

UNIQUE CONTINUATION FOR SCHRÖDINGER EVOLUTIONS, WITH APPLICATIONS TO PROFILES OF CONCENTRATION AND TRAVELING WAVES

L. ESCAURIAZA, C. E. KENIG, G. PONCE, AND L. VEGA

ABSTRACT. We prove unique continuation properties for solutions of the evolution Schrödinger equation with time dependent potentials. As an application of our method we also obtain results concerning the possible concentration profiles of blow up solutions and the possible profiles of the traveling waves solutions of semi-linear Schrödinger equations.

1. INTRODUCTION

In this paper we continue our study initiated in [8], [9], [10], and [11] on unique continuation properties of solutions of Schrödinger equations. To begin with we consider the linear equation

$$(1.1) \quad \partial_t u = i(\Delta u + V(x, t)u), \quad (x, t) \in \mathbb{R}^n \times [0, \infty).$$

We shall be interested in finding the strongest possible space decay of global solutions of (1.1). In this direction our first results are the following ones:

Theorem 1. *Let $u \in C([0, \infty) : L^2(\mathbb{R}^n))$ be a solution of the equation (1.1) with a real potential $V \in L^\infty(\mathbb{R}^n \times [0, \infty))$ satisfying that*

$$(1.2) \quad V(x, t) = V_1(x, t) + V_2(x, t),$$

with V_j , $j = 1, 2$ real valued,

$$(1.3) \quad |V_1(x, t)| \leq \frac{c_1}{\langle x \rangle^\alpha} = \frac{c_1}{(1 + |x|^2)^{\alpha/2}}, \quad 0 \leq \alpha < 1/2,$$

and V_2 supported in $\{(x, t) : |x| \geq 1\}$ such that

$$(1.4) \quad -(\partial_r V_2(x, t))^- \leq \frac{c_2}{|x|^{2\alpha}}, \quad a^- = \min\{a; 0\}.$$

Then there exists a constant $\lambda_0 = \lambda_0(\|V\|_{L^\infty(\mathbb{R}^n \times [0, \infty))}; c_1; c_2; \alpha) > 0$ such that if

$$(1.5) \quad \sup_{t \geq 0} \int_{\mathbb{R}^n} e^{\lambda_0 |x|^p} |u(x, t)|^2 dx < \infty, \quad \text{with} \quad p = (4 - 2\alpha)/3,$$

then

$$(1.6) \quad u \equiv 0.$$

As an immediate consequence of Theorem 1 we have:

1991 *Mathematics Subject Classification.* Primary: 35Q55.

Key words and phrases. Schrödinger evolutions.

The first and fourth authors are supported by MEC grant, MTM2004-03029, the second and third authors by NSF grants DMS-0456583 and DMS-0456833 respectively.

Corollary 1. *Let $u \in C([0, \infty) : L^2(\mathbb{R}^n))$ be a solution of the equation (1.1) with a real potential $V \in L^\infty(\mathbb{R}^n \times [0, \infty))$. If*

$$(1.7) \quad |V(x, t)| \leq \frac{c_1}{\langle x \rangle^\alpha} = \frac{c_1}{(1 + |x|^2)^{1/2}},$$

and for some $p > 1$ and $\lambda_0 > 0$

$$(1.8) \quad \sup_{t \geq 0} \int_{\mathbb{R}^n} e^{\lambda_0 |x|^p} |u(x, t)|^2 dx < \infty,$$

then $u \equiv 0$.

Theorem 2. *Let $u \in C([0, \infty) : L^2(\mathbb{R}^n))$ be a solution of the equation (1.1) with a real potential $V \in L^\infty(\mathbb{R}^n \times [0, \infty))$ satisfying that*

$$(1.9) \quad V(x, t) = V_1(x, t) + V_2(x, t),$$

with V_j , $j = 1, 2$ real valued,

$$(1.10) \quad |V_1(x, t)| \leq \frac{c_1}{\langle x \rangle^{1/2+\epsilon_0}} = \frac{c_1}{(1 + |x|^2)^{1/4+\epsilon_0/2}}, \quad \epsilon_0 > 0,$$

and V_2 supported in $\{(x, t) : |x| \geq 1\}$ such that

$$(1.11) \quad -(\partial_r V_2(x, t))^- \leq \frac{c_2}{|x|^{1+\epsilon_0}}, \quad a^- = \min\{a; 0\}.$$

Then there exists a constant $\lambda_0 = \lambda_0(\|V\|_{L^\infty(\mathbb{R}^n \times [0, \infty))}; c_1; c_2; \epsilon_0) > 0$ such that if

$$(1.12) \quad \sup_{t \geq 0} \int_{\mathbb{R}^n} e^{\lambda_0 |x|} |u(x, t)|^2 dx < \infty,$$

then

$$(1.13) \quad u \equiv 0.$$

Using the results in [9] and [10] one sees that it suffices to assume that the hypothesis (1.5) and (1.12) in Theorem 1 and Theorem 2 respectively, hold for a sequence of times $\{\tilde{T}_j = T_0 + jL : j \in \mathbb{Z}^+\}$ for some $T_0 \geq 0$ and $L > 0$.

The hypothesis on the real character on the potential in these theorems is used to guarantee that the L^2 -norm of the solution of the equation (1.1) is time independent. However, it suffices to have the L^2 -norm of the solution bounded below for all time $t \in [0, \infty)$ by a positive constant, provided that $u(0) \neq 0$. Therefore, Theorem 1 still holds for potentials $V(x, t)$ which can be written as

$$V(x, t) = V_1(x, t) + V_2(x, t) + V_3(x, t),$$

with V_1 and V_2 as before and V_3 complex valued satisfying (1.3) and such that

$$\|V_3\|_{L^1([0, \infty); L^\infty(\mathbb{R}^n))} = \int_0^\infty \|V_3(\cdot, t)\|_\infty dt < \infty.$$

A similar remark applies to Theorem 2.

Next, we define the ‘‘hyperbolic’’ or ‘‘ultra-hyperbolic’’ operator

$$(1.14) \quad \mathcal{L}_k = \partial_{x_1}^2 + \dots + \partial_{x_k}^2 - \partial_{x_{k+1}}^2 - \dots - \partial_{x_n}^2, \quad k \in \{2, \dots, n-1\},$$

and study the linear dispersive equation

$$(1.15) \quad \partial_t u = i(\mathcal{L}_k u + V(x, t)u), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Nonlinear models with a non-degenerate non-elliptic operator \mathcal{L}_k describing the dispersive relation arise in several mathematical and physical contexts. For example, the Davey-Stewartson system [5]

$$(1.16) \quad \begin{cases} i\partial_t u \pm \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x \varphi, & t, x, y \in \mathbb{R}, \\ \partial_x^2 \varphi \pm \partial_y^2 \varphi = \partial_x |u|^2, \end{cases}$$

with $u = u(x, y, t)$ a complex-valued function, $\varphi = \varphi(x, y, t)$ a real-valued function and c_1, c_2 real parameters. The system (1.16) appears as a model in wave propagations [5] and independently as a two dimensional completely integrable system which generalizes the integrable cubic 1-dimensional Schrödinger equation [1]. Also one has the Ishimori system [13]

$$(1.17) \quad \begin{cases} \partial_t S = S \wedge (\partial_x^2 S \pm \partial_y^2 S) + b(\partial_x \phi \partial_y S + \partial_y \phi \partial_x S), & t, x, y \in \mathbb{R}, \\ \partial_x^2 \phi \mp \partial_y^2 \phi = \mp 2S \cdot (\partial_x S \wedge \partial_y S), \end{cases}$$

where $S(\cdot, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with $\|S\| = 1$, $S \rightarrow (0, 0, 1)$ as $\|(x, y)\| \rightarrow \infty$, and \wedge denotes the wedge product in \mathbb{R}^3 . This model was first proposed as a two dimensional generalization of the Heisenberg equation in ferromagnetism. For $b = 1$ the system (1.17) has been shown to be completely integrable (see [1] and references therein).

The arguments used in the proofs of Theorems 1-2 do not rely on the elliptic character of the laplacian in (1.1), so we have:

Theorem 3. *Theorems 1-2 and Corollary 1 still hold for solutions $u \in C([0, \infty) : L^2(\mathbb{R}^n))$ of the equation (1.15) with a potential V verifying the same hypotheses.*

Remarks (i) It is interesting to relate our results with those due to V. Z. Meshkov in [14]:

Theorem. *Let $w \in H_{loc}^2(\mathbb{R}^n)$ be a solution of*

$$(1.18) \quad \Delta w + \tilde{V}(x)w = 0, \quad x \in \mathbb{R}^n, \quad \text{with } \tilde{V} \in L^\infty(\mathbb{R}^n).$$

$$(1.19) \quad \text{If } \int e^{2a|x|^{4/3}} |w|^2 dx < \infty, \quad \forall a > 0, \quad \text{then } w \equiv 0.$$

It was also proved in [14] that for complex valued potentials \tilde{V} the exponent $4/3$ in (1.19) is optimal.

We observe that if the potential in (1.1) $V(x, t)$ is time independent $V = \tilde{V}(x)$, then a solution of $w(x)$ of (1.18) is a stationary solutions of the IVP (1.1). Also for time independent potential $V(x, t) = \tilde{V}(x)$, if $w(x)$ is an H^1 -solution of the eigenvalue problem

$$(1.20) \quad \Delta w + \tilde{V}(x)w = \zeta w,$$

then one has that for $\zeta \in \mathbb{R}$

$$(1.21) \quad v(x, t) = e^{i\zeta t} w(x),$$

is a solution of the IVP (1.1) for which Theorems 1-2 apply. As it was mentioned above the assumption on the real character on the potential in these theorems is only required to guarantee that the L^2 -norm of the solution of the equation (1.1) is time independent. In the case described in (1.20)-(1.21) the solution $v(x, t)$

preserves the L^2 -norm and so the proof of Theorems 1-2 can be carried out. Hence, taking $V_2 \equiv 0$ one has the following results which recovers that in [14] mentioned above, and improves and generalizes those in [4]:

Theorem 4. *Let $w \in H^1(\mathbb{R}^n)$ be a solution of the equation (1.20) with a complex potential $\tilde{V} \in L^\infty(\mathbb{R}^n)$ satisfying*

$$(1.22) \quad \tilde{V}(x) = \tilde{V}_1(x) + \tilde{V}_2(x),$$

such that

$$(1.23) \quad |\tilde{V}_1(x)| \leq \frac{c_1}{\langle x \rangle^\alpha} = \frac{c_1}{(1 + |x|^2)^{\alpha/2}}, \quad 0 \leq \alpha < 1/2,$$

and \tilde{V}_2 real valued and supported in $\{x \in \mathbb{R}^n : |x| \geq 1\}$ such that

$$(1.24) \quad -(\partial_r \tilde{V}_2(x))^- \leq \frac{c_2}{|x|^{2\alpha}}, \quad a^- = \min\{a; 0\}.$$

Then there exists a constant $\lambda_0 = \lambda_0(\|\tilde{V}\|_{L^\infty(\mathbb{R}^n)}; c_1; c_2; \alpha) > 0$ such that if

$$(1.25) \quad \int_{\mathbb{R}^n} e^{\lambda_0 |x|^p} |w(x)|^2 dx < \infty, \quad \text{with} \quad p = (4 - 2\alpha)/3,$$

then

$$(1.26) \quad w \equiv 0.$$

Moreover, if (1.23) and (1.24) holds $\alpha > 1/2$ and (1.25) holds with $p = 1$ and large $\lambda_0 = \lambda_0(\|\tilde{V}\|_{L^\infty(\mathbb{R}^n)}; c_1; \alpha) > 0$, then $w \equiv 0$.

In [4] under the hypotheses $\tilde{V}_2 = 0$, (1.23) and (1.25), but for all $\lambda_0 > 0$, on the complex potential $V(x, t)$ on Theorem 4 it was shown that the eigenfunction $w(x)$ solution of (1.20) corresponding to the real eigenvalue ζ satisfies $w \equiv 0$.

We observe that the conclusion of Corollary 1 applies, i.e. if $\tilde{V}_2 = 0$, $\alpha = 1/2$ in (1.23), and (1.25) holds for some $p > 1$ and $\lambda_0 > 0$, then $w \equiv 0$. In this direction we have the following improvement of the result in Theorem 4 concerning the case $\alpha = 1/2$ in (1.23) and (1.24).

Theorem 5. *Let $w \in H^1(\mathbb{R}^n)$ be a solution of the equation (1.20) with a potential $\tilde{V} \in L^\infty(\mathbb{R}^n)$ satisfying*

$$(1.27) \quad \tilde{V}(x) = \tilde{V}_1(x) + \tilde{V}_2(x),$$

such that \tilde{V}_1 is complex valued with

$$(1.28) \quad |\tilde{V}_1(x)| \leq \frac{c_1}{\langle x \rangle^{1/2}} = \frac{c_1}{(1 + |x|^2)^{1/4}},$$

and \tilde{V}_2 is real valued and supported in $\{x \in \mathbb{R}^n : |x| \geq 1\}$ such that

$$(1.29) \quad -(\partial_r \tilde{V}_2(x))^- \leq \frac{c_2}{|x|}, \quad a^- = \min\{a; 0\}.$$

Then there exists a constant $\lambda_0 = \lambda_0(\|\tilde{V}\|_{L^\infty(\mathbb{R}^n)}; c_1; c_2) > 0$ such that if

$$(1.30) \quad \int_{\mathbb{R}^n} e^{\lambda_0 |x|} |w(x)|^2 dx < \infty,$$

then

$$(1.31) \quad w \equiv 0.$$

We observe that Theorem 5 is a stationary result (not a consequence of the time evolution results in Theorems 1 and 2) in which the ellipticity of the laplacian in (1.20) plays an essential role.

The proof of Theorem 5 will be based in the following Carleman estimate :

Theorem 6. *Let $\rho \in (0, 1]$ and \tilde{V} as in Theorem 5. Then there exists $\tau_0 = \tau_0(\rho; \|\tilde{V}\|_\infty; c_1; c_2) > 0$ such that the inequality*

$$(1.32) \quad \tau^{3/2} \| |x|^{-1/2} e^{\tau|x|} g \|_2 \leq \| e^{\tau|x|} (\Delta g + \tilde{V}g) \|_2$$

holds for any $\tau \geq \tau_0$ and any $g \in C_0^\infty(\mathbb{R}^n - \overline{B_\rho(0)})$.

We return to the consequence of our time evolution results. Thus, combining Theorem 3 and the comments before the statement of Theorem 4 one has that Theorem 4 also applies to the solutions of the non-elliptic eigenvalue problem

$$(1.33) \quad \mathcal{L}_k w + \tilde{V}(x)w = \zeta w,$$

with \mathcal{L}_k as in (1.14) with complex potential \tilde{V} and $\zeta \in \mathbb{R}$.

We shall employ the above results to study the possible profile of the concentration blow up phenomenon in solutions of the initial value problem (IVP) associated to the non-linear Schrödinger equation

$$(1.34) \quad \begin{cases} i\partial_t u + \Delta u \pm |u|^a u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad a > 0, \\ u(x, 0) = u_0(x). \end{cases}$$

We observe that if $u(x, t)$ is a solution of (1.34) then for all $\sigma > 0$

$$(1.35) \quad u_\sigma(x, t) = \sigma^{2/a} u(\sigma x, \sigma^2 t),$$

is also a solution of (1.34) with data $u_\sigma(x, 0) = \sigma^{2/a} u_0(x)$, so

$$(1.36) \quad \|D^s u_\sigma(x, 0)\|_2 = \sigma^{2/a-n/2+s} \|D^s u_0\|_2,$$

where $D^s f(x) = (|\xi|^s \hat{f})^\vee(x)$, $s \in \mathbb{R}$. Thus, if $s_a/2 - 2/a$ the size of the data does not change by the scaling and one says that

$$(1.37) \quad \dot{H}^{n/2-2/a}(\mathbb{R}^n) = D^{n/2-2/a} L^2(\mathbb{R}^n),$$

is a critical space for the IVP (1.34). The following result concerning the local well-posedness of the IVP (1.34) in the critical cases was established in [3].

Theorem. *Let $s_a/2 - 2/a$, $s_a \geq 0$ with $[s_a] \leq a - 1$ if a is not an odd integer, then for each $u_0 \in H^{s_a}(\mathbb{R}^n)$ there exist $T = T(u_0) > 0$ and a unique solution $u = u(x, t)$ of the IVP (1.34) with*

$$(1.38) \quad u \in C([-T, T] : H^{s_a}(\mathbb{R}^n)) \cap L^q([-T, T] : L^p_{s_a}(\mathbb{R}^n)) = Z_T^{s_a}.$$

Moreover, the map data \rightarrow solution is locally continuous from $H^{s_a}(\mathbb{R}^n)$ into $Z_T^{s_a}$.

Above we have introduced the notations :

(a) for $1 < p < \infty$ and $s \in \mathbb{R}$

$$(1.39) \quad L_s^p(\mathbb{R}^n) \equiv (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n), \quad \|\cdot\|_{s,p} \equiv \|(1 - \Delta)^{s/2} \cdot\|_p,$$

with $L_s^2(\mathbb{R}^n) = H^s(\mathbb{R}^n)$,

(b) the indices (q, p) in (1.38) are given by the Strichartz estimate [16], [7] :

$$(1.40) \quad \left(\int_{-\infty}^{\infty} \|e^{it\Delta} u_0\|_p^q dt \right)^{1/q} \leq c \|u_0\|_2,$$

where

$$\frac{n}{2} = \frac{2}{q} + \frac{n}{p}, \quad 2 \leq p \leq \infty, \quad \text{if } n = 1, \quad 2 \leq p < 2n/(n-2), \quad \text{if } n \geq 2.$$

The pseudo-conformal transformation deduced in [7] shows that if $u = u(x, t)$ is a solution of (1.34), then

$$(1.41) \quad v(x, t) = \frac{e^{i\omega|x|^2/4(\nu+\omega t)}}{(\nu+\omega t)^{n/2}} u \left(\frac{x}{\nu+\omega t}, \frac{\gamma+\theta t}{\nu+\omega t} \right), \quad \nu\theta - \omega\gamma = 1,$$

satisfies the equation

$$(1.42) \quad i\partial_t v + \Delta v \pm (\nu+\omega t)^{an/2-2} |v|^a v = 0.$$

Hence, in the L^2 -critical case $a = 4/n$ the equations (1.34) and (1.42) are the same. Also in this case $a = 4/n$ the pseudo-conformal transformation preserves both the space $L^2(\mathbb{R}^n)$ and the space $H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n : |x|^2 dx)$. In particular, if we take $u(x, t) = e^{it} \varphi(x)$ the standing wave solution, i.e. $\varphi(x)$ being the unique positive solution (ground state) of the non-linear elliptic equation

$$(1.43) \quad -\varphi + \Delta\varphi + |\varphi|^{4/n}\varphi = 0, \quad x \in \mathbb{R}^n,$$

it follows that

$$(1.44) \quad v(x, t) = \frac{e^{it/(1-t)} e^{-i|x|^2/4(1-t)}}{(1-t)^{n/2}} \varphi \left(\frac{x}{1-t} \right),$$

is a solution of (1.34) with $a = 4/n$ and $+$ sign in the nonlinear term (focussing case) which blows up at time $t = 1$, i.e.

$$\lim_{t \uparrow 1} \|\nabla v(\cdot, t)\|_2 = \infty,$$

and

$$\lim_{t \uparrow 1} |v(\cdot, t)|^2 = c \delta(\cdot), \quad \text{in the distribution sense.}$$

Since it is known that positive solutions of the elliptic problem (1.43) (in particular the ground state) have exponential decay (see [15], [2]), i.e.

$$\varphi(x) \sim b_1 e^{-b_2|x|}, \quad b_1, b_2 > 0,$$

the blow up solution $v(x, t)$ in (1.44) satisfies

$$(1.45) \quad |v(x, t)| \leq \frac{1}{(1-t)^{n/2}} Q \left(\frac{x}{1-t} \right), \quad t \in (-1, 1),$$

with $Q(x) = b_1 e^{-b_2|x|}$. One may ask if it is possible to have a faster ‘‘concentration profile’’ in a solution of (1.34) with $a = 4/n$ than the one described in (1.45). In other words, whether or not (1.45) can hold with

$$(1.46) \quad Q(x) = b_1 e^{-b_2|x|^p}, \quad b_1, b_2 > 0, \quad p > 1,$$

or

$$(1.47) \quad Q(x) = b_1 e^{-b_3|x|},$$

with b_3 sufficiently large. More generally for $a \geq 4/n$ one may ask if a blow up solution $v(x, t)$ of (1.34) can satisfy

$$(1.48) \quad |v(x, t)| \leq \frac{1}{(1-t)^{2/a}} Q\left(\frac{x}{1-t}\right), \quad t \in (-1, 1),$$

with $Q(\cdot)$ as in (1.46) or as (1.47). Our next result shows that this is not the case.

Theorem 7. *Let $a \geq 4/n$. Let $v \in C((-1, 1) : H^{n/2-2/a}(\mathbb{R}^n))$ be a solution of the equation (1.34). If (1.48) holds with $Q(\cdot)$ as in (1.46) for some $p > 1$ and $b_2 > 0$ or as (1.47), then $v \equiv 0$.*

In [12] we establish the result in Theorem 7 for $a = 4/n$ and $p > 4/3$.

Now we consider the equation in (1.34) with the operator describing the dispersive relation \mathcal{L}_k as in (1.15) being non-degenerate but not elliptic

$$(1.49) \quad i\partial_t u + \mathcal{L}_k u \pm |u|^a u = 0, \quad a > 0.$$

In this case, the local well-posedness theory is similar to that described above for the IVP (1.34). This follows from the fact that the local theory is based on the Strichartz estimates in (1.40) which do not require the ellipticity of the laplacian, i.e. (1.40) holds with \mathcal{L}_k instead of Δ . Hence the results in [3] still holds for the IVP associated to the equation in (1.49). In addition, in this case the pseudo-conformal transformation tells us that if $u = u(x, t)$ is a solution of (1.49), then

$$(1.50) \quad v(x, t) = \frac{e^{i\omega \mathcal{Q}_k(x)/4(\nu+\omega t)}}{(\nu + \omega t)^{n/2}} u\left(\frac{x}{\nu + \omega t}, \frac{\gamma + \theta t}{\nu + \omega t}\right), \quad \nu\theta - \omega\gamma = 1,$$

with

$$(1.51) \quad \mathcal{Q}_k(x) = x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2,$$

verifies the equation

$$(1.52) \quad i\partial_t v + \mathcal{L}_k v \pm (\nu + \omega t)^{an/2-2} |v|^a v = 0.$$

Hence, as in Theorem 7 we have:

Theorem 8. *Let $a \geq 4/n$. Let $v \in C((-1, 1) : H^{n/2-2/a}(\mathbb{R}^n))$ be a solution of the equation in (1.34). If u satisfies (1.48) with $Q(\cdot)$ as in (1.46) or as (1.47), then $u \equiv 0$.*

It should be remarked that the result in Theorem 8 is a conditional one. It assumes that the local solution of the IVP associated to the equation (1.34) blows up (see (1.48)) which is an open problem.

We will adapt our results in Theorems 1 and 2 to study the possible profile of “generalized traveling wave” solutions of a class of equations containing those in (1.34) and (1.49), (see (1.59) and (1.60) below). Roughly, these are solutions $u(x, t)$ for which there exist $\mu \in \mathbb{R}$ and $\vec{e} \in \mathbb{S}^{n-1}$ such that the $L^2(\mathbb{R}^n)$ -norm of $u(x - \mu t \vec{e}, t)$ remains highly concentrated at the origin for all time $t \geq 0$, see (1.54) and (1.57) below.

Corollary 2. *Let $u \in C([0, \infty) : L^2(\mathbb{R}^n))$ be a solution of the equation (1.1) or the equation (1.15) with a real potential $V \in L^\infty(\mathbb{R}^n \times [0, \infty))$.*

(a) *If there exist $\mu \in \mathbb{R}$ and $\vec{e} \in \mathbb{S}^{n-1}$ such that*

$$(1.53) \quad |V(x, t)| \leq \frac{c_1}{(1 + |x + \mu t \vec{e}|^2)^{\alpha/2}},$$

for some constants $c_1 > 0$ and $\alpha \in [0, 1/2)$. Then there exists $\lambda_0(\|V\|_{L^\infty(\mathbb{R}^n \times [0, \infty))}; c_1; \alpha) > 0$ such that if

$$(1.54) \quad \sup_{t \geq 0} \int_{\mathbb{R}^n} e^{\lambda_0 |x + \mu t \vec{e}|^p} |u(x, t)|^2 dx < \infty, \quad \text{with} \quad p = (4 - 2\alpha)/3,$$

then

$$(1.55) \quad u \equiv 0.$$

(b) If there exist $\mu \in \mathbb{R}$ and $\vec{e} \in \mathbb{S}^{n-1}$ such that

$$(1.56) \quad |V(x, t)| \leq \frac{c_1}{(1 + |x + \mu t \vec{e}|^2)^{1/4 + \epsilon_0/2}}, \quad \epsilon_0 > 0,$$

for some constants $c_1 > 0$. Then there exists $\lambda_0(\|V\|_{L^\infty(\mathbb{R}^n \times [0, \infty))}; c_1; \alpha) > 0$ such that if

$$(1.57) \quad \sup_{t \geq 0} \int_{\mathbb{R}^n} e^{\lambda_0 |x + \mu t \vec{e}|} |u(x, t)|^2 dx < \infty,$$

then

$$(1.58) \quad u \equiv 0.$$

As in Corollary 1 we remark that if (1.53) holds with $\alpha = 1/2$ and (1.54) holds for some $p > 1$ and $\lambda_0 > 0$, then $u \equiv 0$.

Finally we shall consider the semi-linear equations of the form

$$(1.59) \quad \partial_t u = i(\Delta u + F(u, \bar{u})u),$$

and

$$(1.60) \quad \partial_t u = i(\mathcal{L}_k u + F(u, \bar{u})u),$$

with \mathcal{L}_k as in (1.14) and $F : \mathbb{C}^2 \rightarrow \mathbb{R}$ (real valued), $F(0, 0) = 0$, and such that there exists $M > 0$ and $j \in \mathbb{Z}^+$ such that

$$(1.61) \quad |F(z, \bar{z})| \leq M(|z| + |z|^j).$$

As a direct consequence of Theorems 1 and 2, Corollary 2, and an appropriate version of the Galilean invariant property for solution of the equations (1.59) and (1.60) we shall establish the following result:

Corollary 3. *Let $u \in C([0, \infty) : L^2(\mathbb{R}^n))$ be a solution of the equation (1.59) or the equation (1.60). If there exist $\mu \in \mathbb{R}$ and $\vec{e} \in \mathbb{S}^{n-1}$ such that*

$$(1.62) \quad |u(x, t)| \leq Q(x + \mu t \vec{e}), \quad \forall x \in \mathbb{R}^n, \quad t > 0,$$

with $Q(\cdot)$ as in (1.46) for some $p > 1$ or as in (1.47), then $u \equiv 0$.

In [6] it was proved that the equation (1.60) with a \mathcal{L}_k non-elliptic operator does not have nontrivial (travelling wave) solutions of the form

$$u(x, t) = e^{i\omega t} \varphi(x + \mu t \vec{e}), \quad \mu \in \mathbb{R}, \quad \vec{e} \in \mathbb{S}^{n-1},$$

with $\varphi \in H^1(\mathbb{R}^n) \cap H_{loc}^2(\mathbb{R}^n)$.

The rest of this paper is organized as follows. Section 2 contains the details of the proof of Theorem 1 in the case $V_2 \equiv 0$ (the proof of Theorems 3, 4, 7, and 8, and Corollaries 2 and 3 follows this approach) and the modifications needed in this proof to obtain the general case. The modifications of this argument required to establish Theorems 2 will be given in section 3. Also section 3 contains some

remarks on the proof of Theorem 3. Theorem 7 will be proved in section 4, and the proofs of Corollaries 2-3 will be outlined in section 5. Finally, Theorems 5 and 6 will be proven in section 6. The appendix is concerned with the existence of the functions φ used in the proofs of Theorem 1 and Theorem 2.

2. PROOF OF THEOREM 1

We begin with two preliminary results. Let \mathcal{S} be a symmetric operator independent of t . Let \mathcal{A} be a skew-symmetric one.

Proposition 1. *For any $T_0, T_1 \in \mathbb{R}$, $T_0 < T_1$ and any suitable function $f(x, t)$ one has*

$$\begin{aligned}
 (2.1) \quad & \int_{T_0}^{T_1} \int [\mathcal{S}; \mathcal{A}]f \bar{f} \, dxdt + \int_{T_0}^{T_1} \int |\mathcal{S}f|^2 \, dxdt \\
 & \leq \int_{T_0}^{T_1} \int |\partial_t f - (\mathcal{S} + \mathcal{A})f|^2 \, dxdt \\
 & \quad + \left| \int \mathcal{S}f(T_1) \overline{f(T_1)} \, dx \right| + \left| \int \mathcal{S}f(T_0) \overline{f(T_0)} \, dx \right|.
 \end{aligned}$$

Proof. Since \mathcal{S} is independent of t one has

$$\begin{aligned}
 (2.2) \quad & \partial_t \langle \mathcal{S}f, f \rangle = \langle \partial_t f, \mathcal{S}f \rangle + \langle \mathcal{S}f, \partial_t f \rangle \\
 & = \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, \mathcal{S}f \rangle + \langle \mathcal{S}f, \partial_t f - (\mathcal{S} + \mathcal{A})f \rangle \\
 & \quad + \langle (\mathcal{S} + \mathcal{A})f, \mathcal{S}f \rangle + \langle \mathcal{S}f, (\mathcal{S} + \mathcal{A})f \rangle \\
 & = 2 \Re \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, \mathcal{S}f \rangle + 2 \langle \mathcal{S}f, \mathcal{S}f \rangle + \langle \mathcal{S}\mathcal{A} - \mathcal{A}\mathcal{S} \rangle f, f.
 \end{aligned}$$

Thus, integrating in the time interval $[T_0, T_1]$ it follows that

$$\begin{aligned}
 & \int_{T_0}^{T_1} \langle [\mathcal{S}; \mathcal{A}]f, f \rangle \, dt + 2 \int_{T_0}^{T_1} \langle \mathcal{S}f, \mathcal{S}f \rangle \, dt \\
 & = -2 \Re \int_{T_0}^{T_1} \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, \mathcal{S}f \rangle + \langle \mathcal{S}f, f \rangle \Big|_{T_0}^{T_1}.
 \end{aligned}$$

Then, using that $2ab \leq a^2 + b^2$ we obtain (2.1). □

Next, for a fixed $T \in \mathbb{R}$ we define $\eta : [T - 1/2, T + 1/2] \rightarrow \mathbb{R}$ as

$$\eta(t) = (t - (T - 1/2))((T + 1/2) - t),$$

so $\eta(T - 1/2) = \eta(T + 1/2) = 0$ and for any $t \in [T - 1/2, T + 1/2]$

$$0 \leq \eta(t) \leq 1/4, \quad |\eta'(t)| \leq 1, \quad \eta''(t) = -2.$$

Proposition 2. *For any $T > 1/2$ one has*

$$\begin{aligned}
 (2.3) \quad & \int_{T-1/2}^{T+1/2} \int \eta(t) (|\mathcal{S}f|^2 + [\mathcal{S}; \mathcal{A}]f \bar{f}) \, dxdt + \int_{T-1/2}^{T+1/2} \int |f|^2 \, dxdt \\
 & \leq 8 \int_{T-1/2}^{T+1/2} \int |\partial_t f - (\mathcal{S} + \mathcal{A})f|^2 \, dxdt + 8 \left| \int |f|^2 \, dx \Big|_{T-1/2}^{T+1/2} \right|.
 \end{aligned}$$

Proof. Since

$$\begin{aligned}
(2.4) \quad \partial_t \langle f, f \rangle &= \langle \partial_t f, f \rangle + \langle f, \partial_t f \rangle \\
&= \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, f \rangle + \langle f, \partial_t f - (\mathcal{S} + \mathcal{A})f \rangle \\
&\quad + \langle (\mathcal{S} + \mathcal{A})f, f \rangle + \langle f, (\mathcal{S} + \mathcal{A})f \rangle \\
&= 2\Re \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, f \rangle + 2\langle \mathcal{S}f, f \rangle,
\end{aligned}$$

multiplying by $\eta'(t)$ and integrating in the time interval $[T - 1/2, T + 1/2]$ one gets

$$\begin{aligned}
(2.5) \quad &-2 \int_{T-1/2}^{T+1/2} \eta'(t) \langle \mathcal{S}f, f \rangle dt \\
&= 2\Re \int_{T-1/2}^{T+1/2} \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, f \rangle \eta'(t) dt - \int_{T-1/2}^{T+1/2} \partial_t \langle f, f \rangle \eta'(t) dt.
\end{aligned}$$

Integration by parts gives

$$(2.6) \quad - \int_{T-1/2}^{T+1/2} \partial_t \langle f, f \rangle \eta'(t) dt = -\langle f, f \rangle \eta'(t) \Big|_{T-1/2}^{T+1/2} + \int_{T-1/2}^{T+1/2} \langle f, f \rangle \eta''(t) dt$$

and

$$(2.7) \quad -2 \int_{T-1/2}^{T+1/2} \eta'(t) \langle \mathcal{S}f, f \rangle dt = 2 \int_{T-1/2}^{T+1/2} \eta(t) \partial_t \langle \mathcal{S}f, f \rangle dt.$$

We recall that from (2.2) one has

$$(2.8) \quad \partial_t \langle \mathcal{S}f, f \rangle = 2\Re \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, \mathcal{S}f \rangle + 2\langle \mathcal{S}f, \mathcal{S}f \rangle + \langle [\mathcal{S}; \mathcal{A}]f, f \rangle,$$

so inserting (2.8) into (2.7), and the result together with (2.6) into (2.5) it follows that

$$\begin{aligned}
(2.9) \quad &4 \int_{T-1/2}^{T+1/2} \eta(t) \langle \mathcal{S}f, \mathcal{S}f \rangle dt + 2 \int_{T-1/2}^{T+1/2} \eta(t) \langle [\mathcal{S}; \mathcal{A}]f, f \rangle dt \\
&= -4\Re \int_{T-1/2}^{T+1/2} \eta(t) \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, \mathcal{S}f \rangle dt \\
&\quad + 2\Re \int_{T-1/2}^{T+1/2} \langle \partial_t f - (\mathcal{S} + \mathcal{A})f, f \rangle \eta'(t) dt \\
&\quad - \langle f, f \rangle \eta'(t) \Big|_{T-1/2}^{T+1/2} + \int_{T-1/2}^{T+1/2} \langle f, f \rangle \eta''(t) dt,
\end{aligned}$$

which combined with the properties of the function η and Cauchy-Schwarz yields the estimates (2.3). \square

Proof of Theorem 1: case $V_2 \equiv 0$.

We fix $\alpha \in [0, 1/2)$ and $p = (4 - 2\alpha)/3 \in (1, 4/3]$. Let $\varphi = \varphi_p$ be a C^4 , radial, strictly convex function on compact sets of \mathbb{R}^n , such that

$$\begin{aligned}
(2.10) \quad &\varphi(r) = r^p + \beta, \quad \text{for } r = |x| \geq 1, \\
&\varphi(0) = 0, \quad \varphi(r) > 0, \quad \text{for } r > 0, \\
&\exists M > 0 \quad \text{s.t. } \varphi(r) \leq Mr^p, \quad \forall r \in [0, \infty).
\end{aligned}$$

The existence of such a function $\varphi = \varphi_p$ will be discussed in the Appendix, part (a).

We recall that

$$(2.11) \quad D^2\varphi = \partial_r^2\varphi \left(\frac{x_j x_k}{r^2} \right) + \frac{\partial_r\varphi}{r} \left(\delta_{jk} - \frac{x_j x_k}{r^2} \right).$$

Therefore,

$$(2.12) \quad \nabla\varphi D^2\varphi \nabla\varphi = \partial_r^2\varphi (\partial_r\varphi)^2 = \frac{c}{|x|^{4-3p}}, \quad \text{for } r = |x| \geq 1,$$

and

$$(2.13) \quad D^2\varphi \geq p(p-1)r^{p-2}I, \quad \text{for } r = |x| \geq 1.$$

Let $f(x, t) = e^{\lambda\varphi(x)}u(x, t)$ where $u(x, t)$ is a solution of the IVP (1.1) so

$$(2.14) \quad e^{\lambda\varphi}(\partial_t - i\Delta)u = e^{\lambda\varphi}(\partial_t - i\Delta)(e^{-\lambda\varphi}f) = \partial_t f - \mathcal{S}f - \mathcal{A}f,$$

where \mathcal{S} is symmetric and \mathcal{A} skew-symmetric both independent of t with

$$(2.15) \quad \mathcal{S} = -i\lambda(2\nabla\varphi \cdot \nabla + \Delta\varphi), \quad \mathcal{A} = i(\Delta + \lambda^2|\nabla\varphi|^2),$$

so that

$$(2.16) \quad [\mathcal{S}; \mathcal{A}] = -\lambda((4\nabla \cdot D^2\varphi \nabla) - 4\lambda^2\nabla\varphi D^2\varphi \nabla\varphi + \Delta^2\varphi).$$

We divide the proof into three steps:

Step 1 : If

$$(2.17) \quad \sup_{t>0} \int e^{\lambda|x|^p} |u(x, t)|^2 dx \leq c_\lambda, \quad p = (4 - 2\alpha)/3.$$

Then there exists $\{T_j : j \in \mathbb{Z}^+\}$ with $T_j \uparrow \infty$ as $j \uparrow \infty$ such that

$$(2.18) \quad \sup_{j \in \mathbb{Z}^+} \int |\mathcal{S}f(x, T_j)|^2 dx \leq \tilde{c}_\lambda,$$

where

$$f = e^{\lambda\varphi(x)}u(x, t),$$

\mathcal{S} as in (2.15), and \tilde{c}_λ denoting a constant depending on c_λ in (2.17), λ , $\|V\|_\infty$ and p .

Proof of step 1 : We combine Proposition 2 with (2.16) passing the term involving $\Delta^2\varphi$ to the right hand side and using that the rest of the commutator in (2.16) is positive to obtain

$$(2.19) \quad \begin{aligned} \int_{T-1/2}^{T+1/2} \int |\mathcal{S}f|^2 \eta(t) dx dt &\leq 8 \left(\int_{T-1/2}^{T+1/2} \int |\partial_t f - \mathcal{S}f - \mathcal{A}f|^2 dx dt \right. \\ &\quad \left. + \lambda \|\Delta^2\varphi\|_\infty \int_{T-1/2}^{T+1/2} \int |f|^2 dx dt + \left| \int |f|^2 dx \Big|_{T-1/2}^{T+1/2} \right| \right) \equiv B. \end{aligned}$$

We use that

$$e^{\lambda\varphi}(\partial_t - i\Delta)u = \partial_t f - \mathcal{S}f - \mathcal{A}f, \quad (\partial_t - i\Delta)u = iVu,$$

to bound the right hand side of (2.19) as

$$(2.20) \quad B \leq c(\lambda \|\Delta^2\varphi\|_\infty + \sup_{t>0} \|V(\cdot, t)\|_\infty^2) \sup_{t>0} \int e^{2\lambda\varphi} |u(x, t)|^2 dx \leq \tilde{c}_\lambda.$$

Inserting this in (2.19) and using that $\eta(t) \geq 3/16$ for $t \in [T - 1/4, T + 1/4]$ one gets that

$$\begin{aligned} \tilde{c}_\lambda &\geq \int_{T-1/2}^{T+1/2} \int |\mathcal{S}f|^2 \eta \, dx dt \geq \int_{T-1/4}^{T+1/4} \int |\mathcal{S}f|^2 \eta \, dx dt \\ &\geq \frac{3}{16} \int_{T-1/4}^{T+1/4} \int |\mathcal{S}f|^2 \, dx dt \geq \frac{3}{32} \int |\mathcal{S}f(x, T^*)|^2 \, dx, \end{aligned}$$

for some $T^* \in [T - 1/4, T + 1/4]$. Hence, we can find a sequence $\{T_j : j \in \mathbb{Z}^+\}$ with $t_j \uparrow \infty$ and $j \uparrow \infty$ such that

$$(2.21) \quad \sup_{j \in \mathbb{Z}^+} \int |\mathcal{S}f(x, T_j)|^2 \, dx \leq \tilde{c}_\lambda.$$

Step 2 : There exists $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$, then for any $j \in \mathbb{Z}^+$,

$$(2.22) \quad \int_{T_1}^{T_j} \int \frac{e^{2\lambda\varphi(x)} |u(x, t)|^2}{\langle x \rangle^{4-3p}} \, dx dt \leq \tilde{c}_\lambda \quad \text{uniformly in } j \in \mathbb{Z}^+.$$

Proof of step 2 : A combination of Proposition 1, the conclusion of step 1, and our hypothesis leads to

$$(2.23) \quad \int_{T_1}^{T_j} \int [\mathcal{S}; \mathcal{A}] f \bar{f} \, dx dt \leq \int_{T_1}^{T_j} \int |e^{\lambda\varphi} V u|^2 \, dx dt + \tilde{c}_\lambda.$$

From our hypothesis (2.10) on φ one has that

$$(2.24) \quad \begin{aligned} \nabla\varphi D^2\varphi \nabla\varphi &\geq \frac{c}{|x|^{4-3p}}, \quad |x| \geq 1, \\ |\Delta^2\varphi(x)| &\leq \frac{c}{\langle x \rangle^2}, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Thus, from our decay hypothesis on the potential (1.3) it follows that there exists $\tilde{\lambda} > 0$ such that if $\lambda \geq \tilde{\lambda}$ and $|x| \geq 1$, then

$$(2.25) \quad 2\lambda^2 \nabla\varphi D^2\varphi \nabla\varphi + \Delta^2\varphi - |V|^2 \geq \frac{\lambda}{\langle x \rangle^{4-3p}}.$$

Next, for any $\epsilon \in (0, 1)$ we consider the domain $\{x : \epsilon \leq |x| \leq 1\}$. In this set we have that

$$(2.26) \quad \nabla\varphi D^2\varphi \nabla\varphi \geq c_{\varphi, \epsilon}, \quad \text{for } \epsilon \leq |x| \leq 1.$$

Therefore, for large enough $\lambda \geq \lambda_\epsilon$

$$(2.27) \quad \lambda^2 \nabla\varphi D^2\varphi \nabla\varphi + \Delta^2\varphi - |V|^2 \geq \lambda, \quad \text{for } \epsilon \leq |x| \leq 1.$$

Hence from (2.23)

$$(2.28) \quad \begin{aligned} &4\lambda \int_{T_1}^{T_j} \int \nabla f D^2\varphi \nabla \bar{f} \, dx dt + 2\lambda^3 \int_{T_1}^{T_j} \int \nabla\varphi D^2\varphi \nabla\varphi |f|^2 \, dx dt \\ &\leq \tilde{c}_\lambda + c' (\lambda \|\Delta^2\varphi\|_\infty + \|V\|_\infty^2) \int_{T_1}^{T_j} \int_{|x| \leq \epsilon} |f|^2 \, dx dt. \end{aligned}$$

In the domain $\{x : |x| \leq \epsilon\}$ we shall use that φ is strictly convex in $r = |x| \leq 2$ to get from (2.28) that

$$(2.29) \quad \begin{aligned} & 4c_\varphi \lambda \int_{T_1}^{T_j} \int_{|x| \leq 2\epsilon} |\nabla f|^2 dx dt + 2\lambda^3 \int_{T_1}^{T_j} \int \nabla \varphi D^2 \varphi \nabla \varphi |f|^2 dx dt \\ & \leq \tilde{c}_\lambda + c' (\lambda \|\Delta^2 \varphi\|_\infty + \|V\|_\infty^2) \int_{T_1}^{T_j} \int_{|x| \leq \epsilon} |f|^2 dx dt, \end{aligned}$$

with c_φ and c' independent of $\epsilon \in (0, 1]$. Now, we pick $\theta \in C^\infty(\mathbb{R}^n)$ such that $\theta(x) \equiv 1$ for $|x| \leq \epsilon$ with $\text{supp } \theta \subset \{x : |x| \leq 2\epsilon\}$ and use Poincare's inequality to get that for each $t \in [T_1, T_j]$

$$(2.30) \quad \begin{aligned} & \int_{|x| \leq \epsilon} |f|^2 dx \leq \int_{|x| \leq 2\epsilon} |\theta f|^2 dx \leq c_\varphi \epsilon^2 \int_{|x| \leq 2\epsilon} |\nabla(\theta f)|^2 dx \\ & \leq c_\varphi \epsilon^2 \int_{|x| \leq 2\epsilon} |\nabla f|^2 dx + c_\varphi \int_{\epsilon \leq |x| \leq 2\epsilon} |f|^2 dx. \end{aligned}$$

Fixing ϵ sufficiently small and then λ large enough it follows from this that

$$(2.31) \quad \begin{aligned} & \lambda \int_{T_1}^{T_j} \int \nabla f D^2 \varphi \nabla \bar{f} dx dt + \lambda^2 \int_{T_1}^{T_j} \int_{|x| \leq 1} |f|^2 dx dt \\ & + \lambda^3 \int_{T_1}^{T_j} \int \nabla \varphi D^2 \varphi \nabla \varphi |f|^2 dx dt \leq \tilde{c}_\lambda. \end{aligned}$$

In particular, for $\lambda_0 \geq \tilde{\lambda}$ sufficiently large we have

$$(2.32) \quad \int_{T_1}^{T_j} \int \frac{|f|^2}{\langle x \rangle^{4-3p}} dx dt \leq \tilde{c}_\lambda, \quad \text{uniformly in } j \in \mathbb{Z}_+,$$

which completes the proof of this step.

We fix $\lambda = \lambda_0$ above for the rest of the proof.

Step 3 : $u(x, t) \equiv 0$.

Proof of step 3 : On the one hand, since the potential $V = V(x, t)$ is real, then the L^2 -norm of the solution $u(x, t)$ of (1.1) is preserved, i.e. for all $t \in \mathbb{R}$

$$\|u(\cdot, t)\|_2 = \|u_0\|_2.$$

On the other hand, from step 2 inequality (2.22) one has

$$\begin{aligned} (T_j - T_1) \|u_0\|_2^2 &= \int_{T_1}^{T_j} \int |u(x, t)|^2 dx dt \\ &= \int_{T_1}^{T_j} \int |u(x, t)|^2 \frac{e^{2\lambda\varphi}}{\langle x \rangle^{4-3p}} \langle x \rangle^{4-3p} e^{-2\lambda\varphi} dx dt \\ &\leq \sup_{x \in \mathbb{R}^n} (\langle x \rangle^{4-3p} e^{-2\lambda\varphi}) \int_{T_1}^{T_j} \int |u(x, t)|^2 \frac{e^{2\lambda\varphi}}{\langle x \rangle^{4-3p}} dx dt \leq \tilde{c}_{\lambda_0}, \end{aligned}$$

which completes the proof of Theorem 1 in the case $V_2 \equiv 0$.

Proof of Theorem 1: general case.

The argument is similar to that presented above in the case $V_2 \equiv 0$, so we sketch it. The step 1 is similar so it will be omitted. In the step 2 we divide the potential

$V(x, t)$ as in (1.2),

$$V(x, t) = V_1(x, t) + V_2(x, t),$$

and define

$$(2.33) \quad \mathcal{S} = -i\lambda(2\nabla\varphi \cdot \nabla + \Delta\varphi), \quad \mathcal{A} = i(\Delta + V_2 + \lambda^2|\nabla\varphi|^2),$$

so that

$$(2.34) \quad [\mathcal{S}; \mathcal{A}] = -\lambda((4\nabla \cdot D^2\varphi\nabla) - 4\lambda^2\nabla\varphi D^2\varphi\nabla\varphi + \Delta^2\varphi) + 2\lambda\nabla\varphi \cdot \nabla V_2 = D_1 + D_2.$$

We notice that D_1 is similar to the term handled in the proof of Theorem 1 in the case $V_2 \equiv 0$, and that since φ is radial and convex one has

$$(2.35) \quad D_2 = 2\lambda\nabla\varphi \cdot \nabla V_2 = 2\lambda\partial_r\varphi\partial_r V_2 \geq 2\lambda\partial_r\varphi(\partial_r V_2)^-.$$

Thus, from our decay hypothesis on the potential it follows that there exists $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$ and $|x| \geq 1$, then

$$(2.36) \quad 2\lambda^2\nabla\varphi D^2\varphi\nabla\varphi + \Delta^2\varphi - |V_1|^2 + 2\partial_r\varphi(\partial_r V_2)^- \geq \frac{\lambda}{\langle x \rangle^{4-3p}}.$$

For $|x| \leq 1$ we apply the argument in the proof of Theorem 1 in the case $V_2 \equiv 0$. Therefore combining these estimates we obtain the proof of the step 2 : There exists $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$, then for any $j \in \mathbb{Z}^+$

$$(2.37) \quad \int_{T_1}^{T_j} \int \frac{e^{2\lambda\varphi(x)} |u(x, t)|^2}{\langle x \rangle^{4-3p}} dx dt \leq \tilde{c}_\lambda \quad \text{independent of } j \in \mathbb{Z}^+.$$

Once (2.37) has been established the rest of the proof follows the same argument given in the step 3 of the proof of Theorem 1 in the case $V_2 \equiv 0$.

3. PROOFS OF THEOREM 2 AND THEOREM 3

Proof of Theorem 2: case $V_2 \equiv 0$.

We shall follow the argument provided in the proof Theorem 1. A main difference is the choice of the function φ in (2.10). In this case we take $\varphi \in C^4$ to be a radial, strictly convex function on compact sets of \mathbb{R}^n , such that

$$(3.1) \quad \varphi(r) = 3r - \int_1^r \frac{dr}{1 + \log r} + \beta, \quad r = |x| \geq 1,$$

so

$$(3.2) \quad \partial_r\varphi(x) = 3 - \frac{1}{1 + \log r}, \quad \partial_r^2\varphi(x) = \frac{1}{r(1 + \log r)^2}, \quad r = |x| \geq 1,$$

and

$$(3.3) \quad \begin{aligned} \varphi(0) &= 0, \quad \varphi(r) > 0, \quad \text{for } r > 0, \\ \exists M > 0 \quad \text{s.t.} \quad \varphi(r) &\leq Mr, \quad \forall r \in [0, \infty). \end{aligned}$$

The existence of such a function φ will be proven in the Appendix, part (b). Since

$$(3.4) \quad D^2\varphi = \partial_r^2\varphi \left(\frac{x_j x_k}{r^2} \right) + \frac{\partial_r\varphi}{r} \left(\delta_{jk} - \frac{x_j x_k}{r^2} \right),$$

for $|x| \geq 1$ one has

$$(3.5) \quad \nabla\varphi D^2\varphi\nabla\varphi = \partial_r^2\varphi(\partial_r\varphi)^2 > \frac{1}{r(1 + \log r)^2},$$

and

$$(3.6) \quad D^2\varphi \geq \partial_r^2\varphi(x)I.$$

The step 1 is similar to that in the proof of Theorem 1, with the appropriate modifications, hence we shall start with step 2.

Step 2 : There exists $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$, then for any $j \in \mathbb{Z}^+$

$$(3.7) \quad \int_{T_1}^{T_j} \int \frac{e^{2\lambda\varphi(x)} |u(x,t)|^2}{\langle x \rangle (\log \langle x \rangle)^2} dxdt \leq \tilde{c}_\lambda \quad \text{independent of } j \in \mathbb{Z}^+.$$

Proof of step 2 : A combination of Proposition 1, the conclusion of step 1, and our hypothesis leads to

$$(3.8) \quad \int_{T_1}^{T_j} \int [\mathcal{S}; \mathcal{A}] f \bar{f} dxdt \leq \int_{T_1}^{T_j} \int |e^{\lambda\varphi} V u|^2 dxdt + \tilde{c}_\lambda.$$

From our assumptions on φ it follows that

$$(3.9) \quad |\Delta^2\varphi(x)| \leq \frac{c}{\langle x \rangle^2}, \quad \forall x \in \mathbb{R}^n.$$

Using the decay hypothesis on the potential (1.10) one has that there exists $\tilde{\lambda} > 0$ such that if $\lambda \geq \tilde{\lambda}$ and $|x| \geq 1$, then

$$(3.10) \quad 2\lambda^2 \nabla\varphi D^2\varphi \nabla\varphi + \Delta^2\varphi - |V|^2 \geq \frac{\lambda}{r(1 + \log r)^2}.$$

Thus, from (3.8) and $\lambda \gg 1$

$$(3.11) \quad \begin{aligned} & 4\lambda \int_{T_1}^{T_j} \int \nabla f D^2\varphi \nabla \bar{f} dxdt + 2\lambda^3 \int_{T_1}^{T_j} \int \nabla\varphi D^2\varphi \nabla\varphi |f|^2 dxdt \\ & \leq \tilde{c}_\lambda + c(\lambda \|\Delta^2\varphi\|_\infty + \|V\|_\infty) \int_{T_1}^{T_j} \int_{|x| \leq 1} |f|^2 dxdt \\ & \leq \tilde{c}_\lambda + c\lambda \int_{T_1}^{T_j} \int_{|x| \leq 1} |f|^2 dxdt. \end{aligned}$$

Next, for a fixed $\epsilon \in (0, 1)$ we consider the domain $\{x : \epsilon \leq |x| \leq 1\}$. In this region

$$(3.12) \quad \nabla\varphi D^2\varphi \nabla\varphi \geq c_{\varphi, \epsilon}, \quad \text{for } \epsilon \leq |x| \leq 1.$$

Therefore, for large enough $\lambda \geq \lambda_\epsilon$

$$(3.13) \quad \lambda^2 \nabla\varphi D^2\varphi \nabla\varphi + \Delta^2\varphi - |V|^2 \geq \lambda, \quad \text{for } \epsilon \leq |x| \leq 1.$$

Hence

$$(3.14) \quad \begin{aligned} & 4\lambda \int_{T_1}^{T_j} \int \nabla f D^2\varphi \nabla \bar{f} dxdt + 2\lambda^3 \int_{T_1}^{T_j} \int \nabla\varphi D^2\varphi \nabla\varphi |f|^2 dxdt \\ & \leq \tilde{c}_\lambda + c'\lambda \int_{T_1}^{T_j} \int_{|x| \leq \epsilon} |f|^2 dxdt, \end{aligned}$$

with c' independent of $\epsilon \in (0, 1]$. In the domain $\{x : |x| \leq \epsilon\}$ we shall use that φ is strictly convex in $r = |x| \leq 2$ to get from (3.11) that

$$(3.15) \quad \begin{aligned} & 2c_\varphi \lambda \int_{T_1}^{T_j} \int_{|x| \leq 2\epsilon} |\nabla f|^2 dx dt + \lambda^3 \int_{T_1}^{T_j} \int \nabla \varphi D^2 \varphi \nabla \varphi |f|^2 dx dt \\ & \leq \tilde{c}_\lambda + c' \lambda \int_{T_1}^{T_j} \int_{|x| \leq \epsilon} |f|^2 dx dt, \end{aligned}$$

with c_φ and c' independent of $\epsilon \in (0, 1]$. Choosing $\theta \in C^\infty(\mathbb{R}^n)$ such that $\theta(x) \equiv 1$ for $|x| \leq \epsilon$ with $\text{supp } \theta \subset \{x : |x| \leq 2\epsilon\}$ and using Poincaré's inequality to get that for each $t \in [T_1, T_j]$ it follows that

$$(3.16) \quad \begin{aligned} \int_{|x| \leq \epsilon} |f|^2 dx & \leq \int_{|x| \leq 2\epsilon} |\theta f|^2 dx \leq c_\varphi \epsilon^2 \int_{|x| \leq 2\epsilon} |\nabla(\theta f)|^2 dx \\ & \leq c_\varphi \epsilon^2 \int_{|x| \leq 2\epsilon} |\nabla f|^2 dx + c_\varphi \int_{\epsilon \leq |x| \leq 2\epsilon} |f|^2 dx. \end{aligned}$$

Gathering the above estimates by fixing ϵ sufficiently small and then $\lambda > \tilde{\lambda}$ large enough one concludes that

$$(3.17) \quad \begin{aligned} & \lambda \int_{T_1}^{T_j} \int \nabla f D^2 \varphi \nabla f dx dt + \lambda^2 \int_{T_1}^{T_j} \int_{|x| \leq 1} |f|^2 dx dt \\ & + \lambda^3 \int_{T_1}^{T_j} \int \nabla \varphi D^2 \varphi \nabla \varphi |f|^2 dx dt \leq \tilde{c}_\lambda. \end{aligned}$$

In particular

$$(3.18) \quad \int_{T_1}^{T_j} \int \frac{|f|^2}{\langle x \rangle (\log \langle x \rangle)^2} dx dt \leq \tilde{c}_\lambda, \quad \text{independent of } j \in \mathbb{Z}^+,$$

which completes the proof of the step 2.

We fixed $\lambda = \lambda_0$ above for the rest of the proof.

Step 3 : $u(x, t) \equiv 0$

Proof of step 3 : On one hand, since the potential $V = V(x, t)$ is real, then the L^2 -norm of the solution $u(x, t)$ of (1.1) is preserved, i.e. for all $t \in \mathbb{R}$

$$\|u(\cdot, t)\|_2 = \|u_0\|_2.$$

On the other hand, from step 2 (3.7)

$$\begin{aligned} (T_j - T_1) \|u_0\|_2^2 & = \int_{T_1}^{T_j} \int |u(x, t)|^2 dx dt \\ & = \int_{T_1}^{T_j} \int |u(x, t)|^2 \frac{e^{2\lambda\varphi}}{\langle x \rangle (\log \langle x \rangle)^2} \langle x \rangle (\log \langle x \rangle)^2 e^{-2\lambda\varphi} dx dt \\ & \leq \sup_{x \in \mathbb{R}^n} (\langle x \rangle (\log \langle x \rangle)^2 e^{-2\lambda\varphi}) \int_{T_1}^{T_j} \int |u(x, t)|^2 \frac{e^{2\lambda\varphi}}{\langle x \rangle (\log \langle x \rangle)^2} dx dt \\ & \leq \tilde{c}_{\lambda_0}, \end{aligned}$$

which completes the proof of Theorem 2 in the case $V_2 \equiv 0$.

The proof in the general case follows the same argument already explained in the proof of Theorem 1 so it will be omitted.

Proof of Theorem 3

The only differences with the previous cases are following computations:

$$\begin{aligned}\mathcal{S} &= -i\lambda(2\nabla\varphi \cdot \tilde{\nabla} + \mathcal{L}_k), & \tilde{\nabla} &= (\partial_{x_1}, \dots, \partial_{x_k}, -\partial_{x_{k+1}}, \dots, -\partial_{x_n}), \\ \mathcal{A} &= i(\mathcal{L}_k + \lambda^2((\partial_{x_1}\varphi)^2 + \dots + (\partial_{x_k}\varphi)^2 - (\partial_{x_{k+1}}\varphi)^2 - (\partial_{x_n}\varphi)^2),\end{aligned}$$

so

$$[\mathcal{S}; \mathcal{A}] = -\lambda(4\tilde{\nabla} \cdot D^2\varphi\tilde{\nabla}) - 4\lambda^2\tilde{\nabla}\varphi D^2\varphi\tilde{\nabla}\varphi + \mathcal{L}_k\mathcal{L}_k\varphi.$$

Hence, the method of proof used in Theorems 1-2 for the elliptic case $\mathcal{L}_k = \Delta$ can be applied to obtain the same results in this non-degenerate case.

4. PROOF OF THEOREM 7

The conformal transformation (1.41) with $\nu = \omega = \theta = 1$ and $\gamma = 0$ tells us that

$$(4.1) \quad w(x, t) = \frac{1}{(1+t)^{n/2}} e^{i|x|^2/4(1+t)} v\left(\frac{x}{1+t}, \frac{t}{1+t}\right),$$

solves the equation

$$(4.2) \quad i\partial_t w + \Delta w \pm (1+t)^{an/2-2}|w|^a w = 0,$$

in the time interval $t \in [0, \infty)$. Thus, from the hypotheses (1.45) it follows that the solution $w(x, t)$ satisfies

$$(4.3) \quad \begin{aligned}|w(x, t)| &= \frac{1}{(1+t)^{n/2}} \left| v\left(\frac{x}{1+t}, \frac{t}{1+t}\right) \right| \\ &\leq \frac{1}{(1+t)^{n/2}} \frac{1}{(1-\frac{t}{1+t})^{2/a}} Q\left(\frac{\frac{x}{1+t}}{1-\frac{t}{1+t}}\right) = \frac{1}{(1+t)^{n/2-2/a}} Q(x).\end{aligned}$$

Since the potential $V(x, t)$ has the form

$$V(x, t) = \pm(1+t)^{an/2-2}|w(x, t)|^a,$$

from (4.3) one sees that it verifies that

$$(4.4) \quad |V(x, t)| \leq (1+t)^{an/2-2} \left(\frac{1}{(1+t)^{n/2-2/a}} \right)^a Q^a(x) = Q^a(x).$$

Therefore, since $a \geq 4/n > 0$ from our hypothesis (1.46) or (1.47) it follows that the potential in (4.2) satisfies the hypothesis in Theorem 1 and Theorem 2 with $V_2 \equiv 0$. Since the L^2 -norm of the solution $w(x, t)$ is preserved for all $t \geq 0$, Theorem 1 and Theorem 2 yield the desired result.

5. PROOFS OF COROLLARIES 2 AND COROLLARY 3

Proof of Corollary 2.

We observe that if $u(x, t)$ solves the equation in (1.1), then

$$(5.1) \quad w(x, t) = u(x - \mu t \vec{e}, t) e^{i(\frac{\mu}{2}x \cdot \vec{e} - \frac{\mu^2 t}{4})},$$

is a solution of the equation

$$(5.2) \quad \partial_t w = i(\Delta w + V(x - \mu t \vec{e}, t) w).$$

Thus, from hypothesis (1.53) and (1.56) the potential in (5.2)

$$(5.3) \quad W(x, t) \equiv V(x - \mu t \vec{e}, t)$$

satisfies the conditions on Theorems 1 and 2, respectively. Therefore, they can be applied to the equation (5.2) to obtain the result.

In the case of the equation (1.15) the transformation (5.1) reads

$$(5.4) \quad w(x, t) = u(x - \mu t \vec{e}, t) e^{i(\frac{\mu}{2}x \cdot \vec{e}(k) - \frac{\mu^2 t}{4} \mathcal{Q}_k(\vec{e}))},$$

with

$$(5.5) \quad \vec{e}(k) = (e_1, \dots, e_k, -e_{k+1}, \dots, -e_n), \quad \text{if} \quad \vec{e} = (e_1, \dots, e_n),$$

and \mathcal{Q}_k as in (1.51). The function $w(x, t)$ satisfies the equation

$$(5.6) \quad \partial_t w = i(\mathcal{L}_k w + V(x - \mu t \vec{e}, t) w).$$

Hence, the potential

$$(5.7) \quad W(x, t) \equiv V(x - \mu t \vec{e}, t)$$

and the solution $w(x, t)$ of (5.6) satisfies the requirements in Theorem 3.

Proof of Corollary 3.

If $u(x, t)$ is a solution of the equation (1.59)

$$\partial_t u = i(\Delta u + F(u, \bar{u}) u),$$

then

$$(5.8) \quad v(x, t) = u(x - \mu t \vec{e}, t) e^{i(\frac{\mu}{2}x \cdot \vec{e} - \frac{\mu^2 t}{4})},$$

satisfies the equation

$$(5.9) \quad \partial_t v = i(\Delta v + F(e^{-i(\frac{\mu}{2}x \cdot \vec{e} - \frac{\mu^2 t}{4})} v, e^{i(\frac{\mu}{2}x \cdot \vec{e} - \frac{\mu^2 t}{4})} \bar{v}) v).$$

So in this case from the hypothesis on $F(z, \bar{z})$ the potential

$$(5.10) \quad W(x, t) \equiv F(e^{-i(\frac{\mu}{2}x \cdot \vec{e} - \frac{\mu^2 t}{4})} v, e^{i(\frac{\mu}{2}x \cdot \vec{e} - \frac{\mu^2 t}{4})} \bar{v}),$$

verifies that

$$|W(x, t)| \leq M(|v(x, t)| + |v(x, t)|^j) = M(|u(x - 2\mu \vec{e}t, t)| + |u(x - 2\mu \vec{e}t, t)|^j).$$

Thus, the assumption (1.62) guarantees that we can use Corollary 1 and Theorem 2 to achieve the result.

In the case of the equation (1.60)

$$\partial_t u = i(\mathcal{L}_k u + F(u, \bar{u}) u),$$

one just needs to define $v(x, t)$ as

$$(5.11) \quad v(x, t) = u(x - \mu t \vec{e}, t) e^{i(\frac{\mu}{2}x \cdot \vec{e}(k) - \frac{\mu^2 t}{4} \mathcal{Q}_k(\vec{e}))},$$

with $\vec{e}(k)$ as in (5.5) and \mathcal{Q}_k as in (1.51). Since $v(x, t)$ solves the equation

$$(5.12) \quad \partial_t v = i(\mathcal{L}_k v + F(e^{-i(\frac{\mu}{2}x \cdot \vec{e}(k) - \frac{\mu^2 t}{4} \mathcal{Q}_k(\vec{e}))} v, e^{i(\frac{\mu}{2}x \cdot \vec{e}(k) - \frac{\mu^2 t}{4} \mathcal{Q}_k(\vec{e}))} \bar{v}) v),$$

one just needs to follow the argument given in the case of the equation (1.59) to obtain the desired result.

6. PROOFS OF THEOREM 5 AND THEOREM 6

Proof of Theorem 6.

We have

$$e^{\tau|x|}(\Delta + \tilde{V}_2)e^{-\tau|x|} = \mathcal{S} + \mathcal{A},$$

where

$$(6.1) \quad \mathcal{S} = \Delta + \tilde{V}_2 + \tau^2, \quad \mathcal{A} = -\frac{\tau}{|x|} (2x \cdot \nabla + n - 1).$$

Hence, the commutator of \mathcal{S} and \mathcal{A} is

$$[\mathcal{S}; \mathcal{A}] = -4\tau \partial_j \cdot \left(\left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right) \partial_k \right) + \frac{\tau(n-1)(n-3)}{|x|^3} + \tau \partial_r \tilde{V}_2.$$

Let $g \in C_0^\infty(\mathbb{R}^n \setminus \overline{B_\rho})$ and set $f = e^{\tau|x|}g$. Then,

$$(6.2) \quad \begin{aligned} \|e^{\tau|x|}(\Delta + \tilde{V}_2)g\|_2^2 &= \|\mathcal{S}f\|_2^2 + \|\mathcal{A}f\|_2^2 + \int_{\mathbb{R}^n} [\mathcal{S}; \mathcal{A}]f\bar{f} dx \\ &= \|\mathcal{S}f\|_2^2 + \|\mathcal{A}f\|_2^2 + \tau \int_{\mathbb{R}^n} \frac{4}{|x|} (|\nabla f|^2 - |\partial_r f|^2) \\ &\quad + \tau \int_{\mathbb{R}^n} \left(\frac{(n-1)(n-3)}{|x|^3} + \partial_r \tilde{V}_2 \right) |f|^2 dx, \end{aligned}$$

with $\partial_r f = \frac{x}{|x|} \cdot \nabla f$ and

$$(6.3) \quad \begin{aligned} \|\mathcal{A}f\|_2 &= \tau \|2\partial_r f + \frac{n-1}{|x|} f\|_2 \geq \sqrt{\tau} \|2\partial_r f + \frac{n-1}{|x|} f\|_2 \\ &\geq 2\sqrt{\tau} \|\partial_r f\|_2 - \sqrt{\tau}(n-1) \| |x|^{-1} f \|_2 \\ &\geq \sqrt{\tau\rho} \| |x|^{-1/2} \partial_r f \|_2 - \sqrt{\tau/\rho} \| |x|^{-1/2} f \|_2 \end{aligned}$$

for $\tau \geq 1$. Combining our hypotheses on the potential (1.27)-(1.29), (6.2) and (6.3) one gets that

$$(6.4) \quad \|\mathcal{S}f\|_2 + \sqrt{\tau\rho} \| |x|^{-1/2} \nabla f \|_2 \leq \|e^{\tau|x|}(\Delta + \tilde{V}_2)g\|_2 + \sqrt{\tau/\rho} \| |x|^{-1/2} f \|_2.$$

Thus using (6.1) it follows that

$$(6.5) \quad \begin{aligned} \tau^3 \int_{\mathbb{R}^n} \frac{|f|^2}{|x|} dx &= \tau \Re \int_{\mathbb{R}^n} \frac{1}{|x|} [\mathcal{S}f\bar{f} - \Delta f\bar{f} - \tilde{V}_2|f|^2] dx \\ &= \tau \Re \int_{\mathbb{R}^n} \frac{1}{|x|} \mathcal{S}f\bar{f} dx - \tau \int_{\mathbb{R}^n} \frac{1}{|x|} \left[\frac{1}{2} \Delta |f|^2 - |\nabla f|^2 + \tilde{V}_2 |f|^2 \right] dx \\ &= \tau \Re \int_{\mathbb{R}^n} \frac{1}{|x|} \left[\mathcal{S}f\bar{f} + |\nabla f|^2 + \frac{(n-3)}{2} \frac{|f|^2}{|x|^2} - \tilde{V}_2 |f|^2 \right] dx. \end{aligned}$$

The last identity, our hypotheses on the potential (1.27)-(1.29), (6.4) and the Cauchy-Schwarz inequality show that Theorem 6 holds for $\tau \geq \tau_0$ with $\tau_0 = \tau_0(n, \|\tilde{V}\|_\infty; c_1; c_2; \rho)$.

Proof of Theorem 5.

We fix $\phi \in C_0^\infty(\mathbb{R}^n)$ such that ϕ is positive, with $\phi(x) = 1$, $|x| \leq 1$ and $\text{supp } \phi \subset \{x : |x| \leq 2\}$ and rewrite the equation (1.20) as

$$(6.6) \quad \Delta u + \tilde{V}(x)u - \zeta u = \Delta u + \tilde{V}(x)u = \Delta u + \tilde{V}_1(x)u + \tilde{V}_2(x)u = 0,$$

with

$$(6.7) \quad \tilde{V}_1(x) = \tilde{V}(x) - \zeta\phi(x), \quad \tilde{V}_2(x) = -\zeta(1 - \phi(x)).$$

Thus, \tilde{V}_1, \tilde{V}_2 satisfy the hypotheses of Theorems 5 and 6. We shall define ϕ_L as

$$\phi_L(x) = \phi(x/L), \quad L > 0.$$

Claim : There exist $\rho_0 \in [0, 1)$ and $M = M(n)$ such that

$$(6.8) \quad \begin{aligned} \|u\|_{L^2(B_{4\rho_0})}^2 &= \int_{|x| \leq 4\rho_0} |u(x)|^2 dx \\ &\leq M \|u\|_{L^2(B_{10\rho_0} - B_{5\rho_0})}^2 = \int_{5\rho_0 \leq |x| \leq 10\rho_0} |u(x)|^2 dx. \end{aligned}$$

Proof of the claim : Multiplying the equation (6.6) by $u\phi_{5\rho}^2$, with ρ to be determined and integrating the result one gets

$$(6.9) \quad - \int |\nabla u|^2 \phi_{5\rho}^2 dx + \int |u|^2 (2|\nabla \phi_{5\rho}|^2 + \phi_{5\rho} \Delta \phi_{5\rho}) dx + \int \Re(\tilde{V}) |u|^2 \phi_{5\rho}^2 dx.$$

Combining (6.9) and Poincare inequality one has that

$$(6.10) \quad \begin{aligned} &\int |u\phi_{5\rho}|^2 dx \leq (10\rho)^2 \int |\nabla(u\phi_{5\rho})|^2 dx \\ &\leq (10\rho)^2 \int |\nabla u|^2 \phi_{5\rho}^2 dx + c_n \int |u|^2 \phi_{5\rho} |\nabla \phi_{5\rho}| dx \\ &\leq (10\rho)^2 (c_n \int_{B_{10\rho} - B_{5\rho}} |u|^2 dx + \|\tilde{V}\|_\infty \int |u\phi_{5\rho}|^2 dx) + c_n \int_{B_{10\rho} - B_{5\rho}} |u|^2 dx. \end{aligned}$$

Fixing ρ_0 small enough, depending on the $\|\tilde{V}\|_\infty$, we establish the claim (6.8).

Next, we apply Theorem theorem20a to $u\Phi = u\Phi_{\rho,R}$ where $\Phi \in C_0^\infty(\mathbb{R}^n)$ with $\Phi(x) = 1$, $4\rho \leq |x| \leq R$, $\Phi(x) = 0$, $|x| \geq 2R$, $\Phi(x) = 0$, $|x| \leq 2\rho$ with $R > 10$ and $\rho \in (0, 1)$ to get that

$$(6.11) \quad \begin{aligned} \tau^3 \||x|^{-1/2} e^{\tau|x|} (u\Phi)\|_2^2 &\leq \|e^{\tau|x|} (\Delta + \tilde{V})(u\Phi)\|_2^2 \\ &\leq 4\|e^{\tau|x|} \nabla u \cdot \nabla \Phi\|_2^2 + 2\|e^{\tau|x|} u \Delta \Phi\|_2^2 \\ &\leq 4\|e^{\tau|x|} \nabla u \cdot \nabla \Phi\|_2^2 + 2c_n \|e^{\tau|x|} u\|_{L^2((B_{2R} - B_R) \cup (B_{4\rho} - B_{2\rho}))}^2. \end{aligned}$$

Using integrations by part and the equation (6.6) one gets that that

$$(6.12) \quad \|e^{\tau|x|} \nabla u \cdot \nabla \Phi\|_2^2 \leq c_n (\|\tilde{V}\|_\infty + \tau^2 + \frac{\tau}{\rho}) \|e^{\tau|x|} u \cdot \nabla \Phi\|_2^2.$$

Therefore

$$A_1 \equiv \tau^3 \||x|^{-1/2} e^{\tau|x|} (u\Phi)\|_2^2 \leq c_n (\|\tilde{V}\|_\infty + \tau^2 + \frac{\tau}{\rho}) \|e^{\tau|x|} u\|_{L^2((B_{2R} - B_R) \cup (B_{4\rho} - B_{2\rho}))}^2 \equiv A_2.$$

On one hand one has that

$$A_1 \geq \tau^3 \|\frac{e^{\tau|x|} u}{|x|^{1/2}}\|_{L^2(B_R - B_{2\rho})}^2 \geq c_n \frac{\tau^3}{\rho} \|e^{\tau|x|} u\|_{L^2(B_{10\rho} - B_{5\rho})}^2 \geq c_n \frac{\tau^3}{\rho} e^{10\tau\rho} \|u\|_{L^2(B_{10\rho} - B_{5\rho})}^2.$$

On the other hand,

$$A_2 \leq c_n (\|\tilde{V}\|_\infty + \tau^2 + \frac{\tau}{\rho}) e^{8\tau\rho} \|u\|_{L^2(B_{4\rho})}^2 + c_n (\|\tilde{V}\|_\infty + \tau^2 + \frac{\tau}{\rho}) e^{4\tau R} \|u\|_{L^2(B_{2R}-B_R)}^2.$$

Therefore, fixing $\rho = \rho_0$ as in the claim it follows that

$$(6.13) \quad \begin{aligned} M \frac{\tau^3}{\rho_0} e^{10\tau\rho_0} \|u\|_{L^2(B_{4\rho_0})}^2 &\leq \frac{\tau^3}{\rho_0} e^{10\tau\rho_0} \|u\|_{L^2(B_{10\rho_0}-B_{5\rho_0})}^2 \\ &\leq c_n (\|\tilde{V}\|_\infty + \tau^2 + \frac{\tau}{\rho_0}) e^{8\tau\rho_0} \|u\|_{L^2(B_{4\rho_0})}^2 + c_n (\|\tilde{V}\|_\infty + \tau^2 + \frac{\tau}{\rho_0}) e^{4\tau R} \|u\|_{L^2(B_{2R}-B_R)}^2. \end{aligned}$$

Therefore, for τ sufficiently large but independently of $R > 10$ it follows that

$$\|u\|_{L^2(B_{2R}-B_R)}^2 \geq c_n e^{10\tau\rho_0} e^{-4\tau R} \|u\|_{L^2(B_{4\rho_0})}^2.$$

Finally, taking $\lambda_0 > 2\tau$ one has

$$(6.14) \quad \begin{aligned} \infty &> \int e^{2\lambda_0|x|} |u(x)|^2 dx \geq \sum_{k=1}^{\infty} \int_{2^{k-1}R \leq |x| \leq 2^k} e^{2\lambda_0|x|} |u(x)|^2 dx \\ &\geq \sum_{k=1}^{\infty} e^{2^k \lambda_0 R} \int_{2^{k-1}R \leq |x| \leq 2^k} |u(x)|^2 dx \\ &\geq \sum e^{2^k R \lambda_0} e^{-2^{k+1} \tau R} e^{8\tau\rho_0} \|u\|_{L^2(B_{4\rho_0})}^2, \end{aligned}$$

which gives a contradiction except if $\|u\|_{L^2(B_{4\rho_0})}^2 = 0$.

7. APPENDIX

Part (a): We recall that $p \in (1, 4/3]$. The aim is to find

$$(7.1) \quad \varphi(r) = a_0 + a_1 r^2 + a_2 r^4 + a_3 r^6 + a_4 r^8, \quad r \in [0, 1],$$

such that

$$(7.2) \quad \begin{aligned} \varphi(1) = d_0, \quad \varphi'(1) = d_1, \quad \varphi^{(2)}(1) = d_2 > 0, \\ \varphi^{(3)}(1) = d_3 < 0, \quad \varphi^{(4)}(1) = d_4 > 0. \end{aligned}$$

for prescribed values d_0, \dots, d_4 such that $\varphi(0) = 0$ and φ is strictly convex for $r \in [0, 1]$. Since in Theorem 1 $\varphi(r) = r^p + \beta$, $r \geq 1$ one has

$$(7.3) \quad \begin{aligned} d_0 = 1 + \beta, \quad d_1 = p > 0, \quad d_2 = p(p-1) > 0, \\ d_3 = p(p-1)(p-2) < 0, \quad d_4 = p(p-1)(p-2)(p-3) > 0. \end{aligned}$$

So we solve the system

$$(7.4) \quad \left\{ \begin{aligned} a_0 + a_1 + a_2 + a_3 + a_4 &= d_0 = 1 + \beta, \\ 2a_1 + 4a_2 + 6a_3 + 8a_4 &= d_1 = p, \\ 2a_1 + 12a_2 + 30a_3 + 56a_4 &= d_2 = p(p-1), \\ 24a_2 + 120a_3 + 336a_4 &= d_3 = p(p-1)(p-2), \\ 24a_2 + 360a_3 + 1680a_4 &= d_4 = p(p-1)(p-2)(p-3). \end{aligned} \right.$$

After some computations one sees that

$$(7.5) \quad \begin{aligned} a_1 &= \frac{p}{6 \cdot 16} (192 - 104p + 18p^2 - p^3) > \frac{p}{2}, & a_2 &= \frac{p(p-2)}{4 \cdot 16} (p-6)(p-8), \\ a_3 &= \frac{-p(p-2)}{6 \cdot 16} (p-4)(p-8), & a_4 &= \frac{p(p-2)}{24 \cdot 16} (p-4)(p-6). \end{aligned}$$

Next, we shall see that this φ is convex in $r \in [0, 1]$. From (7.4) and (7.5) one has

$$(7.6) \quad \varphi^{(2)}(1) = p, \quad \varphi^{(2)}(0) = 2a_1 > p,$$

so it will suffice to show that

$$(7.7) \quad \begin{aligned} \varphi^{(3)}(r) &= 24r(a_2 + 5a_3r^2 + 14a_4r^4) \\ &= 24r \frac{p(p-2)}{12 \cdot 16} (3(p-6)(p-8) - 10(p-4)(p-8)r^2 + 7(p-2)(p-6)r^4) \end{aligned}$$

has no critical points in $(0, 1)$. After some computations one finds that the discriminant \mathcal{D} of the quadratic equation (in r^2) in (7.7) is

$$(7.8) \quad \begin{aligned} \mathcal{D} &= (p-4)(p-8) (10^2(p-4)(p-8) - 84(p-6)^2) \\ &= 16(p-1)(p-4)(p-8)(p-11) < 0, \end{aligned}$$

because $p \in (1, 4/3)$. Since $\varphi^{(3)}$ has no critical points (7.6) tells us that φ is strictly convex in $[0, 1]$. Taking β in (7.4) as

$$\beta = a_1 + a_2 + a_3 + a_4 - 1,$$

it follows that $\varphi(0) = a_0 = 0$. Finally, if $\phi(r) = r^p$

$$\varphi(0) = \varphi'(0) = \phi(0) = \phi'(0) = 0, \quad \phi^{(2)}(r) = p(p-1)r^{p-2} \geq p(p-1) \quad r \in (0, 1).$$

Thus, there exists $M_0 > 0$ such that

$$M_0 p(p-1) \geq \sup_{0 \leq r \leq 1} |\varphi^{(2)}(r)|.$$

Finally, taking $M = \max\{M_0; \beta\}$ one gets that

$$\varphi(r) \leq Mr^p, \quad \forall r \geq 0,$$

which completes the proof.

Part (b): As in the proof of Theorem 2 we choose

$$\varphi(r) = 3r - \int_1^r \frac{dt}{1 + \log t} + \beta,$$

so in this case we have

$$(7.9) \quad d_0 = 3 + \beta, \quad d_1 = 2, \quad d_2 = 1, \quad d_3 = -3, \quad d_4 = 14.$$

Solving the system (7.4) with these values of (d_0, d_1, \dots, d_4) one gets

$$(7.10) \quad \varphi(r) = a_0 + \frac{103}{96}r^2 + \frac{9}{64}r^4 - \frac{17}{96}r^6 + \frac{17}{24 \cdot 16}r^8, \quad r \in [0, 1].$$

To show that φ is convex in $[0, 1]$, we consider

$$(7.11) \quad \varphi^{(2)}(r) = \frac{1}{48}(103 + 81r^2 - 225r^4 + 119r^6), \quad r \in [0, 1],$$

and recall that

$$(7.12) \quad \varphi^{(2)}(0) = 103/48, \quad \varphi^{(2)}(1) = 1.$$

We look for critical points of

$$(7.13) \quad \varphi^{(3)}(r) = \frac{r}{8}(27 - 150r^2 + 119r^4), \quad r \in (0, 1).$$

There is only one critical point the point $r_0 \in (0, 1]$ with

$$(7.14) \quad r_0^2 = \frac{150 - \sqrt{(150)^2 - 4 \cdot 119 \cdot 27}}{2 \cdot 119} = \frac{150 - \sqrt{9648}}{238} \in (0, 1).$$

Since

$$(7.15) \quad \varphi^{(2)}(r_0) \geq 110/48,$$

combining (7.15) and (7.12) it follows that φ is convex in $[0, 1]$. Finally, taking β in (7.9) such that

$$\beta = a_1 + a_2 + a_3 + a_4 - 1,$$

it follows that $\varphi(0) = a_0 = 0$. Finally, an argument similar to that at the end of part (a) shows

$$\varphi(r) \leq Mr, \quad \forall r \geq 0,$$

which provides the desired result.

ACKNOWLEDGMENT : The authors would like to thank J. C. Saut for fruitful conversations concerning this work.

REFERENCES

- [1] M. J. Ablowitz, R. Haberman, *Nonlinear evolution equations in two and three dimensions*, Phys. Rev. Lett., **35** (1975), 1185–1188.
- [2] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations*, Arch. Rational Mech. Anal., **82** (1983), 313–375.
- [3] T. Cazenave, F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s* , Nonlinear Analysis TMA **14** (1990) 807–836.
- [4] J. Cruz-Uribe, *Unique continuation at infinity of solutions to Schrödinger equations with complex potentials*, Proc. Edinburgh Math. Soc., **42**, (1999), 143–153.
- [5] A. Davey, K. Stewartson, *On three-dimensional packets of surface waves*, Proc. Royal London Soc. A **338** (1974), 101–110.
- [6] J. M. Ghidaglia, J. C. Saut, *Nonexistence of travelling wave solutions to nonelliptic nonlinear Schrödinger equations*, J. Nonlinear Sci. **6** (1996), 139145.
- [7] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations*, J. Funct. Anal. **32** (1979), 1–71.
- [8] L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega, *On Uniqueness Properties of Solutions of Schrödinger Equations*, Comm. PDE. **31**, 12 (2006), 1811–1823.
- [9] L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega, *Convexity of Free Solutions of Schrödinger Equations with Gaussian Decay*, Math. Res. Letters, **15**, (2008), 957–972.
- [10] L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega, *Hardy’s uncertainty principle, convexity and Schrödinger equations*, Journal European Math. Soc. **10** (2008), 882–907.
- [11] L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega, *The sharp Hardy Uncertainty Principle for Schrödinger evolutions*, to appear in Duke Math. J.
- [12] L. Escauriaza, C. E. Kenig, G. Ponce, L. Vega, *Uncertainty Principle of Morgan type and Schrödinger Evolutions*, to appear in Journal London Math. Soc.
- [13] Y. Ishimori, *Multi vortex solutions of a two dimensional nonlinear wave equation*, Progr. Theor. Phys., **72** (1984), 33–37
- [14] V. Z. Meshkov, *On the possible rate of decay at infinity of solutions of second-order partial differential equations*, Math. USSR Sbornik **72** (1992), 343361.
- [15] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. math. Phys., **55** (1977), 149–162.

- [16] R. S. Strichartz, R. S., Restriction of Fourier transforms to quadratic surface and decay of solutions of wave equations, *Duke Math. J.* **44** (1977) 705-714.

(L. Escauriaza) UPV/EHU, DPTO. DE MATEMÁTICAS, APTO. 644, 48080 BILBAO, SPAIN.
E-mail address: `luis.escauriaza@ehu.es`

(C. E. Kenig) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL. 60637, USA.

E-mail address: `cek@math.uchicago.edu`

(G. Ponce) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA.

E-mail address: `ponce@math.ucsb.edu`

(L. Vega) UPV/EHU, DPTO. DE MATEMÁTICAS, APTO. 644, 48080 BILBAO, SPAIN.

E-mail address: `luis.vega@ehu.es`