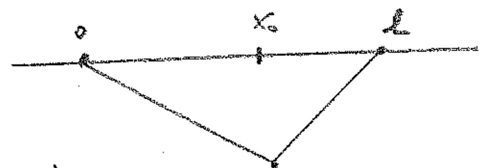
We want $G(x,x_0)$ s.t.

(1) G continuous (smooth for $x \neq x_0$)

(2) $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$ so $\partial_x^2 G(x,x_0) = \delta(x-x_0)$

(3) $G(x,x_0) = 0$ if $x = 0$ or l.



$$G(x,x_0) = \begin{cases} ax & \text{for } 0 < x < x_0 < l \\ b(x-l) & \text{for } x_0 < x < l. \end{cases}$$

st (1) $a x_0 = b(x_0 - l)$ $\Leftrightarrow$ $\begin{cases} (b-a)x_0 = bl \\ b-a = 1 \end{cases}$

(2) $b - a = 1$

Hence $b = \frac{x_0}{l}$ $a = \frac{x_0 - l}{l}$

and

$$G(x,x_0) = \begin{cases} (x_0 - l)\frac{x}{l} & 0 < x \leq x_0 \leq l \\ \frac{x_0}{l}(x-l) & x_0 \leq x \leq l. \end{cases}$$

$G(x,x_0)$ satisfies (1), (2) & (3). Finally let us see that

$G(x,x_0) - \frac{1}{2}|x-x_0|$ is harmonic

$$G(x,x_0) - \frac{1}{2}|x-x_0| = \begin{cases} (x_0 - l)\frac{x}{l} - \frac{(x_0 - x)}{2}, & x < x_0 \\ \frac{x_0}{l}(x-l) - \frac{(x-x_0)}{2}, & x_0 < x_0 \end{cases}$$

Since $G(x;x_0) - \frac{1}{2}|x-x_0|$ is piecewise linear

it suffices to see that is C^2 at $x = x_0$

$$G(x_0^\pm, x_0) - \frac{1}{2}(x_0^\pm, x_0) = 0$$

$$\partial_x \left(G(x, x_0) - \frac{1}{2}|x-x_0| \right) \Big|_{x = x_0^\pm} = \begin{cases} \frac{x_0}{e} - \frac{1}{2} & \text{at } x_0^+ \\ \frac{x_0}{e} - 1 + \frac{1}{2} & \text{at } x_0^- \end{cases}$$

and

$$\partial_x^2 \left(G(x, x_0) - \frac{1}{2}|x-x_0| \right) \equiv 0 .$$

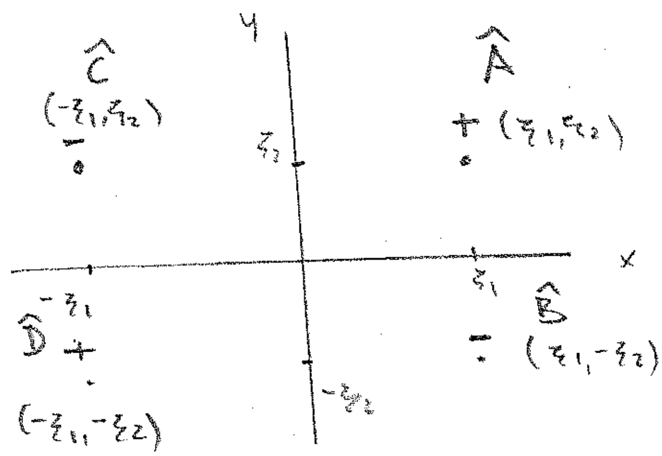
qed.

chapter 7 EXERCISE 17 W. STRAUSS' book.

(a) A fundamental solution of the Laplacian Δ in $\mathbb{R}^2$

is $\dfrac{\log|\vec{x}-\vec{\xi}|}{2\pi}$ $\quad \vec{x}=(x,y) \quad \vec{\xi}=(\xi_1,\xi_2)$

s.t. $\quad \Delta = \partial_x^2+\partial_y^2 \qquad \Delta\left(\dfrac{\log|\vec{x}-\vec{\xi}|}{2\pi}\right) = \delta(\vec{x}-\vec{\xi}).$



on $Q=\{(x,y)\ /\ x,y>0\}$ we have (by symmetry.

$$G\big((x,y),(\xi_1,\xi_2)\big) = \frac{1}{2\pi}\Big[\log\big((x-\xi_1)^2+(y-\xi_2)^2\big)^{1/2} - \log\big((x-\xi_1)^2+(y+\xi_2)^2\big)^{1/2}$$

$$-\log\big((x+\xi_1)^2+(y-\xi_2)^2\big)^{1/2} + \log\big((x+\xi_1)^2+(y+\xi_2)^2\big)\Big]$$

$$= \frac{1}{2\pi}\log\left[\frac{\big((x-\xi_1)^2+(y-\xi_2)^2\big)\cdot\big((x+\xi_1)^2+(y+\xi_2)^2\big)}{\big((x+\xi_1)^2+(y-\xi_2)^2\big)\cdot\big((x-\xi_1)^2+(y+\xi_2)^2\big)}\right]^{1/2} = (*)$$

$A=(x-\xi_1)^2+(y-\xi_2)^2 \qquad B=(x-\xi_1)^2+(y+\xi_2)^2$

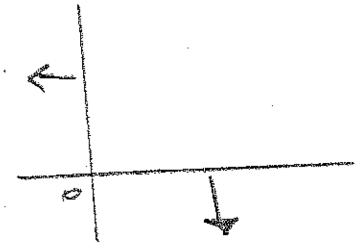
$C=(x+\xi_1)^2+(y-\xi_2)^2 \qquad D=(x+\xi_1)^2+(y+\xi_2)^2.$

So

$$(*) = \frac{1}{4\pi} \log\left[\frac{A\,D}{B\,C}\right]$$

but $\quad A = B - 4y\xi_2 = C - 4x\xi_1$

$\quad D = C + 4y\xi_2 = B + 4x\xi_1$



$$= \frac{1}{4\pi} \log\left(\frac{(B - 4y\xi_2)(C + 4y\xi_2)}{B\,C}\right) = \frac{1}{4\pi} \log\left(\frac{(C - 4x\xi_1)(B + 4x\xi_1)}{BC}\right)$$

(b) Solve the equation
$$\begin{cases} \Delta u = 0 & \text{in } Q \\ u(0,y) = g(y) & y > 0 \\ u(x,0) = h(x) & x > 0 \end{cases}$$

$$u(x,y) = \int_{\partial\Omega} \partial_n G((x,y)(\xi_1,\xi_2)) \text{ data } dS_\xi .$$

$$= \int_0^\infty -\partial_{\xi_2} G((x,y),(\xi_1,\xi_2))\Big|_{\xi_2=0} h(\xi_1)\, d\xi_1$$

$$+ \int_0^\infty -\partial_{\xi_1} G((x,y),(\xi_1,\xi_2))\Big|_{\xi_1=0} g(\xi_2)\, d\xi_2$$

$$G((x,y),(\xi_1,\xi_2)) = \frac{1}{4\pi}\log\left(1 - \frac{4y\xi_2}{B} + \frac{4y\xi_2}{C} - \frac{16y^2\xi_2^2}{BC}\right)$$

$$\partial_{\xi_2} G((x,y),(\xi_1\,\xi_2)) = \frac{1}{4\pi}\left(\frac{1}{1 - \frac{4y\xi_2}{B} + \frac{4y\xi_2}{C} - \frac{16y^2\xi_2^2}{BC}}\right)\cdot$$

$$\cdot\left(-\partial_{\xi_2}\left(\frac{4y\xi_2}{B}\right) + \partial_{\xi_2}\left(\frac{4y\xi_2}{C}\right) - \partial_{\xi_2}\left(\frac{16y^2\xi_2^2}{BC}\right)\right)$$

$$\partial_{\xi_2}\left(\frac{4y\xi_2}{B}\right) = \frac{4y}{B} - \frac{4y\xi_2\, 2(y+\xi_2)}{B^2}$$

$$\partial_{\xi_2}\left(\frac{4y\xi_2}{C}\right) = \frac{4y}{C} - \frac{4y\xi_2(-2)(y-\xi_2)}{C^2}$$

$$\partial_{\xi_2}\left(\frac{16y^2\xi_2^2}{BC}\right) = \frac{32y^2\xi_2}{BC} - \frac{(16y^2\xi_2^2)\left[B(-2)(y-\xi_2) + C\,2(y+\xi_2)\right]}{B^2C^2}$$

Hence

$$-\partial_{\xi_2} G((x,y)(\xi_1\,\xi_2))\Big|_{\xi_2=0} = \frac{-1}{4\pi}\left[-\frac{4y}{(x-\xi_1)^2+y^2} + \frac{4y}{(x+\xi_1)^2+y^2}\right]$$

by symmetry

$$-\partial_{\xi_1} G((x,y)(\xi_1\,\xi_2))\Big|_{\xi_1=0} = -\frac{1}{4\pi}\left[-\frac{4x}{x^2+(y-\xi_2)^2} + \frac{4x}{x^2+(y+\xi_2)^2}\right]$$

Hence.

$$u(x,y) = \int_0^\infty \left(\frac{x}{x^2+(y-\xi_2)^2} - \frac{x}{x^2+(y+\xi_2)^2} \right) \frac{g(\xi_2)}{\pi} \, d\xi_2$$

$$+ \int_0^\infty \left(\frac{y}{(x-\xi_1)^2+y^2} - \frac{y}{(x+\xi_1)^2+y^2} \right) \frac{h(\xi_1)}{\pi} \, d\xi_2$$

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F John Cap 4- GUSTAVO PONCE
Sección 1

(2) Let $L = \Delta + C$ in $n=3$ dimensions, where c = constant.
(a) Find all solutions of $Lu=0$ with spherical symmetry.

Proof

$$L = \Delta + C = [\partial_x^2 + \partial_y^2 + \partial_z^2 + c]$$

we consider

$$v(x,y,z) = \psi(r) \qquad r = \sqrt{(x-\xi_1)^2 + (y-\xi_2)^2 + (z-\xi_3)^2} = \|(x,y,z) - (\xi_1,\xi_2,\xi_3)\|.$$

$\xi = (\xi_1, \xi_2, \xi_3)$.

$$\partial_x \psi(r) = \psi' . r_x \qquad r_x = \frac{x-\xi_1}{r} \qquad r_{xx} = \frac{r - (x-\xi)^2/r}{r^2} = \frac{r^2 - (x-\xi)^2}{r^3}$$

$$\partial_x^2 \psi(r) = \psi''(r_x)^2 + \psi' . r_{xx}.$$

(equal for y, z by symmetry).

Then

$$(\Delta + c)\psi(r) = \psi'' + \frac{2}{r}\psi' + c\psi = 0.$$

changed variable. $\psi = \frac{\varphi}{r}$ or $\varphi = \psi . r$.

$$\left[\psi' = \frac{\varphi' r - \varphi}{r^2}\right] \qquad \left[\psi'' = \frac{\varphi'' r^2 - 2\varphi' r + 2\varphi}{r^3}\right] + \text{simplifying:}$$

then $\psi'' + \frac{2}{r}\psi' + c\psi = \frac{1}{r}\left(\varphi'' + c\varphi\right) = 0$

then $\varphi(t) = A\cos(\sqrt{c}\,t) + B\,\text{sen}(\sqrt{c}\,t)$.

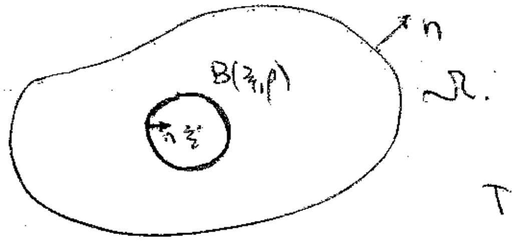
And

$$\psi(r) = \frac{1}{r}\left[A.\cos(\sqrt{c}\,r) + B.\,\text{sen}(\sqrt{c}\,r)\right].$$

(b) Prove that $K(x,\xi) = -\dfrac{\cos(\sqrt{c}\,r)}{4\pi r}$ $\quad r = \|\bar{x}-\bar{\xi}\|$
is a fundamental
solution for L with pole ξ.

Proof.
we have, the special case of Green's identity.

$$\int_{\Omega} [v(\Delta u + cu) - u(\Delta v + cv)]\,dx = \int_{\partial\Omega} \left(v\frac{du}{dn} - u\frac{dv}{dn} \right) ds.$$



$\Omega_\rho = \Omega - B(\xi,\rho)$
Taking $v = \psi(r)$ part (a).
$L(v) = 0$.

$$\int_{\Omega_\rho} v(\Delta u + cu)\,dx = \int_{\partial\Omega} + \int_{S(\xi\rho)} \left(v\frac{du}{dn} - u\frac{dv}{dn} \right) . ds.$$

Since $v = \psi(\rho) = \psi(r)$ $\qquad \dfrac{dv}{dn} = -\psi'(\rho)$ on $S(\xi,\rho)$.

then. $\displaystyle\int_{S(\xi\rho)} v\frac{du}{dn}\,ds = \psi(\rho)\int_{S(\xi\rho)} \frac{du}{dn}\,ds.$

but $\quad \psi(\rho) = O(1/\rho)$. and $\displaystyle\int_{S(\xi,\rho)} \frac{du}{dn}\,ds = O(\rho^2)$.

then $\quad \psi(\rho) . \displaystyle\int_{S(\xi\rho)} \frac{du}{dn} . ds \downarrow 0$ as $\rho \downarrow 0$.

TAKING $A = \frac{-1}{4\pi}$ $B = 0$. (We can take other B.)

we have the solution $\boxed{\psi(r) = -\frac{1}{4\pi r} \cos\sqrt{c}\, r.}$

Then. (Taken $\lim_{\rho \downarrow 0}$). ✓

$$\int_{\Omega} v(\Delta u + cu)\,dx = \int_{\partial\Omega} \left(v \frac{du}{dn} - u \frac{dv}{dn} \right) + u(\xi).$$

Taking the notation $K(x,\xi) = v(r) = \psi(r) = \psi(\|\bar{x}-\xi\|)$.

we have. $u(\xi) = \int_{\Omega} K(x,\xi).(\Delta u + cu)\,dx -$

$$\int_{\partial\Omega} \left(K(x,\xi) \frac{du}{dn} - u(x) \frac{dK(x,\xi)}{dn} \right) dS. \quad (**)$$

let $\phi \in C_0^{\infty}(\Omega)$, putting this function in the formula we have.

$$\phi(\xi) = \int_{\Omega} K(x,\xi)\,(\Delta\phi + c\phi)\,dx - \int_{\partial\Omega} \left(K(x,\xi) \frac{d\phi}{dn} - \phi \frac{dK(x,\xi)}{dn} \right) ds. =$$

$$= \int_{\Omega} K(x,\xi).L\phi\,dx = v.[L\phi] = \delta_{\xi}\phi.$$

The other side

$$\int_{S(\xi,\rho)} u \,\frac{d\sigma}{dn}\, ds = -\left[\frac{A\cos(\sqrt{c}\,r) + B\operatorname{sen}(\sqrt{c}\,r)}{r}\right]'_{r=\rho} \cdot \int_{S(\xi,\rho)} u\, ds.$$

$$= -\frac{\left[-\sqrt{c}\,A\operatorname{sen}(\sqrt{c}\,r) + B\sqrt{c}\cos(\sqrt{c}\,r)\right]r - \left[A\cos(\sqrt{c}\,r) + B\operatorname{sen}(\sqrt{c}\,r)\right]}{r^2}\Bigg|_{r=\rho} \int_{S(\xi,\rho)} u\, ds$$

TAKING $\lim_{r=\rho\downarrow 0}$ $\underbrace{\frac{\left[-\sqrt{c}\,A\operatorname{sen}(\sqrt{c}\,r) + B\sqrt{c}\cos\sqrt{c}\,r\right]}{r}\Big|_{\rho}}_{\alpha(\rho)}\int_{S(\xi,\rho)} u\, ds \downarrow 0$

$[r=\rho]$

because. $\alpha(\rho) = O\left(\frac{1}{\rho}\right)$

$$\int_{S(\xi,\rho)} u\, ds = O(\rho^2).$$

then we have.

$$\frac{A\cos(\sqrt{c}\,r) - B\operatorname{sen}(\sqrt{c}\,r)}{r^2}\Bigg|_{r=\rho} \cdot \int_{S(\xi,\rho)} u.\, ds.$$

Taking $\lim_{\rho\downarrow 0}$

$$\frac{A\cos(\sqrt{c}\,\rho)}{\rho^2}\int_{S(\xi,\rho)} u\, ds \longrightarrow A.4\pi.u(\xi).$$

(Theorem of Lebesgue).

$$-\frac{B\operatorname{sen}(\sqrt{c}\,\rho)}{\rho^2}\int_{S(\xi,\rho)} u\, ds \longrightarrow 0$$

If we choose k_0 s.t. $G\big|_{\partial\Omega} \equiv 0$.

this is

$r=\rho \qquad -\dfrac{\cos(\sqrt{c}\,\rho)}{4\pi\rho} + k_0 \dfrac{\operatorname{sen}(\sqrt{c}\,\rho)}{\rho} = 0 \Rightarrow k_0 = \dfrac{\cos(\sqrt{c}\rho)}{4\pi \operatorname{sen}(\sqrt{c}\rho)}$.

(assuming $\rho \neq \theta\pi$ $\theta\in\mathbb{Z}$ is $c\geq 0$).

then.

$$u(\xi) = \int_{\partial\Omega} \left(u(x) \frac{dG}{dn_x}(x,\xi) \right) . ds_x .$$

Now we calcule $\dfrac{dG(x,\xi)}{dn_x}$

$$G(x,\xi) = -\frac{\cos(\sqrt{c}\,r)}{4\pi r} + \frac{k_0 \operatorname{sen}(\sqrt{c}\,r)}{r}$$

$$\frac{dG}{dn_x}(x,\xi) = -\frac{dG}{dr}\Big|_{r=\rho} \quad \text{on } \partial\Omega . = \{\|x-\xi\| = \rho\}.$$

$$= \frac{4\pi r \operatorname{sen}(\sqrt{c}r)\sqrt{c} + 4\pi\cos(\sqrt{c}r)}{(4\pi r)^2}\Big|_{r=\rho} + k_0 \left[\frac{\sqrt{c}\cos(\sqrt{c}r).r - \operatorname{sen}\sqrt{c}r}{r^2}\right]_{r=\rho}$$

$$= \frac{4\pi\rho\sqrt{c}\operatorname{sen}(\sqrt{c}\rho) + \cancel{4\pi\cos(\sqrt{c}\rho)} + 4\pi\sqrt{c}\rho\dfrac{\cos^2(\sqrt{c}\rho)}{\operatorname{sen}(\sqrt{c}\rho)} - \cancel{4\pi\cos(\sqrt{c}\rho)}}{(4\pi\rho)^2}$$

$$= \frac{4\pi\rho\sqrt{c}\operatorname{sen}^2(\sqrt{c}\rho) + 4\pi\rho\sqrt{c}\cos^2(\sqrt{c}\rho)}{(4\pi\rho)^2\operatorname{sen}(\sqrt{c}\rho)} = \frac{4\pi\rho\sqrt{c}\,[\cos^2\sqrt{c}\rho + \operatorname{sen}^2\sqrt{c}\rho]}{(4\pi\rho)^2 \operatorname{sen}\sqrt{c}\rho}$$

Then $L\sigma = \delta_\xi$. is a fundamental solution.
with pole at ξ.
(Consider $L = \tilde{L}$ self joint).?

— ⓒ Show that a solution of $Lu = 0$ in the ball $|x-\xi| \leq \rho$ has the modified mean value property

$$u(\xi) = \frac{\sqrt{c}.\rho}{\sin(\sqrt{c}\rho)} \; \frac{1}{4\pi\rho^2} \int_{|x-\xi|=\rho} u(x)\, dS_x.$$

Proof

We consider the more general solution with spherical symmetry, which is also a fundamental solution for L with pole ξ.

(K₀ constant)

$$G(x,\xi) = k(x,\xi) + K_0 \frac{\sin(\sqrt{c}.r)}{r} = -\frac{\cos(\sqrt{c}.r)}{4\pi r} + \frac{K_0 \sin(\sqrt{c}.r)}{r}.$$

replacing k by G in the formula. and $Lu = 0$.
on $(**)$

$$u(\xi) = \int_\Omega k(x,\xi)(\Delta u + cu)\,dx - \int_{\partial\Omega} \left(k(x,\xi)\frac{du}{dn} - u(x)\frac{dk(x,\xi)}{dn} \right) dS$$

we have

$$u(\xi) = -\int_{\partial\Omega} \left(G(x,\xi)\,\frac{du}{dn} - u(x)\,\frac{dG(x,\xi)}{dn} \right) dS.$$

$$= \frac{\sqrt{c}}{\sin\sqrt{c}\rho} \cdot \frac{1}{4\pi\rho^2} = \frac{d\, G(x,\xi)}{dn_x}.$$

then.

$$u(\xi) = \frac{\sqrt{c}}{\sin\sqrt{c}\rho} \frac{1}{4\pi\rho^2} \int_{|x-\xi|=\rho} u(x)\, dS_x .$$

qed

(d) Try $u \equiv \phi(x) = \phi(r) = \dfrac{\sin(\sqrt{c}\, r)}{r}$ for $c > 0$

for $c < 0$ use the maximum principle to show that such u does not exist!

Thus no solution for $c < 0$

Solutions for $\sqrt{c} = k\pi \quad k \in \mathbb{Z}^+$

$$C = (k\pi)^2 \; . \; . \; . \; .$$