UNCERTAINTY PRINCIPLE OF MORGAN TYPE AND SCHRÖDINGER EVOLUTIONS

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Abstract. We prove unique continuation properties for solutions of evolution Schrödinger equation with time dependent potentials. In the case of the free solution these correspond to uncertainly principles referred to as being of Morgan type. As an application of our method we also obtain results concerning the possible concentration profiles of solutions of semi-linear Schrödinger equations.

1. Introduction

In this paper we continue our study initiated in [5] [6], and [7] on unique continuation properties of solutions of Schrödinger equations of the form

$$i\partial_t u + \triangle u = V(x,t)u, \qquad (x,t) \in \mathbb{R}^n \times [0,1]. \tag{1.1}$$

The goal is to obtain sufficient conditions on the behavior of the solution u at two different times and on the potential V which guarantee that $u \equiv 0$ in $\mathbb{R}^n \times [0,1]$. Under appropriate assumptions this result will allow us to extend these conditions to the difference $v = u_1 - u_2$ of two solutions $u_1,\ u_2$ of semi-linear Schrödinger equation

$$i\partial_t u + \triangle u = F(u,\overline{u}), \tag{1.2}$$

from which one can infer that $u_1 \equiv u_2$, see [3].

Defining the Fourier transform of a function f as

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\xi\cdot x} f(x)\,dx,$$

the identity

$$\begin{aligned} e^{it\Delta}u_0(x) &= u(x,t) \\ &= (4\pi i t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} u_0(y)\,dy = (2\pi i t)^{-\frac{n}{2}}\, e^{\frac{i|x|^2}{4t}}\, \widehat{e^{\frac{i|\,\cdot\,|^2}{4t}} u_0}\left(\frac{x}{2t}\right), \end{aligned} \tag{1.3}$$

tells us that this kind of results for the free solution of the Schrödinger equation with data u_0

$$i\partial_t u + \triangle u = 0, \qquad u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

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is related to uncertainty principles. In this regard, one has the well known result of G. H. Hardy [9] for $n=1$ and its extension to higher dimensions $n\geq 2$ established in [16] :

$$\text{If}\quad f(x)=O(e^{-\frac{|x|^2}{\beta^2}}),\ \ \widehat{f}(\xi)=O(e^{-\frac{4|\xi|^2}{\alpha^2}}),\ \text{ and }\ \alpha\,\beta<4,\ \text{then}\ f\equiv 0. \tag{1.4}$$

Moreover, if $\alpha\,\beta=4$, then $f(x)=c\,e^{-\frac{|x|^2}{\beta^2}}$.

Using (1.3), (1.4) can be rewritten in terms of the free solution of the Schrödinger equation :

$$\text{If}\quad u_0(x)=O(e^{-\frac{|x|^2}{\beta^2}}),\ \ e^{it\Delta}u_0(x)=O(e^{-\frac{|x|^2}{\alpha^2}}),\ \text{ and }\ \alpha\,\beta<4t,\ \text{then}\ u_0\equiv 0.$$

Also, if $\alpha\,\beta=4t$, then $u_0(x)=c\,e^{-i\frac{|x|^2}{4t}}\,e^{-\frac{|x|^2}{\beta^2}}$.

The corresponding result in terms of the $L^2(\mathbb{R}^n)$-norm was established by Sitaram, Sundari, and Thangavelu in [16]:

$$\text{If}\quad e^{\frac{|x|^2}{\beta^2}}\,f(x),\ \ e^{\frac{4|\xi|^2}{\alpha^2}}\,\widehat{f}(\xi)\in L^2(\mathbb{R}^n),\ \text{ and }\ \alpha\,\beta\leq 4,\ \text{then}\ f\equiv 0.$$

In terms of the free solution of the Schrödinger equation the L^2-version of Hardy Uncertainty Principle says :

$$\text{If}\quad e^{\frac{|x|^2}{\beta^2}}\,u_0(x),\ \ e^{\frac{|x|^2}{\alpha^2}}\,e^{it\Delta}\,u_0(x)\in L^2(\mathbb{R}^n),\ \text{ and }\ \alpha\,\beta\leq 4t,\ \text{then}\ u_0\equiv 0. \tag{1.5}$$

In [7] we proved the following result:

Theorem 1. *Given any solution* $u\in C([0,T]:L^2(\mathbb{R}^n))$ *of*

$$\partial_t u=i\,(\triangle u+V(x,t)u)\,,\quad \text{in}\ \ \mathbb{R}^n\times[0,T], \tag{1.6}$$

with $V=V(x,t)$ *complex valued, bounded (i.e.* $\|V\|_{L^\infty(\mathbb{R}^n\times[0,T])}\leq C$*) and*

$$\lim_{R\to+\infty}\|V\|_{L^1([0,T]:L^\infty(\mathbb{R}^n\setminus B_R)}=0, \tag{1.7}$$

or $V(x,t)=V_1(x)+V_2(x,t)$ *with* V_1 *real valued and* V_2 *complex valued with*

$$\sup_{t\in[0,T]}\|e^{k|x|^2}\,V_2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}<\infty,\qquad \forall k\in\mathbb{Z}^+,$$

satisfying that for some $t\in(0,T]$

$$e^{\frac{|x|^2}{\beta^2}}\,u_0,\quad e^{\frac{|x|^2}{\alpha^2}}\,u(x,t)\in L^2(\mathbb{R}^n),$$

with $\alpha\,\beta<4t$*, then* $u_0\equiv 0$.

Notice that Theorem 1 recovers the L^2-version of Hardy Uncertainty Principle (1.5) for solutions of the IVP (1.6), except for the limiting case $\alpha\,\beta=4t$ for which we prove that the corresponding result fails. More precisely, in [7] it was shown that there exist (complex-valued) bounded potentials $V(x,t)$ satisfying (1.7) for which there exist nontrivial solutions $u\in C([0,T]:L^2(\mathbb{R}^n))$ of (1.6) satisfying

$$e^{\frac{|x|^2}{\beta^2}}\,u_0,\quad e^{\frac{|x|^2}{\alpha^2}}\,u(x,T)\in L^2(\mathbb{R}^n),$$

with $\alpha\,\beta=4T$.

This work is motivated by a different kind of uncertainty principles written in terms of the free solution of the Schrödinger equation. As it was mentioned, we are interested in its extentions to solutions of the equation in (1.1), and to the difference of two solutions of the nonlinear equation (1.2).

First, one has the result due to Beurling-Hörmander [10]: If $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||\widehat{f}(\xi)|e^{|x\,\xi|}\,dx\,d\xi < \infty, \quad \text{then} \quad f \equiv 0. \tag{1.8}$$

This was extended to higher dimensions $n \geq 2$ in [2] and [15] : If $f \in L^2(\mathbb{R}^n), n \geq 2$ and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)||\widehat{f}(\xi)|e^{|x\cdot\xi|}\,dx\,d\xi < \infty, \quad \text{then} \quad f \equiv 0. \tag{1.9}$$

We observe that (1.8), (1.9) implies : If $p \in (1,2)$, $1/p + 1/q = 1$, α, $\beta > 0$, and

$$\int_{\mathbb{R}^n} |f(x)|\,e^{\frac{\alpha^p|x|^p}{p}}dx \,+ \int_{\mathbb{R}^n} |\widehat{f}(\xi)|\,e^{\frac{\beta^q|\xi|^q}{q}}d\xi < \infty, \;\; \alpha\beta \geq 1 \;\Rightarrow\; f \equiv 0, \tag{1.10}$$

or in terms of the solution of the free Schrödinger equation :

If $u_0 \in L^1(\mathbb{R})$ or $u_0 \in L^2(\mathbb{R}^n)$, if $n \geq 2$, and for some $t \neq 0$

$$\int_{\mathbb{R}^n} |u_0(x)|\,e^{\frac{\alpha^p|x|^p}{p}}dx \,+ \int_{\mathbb{R}^n} |\,e^{it\Delta}u_0(x)|\,e^{\frac{\beta^q|x|^q}{q(2t)^q}}dx < \infty, \quad \alpha\,\beta \geq 1, \tag{1.11}$$

then $u_0 \equiv 0$.

The following related (and stronger in one dimension) result was established by Bonami, Demange and Jaming [2] (for further results see [1] and references therein): Let $f \in L^2(\mathbb{R}^n)$, $1 < p < 2$ and $1/p + 1/q = 1$ such that for some $j = 1, .., n$,

$$\int_{\mathbb{R}^n} |f(x)|e^{\frac{\alpha^p|x_j|^p}{p}}dx < \infty \;\; + \;\; \int_{\mathbb{R}^n} |\widehat{f}(\xi)|e^{\frac{\beta^q|\xi_j|^q}{q}}d\xi < \infty. \tag{1.12}$$

If $\alpha\,\beta > |\cos(p\pi/2)|^{1/p}$, then $f \equiv 0$. If $\alpha\beta < |\cos(p\pi/2)|^{1/p}$ there exist non-trivial functions satisfying (1.12) for all $j = 1, .., n$.

This kind of uncertainty principles involving conjugate exponent p, q were first studied by G. W. Morgan in [14].

In [8] Gel'fand and Shilov considered the class Z_p^p, $p \geq 1$ defined as the space of all functions $\varphi(z_1, .., z_n)$ which are analytic for all values of $z_1, .., z_n \in \mathbb{C}$ and such that

$$|\varphi(z_1, .., z_n)| \leq C_0\,e^{\sum_{j=1}^n \epsilon_j\,C_j\,|z_j|^p},$$

where the C_j, $j = 0, 1, .., n$ are positive constants and $\epsilon_j = 1$ for z_j non-real and $\epsilon_j = -1$ for z_j real, $j = 1, .., n$, and showed that the Fourier transform of the function space Z_p^p is the space Z_q^q, with $1/p + 1/q = 1$.

Notice that the class Z_p^p with $p \geq 2$ is closed respect to multiplication by $e^{ic|x|^2}$. Thus, if $u_0 \in Z_p^p$, $p \geq 2$, then by (1.3) one has that $|e^{it\Delta}u_0(x)| \leq d(t)\,e^{-a(t)|x|^q}$, for some functions d, a : $\mathbb{R} \to (0, \infty)$.

Our main result in this paper is the following:

Theorem 2. *Given $p \in (1,2)$ there exists $M_p > 0$ such that for any solution $u \in C([0,1] : L^2(\mathbb{R}^n))$ of*

$$\partial_t u = i\,(\triangle u + V(x,t)u)\,, \quad in \quad \mathbb{R}^n \times [0,1],$$

with $V = V(x,t)$ complex valued, bounded (i.e. $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])} \leq C$) and

$$\lim_{R\to+\infty} \|V\|_{L^1([0,1]:L^\infty(\mathbb{R}^n\setminus B_R))} = 0, \tag{1.13}$$

satisfying for some constants $a_0,\, a_1,\, a_2 > 0$

$$\int_{\mathbb{R}^n} |u(x,0)|^2\, e^{2a_0|x|^p} dx < \infty, \tag{1.14}$$

and for any $k \in \mathbb{Z}^+$

$$\int_{\mathbb{R}^n} |u(x,1)|^2\, e^{2k|x|^p} dx < a_2 e^{2a_1 k^{q/(q-p)}}, \tag{1.15}$$

$1/p + 1/q = 1$, *if*

$$a_0\, a_1^{(p-2)} > M_p, \tag{1.16}$$

then $u \equiv 0$.

Corollary 1. *Given* $p \in (1,2)$ *there exists* $N_p > 0$ *such that if* $u \in C([0,1] : L^2(\mathbb{R}^n))$ *is a solution of*

$$\partial_t u = i(\Delta u + V(x,t)u),$$

with $V = V(x,t)$ *complex valued, bounded (i.e.* $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])} \le C$*) and*

$$\lim_{R\to\infty} \int_0^1 \sup_{|x|>R} |V(x,t)| dt = 0,$$

and there exist $\alpha,\, \beta > 0$

$$\int_{\mathbb{R}^n} |u(x,0)|^2 e^{2\,\alpha^p\,|x|^p/p} dx \;+\; \int_{\mathbb{R}^n} |u(x,1)|^2 e^{2\,\beta^q\,|x|^q/q} dx < \infty, \tag{1.17}$$

$1/p + 1/q = 1$, *with*

$$\alpha\,\beta > N_p, \tag{1.18}$$

then $u \equiv 0$.

As a direct consequence of Corollary 1 we get the following result regarding the uniqueness of solutions for non-linear equations of the form (1.2).

Theorem 3. *Given* $p \in (1,2)$ *there exists* $N_p > 0$ *such that if*

$$u_1,\, u_2 \in C([0,1] : H^k(\mathbb{R}^n)),$$

are strong solutions of (1.2) *with* $k \in \mathbb{Z}^+$, $k > n/2$, $F : \mathbb{C}^2 \to \mathbb{C}$, $F \in C^k$ *and* $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$, *and there exist* $\alpha,\, \beta > 0$ *such that*

$$e^{\alpha^p\,|x|^p/p}\,(u_1(0) - u_2(0))\,, \quad e^{\beta^q\,|x|^q/q}\,(u_1(1) - u_2(1)) \in L^2(\mathbb{R}^n), \tag{1.19}$$

$1/p + 1/q = 1$, *with*

$$\alpha\,\beta > N_p, \tag{1.20}$$

then $u_1 \equiv u_2$.

Notice that the conditions (1.16) and (1.18) are independent of the size of the potential and that we do not assume any regularity on the potential $V(x,t)$.

It will be clear from our proof of Theorem 2 that the result in [2] (1.12) can be extended to our setting with an unsharp constant. More precisely,

Corollary 2. *The results in Corollary 1 still hold with a different constant $N_p > 0$ if one replaces the hypothesis* (1.17) *by the one dimensional version*

$$\int_{\mathbb{R}^n} |u(x,0)|^2 e^{2\,\alpha^p\,|x_j|^p/p} dx < \infty \quad + \quad \int_{\mathbb{R}^n} |u(x,1)|^2 e^{2\,\beta^q\,|x_j|^q/q} dx < \infty \tag{1.21}$$

for some $j = 1, .., n$.

Remarks (i) Similarly, the non-linear version of Theorem 3 still holds, with different constant $N_p > 0$, if one replaces the hypothesis (1.19) by

$$e^{\alpha^p\,|x_j|^p/p}\,(u_1(0) - u_2(0)), \quad e^{\beta^q\,|x_j|^q/q}\,(u_1(1) - u_2(1)) \in L^2(\mathbb{R}^n),$$

for $j = 1, .., n$.

(ii) In this work, we do not try to give an estimate of the universal constant N_p. In fact, we may remark that the corresponding version of the sharp one dimensional condition $\alpha\,\beta > |\cos(p\pi/2)|^{1/p}$ for (1.10) established in [2] is unknown in higher dimensions $n \geq 2$.

(iii) We do not consider here possible versions of the limiting case $p = 1$. One can conjecture, for example that, if $u(x,t)$ is a solution of (1.1) with $u(x,0) = u_0(x)$ having compact support and $u(\cdot,t) \in L^1(e^{\epsilon|x|})$ for some $\epsilon > 0$ and $t \neq 0$, then $u_0 \equiv 0$.

(iv) As in some of our our previous works the main idea in the proof is to combine an upper estimate, based on the decay hypothesis at two different times (see Lemma 1), with a lower estimate based on the positivity of the commutator operator obtained by conjugating the equation with the appropiate exponential weight (see Lemma 2). In previous works we have been able to establish the upper bound estimates from assumptions that at time $t = 0$ and $t = 1$ involve the same weight. However, in our case (Corollary 1) we have different weights at time $t = 0$ and $t = 1$. To overcome this difficulty, we carry out the details with the weight $e^{a_j|x|^p}$, $1 < p < 2$, $j = 0$ at $t = 0$ and $j = 1$ at $t = 1$, with a_0 fixed and $a_1 = k \in \mathbb{Z}^+$ as in (1.15). Although the powers $|x|^p$ in the exponential are equal at time $t = 0$ and $t = 1$ to apply our estimate (Lemma 1) we also need to have the same constant in front of them. To achieve this we apply the conformal or Appel tranformation, to get solutions and potentials, whose bounds depend on $k \in \mathbb{Z}^+$. Thus we have to consider a family of solutions and obtain estimates on their asymptotic value as $k \uparrow \infty$.

Next, we shall extend the method used in the proof Theorem 2 to study the possible profile of the concentration blow up phenomenon in solutions of non-linear Schrödinger equations

$$i\partial_t u + \triangle u + F(u,\overline{u})u = 0. \tag{1.22}$$

To illustrate the problem consider the focussing L^2-critical Schrödinger equation

$$i\partial_t u + \triangle u + |u|^{4/n} u = 0. \tag{1.23}$$

From the pseudo-conformal transformation one has that if $u = u(x,t)$ is a solution of (1.22), then

$$v(x,t) = \frac{e^{-i|x|^2/4(1-t)}}{(1-t)^{n/2}}\, u\left(\frac{x}{1-t}, \frac{t}{1-t}\right), \tag{1.24}$$

is also a solution of (1.23) in its domain of definition.

We recall that the pseudo-conformal transformation preserves both the space $L^2(\mathbb{R}^n)$ and the space $H^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n : |x|^2dx)$. In particular, if we take $u(x,t) = e^{it}\,\varphi(x)$ the standing wave solution, i.e. $\varphi(x)$ being the positive ground state of the non-linear elliptic equation

$$-\Delta\varphi + \varphi = |\varphi|^{4/n}\varphi, \quad x\in\mathbb{R}^n,$$

it follows that

$$v(x,t) = \frac{e^{-i(|x|^2-4)/4(1-t)}}{(1-t)^{n/2}}\,\varphi\left(\frac{x}{1-t}\right), \tag{1.25}$$

is a solution of (1.23) which blows up at time $t=1$, i.e.

$$\lim_{t\uparrow 1}\|\nabla\, v(\cdot,t)\|_2 = \infty,$$

and

$$\lim_{t\uparrow 1}|v(\cdot,t)|^2 = c\,\delta(\cdot), \quad \text{in the distribution sense.}$$

Since it is known that the ground state φ has exponential decay, i.e.

$$\varphi(x) \le b_1 e^{-b_2|x|}, \qquad b_1,\, b_2 > 0,$$

then one has that the blow up solution $v(x,t)$ in (1.25) satisfies

$$|v(x,t)| \le \frac{1}{(1-t)^{n/2}}\,Q\left(\frac{|x|}{1-t}\right), \qquad t\in(-1,1), \tag{1.26}$$

in this case with $Q(x) = b_1\,e^{-b_2|x|}$.

Therefore, for (1.22) with

$$|F(z,\overline{z})| \le b_0\,|z|^{\theta}, \quad \text{with} \quad b_0,\,\theta>0 \quad \text{for} \quad |z|>>1, \tag{1.27}$$

one may ask if it is possible to have a faster "concentration profile" than the one described in (1.26). More precisely, whether or not (1.26) can hold with

$$Q(x) = b_1\,e^{-b_2|x|^p}, \qquad b_1,\,b_2>0, \quad p>1. \tag{1.28}$$

Our next result shows that this is not the case at least for $p > p(\theta)$.

Theorem 4. *Let $u\in C((-1,1) : L^2(\mathbb{R}^n))$ be a solution of the equation (1.22) with $|F(z,\overline{z})|$ as in (1.27). Assume that the L^2-norm of the solution $u(x,t)$ is preserved*

$$\|u(\cdot,t)\|_2 = \|u(\cdot,0)\|_2 = \|u_0\|_2 = a, \quad t\in(-1,1), \tag{1.29}$$

and that (1.26) holds with $Q(\cdot)$ as in (1.28). If $p > p(\theta) = 2(\theta n-2)/(\theta n-1)$, then $a=0$.

Remarks (i) We shall restrict to the case $p\in(1,2)$, and observe that if $\theta = 4/n$ then $p(\theta) = 4/3$. The value $4/3$ is related with the following result due to V. Z. Meshkov [13]: Let $w\in H^2_{loc}(\mathbb{R}^n)$ be a solution of

$$\Delta w - V(x)w = 0, \quad x\in\mathbb{R}^n, \quad \text{with} \quad V\in L^\infty(\mathbb{R}^n).$$

$$\text{If} \quad \int |w(x)|^2\,e^{2a|x|^{4/3}}dx < \infty, \quad \forall\, a>0, \;\text{ then } \quad w\equiv 0.$$

It was also proved in [13] that for complex valued potentials V the exponent $4/3$ is sharp. For further comments see the remark after the proof of Theorem 4.

The rest of this paper is organized as follows. In section 2 we establish the upper bounds needed in the proof of Theorem 2. The lower bounds as well as the conclusion of the proof of Theorem 2 are carried out in section 3. Corollary 1 and Theorem 3 are proved in section 4. Section 5 contains the proof of Theorem 4. Finally, in the Appendix we establish some identities used in the paper.

2. Proof of Theorem 2 : Upper bounds

In this paper c_n will denote a constant which may depend only on the dimension n, which may change from line to line. Similarly, c_p will denote a constant depending only on the values of p and n, and in sections 2-3 c^* will denote a constant depending of the initial parameters, i.e. the norms of u and V in the hypothesis, and on the values of p, n, a_0 and a_1, whose exact value will be irrelevant to our estimate when we take k tending to infinity.

We recall the conformal or Appell transformation. If $u(y,s)$ verifies

$$\partial_s u = i\left(\triangle u + V(y,s)u + F(y,s)\right), \qquad (y,s)\in\mathbb{R}^n\times[0,1], \tag{2.1}$$

and α and β are positive, then

$$\widetilde{u}(x,t) = \left(\tfrac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}} u\left(\tfrac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}, \tag{2.2}$$

verifies

$$\partial_t\widetilde{u} = i\left(\triangle\widetilde{u} + \widetilde{V}(x,t)\widetilde{u} + \widetilde{F}(x,t)\right), \ \text{ in } \mathbb{R}^n\times[0,1], \tag{2.3}$$

with

$$\widetilde{V}(x,t) = \tfrac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}\, V\left(\tfrac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right), \tag{2.4}$$

and

$$\widetilde{F}(x,t) = \left(\tfrac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}+2} F\left(\tfrac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}. \tag{2.5}$$

In our case, we shall chose $\beta=\beta(k)$. By hypothesis

$$\begin{aligned} &\|e^{a_0|x|^p}u(x,0)\|_2 \equiv A_0,\\ &\|e^{k|x|^p}u(x,1)\|_2 \equiv A_k \le a_2\, e^{a_1 k^{q/(q-p)}} = a_2\, e^{a_1 k^{1/(2-p)}}. \end{aligned} \tag{2.6}$$

Thus, for $\gamma=\gamma(k)\in[0,\infty)$ to be chosen later, one has

$$\|e^{\gamma|x|^p}\widetilde{u}_k(x,0)\|_2 = \|e^{\gamma\left(\frac{\alpha}{\beta}\right)^{p/2}|x|^p}u(x,0)\|_2 = B_0, \tag{2.7}$$

and

$$\|e^{\gamma|x|^p}\widetilde{u}_k(x,1)\|_2 = \|e^{\gamma\left(\frac{\beta}{\alpha}\right)^{p/2}|x|^p}u(x,1)\|_2 = A_k. \tag{2.8}$$

To match our hypothesis we take

$$\gamma\left(\frac{\alpha}{\beta}\right)^{p/2} = a_0 \quad \text{and} \quad \gamma\left(\frac{\beta}{\alpha}\right)^{p/2} = k.$$

Therefore,

$$\gamma = (k\,a_0)^{1/2}, \qquad \beta = k^{1/p}, \qquad \alpha = a_0^{1/p}. \tag{2.9}$$

From (2.3), defining

$$M = \int_0^1 \|\Im V(t)\|_\infty dt = \int_0^1 \|\Im\widetilde{V}(s)\|_\infty ds, \tag{2.10}$$

it follows, using energy estimates, that

$$\|u(0)\|_2\, e^{-M} \le \|u(t)\|_2 = \|\,\widetilde{u}(s)\|_2 \le \|u(0)\|_2\, e^{M}, \quad t,\, s \in [0,1]. \tag{2.11}$$

where $s = \beta t/(\alpha(1-t)+\beta t)$.

Next, we shall combine the estimate : for any $x \in \mathbb{R}^n$

$$e^{\gamma |x|^p/p} \simeq \int_{\mathbb{R}^n} e^{\gamma^{1/p}\lambda\cdot x - |\lambda|^q/q}\, |\lambda|^{n(q-2)/2}\, d\lambda, \tag{2.12}$$

whose proof will be given in the appendix, with the following result found in [11] :

Lemma 1. *There exists $\epsilon_0 > 0$ such that if*

$$\mathbb{V} : \mathbb{R}^n \times [0,1] \to \mathbb{C}, \quad \textit{with} \quad \|\mathbb{V}\|_{L^1_t L^\infty_x} \le \epsilon_0, \tag{2.13}$$

and $u \in C([0,1] : L^2(\mathbb{R}^n))$ is a strong solution of the IVP

$$\begin{cases} \partial_t u = i(\Delta + \mathbb{V}(x,t))u + \mathbb{F}(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{2.14}$$

with

$$u_0,\, u_1 \equiv u(\,\cdot\,,1) \in L^2(e^{2\lambda\cdot x}dx),\ \mathbb{F} \in L^1([0,1] : L^2(e^{2\lambda\cdot x}dx)), \tag{2.15}$$

for some $\lambda \in \mathbb{R}^n$, then there exists c_n independent of λ such that

$$\sup_{0\le t\le 1} \|e^{\lambda\cdot x}u(\,\cdot\,,t)\|_2 \le c_n\Big(\|e^{\lambda\cdot x}u_0\|_2 + \|e^{\lambda\cdot x}u_1\|_2 + \int_0^1 \|e^{\lambda\cdot x}\,\mathbb{F}(\cdot,t)\|_2 dt\Big). \ \square \tag{2.16}$$

We want to apply Lemma 1 to a solution of the equation (2.3). Since $0 < \alpha < \beta = \beta(k)$ for $k \ge k_0(c^*)$ it follows that for any $t \in [0,1]$

$$\alpha \le \alpha(1-t) + \beta t \le \beta,$$

therefore if $y = \sqrt{\alpha\beta}\, x/(\alpha(1-t)+\beta t)$, then from (2.9)

$$\sqrt{\frac{\alpha}{\beta}}\,|x| = \frac{a_0^{1/2p}}{k^{1/2p}}\,|x| \le |y| \le \sqrt{\frac{\beta}{\alpha}}\,|x| = \frac{k^{1/2p}}{a_0^{1/2p}}\,|x|. \tag{2.17}$$

Thus,

$$\Big|\tfrac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}\, V\Big(\tfrac{\sqrt{\alpha\beta}\,x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\Big)\Big| \le \frac{\beta}{\alpha}\|V\|_\infty = \left(\frac{k}{a_0}\right)^{1/p} \|V\|_\infty, \tag{2.18}$$

and so

$$\|\widetilde{V}\|_\infty \le \left(\frac{k}{a_0}\right)^{1/p} \|V\|_\infty. \tag{2.19}$$

Also, if

$$s = \beta t/(\alpha(1-t)+\beta t)$$

$$\frac{ds}{dt} = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}, \qquad \text{or} \quad dt = \frac{(\alpha(1-t)+\beta t)^2}{\alpha\beta} ds. \tag{2.20}$$

Therefore,

$$\int_0^1 \|\,\widetilde{V}(\cdot,t)\|_{L^\infty} dt = \int_0^1 \|V(\cdot,s)\|_{L^\infty} ds, \tag{2.21}$$

and from (2.17)

$$\int_0^1 \| \widetilde{V}(\cdot,t)\|_{L^\infty(|x|\geq R)} dt \leq \int_0^1 \|V(\cdot,s)\|_{L^\infty(|y|>\Upsilon)} ds,$$

where by hypothesis

$$\Upsilon = \Big(\frac{a_0}{k}\Big)^{1/2p} R.$$

Thus, if

$$\int_0^1 \|V(\cdot,s)\|_{L^\infty(|y|>\Upsilon)} ds \leq \epsilon_0,$$

then

$$\int_0^1 \| \widetilde{V}(\cdot,t)\|_{L^\infty(|x|\geq R)} dt \leq \epsilon_0, \qquad R = \Upsilon \left(\frac{k}{a_0}\right)^{1/2p}, \tag{2.22}$$

and we can apply Lemma 1 to the equation (2.3) with

$$\mathbb{V} = \widetilde{V}\,\chi_{(|x|>R)}(x), \qquad \mathbb{F} = \widetilde{V}\,\chi_{(|x|\leq R)}(x)\,\widetilde{u}(x,t), \tag{2.23}$$

to get

$$\begin{aligned} &\sup_{[0,1]} \|e^{(2p)^{1/p}\gamma^{1/p}\lambda\cdot x/2}\,\widetilde{u}(t)\|_2 \\ &\leq c_n \Big(\|e^{(2p)^{1/p}\gamma^{1/p}\lambda\cdot x/2}\,\widetilde{u}(0)\|_2 + \|e^{(2p)^{1/p}\gamma^{1/p}\lambda\cdot x/2}\,\widetilde{u}(1)\|_2 \Big) \\ &+ c_n \|\widetilde{V}\|_\infty \|u(0)\|_2\, e^M\, e^{|\lambda|(2p)^{1/p}\gamma^{1/p}R/2}. \end{aligned} \tag{2.24}$$

Now, we square (2.24) to find that

$$\begin{aligned} &\sup_{[0,1]} \int e^{(2p)^{1/p}\gamma^{1/p}\lambda\cdot x}\, |\widetilde{u}(x,t)|^2 dx \\ &\leq c_n \int e^{(2p)^{1/p}\gamma^{1/p}\lambda\cdot x}\, (|\widetilde{u}(x,0)|^2 + |\widetilde{u}(x,1)|^2) dx \\ &+ c_n \|\widetilde{V}\|_\infty^2\, \|u(0)\|_2^2\, e^{2M}\, e^{|\lambda|(2p)^{1/p}\gamma^{1/p}R}, \end{aligned}$$

and multiply the above inequality (for a fixed t) by $e^{-|\lambda|^q/q}\,|\lambda|^{n(q-2)/2}$, integrate in λ and in x, use Fubini theorem and the formula (2.12) to obtain

$$\begin{aligned} &\int_{|x|>1} e^{2\gamma|x|^p} |\widetilde{u}(x,t)|^2 dx \leq c_n \int e^{2\gamma|x|^p} (|\widetilde{u}(x,0)|^2 + |\widetilde{u}(x,1)|^2) dx \\ &+ c_n \|u(0)\|_2^2\, \|\widetilde{V}\|_\infty^2\, R^{c_p}\, e^{2M} e^{2\gamma\, R^p}. \end{aligned} \tag{2.25}$$

Hence, (2.6), (2.9), (2.11), (2.19), and (2.25) lead to

$$\begin{aligned}
&\sup_{[0,1]} \|e^{\gamma|x|^p}\widetilde{u}(t)\|_2 \le c_n(\|e^{\gamma|x|^p}\widetilde{u}(0)\|_2 + \|e^{\gamma|x|^p}\widetilde{u}(1)\|_2)\\
&+ c_n\|u(0)\|_2\, e^{M} e^{\gamma} + c_n\|u(0)\|_2 \left(\frac{k}{a_0}\right)^{c_p} \|V\|_\infty\, e^{M} e^{\Upsilon^p\,\gamma(k/a_0)^{1/2}}\\
&\le c_n(A_0 + A_k) + c_n\|u(0)\|_2\, e^{M} \left(e^{\gamma} + \left(\frac{k}{a_0}\right)^{c_p} \|V\|_\infty\, e^{\Upsilon^p\,k}\right),\\
&\le c^*\, A_k = c^*\, e^{a_1\,k^{1/(2-p)}},
\end{aligned} \tag{2.26}$$

for $k \ge k_0(c^*)$ sufficiently large, since $1/(2-p) > 1$.

Next, we shall obtain bounds for the $\nabla\widetilde{u}$. Let

$$\widetilde{\gamma} = \gamma/2, \tag{2.27}$$

and φ be a strictly convex function on compact sets of $\mathbb{R}^n$, radial such that (see [6] Lemma 3)

$$\begin{aligned}
&D^2\varphi \ge p(p-1)|x|^{(p-2)}I, \quad \text{for} \quad |x| \ge 1,\\
&0 \le \varphi, \quad \|\partial^\alpha\varphi\|_{L^\infty} \le c \quad 2 \le |\alpha| \le 4, \;\; \|\partial^\alpha\varphi\|_{L^\infty(|x|\le 2)} \le c \quad |\alpha| \le 4,\\
&\varphi(x) = |x|^p + O(|x|), \quad \text{for} \quad |x| > 1.
\end{aligned} \tag{2.28}$$

We shall use the equation

$$\partial_t\widetilde{u} = i\Delta\widetilde{u} + iF, \qquad F = \widetilde{V}\,\widetilde{u}, \tag{2.29}$$

and let

$$f(x,t) = e^{\widetilde{\gamma}\varphi}\,\widetilde{u}(x,t).$$

Then f verifies (see Lemma 3 in [6])

$$\partial_t f = \mathcal{S}f + \mathcal{A}f + i\,e^{\widetilde{\gamma}\varphi}F, \quad \text{in} \quad \mathbb{R}^n \times [0,1], \tag{2.30}$$

with symmetric and skew-symmetric operators $\mathcal{S}$ and $\mathcal{A}$

$$\begin{aligned}
\mathcal{S} &= -\, i\widetilde{\gamma}\,(2\nabla\varphi\cdot\nabla + \triangle\varphi)\,,\\
\mathcal{A} &= i\left(\triangle + \widetilde{\gamma}^2|\nabla\varphi|^2\right).
\end{aligned} \tag{2.31}$$

A calculation shows that (see (2.14) in [6]),

$$\mathcal{S}_t + [\mathcal{S},\mathcal{A}] = -\widetilde{\gamma}\left[4\nabla\cdot\left(D^2\varphi\nabla\ \right) - 4\widetilde{\gamma}^2 D^2\varphi\nabla\varphi\cdot\nabla\varphi + \triangle^2\varphi\right]. \tag{2.32}$$

By Lemma 2 in [6]

$$\begin{aligned}
\partial_t^2 H = \partial_t^2\,(f,f) =& 2\partial_t Re\,(\partial_t f - \mathcal{S}f - \mathcal{A}f, f) + 2\,(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]\,f, f)\\
&+ \|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2,
\end{aligned} \tag{2.33}$$

so

$$\begin{aligned}
\partial_t^2 H \ge\;& 2\partial_t Re\,(\partial_t f - \mathcal{S}f - \mathcal{A}f, f)\\
&+ 2\,(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]\,f, f) - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2.
\end{aligned} \tag{2.34}$$

Multiplying (2.34) by $t(1-t)$ and integrating in t we obtain

$$\begin{aligned} &2\int_0^1 t(1-t)\left(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]\, f, f\right) dt \\ &\le c_n \sup_{[0,1]} \|e^{\widetilde{\gamma}\,\varphi}\,\widetilde{u}(t)\|_2^2 + c_n \sup_{[0,1]} \|e^{\widetilde{\gamma}\,\varphi} F(t)\|_2^2. \end{aligned} \tag{2.35}$$

This computation can be justified by parabolic regularization using the fact that we already know the decay estimate for $\widetilde{u}$, (see the proof of Theorem 5 in [6]). Hence, combining (2.26), (2.9) and (2.19) it follows that

$$\begin{aligned} &8\,\widetilde{\gamma}\int_0^1\int t(1-t) D^2\varphi\nabla f\cdot\nabla f dx dt \\ &+ 8\,\widetilde{\gamma}^3\int_0^1\int t(1-t)D^2\varphi\nabla\varphi\cdot\nabla\varphi\,|f|^2 dxdt \\ &\le c_n \sup_{[0,1]}\|e^{\widetilde{\gamma}\,\varphi}\,\widetilde{u}(t)\|_2^2 + c_n\,\|\widetilde{V}\|_{L^\infty}^2 \sup_{[0,1]}\|e^{\widetilde{\gamma}\,\varphi}\,\widetilde{u}(t)\|_2^2 + c_n\widetilde{\gamma}\sup_{[0,1]}\|e^{\widetilde{\gamma}\,\varphi}\,\widetilde{u}(t)\|_2^2 \\ &\le c^*\,k^{c_p}\,A_k^2. \end{aligned} \tag{2.36}$$

We recall that

$$\nabla f = \widetilde{\gamma}\,\nabla\varphi\, e^{\widetilde{\gamma}\varphi}\,\widetilde{u} + e^{\widetilde{\gamma}\varphi}\,\nabla\widetilde{u},$$

and notice that

$$|\,\widetilde{\gamma}^3\,D^2\varphi\,\nabla\varphi\cdot\nabla\varphi\,e^{2\widetilde{\gamma}\,\varphi}| \le c_p\widetilde{\gamma}^3\,e^{2\gamma|x|^p}.$$

Hence, using that

$$D^2\varphi\,\nabla\varphi\cdot\nabla\varphi\,e^{2\widetilde{\gamma}\,\varphi} \le c_n(1+|x|)^{p-2+2(p-1)}\,e^{2\widetilde{\gamma}\,\varphi} \le c_p e^{3\gamma\,\varphi/2}$$

we can conclude that

$$\begin{aligned} &\gamma\int_0^1\int t(1-t)\frac{1}{(1+|x|)^{2-p}}\,|\nabla\widetilde{u}(x,t)|^2 e^{\gamma|x|^p}dxdt + \sup_{[0,1]}\,\|e^{\gamma|x|^p/2}\,\widetilde{u}(t)\|_2^2 \\ &\le c^*\,k^{c_p}\,A_k^2 = c^*\,k^{c_p}\,e^{2a_1\,k^{q/(q-p)}} = c^*\,k^{c_p}\,e^{2a_1\,k^{1/(2-p)}}, \end{aligned} \tag{2.37}$$

for $k \ge k_0(c^*)$ sufficiently large.

3. Proof of Theorem 2 : Lower bounds and conclusion.

First, we deduce a lower bound for

$$\Phi \equiv \int_{|x|<R/2}\int_{3/8}^{5/8} |\widetilde{u}(x,t)|^2 dtdx \ge \|u(0)\|_2\,e^{-M}/10, \tag{3.1}$$

for R sufficiently large. From (2.2)

$$\Phi = \int_{|x|<R/2} \int_{3/8}^{5/8} |\widetilde{u}(x,t)|^2 dtdx$$

$$= \int_{|x|<R/2} \int_{3/8}^{5/8} \left| \left(\tfrac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}} u\left(\tfrac{\sqrt{\alpha\beta}\, x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right)\right|^2 dtdx$$

$$\geq c_n \frac{\beta}{\alpha} \int_{|y|\leq R\,(a_0/k)^{1/2p}} \int_{s(3/8)}^{s(5/8)} |u(y,s)|^2 \frac{dsdy}{s^2}$$

$$\geq c_n \frac{\beta}{\alpha} \int_{|y|\leq R\,(a_0/k)^{1/2p}} \int_{s(3/8)}^{s(5/8)} |u(y,s)|^2 dsdy,$$

where in the t variable we have used that in the interval $t \in [3/8, 5/8]$ (see (2.20))

$$dt = \frac{\beta}{\alpha}\, \frac{t^2}{s^2}\, ds \sim \frac{\beta}{\alpha}\, \frac{1}{s^2}\, ds,$$

since $s(t) = \beta t/(\alpha(1-t)+\beta t)$,

$$s(5/8) - s(3/8) = \frac{\alpha\, \beta\, (5/8 - 3/8)}{(\alpha\, 3/8 + \beta\, 5/8)(\alpha\, 5/8 + \beta\, 3/8)} \sim \frac{\alpha}{\beta},$$

for $k \geq c_n$, and $s(5/8) > s(3/8) \uparrow 1$ as $k \uparrow \infty$ with $s(3/8) \geq 1/2$ for $k \geq c_n$, and in the x variable that

$$y = \sqrt{\alpha\,\beta} x/(\alpha(1-t)+\beta t),$$

so for $t \in [3/8, 5/8]$

$$y \sim \sqrt{\frac{\alpha}{\beta}}\, |x| = \left(\frac{a_0}{k}\right)^{1/2p} |x|.$$

Thus, taking

$$R \geq \iota (k/a_0)^{1/2p}, \tag{3.2}$$

with $\iota = \iota(u)$ a constant to be determined, it follows that

$$\Phi \geq c_n \frac{\beta}{\alpha} \int_{|y|\leq \iota} \int_I |u(y,s)|^2 dsdy,$$

where the interval $I = I_k = [s(3/8), s(5/8)]$ satisfies $I \subset [1/2, 1]$ and $|I| \sim \alpha/\beta$ for k sufficiently large. Moreover, given $\epsilon > 0$ there exists $k_0(\epsilon) > 0$ such that for any $k \geq k_0$ one has that $I_k \subset [1-\epsilon, 1]$.

By hypothesis on $u(x,t)$, i.e. the continuity of $\|u(\cdot, s)\|_2$ at $s = 1$, it follows that there exists $\iota >> 1$ and $K_0 = K_0(u)$ such that for any $k \geq K_0$ and for any $s \in I_k$

$$\int_{|y|\leq \iota} |u(y,s)|^2 dy \geq \|u(0)\|_2\, e^{-M}/10,$$

which yields the desired result. Below we will fix $R \sim k^{1/2(2-p)} >> k^{1/2p}$, $p > 1$ (see (3.13)), so we could have taken $\iota \sim k^l$, $l = 1/2(2-p) - 1/2p = 1/p(2-p) > 0$, and take ι independent of u when $k \uparrow \infty$.

Next, we deduce an upper bound for

$$\int_{|x|<R}\int_{1/32}^{31/32}(|\widetilde{u}|^2+|\nabla\widetilde{u}|^2)(x,t)dtdx. \tag{3.3}$$

From (2.11) and the fact that

$$\|\widetilde{u}(t)\|_2=\|u(s)\|_2,\qquad s=\beta t/(\alpha(1-t)+\beta t),$$

we have

$$\int_{|x|<R}\int_{1/32}^{31/32}|\widetilde{u}(x,t)|^2dtdx\le\|u(0)\|_2^2\,e^{2M},$$

and from (2.37)

$$\int_{1/32}^{31/32}\int_{|x|\le R}|\nabla\widetilde{u}(x,t)|^2dxdt$$

$$\le c_n\int_{1/32}^{31/32}\int_{|x|\le R}t(1-t)\frac{(1+|x|)^{2-p}}{(1+|x|)^{2-p}}\,e^{\gamma|x|^p}\,|\nabla\widetilde{u}(x,t)|^2dxdt \tag{3.4}$$

$$\le c_n\gamma^{-1}\,R^{2-p}\,c^*\,k^{c_p}\,A_k^2\le c^*\,k^{c_p}\,e^{2a_1\,k^{q/(q-p)}}=c^*\,k^{c_p}\,e^{2a_1\,k^{1/(2-p)}},$$

using that R is a power of k. Hence,

$$\int_{|x|<R}\int_{1/32}^{31/32}(|\widetilde{u}|^2+|\nabla\widetilde{u}|^2)(x,t)dtdx\le c^*\,k^{c_p}\,e^{2a_1\,k^{1/(2-p)}}, \tag{3.5}$$

for $k\ge k_0(c^*)$ sufficiently large.

We now recall Lemma 3.1 in [3].

Lemma 2. *Assume that $R>0$ and $\varphi:[0,1]\longrightarrow\mathbb{R}$ is a smooth function. Then, there exists $c=c(n,\|\varphi'\|_\infty+\|\varphi''\|_\infty)>0$ such that, the inequality*

$$\frac{\sigma^{3/2}}{R^2}\,\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}g\|_{L^2(dxdt)}\le c\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}(i\partial_t+\Delta)g\|_{L^2(dxdt)} \tag{3.6}$$

holds, when $\sigma\ge cR^2$ and $g\in C_0^\infty(\mathbb{R}^{n+1})$ has its support contained in the set

$$\{(x,t):|\tfrac{x}{R}+\varphi(t)e_1|\ge1\}\ .$$

First, we need to show that

$$R>>\|\widetilde{V}\|_{L^\infty(\mathbb{R}^n\times[1/32,31/32]}. \tag{3.7}$$

But in this domain from (2.4) one sees that

$$|\widetilde{V}(x,t)|\le(32)^2\,\frac{\alpha}{\beta}\,\|V\|_\infty\le(32)^2\,\frac{a_0^{1/p}}{k^{1/p}}\,\|V\|_\infty,$$

from (3.2) it is clear that (3.7) holds. Define

$$\delta(R)=\left(\int_{1/32}^{31/32}\int_{R-1\le|x|\le R}(|\widetilde{u}|^2+\nabla\widetilde{u}|^2)(x,t)dxdt\right)^{1/2}, \tag{3.8}$$

We choose $\varphi\in C^\infty([0,1])$ and $\theta_R,\ \theta\in C_0^\infty(\mathbb{R}^n)$ functions verifying

$$0\le\varphi\le3,\quad\varphi(t)=3,\ \text{ if }\ t\in[3/8,5/8],\quad\varphi(t)=0,\ \text{ if }\ t\in[0,1/4]\cup[3/4,1],$$

$$\theta_R(x)=1,\ \text{ if }\ |x|\le R-1,\quad \theta_R(x)=0,\ \text{ if }\ |x|>R,$$

and

$$\theta(x)=0,\ \text{ if }\ |x|\le 1,\quad \theta(x)=1,\ \text{ if }\ |x|\ge 2.$$

We define

$$g(x,t)=\theta_R(x)\,\theta(\tfrac{x}{R}+\varphi(t)e_1)\,\widetilde{u}(x,t), \tag{3.9}$$

and make the following remarks on $g(x,t)$:

- if $|x|\le R/2,\ t\in[3/8,5/8]$, then $|x/R+\varphi(t)e_1|\ge 3-1/2=\frac{5}{2}>2$,

so $g(x,t)=\widetilde{u}(x,t)$, and in this set $e^{\sigma\,|x/R+\varphi(t)e_1|^2}\ge e^{25\sigma/4}$.

-if $|x|\ge R$ or $t\in[0,1/4]\cup[3/4,1]$, then $g(x,t)=0$, so

$$\mathit{supp}\, g\subseteq\{|x|\le R\times[1/4,3/4]\}\cap\{|x/R+\varphi(t)e_1|\ge 1\}.$$

Then, if $\xi=x/R+\varphi(t)e_1$ we have

$$\begin{aligned}(i\partial_t+\Delta+\widetilde{V})g&=[\theta(\xi)(2\nabla\theta_R(x)\cdot\nabla\widetilde{u}+\widetilde{u}\Delta\theta_R(x))+2\nabla\theta(\xi)\cdot\nabla\theta_R\,\widetilde{u}\,]\\&+\theta_R(x)\left[2R^{-1}\nabla\theta(\xi)\cdot\nabla\widetilde{u}+R^{-2}\widetilde{u}\Delta\theta(\xi)+i\varphi'\partial_{x_1}\theta(\xi)\widetilde{u}\right]\ =B_1+B_2.\end{aligned} \tag{3.10}$$

Note that

$$\mathit{supp}\ B_1\subset\{(x,t)\in\mathbb{R}^n\times[0,1]\ :R-1\le|x|\le R\times[1/32,31/32]\},$$

and

$$\mathit{supp}\ B_2\subset\{(x,t)\in\mathbb{R}^n\times[0,1]\ :\ 1\le|x/R+\varphi(t)e_1|\le 2\}.$$

Now applying Lemma 2 choosing

$$\sigma=d_nR^2,\qquad d_n^2\ge\|\varphi''\|_\infty+\|\varphi'\|_\infty^2,$$

it follows that

$$R\,\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}g\|_{L^2(dxdt)}\le c_n\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}(i\partial_t+\Delta)g\|_{L^2(dxdt)}$$

$$\le c_n\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}\,\widetilde{V}g\|_{L^2(dxdt)}+c_n\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}\,B_1\|_{L^2(dxdt)} \tag{3.11}$$

$$+\,c_n\|e^{\sigma|\frac{x}{R}+\varphi(t)e_1|^2}\,B_2\|_{L^2(dxdt)}\equiv D_1+D_2+D_3.$$

Since $R>>\|\widetilde{V}\|_\infty$, we can absorb D_1 in the left hand side of (3.11). On the support of B_1 one has $|x/R+\varphi(t)e_1|\le 4$, thus

$$D_2\le c_n\,\delta(R)\,e^{16\sigma}.$$

On the support of B_2, one has $|x|\le R,\ t\in[1/32,31/32]$, and $1\le|x/R+\varphi(t)e_1|\le 2$, so

$$D_3\le c_n\,e^{4\,\sigma}\|\,|\widetilde{u}|+|\nabla\widetilde{u}|\|_{L^2(|x|\le R\times[1/32,31/32])}.$$

Combining this information, (3.1), and (3.5) we have

$$c_n\, e^{25\sigma/4}\, \|u(0)\|_2\, e^{-M} \le R\, e^{25\sigma/4}\, (\int_{|x|<R/2} \int_{3/8}^{5/8} |\widetilde{u}(x,t)|^2 dt dx)^{1/2}$$

$$\le c_n\, \delta(R)\, e^{16\sigma} + c_n\, e^{4\,\sigma} \| |\widetilde{u}| + |\nabla \widetilde{u}| \|_{L^2(|x|\le R\times[1/32,31/32])} \tag{3.12}$$

$$\le c_n\, \delta(R)\, e^{16\sigma} + c^*\, k^{c_p}\, e^{4\,\sigma}\, e^{2a_1\, k^{1/(2-p)}}.$$

Fixing

$$\sigma = d_n\, R^2 = 2\, a_1 k^{1/(2-p)}, \tag{3.13}$$

it follows from (3.12) that, if $\|u(0)\|_2 \neq 0$, for k large,

$$\delta(R) \ge c_n \|u(0)\|_2\, e^{-M}\, e^{-10\sigma} = c_n \|u(0)\|_2\, e^{-M}\, e^{-20\, a_1\, k^{1/(2-p)}}, \tag{3.14}$$

for $k \ge k_0(c^*)$ sufficiently large.

We now use our upper bounds for $\delta(R)$ deduced in (2.37)

$$\delta^2(R) = \int_{\frac{1}{32}}^{\frac{31}{32}} \int_{R-1\le|x|\le R} (|\widetilde{u}|^2 + \nabla \widetilde{u}|^2)(x,t) dx dt$$

$$\le \int_{\frac{1}{32}}^{\frac{31}{32}} \int_{R-1\le|x|\le R} e^{\gamma|x|^p} e^{-\gamma|x|^p}\, |\widetilde{u}|^2(x,t) dx dt$$

$$+ c_n \int_{\frac{1}{32}}^{\frac{31}{32}} \int_{R-1\le|x|\le R} t(1-t)\, \frac{(1+|x|)^{2-p}}{(1+|x|)^{2-p}}\, e^{\gamma|x|^p} e^{-\gamma|x|^p}\, |\nabla \widetilde{u}|^2\, dx dt \tag{3.15}$$

$$\le c_n\, e^{-\gamma\,(R-1)^p} \sup_{[0,1]} \|e^{\gamma|x|^p/2} \widetilde{u}(t)\|_2^2$$

$$+ c_n\, \gamma^{-1}\, R^{2-p}\, e^{-\gamma\,(R-1)^p} \int_{\frac{1}{32}}^{\frac{31}{32}} \int_{R-1\le|x|\le R} \frac{t(1-t)}{(1+|x|)^{2-p}} e^{\gamma|x|^p}\, |\nabla \widetilde{u}|^2\, dx dt$$

$$\le c^*\, k^{c_p}\, e^{2a_1 k^{1/(2-p)}}\, e^{-\gamma\,(R-1)^p}.$$

Gathering the information in (3.14), (3.15), (3.13), and (2.9) one obtains that

$$c_n\, \|u(0)\|_2^2\, e^{-2M} \le c^*\, k^{c_p}\, e^{42\, a_1\, k^{1/(2-p)} - \gamma\,(R-1)^p}$$

$$\le c^*\, k^{c_p}\, e^{42\, a_1\, k^{1/(2-p)} - a_0^{1/2}\, (2a_1/d_n)^{p/2}\, k^{1/(2-p)} + O(k^{(1/2(2-p)})}. \tag{3.16}$$

Hence, if

$$42\,a_1 < a_0^{1/2}\,a_1^{p/2}\,(2/d_n)^{p/2}, \qquad \text{i.e.} \qquad (42)^2\,(d_n/2)^p < a_0\,a_1^{p-2}, \tag{3.17}$$

by letting k tends to infinity it follows from (3.16) that $\|u(0)\|_2 \equiv 0$, which completes our proof.

4. Proof of Corollary 1 and Theorem 3

Proof of Corollary 1. Since

$$\int |u(x,1)|^2\,e^{2\,b\,|x|^q}\,dx < \infty, \qquad \text{with} \quad b = \beta^q/q,$$

one has that

$$\int |u(x,1)|^2\,e^{2\,k\,|x|^p}\,dx \le \|e^{2k|x|^p-2b|x|^q}\|_\infty \int |u(x,1)|^2\,e^{2\,b\,|x|^q}\,dx.$$

A simple calculation shows that

$$\|e^{2k|x|^p-2b|x|^q}\|_\infty = e^{2k|x|^p-2b|x|^q}\big|_{|x|=r_0},$$

where

$$r_0 = \left(\frac{pk}{bq}\right)^{1/(q-p)}.$$

Thus,

$$\|e^{2k|x|^p-2b|x|^q}\|_\infty = e^{2\,a_1\,k^{q/(q-p)}},$$

with

$$a_1 = \frac{1}{b^{p/(q-p)}}\left[\left(\frac{p}{q}\right)^{p/(q-p)} - \left(\frac{p}{q}\right)^{q/(q-p)}\right] = c_p\,\frac{1}{b^{p/(q-p)}}. \tag{4.1}$$

Inserting this value in the hypothesis (1.16) of Therem 1 we obtain

$$a\left(\frac{1}{b^{p/(q-p)}}\right)^{p-2} > c_p^{2-p}\,M_p,$$

with $a = \alpha^p/p$ and $b = \beta^q/q$. Since $q(2-p)/(q-p) = 1$ this gives us

$$\alpha\,\beta > N_p,$$

which yields the result.

□

Proof of Theorem 3. We just apply Corollary 1 with

$$u(x,t) = (u_1 - u_2)(x,t),$$

and

$$V(x,t) = \frac{F(u_1,\overline{u}_1) - F(u_2,\overline{u}_2)}{u_1 - u_2}.$$

□

5. Proof of Theorem 4

We shall follow closely the argument used in the proof of Theorem 2, sections 2-3. Thus, we divide the reasoning in steps.

First, we deduce the corresponding upper bounds. Assume $\|u(t)\|_2 = a \neq 0$. Fix $\bar{t} \in (0,1)$ near 1, and let

$$v(x,t) = u(x, t-1+\bar{t}), \quad t \in [0,1], \tag{5.1}$$

which satisfies the same equation (1.22) with

$$\begin{aligned} |v(x,0)| &\leq \frac{b_1}{(2-\bar{t})^{n/2}}\, e^{-\frac{b_2|x|^p}{(2-\bar{t})^p}}, \\ |v(x,1)| &\leq \frac{b_1}{(1-\bar{t})^{n/2}}\, e^{-\frac{b_2|x|^p}{(1-\bar{t})^p}}. \end{aligned} \tag{5.2}$$

So using the notation

$$A_0 = \frac{b_2}{(2-\bar{t})^p}, \qquad A_1 = \frac{b_2}{(1-\bar{t})^p}, \tag{5.3}$$

one has that

$$\begin{aligned} \int |v(x,0)|^2\, e^{A_0|x|^p} dx &\equiv a_0^2, \\ \int |v(x,1)|^2\, e^{A_1|x|^p} dx &\equiv a_1^2. \end{aligned} \tag{5.4}$$

In the remainder of this section c will denote a constant which may depend on n, b_0, b_1 and b_2, but is independent of the value of $\bar{t}$.

$$V(x,t) = F(u, \overline{u}), \tag{5.5}$$

so by hypothesis

$$|V(x,t)| \leq c\, |u(x, t-1+\bar{t})|^\theta \leq \frac{c}{(2-t-\bar{t})^{\theta n/2}}\, e^{-\frac{c|x|^p}{(2-t-\bar{t})^p}}. \tag{5.6}$$

We use the conformal or Appell transformation. If $v(y,s)$ verifies

$$\partial_s v = i\left(\triangle v + V(y,s)v\right), \qquad (y,s) \in \mathbb{R}^n \times [0,1], \tag{5.7}$$

and α and β are positive, then

$$\widetilde{u}(x,t) = \left(\tfrac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}} v\left(\tfrac{\sqrt{\alpha\beta}\, x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}, \tag{5.8}$$

verifies

$$\partial_t \widetilde{u} = i\left(\triangle\widetilde{u} + \widetilde{V}(x,t)\widetilde{u} + \widetilde{F}(x,t)\right), \ \text{ in } \mathbb{R}^n \times [0,1], \tag{5.9}$$

with

$$\widetilde{V}(x,t) = \tfrac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}\, V\left(\tfrac{\sqrt{\alpha\beta}\, x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right), \tag{5.10}$$

$$\widetilde{F}(x,t) = \left(\tfrac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac{n}{2}+2} F\left(\tfrac{\sqrt{\alpha\beta}\, x}{\alpha(1-t)+\beta t}, \tfrac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t)+\beta t)}}, \tag{5.11}$$

and

$$\begin{aligned} \|e^{\gamma|x|^p}\, \widetilde{u}(x,0)\|_2 &= \|e^{\gamma(\alpha/\beta)^{p/2}\,|x|^p}\, v(x,0)\|_2 = a_0, \\ \|e^{\gamma|x|^p}\, \widetilde{u}(x,1)\|_2 &= \|e^{\gamma(\beta/\alpha)^{p/2}\,|x|^p}\, v(x,1)\|_2 = a_1. \end{aligned} \tag{5.12}$$

We want

$$2\,\gamma(\alpha/\beta)^{p/2} = A_0, \quad 2\,\gamma(\beta/\alpha)^{p/2} = A_1,$$

therefore

$$2\,\gamma = (A_0A_1)^{1/2}, \quad \alpha = A_0^{1/p}, \quad \beta = A_1^{1/p}. \tag{5.13}$$

Since

$$A_0 \simeq 1, \qquad A_1 \simeq \frac{1}{(1-\bar{t})^p},$$

it follows that

$$\gamma \simeq \frac{1}{(1-\bar{t})^{p/2}}, \qquad \beta \simeq \frac{1}{(1-\bar{t})}, \qquad \alpha \simeq 1. \tag{5.14}$$

Notice that here the factor $1/(1-\bar{t})$ with $\bar{t}\uparrow 1$ plays the role of $k\in\mathbb{Z}^+$ with $k\uparrow\infty$ in the proof of Theorem 2 in sections 2 and 3.

Next, we shall estimate $\|\widetilde{V}\|_{L^1_tL^\infty_{|x|>R}}$. Thus,

$$|\widetilde{V}(x,t)| \le \frac{\beta}{\alpha}\,|V(y,s)| \le \frac{\beta}{\alpha}\,\frac{c}{(1-\bar{t})^{\theta n/2}}\,e^{-c|y|^p}, \tag{5.15}$$

with

$$|y| = \frac{\sqrt{\alpha\beta}|x|}{\alpha(1-t)+\beta t} \ge \frac{R\sqrt{\alpha}}{\sqrt{\beta}} \simeq \frac{R}{\sqrt{\beta}} = cR(1-\bar{t})^{1/2}.$$

Thus,

$$\|\widetilde{V}\|_\infty \le \frac{\beta}{\alpha}\,\|V\|_\infty \le \frac{c}{1-\bar{t}}\,\|V\|_\infty \le \frac{c}{(1-\bar{t})^{1+\theta n/2}},$$

and

$$\|\widetilde{V}(x,t)\|_{L^1_tL^\infty(|x|\ge R)} \le \|V(y,s)\|_{L^1_sL^\infty(|y|\ge cR/\sqrt{\beta})} \le \frac{c}{(1-\bar{t})^{\theta n/2}}e^{-cR^p(1-\bar{t})^{p/2}}. \tag{5.16}$$

To apply Lemma 1 we need

$$\|\widetilde{V}(x,t)\|_{L^1_tL^\infty(x|\ge R)} \le \frac{c}{(1-\bar{t})^{\theta n/2}}\,e^{-cR^p(1-\bar{t})^{p/2}} \le \epsilon_0,$$

so we take in the upper bounds part of the proof

$$R^p \simeq \frac{c}{(1-\bar{t})^{p/2}}\,log\left(\frac{c}{\epsilon_0\,(1-\bar{t})^{\theta n/2}}\right), \tag{5.17}$$

or

$$R \simeq \frac{c}{(1-\bar{t})^{1/2}}\,log^{1/p}\left(\frac{c}{\epsilon_0\,(1-\bar{t})^{\theta n/2}}\right). \tag{5.18}$$

Therefore, splitting the term $\widetilde{V}\widetilde{u}$ as in section 2

$$\mathbb{V} = \widetilde{V}\,\chi_{(|x|>R)}(x), \qquad \mathbb{F} = \widetilde{V}\,\chi_{(|x|\le R)}(x)\,\widetilde{u}(x,t), \tag{5.19}$$

a combination of Lemma 1 and the identity (2.12) yields the estimate

$$\sup_{[0,1]} \|e^{\gamma|x|^p}\tilde{u}(t)\|_2^2 \leq c(\|e^{\gamma|x|^p}\tilde{u}(0)\|_2^2 + \|e^{\gamma|x|^p}\tilde{u}(1)\|_2^2 + ca^2\,\|\widetilde{V}\|_{L^\infty}^2\, e^{c\gamma R^p})$$

$$\leq c\,(a_0^2 + a_1^2 + c\,a^2\,\frac{1}{(1-\bar{t})^{2+\theta n}}\,e^{cR^p/(1-\bar{t})^{p/2}}) \tag{5.20}$$

$$\leq c\,a^2\,e^{\frac{c}{(1-\bar{t})^p}\,log(\frac{c}{\epsilon_0(1-\bar{t})^{\theta n/2}})},$$

where $a = \|u_0\|_2$.

We observe that in this case, the upper bound in (5.20) is coming from the "external force" term $F = \widetilde{V}\,\chi_{(|x|\leq R)}(x)\,\widetilde{u}(x,t)$, and not from the data at time $t = 0,\,1$ of $\widetilde{u}$ as in the proof of Theorem 2.

Next, using the same argument given in section 2, (2.27)-(2.37), one finds that

$$\gamma\int_0^1\int t(1-t)\frac{1}{(1+|x|)^{2-p}}\,|\nabla\widetilde{u}(x,t)|^2 e^{\gamma|x|^p}dxdt$$

$$\leq c\,a^2\,e^{\frac{c}{(1-\bar{t})^p}\,log(\frac{c}{\epsilon_0(1-\bar{t})^{\theta n/2}})}. \tag{5.21}$$

Now we turn to the lower bounds estimates. Since they are similar to those given in detail in section 3 we just sketch them. As in (3.7) we first need

$$\|\widetilde{V}\|_{L^\infty(\mathbb{R}^n\times[1/32,31/32])} << R. \tag{5.22}$$

We take in this lower bound part

$$\sigma = d_nR^2, \qquad R = \frac{c}{(1-\bar{t})^{p/2(2-p)}}. \tag{5.23}$$

Since in the time interval $[1/32, 31/32]$ one has that $\alpha(1-t)+\beta t \simeq \beta$, it follows that

$$\|\widetilde{V}\|_{L^\infty(\mathbb{R}^n\times[1/32,31/32])} \leq c\,\frac{\alpha}{\beta}\,\|V\|_\infty \leq \frac{c}{(1-\bar{t})^{\theta n/2-1}}. \tag{5.24}$$

Therefore, (5.22) holds if

$$\frac{\theta n}{2} - 1 < \frac{p}{2(2-p)},$$

or equivalently,

$$p > \frac{2(\theta n - 2)}{\theta n - 1},$$

which is exactly our hypothesis, so (5.22) holds

Finally, we also need that $e^{c\gamma R^p}$ to be larger than our upper bound, i.e.

$$e^{\frac{c}{(1-\bar{t})^p}\,log(\frac{c}{\epsilon_0(1-\bar{t})^{\theta n/2}})} << e^{c\gamma R^p} = e^{\frac{c}{(1-\bar{t})^{p/2}}\,\frac{c}{(1-\bar{t})^{p\cdot p/2(2-p)}}}. \tag{5.25}$$

So it suffices to have

$$p < p/2 + p^2/2(2-p),$$

which holds if $p > 1$. This completes the proof of Theorem 4.

<u>Remark</u> Let us consider the case $\theta = 4/n$ in Theorem 4, so that our requirement is $p > 4/3$. The proof of Theorem 4, in fact, also gives the following result:

Theorem 5. *Assume that* $u \in C([0,1] : L^2(\mathbb{R}^n))$ *satisfies the equation*

$$i\partial_t u = \Delta u + V(x,t)u, \qquad (x,t) \in \mathbb{R}^n \times [0,1],$$

with $V(x,t)$ *complex valued,*

$$|V(x,t)| \le \frac{c}{(1-t)^2},$$

$$\lim_{R\to\infty} \|V\|_{L^1([0,1]:L^\infty(|x|\ge R))} = 0, \tag{5.26}$$

and

$$|u(x,t)| \le \frac{c_1}{(1-t)^{n/2}}\, Q\left(\frac{x}{1-t}\right), \quad \textit{with} \quad Q(y) = e^{-c_2|y|^p}.$$

If $p > 4/3$*, then* $u \equiv 0$*.*

It turns out that for complex potentials $V(x,t)$ as in Theorem 5, without the hypothesis (5.26), the restriction $p > 4/3$ is indeed necessary. This can be seen by performing a pseudo-conformal transformation to the stationary solution furnished by Meshkov's example in [13].

6. Appendix

We shall prove that for any $n \in \mathbb{Z}^+$ and any $p \in (1,2)$ there exists $c = c(n,p) > 1$ such that for any $x \in \mathbb{R}^n$

$$\frac{1}{c}\, e^{|x|^p/p} \le \int_{\mathbb{R}^n} e^{\lambda\cdot x - |\lambda|^q/q} |\lambda|^{n(q-2)/2} d\lambda \le c\, e^{|x|^p/p},$$

i.e.

$$I_n = \int_{\mathbb{R}^n} e^{\lambda\cdot x - |\lambda|^q/q} |\lambda|^{n(q-2)/2} d\lambda \sim e^{|x|^p/p} \tag{6.1}$$

with $1/q + 1/p = 1$.

We shall consider only the case $n \ge 2$ since the case $n = 1$ follows directly from Proposition 1. Assuming that for any $\mu > 1$

$$\Omega_n = \int_0^\pi e^{\mu\cos(\theta)} (\sin(\theta))^{n-2} d\theta \sim \frac{e^\mu}{\mu^{(n-1)/2}}, \tag{6.2}$$

we shall prove (6.1).

Using polar coordinates in $\mathbb{R}^n$ and (6.2) it follows that

$$I_n = c_n \int_0^\infty \int_0^\pi e^{|x| r\cos(\theta) - r^q/q} r^{n(q-2)/2} r^{n-1} (\sin(\theta))^{n-2} d\theta dr$$

$$\sim \int_0^\infty \frac{e^{|x|r - r^q/q}}{|x|^{(n-1)/2}}\, r^{n(q-2)/2} r^{(n-1)/2} dr$$

$$= \frac{1}{|x|^{(n-1)/2}} \int_0^\infty e^{-|x|(-r + r^q/q|x|)} r^{n(q-1)/2 - 1/2} dr = \widetilde{I}_n.$$

We recall Stirling's formula (see Proposition 2.1, page 323 in [17]).

Proposition 1. *If Ψ is a real valued function such that $\Psi'(x_0) = 0$ and $\Psi''(x) > 0$ for $x \in [a, b]$, and Φ is a smooth function, then*

$$\int_a^b e^{-s\Psi(x)}\Phi(x)dx = e^{-s\Psi(x_0)}\left[\frac{\sqrt{2\pi}}{s^{1/2}}\frac{\Phi(x_0)}{(\Psi''(x_0))^{1/2}} + O\left(\frac{1}{s}\right)\right], \quad as \quad s \to \infty.$$

In our case

$$\Psi(r) = -r + r^q/q|x|, \qquad \Psi'(r) = -1 + r^{q-1}/|x| = 0 \text{ if } r = r_0 = |x|^{1/(q-1)},$$

$$\Psi''(r) = (q-1)r^{q-2}/|x|, \quad \Psi''(r_0) = (q-1)|x|^{-1/(q-1)},$$

$$\Psi(r_0) = -\frac{1}{p}|x|^{1/(q-1)},$$

and

$$\Phi(r) = r^{n(q-1)/2-1/2}, \quad \text{so} \quad \Phi(r_0) = |x|^{n/2-1/2(q-1)}.$$

Therefore,

$$\widetilde{I}_n \sim \frac{1}{|x|^{(n-1)/2}}e^{-|x|\Psi(r_0)}\frac{1}{|x|^{1/2}}\frac{\Phi(r_0)}{(\Psi(r_0))^{1/2}} \sim \frac{e^{|x|^p/p}}{|x|^{n/2}}\frac{|x|^{n/2-1/2(q-1)}}{|x|^{-1/2(q-1)}} \sim e^{|x|^p/p},$$

which proves (6.1).

It remains to prove (6.2). Changing variables and recalling the fact that $\mu > 1$ and $n \geq 2$ we have

$$\Omega_n = \int_0^\pi e^{\mu\cos(\theta)}(\sin(\theta))^{n-3}\sin(\theta)d\theta = \int_{-1}^1 e^{\mu\eta}(1-\eta^2)^{(n-3)/2}d\eta$$

$$\sim \int_0^1 e^{\mu\eta}(1-\eta)^{(n-3)/2}(1+\eta)^{(n-3)/2}d\eta \sim \int_0^1 e^{\mu\eta}(1-\eta)^{(n-3)/2}d\eta$$

$$\sim e^\mu \int_0^1 e^{-\mu\nu}\nu^{(n-3)/2}d\nu = \frac{e^\mu}{\mu^{(n-1)/2}}\int_0^\mu e^{-\rho}\,\rho^{(n-3)/2}\,d\rho \sim \frac{e^\mu}{\mu^{(n-1)/2}},$$

which yields (6.2).

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