

UNIQUENESS PROPERTIES OF SOLUTIONS TO SCHRÖDINGER EQUATIONS

L. ESCAURIAZA, C. E. KENIG, G. PONCE, AND L. VEGA

1. INTRODUCTION

To place the subject of this paper in perspective, we start out with a brief discussion of unique continuation. Consider solutions to

$$(1.1) \quad \Delta u(x) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}(x) = 0,$$

(harmonic functions) in the unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$. When $n = 2$, these functions are real parts of holomorphic functions, and so, if they vanish of infinite order at $x = 0$, they must vanish identically. We call this the strong unique continuation property (s.u.c.p.). The same result holds for $n > 2$, since harmonic functions are still real analytic in $\{x \in \mathbb{R}^n : |x| < 1\}$. In fact, it is well-known that if $P(x, D)$ is a linear elliptic differential operator with real analytic coefficients, and $P(x, D)u = 0$ in a open set $\Omega \subset \mathbb{R}^n$, then u is real analytic in Ω . Hence, the (s.u.c.p.) also holds for such solutions. Through the work of Hadamard [28] on the uniqueness of the Cauchy problem (which is closely related to the strong unique continuation property discussed earlier) it became clear (for applications in nonlinear problems) that it would be desirable to establish the strong unique continuation property for operators whose coefficients are not necessarily real analytic, or even C^∞ . The first results in this direction were found in the pioneering work of Carleman [9] (when $n = 2$) and Müller [47] (when $n > 2$), who proved the (s.u.c.p) for

$$P(x, D) = \Delta + V(x), \quad \text{with} \quad V \in L_{loc}^\infty(\mathbb{R}^n).$$

In order to establish his result, Carleman introduced a method (the method of ‘‘Carleman estimates’’) which has permeated the subject ever since. In this context, an example of a Carleman estimate is :

For $f \in C_0^\infty(\{x \in \mathbb{R}^n : |x| < 1\} - \{0\})$, $\alpha > 0$ and

$$w(r) = r \exp\left(\int_0^r \frac{e^{-s} - 1}{s} ds\right),$$

one has

$$(1.2) \quad \alpha^3 \int w^{-1-2\alpha}(|x|) f^2(x) dx \leq c \int w^{2-2\alpha}(|x|) |\Delta f(x)|^2 dx,$$

with c independent of α

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For a proof of this estimate, see [26], [7]. The (s.u.c.p.) of Carleman-Müller follows easily from (1.2) (see [38] for instance).

In the late 1950's and 1960's there was a great deal of activity on the subject of (s.u.c.p.) and the closely related uniqueness in the Cauchy problem, some highlights being [1] and [8] respectively, both of which use the method of Carleman estimates. These results and methods have had a multitude of applications to many areas of analysis, including to non-linear problems. (For a recent example, see [39] for an application to energy critical non-linear wave equations).

In connection with the Carleman-Müller (s.u.c.p.) a natural question is : How fast is a solution u allowed to vanish, before it must vanish identically?

By considering $n = 2$, $u(x_1, x_2) = \Re(x_1 + ix_2)^N$, we see that to make sense of the question, a normalization is required, for instance

$$\sup_{|x| < 3/4} |u(x)| \geq 1, \quad \|u\|_{L^\infty(|x| < 1)} < \infty.$$

We refer to questions of this type as “quantitative unique continuation”. It is also of interest to consider unique continuation type questions around the point at infinity. For instance, a conjecture of E. M. Landis [41] was : if

$$\Delta u + Vu = 0, \quad x \in \mathbb{R}^n, \quad \text{with } \|V\|_\infty \leq 1, \quad \|u\|_\infty < \infty,$$

and for some $\epsilon > 0$ one has

$$|u(x)| \leq c_\epsilon e^{-c_\epsilon |x|^{1+\epsilon}},$$

then $u \equiv 0$.

For the case of complex valued potentials $V(x)$, this conjecture was disproved by Meshkov [45] who constructed V , u , $u \not\equiv 0$ with

$$|u(x)| \leq c e^{-c|x|^{4/3}}, \quad n \geq 2.$$

Meshkov also showed that if

$$|u(x)| \leq c_\epsilon e^{-c_\epsilon |x|^{4/3+\epsilon}}, \quad \text{for some } \epsilon > 0,$$

then $u \equiv 0$.

It turns out that a “quantitative” formulation of this can also be proved, as it was done in [7], and this was crucial for the resolution in [7] of a long-standing problem in disordered media, namely Anderson localization near the bottom of the spectrum, for the continuous Anderson-Bernoulli model in \mathbb{R}^n , $n \geq 1$.

Next, we turn to versions of unique continuation for evolution equations. We start with parabolic equations and consider solutions of

$$\partial_t u - \Delta u = W \cdot \nabla u + Vu, \quad \text{with } \|W\|_\infty + \|V\|_\infty < \infty,$$

(or equivalently $|\partial_t u - \Delta u| \leq M(|\nabla u| + |u|)$). Using a parabolic analog of the Carleman estimate described earlier, one can show that if

$$|\partial_t u - \Delta u| \leq M(|\nabla u| + |u|), \quad (x, t) \in \{x \in \mathbb{R}^n : |x| < 4R\} \times [t_0, t_1], \quad R > 0,$$

with $|u(x)| \leq A$ and

$$u \equiv 0, \quad (x, t) \in \{x \in \mathbb{R}^n : R < |x| < 4R\} \times [t_0, t_1],$$

then

$$u \equiv 0, \quad (x, t) \in \{x \in \mathbb{R}^n : |x| < R\} \times [t_0, t_1].$$

We call this type of result “unique continuation through spatial boundaries”, (see [26], [55] and references therein for this type of result and strengthenings of it).

This result is closely related to the “elliptic” (s.u.c.p.) discussed before. On the other hand, for parabolic equations, there is also a “backward uniqueness” principle, which is very useful in applications to control theory (see [44] for an early result in this direction) : Consider solutions to

$$|\partial_t u - \Delta u| \leq M(|\nabla u| + |u|), \quad (x, t) \in \mathbb{R}^n \times (0, 1],$$

with $\|u\|_\infty \leq A$. Then, if $u(\cdot, 1) \equiv 0$, we must have $u \equiv 0$. This result is also proved through Carleman estimates (see [44]).

Recently, a strengthening of this result has been obtained in [25], where one considers solutions only defined in $R_+^n \times (0, 1]$, $R_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$, without any assumptions on u at $x_1 = 0$, and still obtains the “backward uniqueness” result. This strengthening had an important application to non-linear equations, allowing the authors of [25] to establish a long-standing conjecture of J. Leray on regularity and uniqueness of solutions to the Navier-Stokes equations (see also [52] for a recent extension).

Finally, we turn to dispersive equations. Typical examples of these are the k -generalized KdV equation

$$(1.3) \quad \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}, \quad k \in \mathbb{Z}^+,$$

and the non-linear Schrödinger equation

$$(1.4) \quad \partial_t u = i(\Delta u \pm |u|^{p-1}u), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \quad p > 1.$$

These equations model phenomena of wave propagation and have been extensively studied in the last 30 years or so.

For these equations, “unique continuation through spatial boundaries” also holds, as it was shown by Saut-Scheurer [51] for the KdV-type equations and by Izakov [36] for Schrödinger type equations. (All of these results were established through Carleman estimates). These equations however are time reversible (no preferred time direction) and so “backward uniqueness” is immediate, unlike in parabolic problems. Once more in connection with control theory, this time for dispersive equations, Zhang [56] showed, for solutions of

$$(1.5) \quad \partial_t u = i(\partial_x^2 u \pm |u|^2 u), \quad (x, t) \in \mathbb{R} \times [0, 1],$$

that if $u(x, t) = 0$ for $(x, t) \in (-\infty, a) \times \{0, 1\}$ (or $(x, t) \in (a, \infty) \times \{0, 1\}$) for some $a \in \mathbb{R}$, the $u \equiv 0$. Zhang’s proof was based on the inverse scattering method which uses that this is a completely integrable model, and did not apply to other non-linearities or dimensions. This type of result was extended to the k -generalized KdV (1.3) and the general non-linear Schrödinger equation in (1.4) in all dimensions (where inverse scattering is no longer available) using suitable Carleman estimates (see [40], [34], [35], and references therein).

For recent surveys of the results presented so far, see [37], [38].

Returning to “backward uniqueness” for parabolic equations, in analogy with Landis’ “elliptic” conjecture mentioned earlier, Landis-Oleinik [43] conjectured that in the “backward uniqueness” result one can replace the hypothesis $u(\cdot, 1) \equiv 0$ with the weaker one

$$|u(x, 1)| \leq c_\epsilon e^{-c_\epsilon |x|^{2+\epsilon}}, \quad \text{for some } \epsilon > 0.$$

This is indeed true and was established in [18] and [49]. Similarly, one can conjecture (as it was done in [20]) that for Schrödinger equations, if

$$|u(x, 0)| + |u(x, 1)| \leq c_\epsilon e^{-c_\epsilon |x|^{2+\epsilon}}, \quad \text{for some } \epsilon > 0,$$

then $u \equiv 0$. This was established in [18].

In analogy with the improvement of “backward uniqueness” in [25], one can show that it suffices to deal with solutions in $\mathbb{R}_+^n \times (0, 1]$ (for parabolic problems) and require

$$|u(x, 1)| \leq c_\epsilon e^{-c_\epsilon x_1^{2+\epsilon}}, \quad x_1 > 0, \quad \text{for some } \epsilon > 0,$$

to conclude that $u \equiv 0$ ([49]), and that for the Schrödinger equations it suffices to have u a solution in $\mathbb{R}_+^n \times [0, 1]$, with

$$|u(x, 0)| + |u(x, 1)| \leq c_\epsilon e^{-c_\epsilon x_1^{2+\epsilon}}, \quad x_1 > 0, \quad \text{for some } \epsilon > 0,$$

to conclude that $u \equiv 0$, as we will prove in section 5 of this paper.

In [16] it was pointed out for the first time (see also [10]) that both the results in [18] and in [16], in the case of the free heat equation

$$\partial_t u = \Delta u,$$

and the free Schrödinger equation

$$\partial_t u = i\Delta u,$$

respectively, are in fact a corollary of the more precise Hardy uncertainty principle for the Fourier transform, which says :

If $f(x) = O(e^{-|x|^2/\beta^2})$, $\widehat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $1/\alpha\beta > 1/4$, then $f \equiv 0$, and if $1/\alpha\beta = 1/4$, $f(x) = ce^{-|x|^2/\beta^2}$ as will be discussed below.

Thus, in a series of papers ([16]-[23], [11]) we took up the task of finding the sharp version of the Hardy uncertainty principle, in the context of evolution equations. The results obtained have already yielded new results on non-linear equations. For instance in [21] and [23] we have found applications to the decay of concentration profiles of possible self-similar type blow-up solutions of non-linear Schrödinger equations and to the decay of possible solitary wave type solutions of non-linear Schrödinger equations.

In the rest of this work we shall review some of our recent results concerning unique continuation properties of solutions of Schrödinger equations of the form

$$(1.6) \quad \partial_t u = i(\Delta u + F(x, t, u, \bar{u})), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

We shall be mainly interested in the case where

$$(1.7) \quad F(x, t, u, \bar{u}) = V(x, t)u(x, t)$$

is describing the evolution of the Schrödinger flow with a time dependent potential $V(x, t)$, and in the semi-linear case

$$(1.8) \quad F(x, t, u, \bar{u}) = F(u, \bar{u}),$$

with $F : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, $F(0, 0) = \partial_u F(0, 0) = \partial_{\bar{u}} F(0, 0) = 0$.

Let us consider a familiar dispersive model, the k -generalized Korteweg-de Vries equation (1.3) and recall a theorem established in [17] :

Theorem 1. *There exists $c_0 > 0$ such that for any pair*

$$u_1, u_2 \in C([0, 1] : H^4(\mathbb{R}) \cap L^2(|x|^2 dx))$$

of solutions of (1.3) such that if

$$(1.9) \quad u_1(\cdot, 0) - u_2(\cdot, 0), \quad u_1(\cdot, 1) - u_2(\cdot, 1) \in L^2(e^{c_0 x_+^{3/2}} dx),$$

then $u_1 \equiv u_2$.

Above we have used the notation: $x_+ = \max\{x; 0\}$.

Notice that taking $u_2 \equiv 0$ Theorem 1 gives a restriction on the possible decay of a non-trivial solution of (1.3) at two different times. The power $3/2$ in the exponent in (1.9) reflects the asymptotic behavior of the Airy function. More precisely, the solution of the initial value problem (IVP)

$$(1.10) \quad \begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x, 0) = v_0(x), \end{cases}$$

is given by the group $\{U(t) : t \in \mathbb{R}\}$

$$U(t)v_0(x) = \frac{1}{\sqrt[3]{3t}} \text{Ai} \left(\frac{\cdot}{\sqrt[3]{3t}} \right) * v_0(x),$$

where

$$\text{Ai}(x) = c \int_{-\infty}^{\infty} e^{ix\xi + i\xi^3} d\xi,$$

is the Airy function which satisfies the estimate

$$|\text{Ai}(x)| \leq c(1 + x_-)^{-1/4} e^{-cx_+^{3/2}}.$$

It was also shown in [17] that Theorem 1 is optimal :

Theorem 2. *There exists $u_0 \in S(\mathbb{R})$, $u_0 \not\equiv 0$ and $\Delta T > 0$ such that the IVP associated to the k -gKdV equation (1.3) with data u_0 has solution*

$$u \in C([0, \Delta T] : \mathbb{S}(\mathbb{R})),$$

satisfying

$$|u(x, t)| \leq \tilde{d} e^{-x^{3/2}/3}, \quad x > 1, \quad t \in [0, \Delta T],$$

for some constant $\tilde{d} > 0$.

In the case of the free Schrödinger group $\{e^{it\Delta} : t \in \mathbb{R}\}$

$$e^{it\Delta} u_0(x) = (e^{-i|\xi|^2 t} \widehat{u_0})^\vee(x) = \frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * u_0(x),$$

the fundamental solution does not decay. However, one has the identity

$$(1.11) \quad \begin{aligned} u(x, t) &= e^{it\Delta} u_0(x) = \int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}} u_0(y) dy \\ &= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-2ix \cdot y/4t} e^{i|y|^2/4t} u_0(y) dy \\ &= \frac{e^{i|x|^2/4t}}{(2it)^{n/2}} (e^{i|\cdot|^2/4t} \widehat{u_0}) \left(\frac{x}{2t} \right), \end{aligned}$$

where

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx.$$

Hence,

$$c_t e^{-i|x|^2/4t} u(x, t) = (e^{i|\cdot|^2/4t} \widehat{u_0}) \left(\frac{x}{2t} \right), \quad c_t = (2it)^{n/2},$$

which tells us that $e^{-i|x|^2/4t} u(x, t)$ is a multiple of the rescaled Fourier transform of $e^{i|y|^2/4t} u_0(y)$. Thus, as we pointed out earlier, the behavior of the solution of the free Schrödinger equation is closely related to uncertainty principles for the Fourier transform. We shall study these uncertainty principles and their relation with the uniqueness properties of the solution of the Schrödinger equation (1.6). In the early 1930's N. Wiener's remark (see [29], [33], and [46]):

“a pair of transforms f and g (\widehat{f}) cannot both be very small”,

motivated the works of G. H. Hardy [29], G. W. Morgan [46], and A. E. Ingham [33] which will be considered in detail in this note. However, before that we shall return to a review of some previous results concerning uniqueness properties of solutions of the Schrödinger equation which we mentioned earlier and which were not motivated by the formula (1.11).

For solutions $u(x, t)$ of the 1-D cubic Schrödinger equation (1.5) B. Y. Zhang [56] showed :

If $u(x, t) = 0$ for $(x, t) \in (-\infty, a) \times \{0, 1\}$ (or $(x, t) \in (a, \infty) \times \{0, 1\}$) for some $a \in \mathbb{R}$, then $u \equiv 0$.

As it was mentioned before, his proof is based on the inverse scattering method, which uses the fact that the equation in (1.5) is a completely integrable model.

In [40] it was proved under general assumptions on F in (1.8) that :

If $u_1, u_2 \in C([0, 1] : H^s(\mathbb{R}^n))$, with $s > \max\{n/2; 2\}$ are solutions of the equation (1.6) with F as in (1.8) such that

$$u_1(x, t) = u_2(x, t), \quad (x, t) \in \Gamma_{x_0}^c \times \{0, 1\},$$

where $\Gamma_{x_0}^c$ denotes the complement of a cone Γ_{x_0} with vertex $x_0 \in \mathbb{R}^n$ and opening $< 180^\circ$, then $u_1 \equiv u_2$.

(For further results in this direction see [40], [34], [35], and references therein).

A key step in the proof in [40] was the following uniform exponential decay estimate:

Lemma 1. *There exists $\epsilon_0 > 0$ such that if*

$$(1.12) \quad \mathbb{V} : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{C}, \quad \text{with} \quad \|\mathbb{V}\|_{L_t^1 L_x^\infty} \leq \epsilon_0,$$

and $u \in C([0, 1] : L^2(\mathbb{R}^n))$ is a strong solution of the IVP

$$(1.13) \quad \begin{cases} \partial_t u = i(\Delta + \mathbb{V}(x, t))u + \mathbb{G}(x, t), \\ u(x, 0) = u_0(x), \end{cases}$$

with

$$(1.14) \quad u_0, u_1 \equiv u(\cdot, 1) \in L^2(e^{2\lambda \cdot x} dx), \quad \mathbb{G} \in L^1([0, 1] : L^2(e^{2\lambda \cdot x} dx)),$$

for some $\lambda \in \mathbb{R}^n$, then there exists c_n independent of λ such that

$$(1.15) \quad \begin{aligned} & \sup_{0 \leq t \leq 1} \|e^{\lambda \cdot x} u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \\ & \leq c_n \left(\|e^{\lambda \cdot x} u_0\|_{L^2(\mathbb{R}^n)} + \|e^{\lambda \cdot x} u_1\|_{L^2(\mathbb{R}^n)} + \int_0^1 \|e^{\lambda \cdot x} \mathbb{G}(\cdot, t)\|_{L^2(\mathbb{R}^n)} dt \right). \end{aligned}$$

Notice that in the above result one assumes the existence of a reference L^2 -solution u of the equation (1.13) and then under the hypotheses (1.12) and (1.14) shows that the exponential decay in the time interval $[0, 1]$ is preserved.

The estimate (1.15) can be combined with the subordination formula

$$(1.16) \quad e^{\gamma|x|^p/p} \simeq \int_{\mathbb{R}^n} e^{\gamma^{1/p}\lambda \cdot x - |\lambda|^q/q} |\lambda|^{n(q-2)/2} d\lambda, \quad \forall x \in \mathbb{R}^n \text{ and } p > 1,$$

to get that for any $\alpha > 0$ and $a > 1$

$$(1.17) \quad \begin{aligned} & \sup_{0 \leq t \leq 1} \|e^{\alpha|x|^a} u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \\ & \leq c_n \left(\|e^{\alpha|x|^a} u_0\|_{L^2(\mathbb{R}^n)} + \|e^{\alpha|x|^a} u_1\|_{L^2(\mathbb{R}^n)} + \int_0^1 \|e^{\alpha|x|^a} \mathbb{G}(\cdot, t)\|_{L^2(\mathbb{R}^n)} dt \right). \end{aligned}$$

Under appropriate assumptions on the potential $V(x, t)$ in (1.7) one writes

$$V(x, t)u = \chi_R V(x, t)u + (1 - \chi_R)V(x, t)u = \mathbb{V}(x, t)u + \mathbb{G}(x, t),$$

with $\chi_R \in C_0^\infty$, $\chi_R(x) = 1$, $|x| < R$, supported in $|x| < 2R$, and applies the estimate (1.17) by fixing R sufficiently large. Also under appropriate hypothesis on F and u a similar argument can be used for the semi-linear equation in (1.8).

The estimate (1.17) gives a control on the decay of the solution in the whole time interval in terms of that at the end points and that of the ‘‘external force’’. As we shall see below a key idea will be to get improvements of this estimate based on logarithmically convex versions of it.

We recall that if one considers the equation (1.6) with initial data $u_0 \in \mathbb{S}(\mathbb{R}^n)$ and a smooth potential $V(x, t)$ in (1.7) or smooth nonlinearity F in (1.8), it follows that the corresponding solution satisfies that $u \in C([-T, T] : \mathbb{S}(\mathbb{R}^n))$. This can be proved using the commutative property of the operators

$$L = \partial_t - i\Delta, \quad \text{and} \quad \Gamma_j = x_j + 2t\partial_{x_j}, \quad j = 1, \dots, n,$$

see [30]-[31]. From the proof of this fact one also has that the persistence property of the solution $u = u(x, t)$ (i.e. if the data $u_0 \in X$, a function space, then the corresponding solution $u(\cdot)$ describes a continuous curve in X , $u \in C([-T, T] : X)$, $T > 0$) with data $u_0 \in L^2(|x|^m)$ can only hold if $u_0 \in H^s(\mathbb{R}^n)$ with $s \geq 2m$. Roughly speaking, for exponential weights one has a more involved argument where the time direction plays a role. Considering the IVP for the one dimensional free Schrödinger equation

$$(1.18) \quad \begin{cases} \partial_t u = i\partial_x^2 u, & x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x) \in L^2(\mathbb{R}), \end{cases}$$

and assuming that $e^{\beta x} u_0 \in L^2(\mathbb{R})$, $\beta > 0$, then one formally has that

$$v(x, t) = e^{\beta x} u(x, t)$$

satisfies the equation

$$\partial_t v = i(\partial_x - \beta)^2 v.$$

Thus,

$$v(x, \pm 1) = e^{\beta x} u(x, \pm 1) \in L^2(\mathbb{R}) \quad \text{if} \quad e^{\pm 2\beta x} \widehat{e^{\beta x} u_0} \in L^2(\mathbb{R}).$$

However, if we knew that $e^{\beta x} u(x, 1)$, $e^{\beta x} u(x, -1) \in L^2(\mathbb{R})$ integrating forward in time the positive frequencies of $e^{\beta x} u(x, t)$ and backward in time the negative

frequencies of $e^{\beta x}u(x, t)$ one gets an estimate similar to that in (1.15) with $\lambda = \beta$ and $\mathbb{G} = 0$. This argument motivates the idea behind Lemma 1 and its proof.

The rest of this paper is organized as follows: section 2 contains the results related to Hardy's uncertainty principle including a short discussion on the version of this principle in terms of the heat flow. Section 3 those concerned with Morgan's uncertainty principle. In section 4 we shall consider the limiting case in section 3. Also, section 4 includes the statements of some related forthcoming results. Earlier in the introduction we have discussed uniqueness results obtained under the assumption that the solution vanishes at two different time in a semi-space (see [56], [34], [35], [20]). In section 2 similar uniqueness results will be established under a Gaussian decay hypothesis, in the whole space. In section 5 we shall obtain a unifying result, i.e. a uniqueness result under Gaussian decay in a semi-space of \mathbb{R}^n at two different times. The appendix contains an abstract lemma and a corollary which will be used in the previous sections.

2. HARDY'S UNCERTAINTY PRINCIPLE

In [29] G. H. Hardy's proved the following one dimensional ($n = 1$) result:

If $f(x) = O(e^{-|x|^2/\beta^2})$, $\widehat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $1/\alpha\beta > 1/4$, then $f \equiv 0$. Also, if $1/\alpha\beta = 1/4$, $f(x)$ is a constant multiple of $e^{-|x|^2/\beta^2}$.

To our knowledge the available proofs of this result and its variants use complex analysis, mainly appropriate versions of the Phragmén-Lindelöf principle. There has also been considerable interest in a better understanding of this result and on extensions of it to other settings: [5], [6], [12], [32], and [53]. In particular, the extension of Hardy's result to higher dimension $n \geq 2$ (via Radon transform) was given in [53].

The formula (1.11) allows us to re-write this uncertainty principle in terms of the solution of the IVP for the free Schrödinger equation

$$\begin{cases} \partial_t u = i\Delta u, & (x, t) \in \mathbb{R}^n \times (0, +\infty), \\ u(x, 0) = u_0(x), \end{cases}$$

in the following manner :

If $u(x, 0) = O(e^{-|x|^2/\beta^2})$, $u(x, T) = O(e^{-|x|^2/\alpha^2})$ and $T/\alpha\beta > 1/4$, then $u \equiv 0$. Also, if $T/\alpha\beta = 1/4$, u has as initial data u_0 equal to a constant multiple of $e^{-(1/\beta^2 + i/4T)|y|^2}$.

The corresponding L^2 -version of Hardy's uncertainty principle was established in [13] :

If $e^{|x|^2/\beta^2} f$, $e^{4|\xi|^2/\alpha^2} \widehat{f}$ are in $L^2(\mathbb{R}^n)$ and $1/\alpha\beta \geq 1/4$, then $f \equiv 0$.

In terms of the solution of the Schrödinger equation it states :

If $e^{|x|^2/\beta^2} u(x, 0)$, $e^{4|\xi|^2/\alpha^2} u(x, T)$ are in $L^2(\mathbb{R}^n)$ and $T/\alpha\beta \geq 1/4$, then $u \equiv 0$.

More generally, it was shown in [13] that :

If $e^{|x|^2/\beta^2} f \in L^p(\mathbb{R}^n)$, $e^{4|\xi|^2/\alpha^2} \widehat{f} \in L^q(\mathbb{R}^n)$, $p, q \in [1, \infty]$ with at least one of them finite and $1/\alpha\beta \geq 1/4$, then $f \equiv 0$.

In [20] we proved a uniqueness result for solutions of (1.6) with F as in (1.7) for bounded potentials V verifying that either,

$$V(x, t) = V_1(x) + V_2(x, t),$$

with V_1 real-valued and

$$\sup_{[0, T]} \|e^{T^2|x|^2/(\alpha t + \beta(T-t))^2} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty,$$

or

$$(2.1) \quad \lim_{R \rightarrow +\infty} \int_0^T \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)} dt = 0.$$

More precisely, it was shown that the only solution $u \in C([0, T], L^2(\mathbb{R}^n))$ to (1.6) with $F = V(x, t)u$, verifying

$$(2.2) \quad \|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

with $T/\alpha\beta > 1/2$ and V satisfying one of the above conditions is the zero solution. Notice that this result differs by a factor of $1/2$ from that for the solution of the free Schrödinger equation given by the L^2 -version of the Hardy uncertainty principle described above ($T/\alpha\beta \geq 1/4$).

In [22] we showed that the optimal version of Hardy's uncertainty principle in terms of L^2 -norms, as established in [13], holds for solutions of

$$(2.3) \quad \partial_t u = i(\Delta u + V(x, t)u), \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

such that (2.2) holds with $T/\alpha\beta > 1/4$ and for many general bounded potentials $V(x, t)$, while it fails for some complex-valued potentials in the end-point case, $T/\alpha\beta = 1/4$.

Theorem 3. *Let $u \in C([0, T]) : L^2(\mathbb{R}^n)$ be a solution of the equation (2.3). If there exist positive constants α and β such that $T/\alpha\beta > 1/4$, and*

$$\|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)}, \quad \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < \infty,$$

and the potential V is bounded and either, $V(x, t) = V_1(x) + V_2(x, t)$, with V_1 real-valued and

$$\sup_{[0, T]} \|e^{T^2|x|^2/(\alpha t + \beta(T-t))^2} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

or

$$\lim_{R \rightarrow +\infty} \|V\|_{L^1([0, T], L^\infty(\mathbb{R}^n \setminus B_R))} = 0.$$

Then, $u \equiv 0$.

We remark that there are no assumptions on the size of the potential in the given class or on the dimension and that we do not assume any decay of the gradient, neither of the solutions or of the time-independent potential or any *a priori* regularity on this potential or the solution.

Theorem 4. *Assume that $T/\alpha\beta = 1/4$. Then, there is a smooth complex-valued potential V verifying*

$$|V(x, t)| \lesssim \frac{1}{1 + |x|^2}, \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

and a nonzero smooth function $u \in C^\infty([0, T], \mathcal{S}(\mathbb{R}^n))$ solution of (2.3) such that

$$(2.4) \quad \|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)}, \quad \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < \infty.$$

Our proof of Theorem 3 does not use any complex analysis, giving, in particular, a new proof (up to the end-point) of the L^2 -version of Hardy's uncertainty principle for the Fourier transform. It is based on Carleman estimates for certain evolutions. More precisely, it is based on the convexity and log-convexity properties present for the solutions of these evolutions. Thus, the convexity and log-convexity of appropriate L^2 -quantities play the role of the Phragmén-Lindelöf principle. We observe that the product of log-convex functions is log-convex which, roughly speaking, replaces the fact that the product of analytic functions is analytic.

In [11] in collaboration with M. Cowling, we gave new proofs, based only on *real variable* techniques, of both the L^2 -version of the Hardy uncertainty principle and the original Hardy's uncertainty principle (L^∞) n -dimensional version for the Fourier transform as stated at the beginning of this section, including the end point case $1/\alpha\beta = 1/4$.

Returning to Theorem 3 as a by product of our proof, we obtain the following optimal interior estimate for the Gaussian decay of solutions to (2.3).

Theorem 5. *Assume that u and V verify the hypothesis in Theorem 3 and $T/\alpha\beta \leq 1/4$. Then,*

$$(2.5) \quad \sup_{[0,T]} \|e^{a(t)|x|^2} u(t)\|_{L^2(\mathbb{R}^n)} + \|\sqrt{t(T-t)} \nabla \left(e^{(a(t) + \frac{i\dot{a}(t)}{8a(t)})|x|^2} u \right)\|_{L^2(\mathbb{R}^n \times [0,T])} \\ \leq N \left[\|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} \right],$$

where

$$a(t) = \frac{\alpha\beta RT}{2(\alpha t + \beta(T-t))^2 + 2R^2(\alpha t - \beta(T-t))^2},$$

R is the smallest root of the equation

$$\frac{T}{\alpha\beta} = \frac{R}{2(1+R^2)}$$

and N depends on T , α , β and the conditions on the potential V in Theorem 3.

One has that $1/a(t)$ is convex and attains its minimum value in the interior of $[0, T]$, when

$$|\alpha - \beta| < R^2(\alpha + \beta).$$

To see the optimality of Theorem 5, we write

$$(2.6) \quad u_R(x, t) = R^{-\frac{n}{2}} \left(t - \frac{i}{R} \right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4i(t - \frac{i}{R})}} = (Rt - i)^{-\frac{n}{2}} e^{-\frac{(R-iR^2t)}{4(1+R^2t^2)}|x|^2},$$

which is a free wave (i.e. $V \equiv 0$, in (2.3)) satisfying in $\mathbb{R}^n \times [-1, 1]$ the corresponding time translated conditions in Theorem 5 with $T = 2$ and

$$\frac{1}{\beta^2} = \frac{1}{\alpha^2} = \mu = \frac{R}{4(1+R^2)} \leq \frac{1}{8}.$$

Moreover

$$\frac{R}{4(1+R^2t^2)},$$

is increasing in the R -variable, when $0 < R \leq 1$ and $-1 \leq t \leq 1$.

Our improvement over the results in [16] and [20] is a consequence of the possibility of extending the following argument (for the case of free waves) to prove Theorem 3 (a non-free wave case).

We recall the conformal or Appell transformation: If $u(y, s)$ verifies

$$(2.7) \quad \partial_s u = i(\Delta u + V(y, s)u + F(y, s)), \quad (y, s) \in \mathbb{R}^n \times [0, 1],$$

and α and β are positive, then

$$(2.8) \quad \tilde{u}(x, t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t} \right)^{\frac{n}{2}} u \left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t} \right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}},$$

verifies

$$(2.9) \quad \partial_t \tilde{u} = i \left(\Delta \tilde{u} + \tilde{V}(x, t)\tilde{u} + \tilde{F}(x, t) \right), \quad \text{in } \mathbb{R}^n \times [0, 1],$$

with

$$(2.10) \quad \tilde{V}(x, t) = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} V \left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t} \right),$$

and

$$(2.11) \quad \tilde{F}(x, t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t} \right)^{\frac{n}{2}+2} F \left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t} \right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}.$$

Thus, to prove Theorem 3 for free waves, it suffices to consider $u \in C([-1, 1], L^2(\mathbb{R}^n))$ being a solution of

$$(2.12) \quad \partial_t u - i\Delta u = 0, \quad (x, t) \in \mathbb{R} \times [-1, 1],$$

and

$$(2.13) \quad \|e^{\mu|x|^2} u(-1)\|_{L^2(\mathbb{R}^n)} + \|e^{\mu|x|^2} u(1)\|_{L^2(\mathbb{R}^n)} < +\infty,$$

for some $\mu > 0$.

The main idea consists of showing that either $u \equiv 0$ or there is a function $\theta_R : [-1, 1] \rightarrow [0, 1]$ such that

$$(2.14) \quad \|e^{\frac{R|x|^2}{4(1+R^2t^2)}} u(t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu|x|^2} u(-1)\|_{L^2(\mathbb{R}^n)}^{\theta_R(t)} \|e^{\mu|x|^2} u(1)\|_{L^2(\mathbb{R}^n)}^{1-\theta_R(t)},$$

where R is the smallest root of the equation

$$\mu = \frac{R}{4(1+R^2)}.$$

This gives the optimal improvement of the Gaussian decay of a free wave verifying (2.13) and we also see that if $\mu > 1/8$, then u is zero.

The proof of these facts relies on new logarithmic convexity properties of free waves verifying (2.13) and on those already established in [20]. In [20, Theorem 3], the positivity of the space-time commutator of the symmetric and skew-symmetric parts of the operator,

$$e^{\mu|x|^2} (\partial_t - i\Delta) e^{-\mu|x|^2},$$

is used to prove that $\|e^{\mu|x|^2} u(t)\|_{L^2(\mathbb{R}^n)}$ is logarithmically convex in $[-1, 1]$. More precisely, defining

$$f(x, t) = e^{\mu|x|^2} u(x, t) = e^{it\Delta} u_0(x),$$

it follows that

$$e^{\mu|x|^2} (\partial_t - i\Delta) u = e^{\mu|x|^2} (\partial_t - i\Delta) (e^{-\mu|x|^2} f) = \partial_t f - \mathcal{S}f - \mathcal{A}f,$$

where \mathcal{S} is symmetric and \mathcal{A} skew-symmetric with

$$\mathcal{S} = -i\mu(4x \cdot \nabla + 2n), \quad \mathcal{A} = i(\Delta + 4\mu^2|x|^2),$$

so that

$$[\mathcal{S}; \mathcal{A}] = -8\mu(\nabla \cdot \nabla) + 16\mu^2|x|^2.$$

Formally, using the abstract Lemma 3 (see the appendix) and the Heisenberg inequality

$$\|f\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{2}{n} \| |x|f \|_{L^2(\mathbb{R}^n)} \| \nabla f \|_{L^2(\mathbb{R}^n)},$$

whose proof follows by integration by parts, one sees that

$$H(t) = \|f(t)\|_{L^2(\mathbb{R}^n)}^2 = \|e^{\mu|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}^2$$

is logarithmically convex so

$$\|e^{\mu|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu|x|^2}u(-1)\|_{L^2(\mathbb{R}^n)}^{\frac{1-t}{2}} \|e^{\mu|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}^{\frac{1+t}{2}},$$

when, $-1 \leq t \leq 1$.

Setting $a_1 \equiv \mu$, we begin an iterative process, where at the k -th step, we have k smooth even functions, $a_j : [-1, 1] \rightarrow (0, +\infty)$, $1 \leq j \leq k$, such that

$$\mu \equiv a_1 < a_2 < \dots < a_k \in (-1, 1),$$

$$F(a_j) > 0, \quad a_j(1) = \mu, \quad j = 1, \dots, k,$$

where

$$F(a) = \frac{1}{a} \left(\ddot{a} - \frac{3\dot{a}^2}{2a} + 32a^3 \right)$$

and functions $\theta_j : [-1, 1] \rightarrow [0, 1]$, $1 \leq j \leq k$, such that for $t \in [-1, 1]$

$$(2.15) \quad \|e^{a_j(t)|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu|x|^2}u(-1)\|_{L^2(\mathbb{R}^n)}^{\theta_j(t)} \|e^{\mu|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}^{1-\theta_j(t)}.$$

These estimates follow from the construction of the functions a_i , while the method strongly relies on the following formal convexity properties of free waves:

$$(2.16) \quad \partial_t \left(\frac{1}{a} \partial_t \log H_b \right) \geq -\frac{2\ddot{b}^2|\xi|^2}{F(a)},$$

$$(2.17) \quad \partial_t \left(\frac{1}{a} \partial_t H \right) \geq \epsilon_a \int_{\mathbb{R}^n} e^{a|x|^2} (|\nabla u|^2 + |x|^2|u|^2) dx,$$

where

$$H_b(t) = \|e^{a(t)|x+b(t)\xi|^2}u(t)\|_{L^2(\mathbb{R}^n)}^2, \quad H(t) = \|e^{a(t)|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}^2,$$

$\xi \in \mathbb{R}^n$ and $a, b : [-1, 1] \rightarrow \mathbb{R}$ are smooth functions with

$$a > 0, \quad F(a) > 0 \quad \text{in} \quad [-1, 1].$$

Once the k -th step is completed, we take $a = a_k$ in (2.16) with a certain choice of $b = b_k$, verifying $b(-1) = b(1) = 0$ and then, a certain test is performed. When the answer to the test is positive, it follows that $u \equiv 0$. Otherwise, the logarithmic convexity associated to (2.16) allows us to find a new smooth function a_{k+1} in $[-1, 1]$ with

$$a_1 < a_2 < \dots < a_k < a_{k+1}, \quad (-1, 1),$$

and verifying the same properties as a_1, \dots, a_k .

When the process is infinite, we have (2.15) for all $k \geq 1$ and there are two possibilities:

$$\text{either} \quad \lim_{k \rightarrow +\infty} a_k(0) = +\infty, \quad \text{or} \quad \lim_{k \rightarrow +\infty} a_k(0) < +\infty.$$

In the first case and (2.15) one has that $u \equiv 0$, while in the second, the sequence a_k is shown to converge to an even function a verifying

$$(2.18) \quad \begin{cases} \ddot{a} - \frac{3\dot{a}^2}{2a} + 32a^3 = 0, & [-1, 1] \\ a(1) = \mu. \end{cases}$$

Because

$$a(t) = \frac{R}{4(1 + R^2 t^2)}, \quad R \in \mathbb{R}^+,$$

are all the possible even solutions of this equation, a must be one of them and

$$\mu = \frac{R}{4(1 + R^2)},$$

for some $R > 0$. In particular, $u \equiv 0$, when $\mu > 1/8$.

As it was already mentioned above, our proof of Theorem 3 (the case of non-zero potentials $V = V(x, t)$), is based on the extension of the above convexity properties to the non-free case.

Theorem 4 establishes the sharpness of the result in Theorem 3 by giving an example of a complex valued potential $V(x, t)$ verifying (2.1) and a non-trivial solution $u \in C([0, T] : L^2(\mathbb{R}^n))$ of (2.3) for which (2.2) holds with $T/\alpha\beta = 1/4$. Thus, one may ask : Is it possible to construct a real valued potential $V(x, t)$ verifying the same properties, i.e. satisfying (2.1) and having a non-trivial solution $u \in C([0, T] : L^2(\mathbb{R}^n))$ of (2.3) such that (2.2) holds with $T/\alpha\beta = 1/4$?

The same question concerning the sharpness of the above result presents itself in the case of time independent potentials $V = V(x)$. In this regard, we consider the stationary problem

$$(2.19) \quad \Delta w + V(x)w = 0, \quad x \in \mathbb{R}^n, \quad V \in L^\infty(\mathbb{R}^n),$$

and recall V. Z. Meshkov's result in [45] :

If $w \in H_{loc}^2(\mathbb{R}^n)$ is a solution of (2.19) such that

$$(2.20) \quad \int_{\mathbb{R}^n} e^{a|x|^{4/3}} |w(x)|^2 dx < \infty, \quad \forall a > 0,$$

then $u \equiv 0$.

Moreover, it was also proved in [45] that for complex potentials V , the exponent $4/3$ in (2.20) is optimal. However, it has been conjectured that for real valued potentials the optimal exponent should be 1, (see also [7] for a quantitative form of these results and applications to Anderson localization of Bernoulli models).

More generally, it was established in [23], (see also [14]) :

If $w \in H_{loc}^2(\mathbb{R}^n)$ is a solution of (2.19) with a complex valued potential V satisfying

$$V(x) = V_1(x) + V_2(x),$$

such that

$$(2.21) \quad |V_1(x)| \leq \frac{c_1}{(1 + |x|^2)^{\alpha/2}}, \quad \alpha \in [0, 1/2),$$

and V_2 real valued supported in $\{x : |x| \geq 1\}$ such that

$$-(\partial_r V_2(x))^- < \frac{c_2}{|x|^{2\alpha}}, \quad a^- = \min\{a; 0\}.$$

Then there exists $a = a(\|V\|_\infty; c_1; c_2; \alpha) > 0$ such that if

$$(2.22) \quad \int_{\mathbb{R}^n} e^{a|x|^r} |w(x)|^2 dx < \infty, \quad r = (4 - 2\alpha)/3,$$

then $u \equiv 0$.

In addition, one can take the value $r = 1$ in (2.20) by assuming $\alpha > 1/2$ in (2.21).

It was also proved in [14] that for complex potentials these results for $\alpha \in [0, 1/2)$ are sharp.

By noticing that given a solution $\phi(x)$ of the eigenvalue problem

$$(2.23) \quad \Delta\phi + \tilde{V}(x)\phi = \lambda\phi, \quad x \in \mathbb{R}^n,$$

with $\lambda \in \mathbb{R}$, then $V(x) = \tilde{V}(x) + \lambda$ satisfies the hypothesis of the previous result and

$$u(x, t) = e^{it\lambda} \phi(x),$$

solves the evolution equation

$$(2.24) \quad \partial_t u = i(\Delta u + V(x)u), \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

one gets a lower bound for the value of the strongest possible decay rate of non-trivial solutions $u(x, t)$ of (2.24) at two different times.

As a direct consequence of Theorem 3 we have the following application concerning the uniqueness of solutions for semi-linear equations of the form (1.6) with F as in (1.8).

Theorem 6. *Let u_1 and u_2 be strong solutions in $C([0, T], H^k(\mathbb{R}^n))$, $k > n/2$ of the equation (1.6) with F as in (1.8) such that $F \in C^k$ and $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$. If there are α and β positive with $T/\alpha\beta > 1/4$ such that*

$$e^{|x|^2/\beta^2} (u_1(0) - u_2(0)), \quad e^{|x|^2/\alpha^2} (u_1(T) - u_2(T)) \in L^2(\mathbb{R}^n),$$

then $u_1 \equiv u_2$.

In Theorem 6 we did not attempt to optimize the regularity assumption on the solutions u_1, u_2 .

By fixing $u_2 \equiv 0$ Theorem 6 provides a restriction on the possible decay at two different times of a non-trivial solution u_1 of equation (1.6) with F as in (1.8). It is an open question to determine the optimality of this kind of result. More precisely, for the standard semi-linear Schrödinger equations

$$(2.25) \quad \partial_t u = i(\Delta u + |u|^{\gamma-1}u), \quad \gamma > 1,$$

one has the *standing wave* solutions

$$u(x, t) = e^{i\omega t} \varphi(x), \quad \omega > 0,$$

where φ is the unique (up to translation) positive solution of the elliptic problem

$$-\Delta\varphi + \omega\varphi = |\varphi|^{\gamma-1}\varphi,$$

which has a linear exponential decay, i.e.

$$\varphi(x) = O(e^{-c|x|}), \quad \text{as } |x| \rightarrow \infty,$$

for an appropriate value of $c > 0$ (see [54], [3], [4], and [42]). Whether or not these standing waves are the solutions of (2.25) having the strongest possible decay at two different times is an open question.

Hardy's uncertainty principle also admits a formulation in terms of the heat equation

$$\partial_t u = \Delta u, \quad t > 0, \quad x \in \mathbb{R}^n,$$

whose solution with data $u(x, 0) = u_0(x)$ can be written as

$$u(x, t) = e^{t\Delta} u_0(x) = \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}} u_0(y) dy.$$

More precisely, Hardy's uncertainty principle can restated in the following equivalent forms :

(i) If $u_0 \in L^2(\mathbb{R}^n)$ and there exists $T > 0$ such that $e^{|x|^2/(\delta^2 T)} e^{T\Delta} u_0 \in L^2(\mathbb{R}^n)$ for some $\delta \leq 2$, then $u_0 \equiv 0$.

(ii) If $u_0 \in \mathcal{S}(\mathbb{R}^n)$ (tempered distribution) and there exists $T > 0$ such that $e^{|x|^2/(\delta^2 T)} e^{T\Delta} u_0 \in L^\infty(\mathbb{R}^n)$ for some $\delta < 2$, then $u_0 \equiv 0$. Moreover, if $\delta = 2$, then u_0 is a constant multiple of the Dirac delta measure.

In fact, applying Hardy's uncertainty principle to $e^{T\Delta} u_0$ one has that $e^{\frac{|x|^2}{\delta^2 T}} e^{T\Delta} u_0$ and $e^{T|\xi|^2} \widehat{e^{T\Delta} u_0} = \widehat{u_0}$ in $L^2(\mathbb{R}^n)$ with $2\delta \leq 4$ implies $e^{\Delta} u_0 \equiv 0$. Then, backward uniqueness arguments (see for example [44, Chapter 3, Theorem 11]) shows that $u_0 \equiv 0$.

In [20] we proved the following weaker extension of this result for parabolic operators with lower order variable coefficients :

Theorem 7. Let $u \in C([0, 1] : L^2(\mathbb{R}^n)) \cap L^2([0, T] : H^1(\mathbb{R}^n))$ be a solution of the IVP

$$\begin{cases} \partial_t u = \Delta u + V(x, t)u, & \text{in } \mathbb{R}^n \times (0, 1], \\ u(x, 0) = u_0(x), \end{cases}$$

where

$$V \in L^\infty(\mathbb{R}^n \times [0, 1]).$$

If

$$u_0 \quad \text{and} \quad e^{\frac{|x|^2}{\delta^2}} u(1) \in L^2(\mathbb{R}^n),$$

for some $\delta < 1$, then $u_0 \equiv 0$.

It is natural to expect that Hardy's uncertainty principle holds in this context with bounded potentials V and with the parameter δ verifying the condition of the free case, i.e. $\delta \leq 2$.

Earlier results in this directions, addressing a question of Landis and Oleinik [43], were obtained in [18] and [49].

3. UNCERTAINTY PRINCIPLE OF MORGAN TYPE

In [46] G. W. Morgan proved the following uncertainty principle:

If $f(x) = O(e^{-\frac{a^p |x|^p}{p}})$, $1 < p \leq 2$ and $\widehat{f}(\xi) = O(e^{-\frac{(b+\epsilon)^q |\xi|^q}{q}})$, $1/p + 1/q = 1$, $\epsilon > 0$, with

$$ab > \left| \cos\left(\frac{p\pi}{2}\right) \right|,$$

then $f \equiv 0$.

In [32] Beurling-Hörmander showed :

If $f \in L^1(\mathbb{R})$ and

$$(3.1) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)| |\widehat{f}(\xi)| e^{|x \cdot \xi|} dx d\xi < \infty, \quad \text{then } f \equiv 0.$$

This result was extended to higher dimensions $n \geq 2$ in [6] and [48] :

If $f \in L^2(\mathbb{R}^n)$, $n \geq 2$ and

$$(3.2) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)| |\widehat{f}(\xi)| e^{|x \cdot \xi|} dx d\xi < \infty, \quad \text{then } f \equiv 0.$$

We observe that from (3.1) and (3.2) it follows that :

If $p \in (1, 2]$, $1/p + 1/q = 1$, $a, b > 0$, and

$$(3.3) \quad \int_{\mathbb{R}^n} |f(x)| e^{\frac{a^p |x|^p}{p}} dx + \int_{\mathbb{R}^n} |\widehat{f}(\xi)| e^{\frac{b^q |\xi|^q}{q}} d\xi < \infty, \quad ab \geq 1 \Rightarrow f \equiv 0.$$

Notice that in the case $p = q = 2$ this gives us an L^1 -version of Hardy's uncertainty result discussed above, and for $p < 2$ an n -dimensional L^1 -version of Morgan's uncertainty principle.

In the one-dimensional case ($n = 1$), the optimal L^1 -version of Morgan's result in (3.3),

$$(3.4) \quad \int_{\mathbb{R}} |f(x)| e^{\frac{a^p |x|^p}{p}} dx + \int_{\mathbb{R}} |\widehat{f}(\xi)| e^{\frac{b^q |\xi|^q}{q}} d\xi < \infty, \quad ab > \left| \cos\left(\frac{p\pi}{2}\right) \right| \Rightarrow f \equiv 0.$$

was established in [6] and [2] (for further results see [5] and references therein). A sharp condition for a, b, p in (3.4) in higher dimension seems to be unknown. However, in [6] it was shown :

If $f \in L^2(\mathbb{R}^n)$, $1 < p \leq 2$ and $1/p + 1/q = 1$ are such that for some $j = 1, \dots, n$,

$$(3.5) \quad \int_{\mathbb{R}^n} |f(x)| e^{\frac{a^p |x_j|^p}{p}} dx < \infty + \int_{\mathbb{R}^n} |\widehat{f}(\xi)| e^{\frac{b^q |\xi_j|^q}{q}} d\xi < \infty.$$

If $ab > \left| \cos\left(\frac{p\pi}{2}\right) \right|$, then $f \equiv 0$.

If $ab < \left| \cos\left(\frac{p\pi}{2}\right) \right|$, then there exist non-trivial functions satisfying (3.5).

Using (1.11) the above result can be stated in terms of the solution of the free Schrödinger equation. In particular, (3.3) can be re-written as :

If $u_0 \in L^1(\mathbb{R})$ or $u_0 \in L^2(\mathbb{R}^n)$, if $n \geq 2$, and for some $t \neq 0$

$$(3.6) \quad \int_{\mathbb{R}^n} |u_0(x)| e^{\frac{a^p |x|^p}{p}} dx + \int_{\mathbb{R}^n} |e^{it\Delta} u_0(x)| e^{\frac{b^q |x|^q}{q(2t)^q}} dx < \infty,$$

with

$$ab > \left| \cos\left(\frac{p\pi}{2}\right) \right| \quad \text{if } n = 1, \quad \text{and } ab > 1 \quad \text{if } n \geq 2,$$

then $u_0 \equiv 0$.

Related with Morgan's uncertainty principle one has the following result due to Gel'fand and Shilov. In [27] they considered the class Z_p^p , $p > 1$, defined as the space of all functions $\varphi(z_1, \dots, z_n)$ which are analytic for all values of $z_1, \dots, z_n \in \mathbb{C}$ and such that

$$|\varphi(z_1, \dots, z_n)| \leq C_0 e^{\sum_{j=1}^n \epsilon_j C_j |z_j|^p},$$

where the C_j , $j = 0, 1, \dots, n$ are positive constants and $\epsilon_j = 1$ for z_j non-real and $\epsilon_j = -1$ for z_j real, $j = 1, \dots, n$, and showed that the Fourier transform of the function space Z_p^p is the space Z_q^q , with $1/p + 1/q = 1$.

Notice that the class Z_p^p with $p \geq 2$ is closed with respect to multiplication by $e^{ic|x|^2}$. Thus, if $u_0 \in Z_p^p$, $p \geq 2$, then by (1.11) one has that

$$|e^{it\Delta} u_0(x)| \leq d(t) e^{-a(t)|x|^q},$$

for some functions $d, a : \mathbb{R} \rightarrow (0, \infty)$.

In [21] the following results were established:

Theorem 8. *Given $p \in (1, 2)$ there exists $M_p > 0$ such that for any solution $u \in C([0, 1] : L^2(\mathbb{R}^n))$ of*

$$\partial_t u = i(\Delta u + V(x, t)u), \quad \text{in } \mathbb{R}^n \times [0, 1],$$

with $V = V(x, t)$ complex valued, bounded (i.e. $\|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])} \leq C$) and

$$(3.7) \quad \lim_{R \rightarrow +\infty} \|V\|_{L^1([0, 1] : L^\infty(\mathbb{R}^n \setminus B_R))} = 0,$$

satisfying that for some constants $a_0, a_1, a_2 > 0$

$$(3.8) \quad \int_{\mathbb{R}^n} |u(x, 0)|^2 e^{2a_0|x|^p} dx < \infty,$$

and for any $k \in \mathbb{Z}^+$

$$(3.9) \quad \int_{\mathbb{R}^n} |u(x, 1)|^2 e^{2k|x|^p} dx < a_2 e^{2a_1 k^{q/(q-p)}},$$

$1/p + 1/q = 1$, if

$$(3.10) \quad a_0 a_1^{(p-2)} > M_p,$$

then $u \equiv 0$.

Corollary 1. *Given $p \in (1, 2)$ there exists $N_p > 0$ such that if $u \in C([0, 1] : L^2(\mathbb{R}^n))$ is a solution of*

$$\partial_t u = i(\Delta u + V(x, t)u),$$

with $V = V(x, t)$ complex valued, bounded (i.e. $\|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])} \leq C$) and

$$\lim_{R \rightarrow \infty} \int_0^1 \sup_{|x| > R} |V(x, t)| dt = 0,$$

and there exist $\alpha, \beta > 0$ such that

$$(3.11) \quad \int_{\mathbb{R}^n} |u(x, 0)|^2 e^{2\alpha^p |x|^p/p} dx + \int_{\mathbb{R}^n} |u(x, 1)|^2 e^{2\beta^q |x|^q/q} dx < \infty,$$

$1/p + 1/q = 1$, with

$$(3.12) \quad \alpha \beta > N_p,$$

then $u \equiv 0$.

As a consequence of Corollary 1 one obtains the following result concerning the uniqueness of solutions for the semi-linear equations (1.6) with F as in (1.8)

$$(3.13) \quad i\partial_t u + \Delta u = F(u, \bar{u}).$$

Theorem 9. *Given $p \in (1, 2)$ there exists $N_p > 0$ such that if*

$$u_1, u_2 \in C([0, 1] : H^k(\mathbb{R}^n)),$$

are strong solutions of (3.13) with $k \in \mathbb{Z}^+$, $k > n/2$, $F : \mathbb{C}^2 \rightarrow \mathbb{C}$, $F \in C^k$ and $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$, and there exist $\alpha, \beta > 0$ such that

$$(3.14) \quad e^{\alpha^p |x|^p/p} (u_1(0) - u_2(0)), \quad e^{\beta^q |x|^q/q} (u_1(1) - u_2(1)) \in L^2(\mathbb{R}^n),$$

$1/p + 1/q = 1$, with

$$(3.15) \quad \alpha \beta > N_p,$$

then $u_1 \equiv u_2$.

Notice that the conditions (3.10) and (3.12) are independent of the size of the potential and there is not any *a priori* regularity assumption on the potential $V(x, t)$.

The result in [6], see (3.5), can be extended to our setting with an non-optimal constant. More precisely,

Corollary 2. *The conclusions in Corollary 1 still hold with a different constant $N_p > 0$ if one replaces the hypothesis (3.11) by the following one dimensional version*

$$(3.16) \quad \int_{\mathbb{R}^n} |u(x, 0)|^2 e^{2\alpha^p |x_j|^p/p} dx < \infty \quad + \quad \int_{\mathbb{R}^n} |u(x, 1)|^2 e^{2\beta^q |x_j|^q/q} dx < \infty,$$

for some $j = 1, \dots, n$.

Similarly, the non-linear version of Theorem 9 still holds, with different constant $N_p > 0$, if one replaces the hypothesis (3.14) by

$$e^{\alpha^p |x_j|^p/p} (u_1(0) - u_2(0)), \quad e^{\beta^q |x_j|^q/q} (u_1(1) - u_2(1)) \in L^2(\mathbb{R}^n),$$

for $j = 1, \dots, n$.

In [21] we did not attempt to give an estimate of the universal constant N_p .

The limiting case $p = 1$ will be considered in the next section.

The main idea in the proof of these results is to combine an upper estimate with a lower one to obtain the desired result. The upper estimate is based on the decay hypothesis on the solution at two different times (see Lemma 1). In previous works we had been able to establish these estimates from assumptions that at time $t = 0$ and $t = 1$ involving the same weight. However, in our case (Corollary 1) we have different weights at time $t = 0$ and $t = 1$. To overcome this difficulty, we carry out the details with the weight $e^{a_j |x|^p}$, $1 < p < 2$, $j = 0$ at $t = 0$ and $j = 1$ at $t = 1$, with a_0 fixed and $a_1 = k \in \mathbb{Z}^+$ as in (3.9). Although the powers $|x|^p$ in the exponential are equal at time $t = 0$ and $t = 1$ to apply our estimate (Lemma 1) we also need to have the same constant in front of them. To achieve this we apply the conformal or Appell transformation described above, to get solutions and potentials, whose bounds depend on $k \in \mathbb{Z}^+$. Thus we have to consider a family of solutions and obtain estimates on their asymptotic value as $k \uparrow \infty$.

The proof of the lower estimate is based on the positivity of the commutator operator obtained by conjugating the equation with the appropriate exponential weight, (see Lemma 3 in the appendix)

4. PALEY-WIENER THEOREM AND UNCERTAINTY PRINCIPLE OF INGHAM TYPE

This section is concerned with the limiting case $p = 1$ in the previous section.

It is easy to see that if $f \in L^1(\mathbb{R}^n)$ is non-zero and has compact support, then \widehat{f} cannot satisfy a condition of the type $\widehat{f}(y) = O(e^{-\epsilon|y|})$ for any $\epsilon > 0$. However, it may be possible to have $f \in L^1(\mathbb{R}^n)$ a non-zero function with compact support, such that $\widehat{f}(\xi) = O(e^{-\epsilon(y)|y|})$, $\epsilon(y)$ being a positive function tending to zero as $|y| \rightarrow \infty$.

In the one-dimensional case ($n = 1$) soon after Hardy's result described above, A. E. Ingham [33] proved the following :

There exists $f \in L^1(\mathbb{R})$ non-zero, even, vanishing outside an interval such that $\widehat{f}(y) = O(e^{-\epsilon(y)|y|})$ with $\epsilon(y)$ being a positive function tending to zero at infinity if and only if

$$\int^{\infty} \frac{\epsilon(y)}{y} dy < \infty.$$

In a similar direction the Paley-Wiener Theorem [50] gives a characterization of a function or distribution with compact support in term of analyticity properties of its Fourier transform.

Regarding our results discussed above it would be interesting to identify a class of potentials $V(x, t)$ for which a result of the following kind holds:

If $u \in C([0, 1] : L^2(\mathbb{R}^n))$ is a non-trivial solution of the IVP

$$(4.1) \quad \begin{cases} \partial_t u = i(\Delta u + V(x, t)u), & (x, t) \in \mathbb{R}^n \times [0, 1], \\ u(x, 0) = u_0(x), \end{cases}$$

with $u_0 \in L^2(\mathbb{R}^n)$ having compact support, then $e^{\epsilon|x|} u(\cdot, t) \notin L^2(\mathbb{R}^n)$ for any $\epsilon > 0$ and any $t \in (0, 1]$.

In this direction we have the following result which will appear in [24]:

Theorem 10. *Assume that $u \in C([0, 1] : L^2(\mathbb{R}^n))$ is a strong solution of the IVP (2.4) with*

$$(4.2) \quad \text{supp } u_0 \subset B_R(0) = \{x \in \mathbb{R}^n : |x| \leq R\},$$

$$(4.3) \quad \int_{\mathbb{R}^n} |e^{2a_1|x|} u(x, 1)|^2 dx < \infty, \quad a_1 > 0,$$

and

$$(4.4) \quad \|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])} = M_0,$$

with

$$(4.5) \quad \lim_{R \rightarrow +\infty} \|V\|_{L^1([0, 1] : L^\infty(\mathbb{R}^n \setminus B_R))} = 0.$$

Then, there exists $b = b(n) > 0$ (depending only on the dimension n) such that if

$$\frac{a_1}{R M_0} \geq b,$$

then $u \equiv 0$.

A similar question can be raised for results of the type described above due to A. E. Ingham in [33] and possible extensions to higher dimensions $n \geq 2$.

It would be interesting to obtain extensions of the above results characterizing the decay of the solution $u(x, t)$ to the equation (1.6) with F as in (1.8) associated to data $u_0 \in L^2(\mathbb{R}^n)$ with compact support or with $u_0 \in C_0^\infty(\mathbb{R}^n)$. In this direction, some results can be deduced as a consequence of Theorem 10, see [24].

5. HARDY'S UNCERTAINTY PRINCIPLE IN A HALF-SPACE

In the introduction we have briefly reviewed some uniqueness results established for solutions of the Schrödinger equation vanishing at two different times in a semi-space of \mathbb{R}^n , (see [56], [15], [34], [35], [20]). In section 2, we have studied uniqueness results gotten under the hypothesis that the solution of the Schrödinger equation at two different times has an appropriate Gaussian decay, in the whole space \mathbb{R}^n . In this section, we shall deduce a unified result, i.e. a uniqueness result under the hypothesis that at two different times the solution of the Schrödinger equation has Gaussian decay in just a semi-space of \mathbb{R}^n .

Theorem 11. *Assume that $u \in C([0, 1] : L^2((0, \infty) \times \mathbb{R}^{n-1}))$ is a strong solution of the IVP*

$$(5.1) \quad \begin{cases} \partial_t u = i(\Delta + V(x, t))u, \\ u(x, 0) = u_0(x), \end{cases}$$

with

$$(5.2) \quad \int_0^1 \int_{1/2}^{3/2} |\partial_{x_1} u(x, t)|^2 dx dt < \infty,$$

$$(5.3) \quad V : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{C}, \quad V \in L^\infty(\mathbb{R}^n \times [0, 1]),$$

and

$$(5.4) \quad \lim_{R \rightarrow +\infty} \int_0^1 \|V(t)\|_{L^\infty(\{x_1 > R\})} dt = 0.$$

Assume that

$$(5.5) \quad \int_{x_1 > 0} e^{c_0 |x_1|^2} |u(x, 0)|^2 dx < \infty,$$

$$\int_{x_1 > 0} e^{c_1 |x_1|^2} |u(x, 1)|^2 dx < \infty,$$

with $c_0, c_1 > 0$ sufficiently large. Then $u \equiv 0$.

Remarks : (a) Note that in Theorem 11, the solution does not need to be defined for $x_1 \leq 0$. In this sense, this is a stronger result than the uniqueness results in [56], [40], [34], [35], and [15], which required that the solution be defined in $\mathbb{R}^n \times [0, 1]$ and be $C([0, 1] : L^2(\mathbb{R}^n))$.

On the other hand, we need to assume the condition (5.2). Note that [40] also needs an extra assumption on ∇u , stronger than (5.2), but that in [34], which among other things removed any extra assumption on ∇u , but still required the solution to be defined in $\mathbb{R}^n \times [0, 1]$ and be in $C([0, 1] : L^2(\mathbb{R}^n))$. If in the setting of

Theorem 11 we know that u is a solution in $\mathbb{R}^n \times [0, 1]$ and is in $C([0, 1] : L^2(\mathbb{R}^n))$, then we can dispose the hypothesis (5.2) as follows:

First as in the first step of the proof of Theorem 11, we can use the Appell transformation to reduce to the case $c_1 = c_2 = 2\gamma$. Then, using $\varphi(x_1)$ a ‘‘regularized’’ convex function which agrees with x_1^+ for $x_1 > 1$, $x_1 < -1$, an application of Lemma 3 and Corollary 3 in the appendix yields the estimate

$$\sup_{0 \leq t \leq 1} \int e^{2\gamma(x_1^+)^2} |u(x, t)|^2 dx + \int_0^1 \int_{x_1 > 2} t(1-t) |\nabla u(x, t)|^2 e^{2\gamma(x_1^+)^2} dx dt < \infty.$$

Once this is obtained, by restricting our attention to

$$(2, \infty) \times \mathbb{R}^{n-1} \times [\delta, 1 - \delta],$$

for each $\delta > 0$, we are in the situation of Theorem 11, and hence $u \equiv 0$ on $\{x_1 > 2\} \times [0, 1]$. Finally, Izakov’s result in [36] concludes that $u \equiv 0$ (more precisely, the version of Izakov’s result proved in [34], which does not require ∇u to exist for $-1 < x_1 < 1$).

(b) We have seen that Theorem 11 includes many of the uniqueness results for solutions vanishing at two different times in a semi-space. In comparison with the results in section 2, since the extra assumption (5.2) can be recovered as in remark (a) when the solution is defined in $\mathbb{R}^n \times [0, 1]$ and is in $C([0, 1] : L^2(\mathbb{R}^n))$, the only weakness is that the provide an optimal estimate for the constants c_1, c_2 , but on the other hand deals with solutions only defined in $(0, \infty) \times \mathbb{R}^{n-1} \times [0, 1]$.

(c) In Theorem 11 the direction \vec{e}_1 can be replaced by any other $\omega \in \mathbb{S}^{n-1}$.

Proof of Theorem 11: The strategy of the proof follows closely the one in [16]. We divide the proof into three steps.

First Step : Reduction to the case $c_0 = c_1 = 2\gamma$.

This follows by using the conformal or Appell transformation introduced in section 2 (see (2.7)-(2.11)), combined with the observation that the set $\{x_1 > 0\}$ remains invariant.

Second Step : Upper Bounds.

We define

$$v(x, t) = \theta(x_1) u(x, t),$$

with $\theta \in C^\infty(\mathbb{R})$, non-decreasing with $\theta(x_1) \equiv 1$ if $x_1 > 3/2$, and $\theta(x_1) \equiv 0$ if $x_1 < 1/2$. Therefore,

$$(5.6) \quad \partial_t v = i \Delta v + i V(x, t) v + i F(x, t), \quad F(x, t) = 2 \partial_{x_1} u \theta'(x_1) + u \theta''(x_1).$$

Using (5.2) we can apply Lemma 1 to get that

$$(5.7) \quad \begin{aligned} & \sup_{0 \leq t \leq 1} \|e^{\lambda \cdot x_1} v(\cdot, t)\|_{L^2(\mathbb{R}^n)} \\ & \leq c_n \left(\|e^{\lambda \cdot x_1} v(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\lambda \cdot x_1} v(1)\|_{L^2(\mathbb{R}^n)} \right. \\ & \quad \left. + \int_0^1 \|e^{\lambda \cdot x_1} F(\cdot, t)\|_{L^2(\mathbb{R}^n)} dt + \int_0^1 \|e^{\lambda \cdot x_1} V \chi_{\{x_1 < R\}} v(\cdot, t)\|_{L^2(\mathbb{R}^n)} dt \right), \end{aligned}$$

for some fixed R sufficiently large. Thus, using (5.2)

$$(5.8) \quad \begin{aligned} & \sup_{0 \leq t \leq 1} \|e^{\lambda \cdot x_1} v(\cdot, t)\|_{L^2(\mathbb{R}^n)} \\ & \leq c_n \left(\|e^{\lambda \cdot x_1} v(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\lambda \cdot x_1} v(1)\|_{L^2(\mathbb{R}^n)} \right. \\ & \quad \left. + c e^{c|\lambda|} + c \|V\|_\infty e^{c|\lambda|R} \right). \end{aligned}$$

Thus, from the formula (1.16) (with $p = 2$ and $n = 1$) and (5.8) we obtain that

$$\begin{aligned} & \sup_{0 \leq t \leq 1} \|e^{\gamma |x_1|^2} v(\cdot, t)\|_{L^2(\mathbb{R}^n)} \\ & \leq \left(\|e^{\gamma |x_1|^2} v(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\gamma |x_1|^2} v(1)\|_{L^2(\mathbb{R}^n)} + c + \|V\|_\infty e^{c\gamma R^2} \right). \end{aligned}$$

Thus,

$$(5.9) \quad \sup_{0 \leq t \leq 1} \|e^{\gamma |x_1|^2} v(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq c_\gamma.$$

Combining this and the equation for v we shall get a smoothing estimate. Using the notation

$$H(t) = \|f\|_{L^2(\mathbb{R}^n)}^2 = \|f\|^2,$$

with

$$f(x, t) = e^{\gamma |x_1|^2} v(x, t)$$

and the abstract Lemma 3 (see the appendix) one formally has that

$$(5.10) \quad \begin{aligned} \partial_t^2 H & \leq 2\partial_t \operatorname{Re}(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) \\ & \quad + 2(\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]f, f) + \|e^{\gamma |x_1|^2} (F + Vv)\|^2, \end{aligned}$$

with

$$e^{\gamma |x_1|^2} (\partial_t - i\Delta)(e^{-\gamma |x_1|^2} f) = \partial_t f - \mathcal{S}f - \mathcal{A}f = e^{\gamma |x_1|^2} (F + Vv),$$

where $\mathcal{S} = -i\gamma(4x_1 \partial_{x_1} + 2)$ is symmetric, $\mathcal{A} = i(\Delta + 4\gamma x_1^2)$ is skew-symmetric, and F as in (5.6). Since,

$$[\mathcal{S}, \mathcal{A}] = -8\gamma \partial_{x_1}^2 + 16\gamma^2 x_1^2.$$

using the inequality

$$\begin{aligned} & \int_{\mathbb{R}^n} (|\partial_{x_1} f|^2 + 4\gamma^2 |x_1|^2 |f|^2) dx = \int_{\mathbb{R}^n} e^{2\gamma |x_1|^2} (|\partial_{x_1} u|^2 - 2\gamma |u|^2) dx \\ & \geq 2\gamma \int_{\mathbb{R}^n} |f|^2 dx. \end{aligned}$$

together with Corollary 3 we conclude that

$$(5.11) \quad \int_0^1 \int |t(1-t)| \partial_{x_1} v(x, t)|^2 e^{2\gamma |x_1|^2} dx dt \leq c_\gamma.$$

Combining and (5.9) and (5.11) one gets that

$$(5.12) \quad \begin{aligned} & \sup_{0 \leq t \leq 1} \|e^{\gamma |x_1|^2} v(\cdot, t)\|_{L^2(\mathbb{R}^n)} \\ & \quad + \int_0^1 \int |t(1-t)| \partial_{x_1} v(x, t)|^2 e^{2\gamma |x_1|^2} dx dt \leq c_\gamma. \end{aligned}$$

Step3

We recall the following result which is a slight variation of that proven in detail in [16] (Lemma 3.1, page 1818) :

Lemma 2. *Assume that $R > 0$ and $\varphi : [0, 1] \rightarrow \mathbb{R}$ is a smooth function. Then, there exists $c = c(n; \|\varphi'\|_\infty + \|\varphi''\|_\infty) > 0$ such that the inequality*

$$(5.13) \quad \frac{\alpha^{3/2}}{R^2} \left\| e^{\alpha \left| \frac{x_1 - x_{0_1}}{R} + \varphi(t) \right|^2} g \right\|_{L^2(dxdt)} \leq c \left\| e^{\alpha \left| \frac{x_1 - x_{0_1}}{R} + \varphi(t) \right|^2} (i\partial_t + \Delta) g \right\|_{L^2(dxdt)}$$

holds when $\alpha > cR^2$ and $g \in C_0^\infty(\mathbb{R}^{n+1})$ is supported in the set

$$\{(x, t) = (x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : \left| \frac{x_1 - x_{0_1}}{R} + \varphi(t) \right| \geq 1\}.$$

Now, we will chose $x_{0_1} = R/2$, $0 \leq \varphi(t) \leq a$, with $a = 3/2 - 1/R$, $\varphi(t) = a$, on $3/8 \leq t \leq 5/8$, $\varphi(t) = 0$, for $t \in [0, 1/4] \cup [3/4, 1]$, and $\theta_R \in C^\infty(\mathbb{R})$ with $\theta_R(x_1) = 1$ on $1 < x_1 < R - 1$, and $\theta_R(x_1) = 0$ for $x_1 < 1/2$ or $x_1 > R$.

Also we chose $\eta \in C^\infty(\mathbb{R})$ with $\eta(x_1) = 0$, $x_1 \leq 1$ and $\eta(x_1) = 1$, $x_1 \geq 1 + 1/2R$.

We notice that up to translation we can assume that

$$(5.14) \quad \int_{3/8}^{5/8} \int_{2 < x_1 < 3} |u(x, t)|^2 dxdt = b \neq 0,$$

otherwise we would have

$$u(x, t) = 0 \quad \text{on} \quad (x, t) \text{ s.t. } (x_1, t) \in (0, \infty) \times (3/8, 5/8),$$

and thus by Izakov's result [36] we would get that $u \equiv 0$.

We let

$$(5.15) \quad g(x, t) = \theta_R(x_1) \eta\left(\frac{x_1 - R/2}{R} + \varphi(t)\right) u(x, t).$$

It is easy to see that g is supported on the set

$$(5.16) \quad \{(x, t) \in \mathbb{R}^{n+1} : 1/2 < x_1 < R, 1/32 < t < 31/32, \left| \frac{x_1 - R/2}{R} + \varphi(t) \right| \geq 1\}.$$

so satisfies the hypothesis of Lemma 2. Also if $(x_1, t) \in (2, 3) \times (3/8, 5/8)$ one has $\varphi = a$, $\eta\left(\frac{x_1 - R/2}{R} + a\right) = 1$ and $\theta_R = 1$, hence in this domain

$$g(x, t) = u(x, t).$$

Thus, from (5.16) it follows that

$$\left| \frac{x_1 - R/2}{R} + \varphi(t) \right| \geq 1 + 1/R,$$

so we have the lower bound of (5.13)

$$\frac{\alpha^{3/2}}{R^2} b e^{\alpha(1+1/R)^2},$$

with b as in (5.14). Now we shall estimate the right hand side of (5.13). Thus,

$$(5.17) \quad \begin{aligned} (i\partial_t - \Delta)g &= -\theta_R(x_1) \eta\left(\frac{x_1 - R/2}{R} + \varphi(t)\right) V(x, t) u(x, t) \\ &+ \eta\left(\frac{x_1 - R/2}{R} + \varphi(t)\right) (2\theta'(x_1) \partial_{x_1} u + u \theta_R''(x_1)) \\ &+ (i\eta'(\cdot) \varphi'(t) + \eta''(\cdot) \frac{1}{R^2}) \theta_R(x_1) u(x, t) \equiv E_1 + E_2 + E_3. \end{aligned}$$

Choosing $R \gg \|V\|_\infty$, and recalling the fact that $\alpha > cR^2$ we see that the contribution of the term E_1 involving the potential V can be absorbed by the term in the left hand side of (5.13).

Next, we notice that the terms in E_2 involve derivatives of θ_R (θ'_R or θ''_R) so they are supported in the $(x, t) \in \mathbb{R}^n \times [0, 1]$ such that

$$1/2 < x_1 < 1, \quad \text{or} \quad R-1 < x_1 < R.$$

But, if $1/2 < x_1 < 1$, it follows that

$$\frac{x_1 - R/2}{R} + \varphi(t) \leq 1/R - 1/2 + 3/2 - 1/R = 1, \quad \text{so} \quad \eta\left(\frac{x_1 - R/2}{R} + \varphi(t)\right) = 0.$$

Thus, we only get contribution from the $(x, t) \in \mathbb{R}^n \times [0, 1]$ such that $R-1 < x_1 < R$, which can be bounded by

$$c \int_{1/32}^{31/32} \int_{R-1 < x_1 < R} (|u|^2 + |\partial_{x_1} u|^2)(x, t) e^{\alpha(2-1/R)^2} dx dt.$$

Finally, we look at the contribution of the term in E_3 in (5.17). In those the derivatives fall on η , thus they are supported in the region

$$1 \leq \frac{x_1 - R/2}{R} + \varphi(t) \leq 1 + \frac{1}{2R}, \quad \frac{1}{2} < x_1 < R, \quad \frac{1}{32} < t < \frac{31}{32}.$$

Hence, their contribution in (5.13) is bounded by

$$c \int_{1/32}^{31/32} \int_{1/2 < x_1 < R} |u(x, t)|^2 e^{\alpha(1+1/(2R))^2} dx dt \leq c_\gamma e^{\alpha(1+1/(2R))^2}.$$

Defining

$$(5.18) \quad \delta(R) = \int_{1/32}^{31/32} \int_{R-1 < x_1 < R} (|u|^2 + |\partial_{x_1} u|^2)(x, t) dx dt,$$

and collecting the above information using that $\alpha = c_n R^2$ we get

$$c R b e^{\alpha(1+1/R)^2} \leq c \delta(R) e^{\alpha(2-1/R)^2} + c_\gamma e^{\alpha(1+1/(2R))^2}.$$

Therefore, for R sufficiently large it follows that (since $b \neq 0$)

$$c R b e^{\alpha(1+1/R)^2} \leq c \delta(R) e^{\alpha(2-1/R)^2},$$

and since $\alpha = c_n R^2$ one has that

$$\delta(R) \geq b e^{-c_n R^2}.$$

To conclude we recall that the upper bounds in (5.12) gave us

$$\delta(R) \leq c e^{-\gamma R^2},$$

hence if $\gamma > c_n/2$ we conclude that $b = 0$, which yields the desired result $u \equiv 0$.

6. APPENDIX

Above we have used the following abstract results established in [20]:

Lemma 3. *Let \mathcal{S} be a symmetric operator, \mathcal{A} be a skew-symmetric one, both allowed to depend on the time variable. Let G be a positive function, $f(x, t)$ a reasonable function,*

$$H(t) = (f, f) = \|f\|_{L^2(\mathbb{R}^n)}^2 = \|f\|^2, \quad D(t) = (\mathcal{S}f, f),$$

$$\partial_t \mathcal{S} = \mathcal{S}_t \quad \text{and} \quad N(t) = \frac{D(t)}{H(t)}.$$

Then,

$$(6.1) \quad \begin{aligned} \partial_t^2 H &= 2\partial_t \operatorname{Re}(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) + 2(\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]f, f) \\ &+ \|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2 \end{aligned}$$

and

$$\dot{N}(t) \geq (\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]f, f) / H - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2 / (2H).$$

Moreover, if

$$(6.2) \quad |\partial_t f - \mathcal{A}f - \mathcal{S}f| \leq M_1 |f| + G, \quad \text{in } \mathbb{R}^n \times [0, 1], \quad \mathcal{S}_t + [\mathcal{S}, \mathcal{A}] \geq -M_0,$$

and

$$M_2 = \sup_{[0,1]} \|G(t)\| / \|f(t)\|$$

is finite, then $\log H(t)$ is “logarithmically convex” in $[0, 1]$ and there is a universal constant N such that

$$(6.3) \quad H(t) \leq e^{N(M_0 + M_1 + M_2 + M_1^2 + M_2^2)} H(0)^{1-t} H(1)^t, \quad \text{when } 0 \leq t \leq 1.$$

By multiplying the formula (6.1) by $t(1-t)$, integrating the result over $[0, 1]$ and using integration by parts, one gets the following “smoothing” inequality

Corollary 3. *With the same hypotheses and notation as in Lemma 3*

$$(6.4) \quad \begin{aligned} &2 \int_0^1 t(1-t) (\mathcal{S}_t f + [\mathcal{S}, \mathcal{A}]f, f) dt + \int_0^1 H(t) dt \leq H(0) + H(1) \\ &+ 2 \int_0^1 (1-2t) \operatorname{Re}(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) dt \\ &+ \int_0^1 t(1-t) \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|_2^2 dt. \end{aligned}$$

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(L. Escauriaza) UPV/EHU, DPTO. DE MATEMÁTICAS, APTO. 644, 48080 BILBAO, SPAIN.
E-mail address: `luis.escauriaza@ehu.es`

(C. E. Kenig) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL. 60637, USA.

E-mail address: `cek@math.uchicago.edu`

(G. Ponce) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA.

E-mail address: `ponce@math.ucsb.edu`

(L. Vega) UPV/EHU, DPTO. DE MATEMÁTICAS, APTO. 644, 48080 BILBAO, SPAIN.

E-mail address: `luis.vega@ehu.es`