f(s)ds$$

is bounded in the sense that

$$\|\Phi f\|_{L^q((-T,T);L^r(\mathbb{R}^n))} \le c\|f\|_{L^{q'}((-T,T);L^{r'}(\mathbb{R}^n))},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.

If we assume for the moment that dispersive blow up holds true for the linear model (6.2), then the Strichartz estimates above are already sufficient to prove dispersive blow up for the nonlinear equation (6.1) in the physically relevant dimensions $n \le 3$.

Proposition 6.3. *Let $n \leq 3$ and $\alpha \in \{0, -1, +1\}$. Assume that the linear fourth-order equation* (6.2) *exhibits dispersive blow up at some point (x_*, t_*) in space-time. Then, for $p < \frac{8}{n} - 1$, so does the fourth-order initial-value problem* (6.1).

Proof. The proof follows closely the one given in [9] for the one-dimensional, second-order nonlinear Schrödinger equation. In particular, the fact that the dispersive estimate in Lemma 6.1 of the fundamental solution Σ_α has temporal behavior that goes like $t \mapsto t^{-n/4}$, which is locally integrable for $n < 4$, is a key point in the proof.

The Duhamel representation of (6.1) is given by

$$\begin{aligned} u(x,t) &= e^{it(\Delta^2 - \alpha\Delta)} u_0(x) + i\lambda \int_0^t e^{i(t-s)(\Delta^2-\alpha\Delta)} |u(x,s)|^p u(x,s)\, ds \\ &= e^{it(\Delta^2 - \alpha\Delta)} u_0(x) + i\lambda I(x,t), \end{aligned}$$

The first term on the right-hand side exhibits dispersive blow up by assumption. To prove that the integral term is continuous and bounded, notice that

$$|I(x,t)| \leq C \int_0^t \int_{\mathbb{R}^n} \frac{1}{(t-s)^{n/4}} |u|^{p+1}(x-y, t-s)\, ds\, dy$$

using Lemma 6.1. Applying Hölder's inequality, with a $\gamma \in (0, 4/n)$ to be determined presently, it is found that

$$|I(t,x)| \leq \left(\int_{\mathbb{R}} \frac{ds}{(t-s)^{n\gamma/4}} \right)^{1/\gamma} \left(\int_{\mathbb{R}} \|u(\cdot,s)\|_{p+1}^{\gamma'(p+1)} ds \right)^{1/\gamma'},$$

with $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Choose an admissible Strichartz pair in the range (6.4) as follows. Take $r = p+1$ so that $q = \frac{8(p+1)}{n(p-1)}$. The condition $\gamma'(p+1) \leq q$ then yields

$$\gamma' = \frac{\gamma}{\gamma - 1} \leq \frac{8}{n(p-1)}.$$

Combined with the condition $\gamma < \frac{4}{n}$, one obtains

$$p < \frac{8}{n}\left(1 - \frac{1}{\gamma}\right) = \frac{8}{n} - 1,$$

and the assertion is proved. $\square$

Remark 6.4. The strategy deployed in this proof does not yield the optimal range of exponents p nor is it valid for $n \geq 4$. This is because it is based only on Strichartz estimates and because $t \mapsto t^{-n/4}$ is locally integrable only for $n < 4$. To extend the proof to higher dimensions $n > 3$, and to more general nonlinearities $p > 0$, one could argue as in the proof of Proposition 4.1. However, an essential ingredient in our argument was the Cazenave-Weissler result recounted in Proposition 3.1. Thus to carry out this line of reasoning successfully, we would need the analog of the Cazenave-Weissler results in the case of fourth-order equations, as well as the corresponding smoothing estimates for the Duhamel term established in Section 4. These tasks will be goal of an upcoming work.

To close the analysis, it is still required to prove that the linear fourth-order equation (6.2) does exhibit dispersive blow up. To this end, we will need a more precise description of the decay of the fundamental solution. This will only be carried out in the one-dimensional case, though the result in higher spatial dimensions does not require new ideas, just more complex computations.

Proposition 6.5. *Let $n = 1$. Given $(x^*, t^*) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, there exists $u_0 \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that the solution $u \in C_{\mathrm{b}}(\mathbb{R}; L^2(\mathbb{R}))$ to (6.2) has the following properties.*

(1) *The solution u blows up at (x_*, t_*) which is to say*
$$\lim_{(x,t)\in\mathbb{R}\to(x_*,t_*)} |u(x,t)| = +\infty.$$
(2) *The function u is continuous on $\big\{(x,t) \text{ on } \mathbb{R} \times \mathbb{R} \setminus \{t_*\}\big\}$.*
(3) *The solution $u(\cdot, t_*)$ is a continuous function on $\mathbb{R} \setminus \{x_*\}$.*

Proof. For $n = 1$, it is readily checked that the fundamental solution of
$$i\partial_t u_t - \alpha\partial_x^2 u + \partial_x^4 u = 0 \tag{6.5}$$
is given by
$$\Sigma_\alpha(x,t) = \frac{1}{(4t)^{1/4}} B\left(2\alpha t^{1/2}, \frac{x}{(4t)^{1/4}}\right)$$
where B is the Pearcey integral defined by
$$B(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} e^{i\left(\frac{1}{4}s^4 + \frac{1}{2}xs^2 + ys\right)}\, ds.$$
The Pearcey integral is a smooth and bounded function of $(x,y) \in \mathbb{R}^2$ which decays in both variables thusly:
$$|B(x,y)| \le c\big(1 + y^2 + |x|^3\big)^{-1/18}\Big(1 + (1 + y^2 + |x|^3)^{-5/9}|(3y)^2 + (2x)^3|\Big)^{-1/4}$$
(see, e.g., [3]). To establish dispersive blow up for (6.5), an asymptotic expansion of the Pearcey integral with respect to the second variable y is helpful. Such an expansion has been established in [33], Formulas (2.14) and (5.12). [1] These results imply that

(i) in the case $\alpha = 0$, one has
$$B(0,y) = C_1 y^{-1/3} + C_2 y^{-5/3} + O(|y|^{-3}),$$
as $|y| \to +\infty$, uniformly for bounded values of x, where C_1 and C_2 are nonzero complex constants and,

(ii) when $\alpha \neq 0$,
$$B(x,y) = 2^{1/6} e^{-i\pi/24} y^{-1/3} e^{-ix^2/6} \exp\Big(-\frac{3i}{4^{4/3}} y^{4/3} + \frac{i}{4^{2/3}} x y^{-2/3}]\Big) \times$$
$$\left(1 + \frac{4^{-1/3}}{3} x y^{-2/3}(1 - \frac{i}{9}x^2) + O(|y|^{-4/3})\right)$$
for $|y| \to +\infty$ and bounded x.

With these results in hand, an argument can be mounted that mimics the case of the usual Schrödinger group. Without loss of generality, take it that $(x_*, t_*) = (0, \frac{1}{4})$ and consider in (6.5) the initial data
$$u_0(x) = \frac{\Sigma_0(-x, \frac{1}{4})}{(1 + x^2)^m} = \frac{B(0, -x)}{(1 + x^2)^m}, \quad \frac{1}{12} < m \le \frac{1}{6},$$

[1] Note the rotation of coordinates in those formulas as compared to the way they are written here.

if $\alpha = 0$, and

$$u_0(x) = \frac{\Sigma_\alpha(-x, \frac{1}{4})}{(1+x^2)^m} = \frac{B(\alpha, -x)}{(1+x^2)^m}, \quad \frac{1}{12} < m \leq \frac{1}{6},$$

if $\alpha = \pm 1$.

The asymptotics of the Pearcey integral can then be brought to bear to prove Proposition 6.5. In particular, one uses these asymptotics to show that the solution of the linear problem blows up at $(0, \frac{1}{4})$, that it is continuous on the complement of this point and that the relevant Duhamel integral is everywhere continuous. We pass over the details. $\square$

The analogous result for general dimensions $n \in \mathbb{N}$ can be established by passing to radial coordinates and reducing it to the one-dimensional case (as in [3] where the first term in the asymptotic expansion is given).

6.2. **Third order NLS in one dimension.** Finally, we briefly discuss the situation for a third-order nonlinear Schroödinger equation in $n = 1$ dimensions. Consider the initial=value problem

$$\partial_t u + i\alpha \partial_x^2 u + \beta \partial_x^3 u + i\gamma |u|^2 u = 0, \quad u\big|_{t=0} = u_0(x), \tag{6.6}$$

where $x \in \mathbb{R}$, and $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ are given parameters. This is a model problem for a more complicated third-order equation that arises in optical wave propagation (see [31, 40]).

Equation (6.6) appears similar to the well known Korteweg-de Vries equation, hence the appearance of dispersive blow up can be established along the same lines as was pursued in [8] for this latter equation. Indeed, the associated linear equation,

$$\partial_t u + i\alpha \partial_x^2 u + \beta \partial_x^3 u = 0, \tag{6.7}$$

admits a fundamental solution of the form

$$\Lambda(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\beta\xi^3 - i\alpha\xi^2 + ix\cdot\xi} \, d\xi.$$

The expansion

$$\left(\xi + \frac{\alpha}{3\beta}\right)^3 = \xi^3 + \frac{\alpha}{\beta}\xi^2 + \frac{\alpha^2 \xi}{3\beta^2} + \frac{\alpha^3}{27\beta^3}$$

allows us to write Λ in the form

$$\Lambda(x,t) = \frac{1}{2\pi (t\beta)^{1/3}} \exp\left(\frac{4it\alpha^3}{27\beta^2}\right) \exp\left(\frac{-i\alpha x}{2\beta^2}\right) \mathrm{Ai}\left(\frac{1}{t^{1/3}\beta^{1/3}}\left(x - \frac{\alpha^2}{3\beta}t\right)\right).$$

Here, Ai denotes the well-known the Airy function defined, for example, by

$$\mathrm{Ai}(z) = \int_{\mathbb{R}} e^{i(\xi^3 + iz\xi)} \, d\xi.$$

In [8], the dispersive properties of the Airy function are the basis for establishing dispersive blow up for the Korteweg-de Vries equation. Following these ideas, dispersive blow up for (6.7) at $(x_*, t_*) = (0, 1)$ is easily obtained by taking initial data of the form

$$u_0(x) = \frac{A(-x)}{(1+x^2)^m}, \quad \text{with } \frac{1}{8} < m \leq \frac{1}{4}$$

and

$$A(x) = \mathrm{Ai}\left(\frac{\alpha^2 + x}{3\beta^{4/3}}\right).$$

The proof of dispersive blow up for this third-order Schrödinger equation can then be accomplished just as in [8]. Indeed, the proof is easier since, contrary to the Korteweg-de Vries equation, the Duhamel representation of (6.6) does not involve any spatial derivatives of the dependent variable. Consequently it can be shown to be bounded by using only Strichartz estimates for the linearized Korteweg-de Vries equation (see [4, 10]).

Remark 6.6. Combining the results of the foregoing sections allows one to deduce dispersive blow up for nonlinear Schrödinger-type equations with anisotropic dispersion, such as

$$i\partial_t u + \alpha\Delta u + i\beta\partial_{x_1}^3 u + \gamma\partial_{x_1}^4 u + |u|^p u = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$. The Cauchy problem for this equation has been studied in [10] (see also [19]).

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