THE IVP FOR THE BENJAMIN-ONO EQUATION IN WEIGHTED SOBOLEV SPACES II

GERMÁN FONSECA, FELIPE LINARES, AND GUSTAVO PONCE

ABSTRACT. In this work we continue our study initiated in [10] on the uniqueness properties of real solutions to the IVP associated to the Benjamin-Ono (BO) equation. In particular, we shall show that the uniqueness results established in [10] do not extend to any pair of non-vanishing solutions of the BO equation. Also, we shall prove that the uniqueness result established in [10] under a hypothesis involving information of the solution at three different times can not be relaxed to two different times.

1. INTRODUCTION

This work is concerned with special decay and uniqueness properties of real solutions of the initial value problem (IVP) for the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\begin{aligned} \mathcal{H}f(x) &= \frac{1}{\pi}\,\text{p.v.}(\frac{1}{x} * f)(x) \\ &= \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y|\geq\epsilon}\frac{f(x-y)}{y}dy = -i\,(\text{sgn}(\xi)\,\widehat{f}(\xi))^{\vee}(x). \end{aligned} \tag{1.2}$$

The BO equation was deduced by Benjamin [3] and Ono [22] as a model for long internal gravity waves in deep stratified fluids. Later, it was also shown that it is a completely integrable system (see [2], [6] and references therein).

The problem of finding the minimal regularity property (measure in the Sobolev scale $H^s(\mathbb{R}) = \left(1-\partial_x^2\right)^{-s/2} L^2(\mathbb{R})$, $s \in \mathbb{R}$) required in the data u_0 which guarantees that the IVP (1.1) is locally wellposed (LWP) or globally wellposed (GWP) has been extensively considered. Thus, one has the following list of works: in [24] $s > 3$ was proven, in [1] and [13] $s > 3/2$, in [23] $s \geq 3/2$, in [17] $s > 5/4$, in [15] $s > 9/8$, in [25] $s \geq 1$, in [4] $s > 1/4$, and in [12] $s \geq 0$. It should be pointed out that the result in [21] (see also [18]) implies that none well-posedness in $H^s(\mathbb{R})$, $s \in \mathbb{R}$ can be established by a solely contraction principle arguments. For further results on uniqueness and comments we refer to [19].

Our study here includes both the regularity and the decay of the solution measure in the L^2 sense. More precisely, we deal with persistence properties (i.e. if the data u_0 belongs to the function space X, then the corresponding solution of (1.1) defined

1991 *Mathematics Subject Classification.* Primary: 35B05. Secondary: 35B60.
Key words and phrases. Benjamin-Ono equation, weighted Sobolev spaces.

a continuous curve on X, $u \in C([0,T] : X))$ of real valued solutions of the IVP (1.1) in the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \qquad s,\, r \in \mathbb{R}, \tag{1.3}$$

and

$$\dot{Z}_{s,r} = \{f \in H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx) \,:\, \widehat{f}(0) = 0\}, \qquad s,\, r \in \mathbb{R}. \tag{1.4}$$

Notice that the conservation laws for real solutions of (1.1)

$$I_1(u_0) = \int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx,$$

guarantees that the property $\widehat{u}_0(0) = 0$ is preserved by the solution flow.

As an extension of the results in [13], [14] the following theorems were proven in [10]:

Theorem A. ([10]) *(i) Let $s \geq 1$, $r \in [0,s]$, and $r < 5/2$. If $u_0 \in Z_{s,r}$, then the solution $u(x,t)$ of the IVP (1.1) satisfies that $u \in C([0,\infty) : Z_{s,r})$.*

(ii) For $s > 9/8$ ($s \geq 3/2$), $r \in [0,s]$, and $r < 5/2$ the IVP (1.1) is LWP (GWP resp.) in $Z_{s,r}$.

(iii) If $r \in [5/2, 7/2)$ and $r \leq s$, then the IVP (1.1) is GWP in $\dot{Z}_{s,r}$.

Theorem B. ([10]) *Let $u \in C(\mathbb{R} : Z_{2,2})$ be a solution of the IVP (1.1). If there exist two different times $t_1, t_2 \in \mathbb{R}$ such that*

$$u(\cdot,t_j) \in Z_{5/2,5/2}, \;\; j=1,2, \;\; \textit{then} \;\; \widehat{u}_0(0) = 0\,, \;\; (\textit{so} \;\; u(\cdot,t) \in \dot{Z}_{5/2,5/2}). \tag{1.5}$$

Theorem C. ([10]) *Let $u \in C(\mathbb{R} : \dot{Z}_{3,3})$ be a solution of the IVP (1.1). If there exist three different times $t_1, t_2, t_3 \in \mathbb{R}$ such that*

$$u(\cdot,t_j) \in Z_{7/2,7/2}, \;\; j=1,2,3, \;\; \textit{then} \;\; u(x,t) \equiv 0. \tag{1.6}$$

Remarks : (a) Theorem A with $s \geq r = 2$, Theorem B with $s \geq r = 3$, and Theorem C with $s = r = 4$ were proved by Iorio, see [13], [14].

(b) Theorem B shows that the condition $\widehat{u}_0(0) = 0$ is necessary for persistence property of the solution to hold in $Z_{s,5/2}$, with $s \geq 5/2$, so in that sence parts (i)-(ii) of Theorem A are optimal. Theorem C affirms that there is an upper limit of the spacial L^2-decay rate of the solution i.e.

$$|x|^{7/2}u(\cdot,t) \notin L^\infty([0,T] : L^2(\mathbb{R})), \qquad \forall\, T > 0,$$

regardless of the decay and regularity of the non-zero initial data u_0. In particular, Theorem C shows that Theorem A part (iii) is sharp.

In view of the results in Theorems A, Theorem B, and Theorem C the following two questions present themselves.

Question 1 : Can these uniqueness results be extended to any pair of solutions u_1, u_2 of the (1.1) with $u_1 \neq 0$, $u_2 \neq 0$?

We recall that the uniqueness results obtained in [8] for the IVP associated to the k-generalized Korteweg-de Vries (k-gKdV) equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \qquad t, x \in \mathbb{R}, \tag{1.7}$$

and those in [9] for the IVP associated to the semi-linear Schrödinger (NLS) equation

$$i\partial_t u + \Delta u = F(u, \bar{u}), \qquad t \in \mathbb{R}, x \in \mathbb{R}^n, \tag{1.8}$$

hold for any pair u_1, u_2 of solutions in a suitable class.

Our first result gives a negative answer to Question 1:

Theorem 1. *There exist $u_1,\ u_2 \in C(\mathbb{R} : Z_{4,2}),\ u_1 \neq 0,\ u_2 \neq 0$ solutions of the IVP (1.1) such that*

$$u_1 \neq u_2$$

and for any $T > 0$

$$u_1 - u_2 \in L^\infty([-T, T] : Z_{4,4}). \tag{1.9}$$

Remarks : (a) Combining the argument presented here and those used in [10] relying on the notion of A_2 weight one can extend the result in Theorem 1 to the index $Z_{5,9/2-}$ by assuming that $u_1,\ u_2 \in C(\mathbb{R} : Z_{5,2})$. This tells us that for a uniqueness result involving any pair of suitable solutions of the BO equation to be valid it should involve a decay index $r \geq 9/2$.

Next, we observe that the hypothesis (1.6) in Theorem C involves requirement of the solution $u(x, t)$ at three different times $t_1 < t_2 < t_3$.

Question 2: Can the assumption (1.6) in Theorem C be reduced to a two different times $t_1 < t_2$?

We recall that the uniqueness results for the k-gKdV in [8], for the NLS in [9], those obtained in [11] for the IVP associated to the Camassa-Holm equation

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, \quad t, x \in \mathbb{R}, \tag{1.10}$$

as well as many other deduced for dispersive models require a condition involving only two different times.

Surprisingly, our second result shows that for the BO this is not the case, the condition involving three different times in Theorem C is necessary:
$\exists\ u \in C(\mathbb{R} : \dot{Z}_{5,7/2-}),\ u \neq 0$, solution of (1.1) for which there are $t_1,\ t_2 \in \mathbb{R},\ t_1 \neq t_2$ such that

$$u(\cdot, t_j) \in \dot{Z}_{5,4} \subset Z_{7/2,7/2-}, \qquad j = 1, 2.$$

More precisely, we shall prove :

Theorem 2. *For any $u_0 \in \dot{Z}_{5,4}$ such that*

$$\int_{-\infty}^{\infty} x\, u_0(x) dx \neq 0, \tag{1.11}$$

the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{5,7/2-})$ of the (1.1) provided by Theorem A part (iii) satisfies that

$$u(\cdot, t^*) \in \dot{Z}_{4,4}, \tag{1.12}$$

where

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{-\infty}^{\infty} x\, u_0(x) dx. \tag{1.13}$$

Remarks : (a) The result in Theorem 2 is due to the relation of the dispersive relation and the structure of the nonlinearity of the BO equation. In particular, one can see that if $u_0 \in \dot{Z}_{5,4}$ verifying (1.11), then the solution $W(t)u_0(x)$ of the associated linear IVP

$$\partial_t u + \mathcal{H}\partial_x^2 u = 0, \qquad u(x,0) = u_0(x),$$

satisfies

$$W(t)u_0(x) = c(e^{-it|\xi|\xi}\widehat{u_0}(\xi))^\vee \in L^2(|x|^{7-}) - L^2(|x|^7), \qquad \forall\, t \neq 0.$$

However, for the same data u_0 one has that the solution $u(x,t)$ of the (1.1) satisfies

$$u(\cdot,0),\ u(\cdot,t^*) \in L^2(|x|^8dx), \ \text{ and } \ u(\cdot,t) \in L^2(|x|^{7-}) - L^2(|x|^7), \ \ \forall\, t \notin \{0,t^*\}.$$

(b) The value of t^* described in (1.13) can be motivated from the identity

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu(x,t)dx = \frac{1}{2}\|u(\cdot,t)\|_2^2 = \frac{1}{2}\|u_0\|_2^2,$$

(using the second conservation law which tells us that the L^2 norm of the real solution is preserved by the solution flow) which describes the time evolution of the first momentum of the solution

$$\int_{-\infty}^{\infty} xu(x,t)dx = \int_{-\infty}^{\infty} xu_0(x)dx + \frac{t}{2}\|u_0\|_2^2.$$

So assuming that

$$\int_{-\infty}^{\infty} x\, u_0(x)dx \neq 0, \tag{1.14}$$

one looks for the times where the average of the first momentum of the solution vanishes, i.e. for t such that

$$\int_0^t\int_{-\infty}^{\infty} x\, u(x,t)dxdt = \int_0^t (\int_{-\infty}^{\infty} x\, u_0(x)dx + \frac{t'}{2}\|u_0\|_2^2)dt' = 0,$$

which under the assumption (1.14) has a unique solution $t = t^*$ given by the formula in (1.13).

(c) To prove Theorem 2 we shall work with the integral equation version of the problem (1.1). Roughly speaking, from the result in [21] one cannot regard the nonlinear term as a perturbation of the linear one. So to obtain our result we use an argument similar to that in [14]. This is based on the special structure of the equation and allows us to reduce the contribution of two terms in the integral equation to just one. Also the use of the integral equation in the proof and the result in [21] explains our assumption $u_0 \in \dot{Z}_{5,4}$ instead of the expected one from the differential equation point of view $u_0 \in \dot{Z}_{4,4}$.

(d) One may ask if it is possible to have a stronger decay at $t = t^*$ than the one described in (1.12). In this regard, our argument shows that for $u_0 \in \dot{Z}_{6,5}$ it follows that

$$u(\cdot,t^*) \in \dot{Z}_{5,5}$$

if and only if

$$\int_0^{t^*}\int_{-\infty}^{\infty} x^2\, u(x,t)dx\, dt = 0. \tag{1.15}$$

However, the time evolution of the second momentum of the solution does not seem to have a simple expression which allows to verify the identity (1.15).

(e) The result in Theorem 2 can be extended to higher powers of the BO equations

$$\partial_t u + \mathcal{H}\partial_x^2 u + u^{2k+1}\partial_x u = 0, \quad k = 0, 1, 2, ...$$

where the formula (1.13) for t^* in this case will be given as the solution of the equation

$$\int_0^{t^*} \Big(\int x\, u_0(x)dx + \frac{t}{2k+2}\, \|u(t)\|_{2k+2}^{2k+2} \Big) dt = 0, \quad k = 1, 2, ...$$

It is clear that if such a time t^* exists it is unique.

(f) A close inspection of the proof of Theorem C in [10] gives us the following result which allows us to establish a uniqueness result with a condition involving only two times $t_1 = 0,\ t_2 \neq 0$ for a suitable class of solutions:

Theorem 3. *Let $u \in C(\mathbb{R} : \dot{Z}_{7/2,3})$ be a solution of the IVP* (1.1) *for which there exist two times $t_1,\, t_2 \in \mathbb{R},\ t_1 \neq t_2$ such that*

$$u(\cdot, t_j) \in \dot{Z}_{7/2,7/2}.$$

If

$$\int x\, u(x, t_1)dx = 0 \qquad or \qquad \int x\, u(x, t_2)dx = 0,$$

then

$$u(x,t) \equiv 0.$$

(g) In a forthcoming work we shall consider the extentions of the results established here and those in [10] to solutions of the IVP for the dispersive model

$$\partial_t u + D_x^{1+a}\partial_x u + u\partial_x u = 0, \qquad t, x \in \mathbb{R}, \quad a \in (0,1),$$

where

$$D_x = (-\partial_x^2)^{1/2} = \mathcal{H}\partial_x.$$

Thus, the cases $a = 0$ and $a = 1$ correspond to the BO equation and the KdV equation, respectively.

As it was mentioned above the proof of Theorem 3 is contained in the proof of Theorem C given in [10], therefore it will be omitted.

We recall that if for a solution $u \in C(\mathbb{R} : H^s(\mathbb{R}))$, $s \geq 0$ of (1.1) one has that $\exists\, t_0 \in \mathbb{R}$ such that $u(x, t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C(\mathbb{R} : H^{s'}(\mathbb{R}))$. So the propagation of the $H^s(\mathbb{R})$ regularity of the solution is not an issue.

The rest of this paper is organized as follows: section 2 contains all the estimates needed in the proof of Theorems 1 and 2. The proof of Theorem 1 will be given in section 3. Theorem 2 will be proven in section 4.

2. Preliminary Estimates

As in [10] we shall use the following generalization of Calderón commutator estimates [5] found in [7]:

Theorem 4. *For any* $p \in (1,\infty)$ *and* $l,\, m \in \mathbb{Z}^+ \cup \{0\}$, $l+m \geq 1$ *exists* $c = c(p;l;m) > 0$ *such that*

$$\|\partial_x^l[\mathcal{H};\, a]\partial_x^m f\|_p \leq c\|\partial_x^{l+m} a\|_\infty \|f\|_p. \tag{2.1}$$

We shall also use the pointwise identities

$$[\mathcal{H};x]\partial_x f = [\mathcal{H};x^2]\partial_x^2 f = 0,$$

and more generally

$$[\mathcal{H};x]f = 0 \quad \text{if and only if} \quad \int f dx = 0,$$

and

$$[\mathcal{H};x^2]f = 0 \quad \text{if and only if} \quad \int f dx = \int x f dx = 0.$$

To justify the finiteness of the quantities involved in the energy estimate used in the proof of Theorem 1 we introduce the truncated weights $w_N(x)$. Using the notation $\langle x\rangle = (1+x^2)^{1/2}$ we define

$$w_N(x) = \begin{cases} \langle x\rangle & \text{if } |x| \leq N, \\ 2N & \text{if } |x| \geq 3N, \end{cases} \tag{2.2}$$

$w_N(x)$ smooth and non-decreasing in $|x|$ with $w_N'(x) \leq 1$ for all $x \geq 0$. We observe that

$$x\, w_N'(x) \leq c w_N(x),$$

where the constant c is independent on N.

3. Proof of Theorem 1

We take two solutions u_1, u_2 of (1.1) whose data $u_{1,0}$, $u_{2,0}$ satisfy

$$u_1(x,0) = u_{1,0}(x),\;\; u_2(x,0) = u_{2,0}(x) \in Z_{4,4}, \tag{3.1}$$

with

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} u_1(x,0)\,dx = \int_{-\infty}^{\infty} u_2(x,0)\,dx, \\ \displaystyle\int_{-\infty}^{\infty} x\,u_1(x,0)\,dx = \int_{-\infty}^{\infty} x\,u_2(x,0)\,dx, \\ \|u_{1,0}\|_2 = \|u_{2,0}\|_2, \qquad u_{1,0} \neq u_{2,0} \\ u_{1,0} \neq 0\,, \qquad u_{2,0} \neq 0. \end{cases} \tag{3.2}$$

Thus, from the result in [10] it follows that

$$u_1,\; u_2 \in C(\mathbb{R} : Z_{4,5/2^-}).$$

Defining
$$v(x,t) = u_1(x,t) - u_2(x,t),$$
one sees that v verifies the linear equation
$$\partial_t v + \mathcal{H}\partial_x^2 v + u_1\partial_x v + \partial_x u_2 v = 0, \tag{3.3}$$
with
$$v \in C(\mathbb{R} : Z_{4,5/2-}), \tag{3.4}$$
and
$$\int_{-\infty}^{\infty} v(x,t)\,dx = \int_{-\infty}^{\infty} x\,v(x,t)\,dx = 0, \quad \forall t \in \mathbb{R}. \tag{3.5}$$
The identities in (3.5) follow by combining our hypothesis (3.2), the first conservation law
$$\int_{-\infty}^{\infty} u_j(x,t)\,dx = \int_{-\infty}^{\infty} u_{j,0}(x)\,dx, \qquad \forall\, t \in \mathbb{R}, \;\; j = 1,2,$$
and the identity
$$\frac{d}{dt}\int_{-\infty}^{\infty} x\,u_j(x,t)\,dx = \frac{1}{2}\|u_j(t)\|_2^2 = \frac{1}{2}\|u_{j,0}\|_2^2, \qquad \forall\, t \in \mathbb{R}, \quad j = 1,2.$$

Now, differentiating the equation in (3.3) and multiplying the result by w_N^2 we get
$$\partial_t(w_N^2\partial_x v) + w_N^2\mathcal{H}\partial_x^2\,\partial_x v + w_N^2\,\partial_x(u_1\partial_x v + v\partial_x u_2) = 0. \tag{3.6}$$
We rewrite the second term in (3.6) as
$$\begin{aligned}
&w_N^2\mathcal{H}\partial_x^2\,\partial_x v\\
&= \mathcal{H}(w_N^2\partial_x^2\,\partial_x v) - [\mathcal{H}; w_N^2]\partial_x^3\, v\\
&= \mathcal{H}\partial_x^2(w_N^2\,\partial_x v) - 2\mathcal{H}(\partial_x w_N^2\partial_x^2 v) - \mathcal{H}(\partial_x^2 w_N^2\,\partial_x v) - [\mathcal{H}; w_N^2]\partial_x^3\, v\\
&= G_1 + G_2 + G_3 + G_4.
\end{aligned} \tag{3.7}$$

Theorem 4 yields the inequality
$$\|G_4\|_2 = \|[\mathcal{H}; w_N^2]\partial_x^3\, v\|_2 \le c\,\|v\|_2,$$
with c denoting a constant independent of N which may change from line to line. Also one has that
$$\|G_3\|_2 = \|\mathcal{H}(\partial_x^2 w_N^2\,\partial_x v)\|_2 \le c\,\|\partial_x v\|_2$$
To control $\|G_2\|_2$ we use integration by part to get that
$$\|w_N\,\partial_x^2 v\|_2^2 \le \|w_N^2\,\partial_x v\|\,\|\partial_x^3 v\|_2 + \|\partial_x v\|_2^2,$$
so
$$\|G_2\|_2 = \|\mathcal{H}(\partial_x w_N^2\,\partial_x^2 v)\|_2 \le \|w_N\,\partial_x^2 v\|_2 \le c(\|w_N^2\partial_x v\|_2 + \|v(t)\|_{3,2}).$$
In the energy estimate the contribution of the term G_1 is null, since inserting it in (3.6), multiplying the equation (3.3) by $w_N^2\partial_x v$, and integrating the result in the space variable after integration by parts it vanishes. It remains to bound the contribution from the third term in (3.6) in the energy estimate, i.e.
$$N_1(t) = |\int_{-\infty}^{\infty} w_N^2\,\partial_x(u_1\partial_x v + v\partial_x u_2)\,w_N^2\partial_x v\,dx|.$$

Using the hypotheses and integration by parts it follows that for any $T>0$

$$\begin{aligned} N_1(t) &\le c_T(\|\partial_x u_1(t)\|_\infty + \|\partial_x u_2(t)\|_\infty)\|w_N^2\partial_x v\|_2^2 \\ &\quad + c_T\|u_1\|_\infty \|x\partial_x v\|_2\|w_N^2\partial_x v\|_2 + c_T\|\partial_x^2 u_2(t)\|_\infty \|x^2 v\|_2 \|w_N^2\partial_x v\|_2 \\ &\le c_T(\|w_N^2\partial_x v(t)\|_2 + \|w_N^2\partial_x v(t)\|_2^2), \qquad \forall t\in[-T,T], \end{aligned}$$

with c_T denoting a constant depending on the initial solutions and on their data but independent of N whose value may change from to line. Collecting the above information we conclude that for any $T>0$

$$\sup_{t\in[-T,T]} \|w_N^2\,\partial_x v(t)\|_2 < c_T.$$

Therefore, taking $N\uparrow\infty$ it follows that for any $T>0$

$$\sup_{t\in[-T,T]} \|x^2\,\partial_x v(t)\|_2 < M_T, \tag{3.8}$$

with M_T denoting a constant depending only on initial parameters and on T and whose value may change from line to line. Since by hypothesis we have

$$\sup_{t\in[-T,T]} \|\partial_x^3 v(t)\|_2 < M_T,$$

by integration by parts one gets that for any $T>0$

$$\sup_{t\in[-T,T]} \|x\,\partial_x^2 v(t)\|_2 < M_T.$$

Next, from the identity

$$x\,\mathcal{H}\,\partial_x^2 v = \mathcal{H}\,(x\,\partial_x^2 v) = \mathcal{H}\partial_x^2(xv) - 2\mathcal{H}\partial_x v,$$

we get the equation for $w_N^2 xv$

$$\begin{aligned} &\partial_t(w_N^2 xv) + \mathcal{H}\partial_x^2(w_N^2 xv) - 2\mathcal{H}(\partial_x w_N^2\partial_x(xv)) - \mathcal{H}(\partial_x^2 w_N^2\,xv) \\ &- [\mathcal{H}; w_N^2]\partial_x^2(xv) - 2w_N^2\mathcal{H}\partial_x v + w_N^2\,x(u_1\partial_x v + v\partial_x u_2) = 0. \end{aligned} \tag{3.9}$$

We recall that for all $t\in\mathbb{R}$

$$\int_{-\infty}^{\infty} v(x,t)\,dx = \int_{-\infty}^{\infty} x\,v(x,t)\,dx = 0,$$

so that

$$xH(v) = H(xv), \quad \text{and} \quad x^2H(v) = H(x^2v). \tag{3.10}$$

The following string of estimates

$$\|\mathcal{H}(\partial_x w_N^2\partial_x(xv))\|_2 \le c(\|w_N x\partial_x v\|_2 + \|w_N v\|_2) \le c(\|x^2\partial_x v\|_2 + \|xv\|_2),$$

$$\|\mathcal{H}(\partial_x^2 w_N^2\,xv)\|_2 \le c\|xv\|_2,$$

(by Theorem 4)

$$\|[\mathcal{H}; w_N^2]\partial_x^2(xv)\|_2 \le c\|\partial_x^2 w_N^2\|_\infty \|xv\|_2 \le c\|xv\|_2,$$

(by (3.10))

$$\begin{aligned} &\|w_N^2\mathcal{H}\partial_x v\|_2 \le \|(1+x^2)\mathcal{H}\partial_x v\|_2 \\ &\le \|\partial_x v\|_2 + \|x^2\mathcal{H}\partial_x v\|_2 \le \|\partial_x v\|_2 + \|x\mathcal{H}(x\partial_x v)\|_2 \\ &\le \|\partial_x v\|_2 + \|\mathcal{H}(x^2\partial_x v)\|_2 \le \|\partial_x v\|_2 + \|x^2\partial_x v\|_2, \end{aligned}$$

and (by integrating by parts)

$$
\begin{aligned}
&|\int w_N^2\, x(u_1\partial_x v + v\partial_x u_2) w_N^2 x v dx| \\
&\le (\|\partial_x u_1\|_\infty + \|\partial_x u_2\|_\infty)\|w_N^2 xv\|_2^2 + \|u_1\|_\infty \|x^2 v\|_2 \|w_N^2 xv\|_2.
\end{aligned}
$$

inserted in the energy estimate for (3.9) together with the result in the previous step (3.8) allows us to conclude that for any $T>0$

$$
\sup_{t\in[-T,T]} \|w_N^2 xv\|_2 \le c_T,
$$

with c_T independent of N. Hence, it follows that

$$
\sup_{t\in[-T,T]} \|x^3\, v\|_2 \le M_T. \tag{3.11}
$$

Now, we shall estimate $w_N^2 x\partial_x v$. From the equation

$$
\partial_t(x\partial_x v) + \mathcal{H}\partial_x^2\,(x\partial_x v) - 2\mathcal{H}\partial_x^2 v + x\,\partial_x(u_1\partial_x v + v\partial_x u_2) = 0,
$$

we obtain that

$$
\begin{aligned}
&\partial_t(w_N^2 x\partial_x v) + \mathcal{H}\partial_x^2\,(w_N^2 x\partial_x v) - 2\mathcal{H}(\partial_x w_N^2 \partial_x(x\partial_x v)) \\
&- \mathcal{H}(\partial_x^2(w_N^2) x\partial_x v) - [\mathcal{H}; w_N^2]\partial_x^2(x\partial_x v) - 2w_N^2 \mathcal{H}\partial_x^2 v \\
&+ w_N^2 x\,\partial_x(u_1\partial_x v + v\partial_x u_2) = 0.
\end{aligned} \tag{3.12}
$$

By integration by part one gets that

$$
\begin{aligned}
&\|w_N \partial_x^3 v\|_2^2 \le c(\|w_N^2\partial_x^2 v\|_2\, \|\partial_x^4 v\|_2 + \|\partial_x^2 v\|_2^2), \\
&\|w_N^2 \partial_x^2 v\|_2^2 \le c(\|w_N^3\partial_x v\|_2\, \|w_N\partial_x^3 v\|_2 + \|w_N\partial_x v\|_2^2).
\end{aligned} \tag{3.13}
$$

with a constant c independent of N. We observe that for each N fixed all the quantities in (3.13) are finite. Hence, from (3.13) it follows that

$$
\begin{aligned}
&\|w_N \partial_x^3 v\|_2 \le c(\|w_N^3\partial_x v\|_2^{1/3}\, \|\partial_x^4 v\|_2^{2/3} + \|w_N\partial_x v\|_2 + \|v\|_{4,2}), \\
&\|w_N^2 \partial_x^2 v\|_2 \le c(\|w_N^3\partial_x v\|_2^{2/3}\, \|\partial_x^4 v\|_2^{1/3} + \|w_N^3\partial_x v\|_2 + \|v\|_{2,2}),
\end{aligned} \tag{3.14}
$$

with c independent of N.

Returning to the equation (3.12) we shall use Theorem 4 to get that

$$
\begin{aligned}
&\|w_N^2\mathcal{H}\partial_x^2 v\|_2 \le \|\mathcal{H}(w_N^2\partial_x^2 v)\|_2 + \|[\mathcal{H}; w_N^2]\partial_x^2 v\|_2 \\
&\le c(\|w_N^2\partial_x^2 v\|_2 + \|\partial_x^2 w_N^2\|_\infty \|v\|_2) = c(D_1 + \|v\|_2).
\end{aligned}
$$

Thus, by combining the second inequality in (3.14) and Young's inequality it follows that

$$
D_1 \le c(\|w_N^2 x\partial_x v\|_2 + \|v\|_{4,2}).
$$

Theorem 4 yields the inequality

$$
\|[\mathcal{H}; w_N^2]\partial_x^2(x\partial_x v)\|_2 \le c\,\|x\partial_x v\|_2,
$$

with c independent of N whose value may change from line to line. Also one has

$$
\|\mathcal{H}(\partial_x^2 w_N^2\, x\partial_x v)\|_2 \le c\,\|x\partial_x v\|_2.
$$

To control the third term in (3.12) we write

$$\begin{aligned}
&\|\partial_x(w_N^2)\,\partial_x(x\partial_x v)\|_2\\
&\le c(\|w_N w_N' x\partial_x^2 v\|_2 + \|w_N w_N'\partial_x v\|_2\\
&\le c(\|w_N^2\partial_x^2 v\|_2 + \|x\partial_x v\|_2) = c(D_2 + \|x\partial_x v\|_2).
\end{aligned}$$

Thus, by using the second inequality in (3.14)

$$D_2 \le c(\|w_N^2 x\partial_x v\|_2 + \|v\|_{4,2}).$$

So besides the first two terms in (3.12) it remains to bound the contribution from the last term in the energy estimate, i.e.

$$N_2(t) = |\int_{-\infty}^{\infty} w_N^2\, x\partial_x(u_1\partial_x v + v\partial_x u_2) w_N^2\, x\partial_x v\,dx|.$$

Using our hypothesis and integration by parts it follows that for any $T>0$

$$\begin{aligned}
N_2(t) \le{}& c(\|\partial_x u_1\|_\infty + \|\partial_x u_2\|_\infty)\|\,w_N^2 x\partial_x v\|_2^2\\
&+ (\|u_1\|_\infty\|x^2\partial_x v\|_2 + \|\partial_x^2 u_2\|_\infty\|x^3 v\|_2)\|\,w_N^2 x\partial_x v\|_2,
\end{aligned}$$

with c_T depending on the initial solutions and their data but independent of N. Collecting the above information we conclude that for any $T>0$

$$\sup_{t\in[-T,T]}\|w_N^2\,x\partial_x v(t)\|_2 < c_T,$$

with c_T depending on the initial solutions u_1, u_2, the initial data, and on T, but independent of N. Therefore, taking $N\uparrow\infty$ it follows that for any $T>0$

$$\sup_{t\in[-T,T]}\|x^3\,\partial_x v(t)\|_2 < M_T, \tag{3.15}$$

with M_T denoting a generic constant which may change line to line but depending only on initial parameters and on T. From (3.14) we have

$$\sup_{t\in[-T,T]}\|x^2\partial_x^2 v(t)\|_2 < M_T,$$

by integration by part one gets that for any $T>0$

$$\sup_{t\in[-T,T]}\|x\,\partial_x^3 v(t)\|_2 < M_T. \tag{3.16}$$

Using the identity

$$x^2\,\mathcal{H}\,\partial_x^2 v = \mathcal{H}\partial_x^2(x^2 v) - 4\mathcal{H}\partial_x(xv) + 2\mathcal{H}v,$$

we get the equation for $w_N^2 x^2 v$

$$\begin{aligned}
&\partial_t(w_N^2 x^2 v) + \mathcal{H}\partial_x^2(w_N^2 x^2 v) - 2\mathcal{H}(\partial_x w_N^2\,\partial_x(x^2 v))\\
&- \mathcal{H}(\partial_x^2 w_N^2\,x^2 v) - [\mathcal{H}; w_N^2]\partial_x^2(x^2 v) - 4w_N^2\mathcal{H}\partial_x(xv)\\
&+ 2w_N^2\mathcal{H}v + w_N^2\,x^2(u_1\partial_x v + v\partial_x u_2) = 0.
\end{aligned}\tag{3.17}$$

We recall that for all $t\in\mathbb{R}$

$$\int_{-\infty}^{\infty} v(x,t)\,dx = \int_{-\infty}^{\infty} x\,v(x,t)\,dx = 0,$$

so that

$$xH(v) = H(xv), \qquad xH(xv) = H(x^2 v), \qquad \text{and} \qquad x^2 H(v) = H(x^2 v). \tag{3.18}$$

We shall deduce the following estimates:
(using (3.18))

$$\|w_N^2\mathcal{H}v\|_2 \le \|(1+x^2)\mathcal{H}v\|_2 \le \|xv\|_2 + \|x^2Hv\|_2 \le \|xv\|_2 + \|x^2v\|_2,$$

(using (3.18) and (3.16))

$$\begin{aligned}&\|w_N^2\mathcal{H}\partial_x(xv)\|_2 \le \|(1+x^2)\mathcal{H}\partial_x(xv)\|_2\\ &\le \|\mathcal{H}\partial_x(xv)\|_2 + \|x^2\mathcal{H}\partial_x(xv)\|_2\\ &\le \|\partial_x(xv)\|_2 + \|x\mathcal{H}(x\partial_x(xv))\|_2\\ &\le \|\partial_x(xv)\|_2 + \|x\mathcal{H}(\partial_x(xv))\|_2 + 2\|x\mathcal{H}(xv)\|_2\\ &\le \|\partial_x(xv)\|_2 + \|x\partial_x(xv)\|_2 + 2\|x^2v\|_2,\end{aligned}$$

(using Theorem 4)

$$\|[\mathcal{H};w_N^2]\partial_x^2(x^2v)\|_2 \le c\|\partial_x^2w_N^2\|_\infty\|x^2v\|_2 \le c\|x^2v\|_2,$$

$$\|\mathcal{H}(\partial_x^2w_N^2x^2v)\|_2 \le \|x^2v\|_2,$$

(using (3.15))

$$\begin{aligned}&\|\mathcal{H}(\partial_xw_N^2\partial_x(x^2v))\|_2 \le \|\partial_xw_N^2\partial_x(x^2v)\|_2\\ &\le 8(\|w_Nw_N'xv\|_2 + \|w_Nw_N'x^2\partial_xv\|) \le 8(\|x^2v\|_2 + \|w_N^3\partial_xv\|_2),\end{aligned}$$

and integrating by parts (for the last term in (3.17))

$$\begin{aligned}&\left|\int w_N^2\,x^2(u_1\partial_xv + v\partial_xu_2)w_N^2x^2v\,dx\right|\\ &\le (\|\partial_xu_1\|_\infty + \|\partial_xu_2\|_\infty)\|w_N^2x^2v\|_2^2 + \|u_1\|_\infty\|w_N^2xv\|_2\|w_N^2x^2v\|_2.\end{aligned}$$

Collecting this information in the energy estimate for (3.17) together with the result in the previous steps (3.16) and (3.15) allows us to conclude that for any $T>0$

$$\sup_{t\in[-T,T]}\|w_N^2x^2v\|_2 \le M_T,$$

with M_T independent of N. Hence, it follows that

$$\sup_{t\in[-T,T]}\|x^3\,v\|_2 \le M_T. \tag{3.19}$$

Hence, for any $T>0$

$$v \in L^\infty([-T,T]:Z_{4,4}),$$

which yields the desired result.

4. Proof of Theorem 2

We introduce the notation

$$F_j(t,\xi,\widehat{u}_0) = \partial_\xi^j(e^{-it|\xi|\xi}\widehat{u}_0), \quad j=0,1,2,3,4. \tag{4.1}$$

Therefore,

$$\begin{aligned}F_3(t,\xi,\widehat{u}_0) &= \partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0)\\ &= e^{-it|\xi|\xi}(8it^3\xi^3\widehat{u}_0 - 12t^2\xi\widehat{u}_0 - 12t^2\xi^2\partial_\xi\widehat{u}_0\\ &\quad - 6it\,\mathrm{sgn}(\xi)\partial_\xi\widehat{u}_0 - 6it|\xi|\partial_\xi^2\widehat{u}_0 - 2it\delta\widehat{u}_0 + \partial_\xi^3\widehat{u}_0).\end{aligned} \tag{4.2}$$

We observe that since the initial data u_0 has zero mean value the term involving the Dirac delta in (4.2) vanishes. Thus, under the assumption that u_0 has zero mean value one finds that

$$
\begin{aligned}
F_4(t,\xi,\widehat{u}_0) &= \partial_\xi^4(e^{-it|\xi|\xi}\widehat{u}_0)\\
&= e^{-it|\xi|\xi}(12t^2\widehat{u}_0 + 48it^3\xi|\xi|\widehat{u}_0 + 16t^4\xi^4\widehat{u}_0\\
&\quad - 48t^2\xi\partial_\xi\widehat{u}_0 - 6it\delta\partial_\xi\widehat{u}_0 + 24it^3|\xi|\xi^2\partial_\xi\widehat{u}_0\\
&\quad - 12it\,\mathrm{sgn}(\xi)\partial_\xi^2\widehat{u}_0 - 24t^2\xi^2\partial_\xi^2\widehat{u}_0 - 8it|\xi|\partial_\xi^3\widehat{u}_0 + \partial_\xi^4\widehat{u}_0)\\
&= E_1(t,\xi,\widehat{u}_0) + \dots\dots + E_{10}(t,\xi,\widehat{u}_0).
\end{aligned}
\tag{4.3}
$$

Hence

$$
\widehat{u}(\xi,t) = F_0(t,\xi,\widehat{u}_0) - \int_0^t F_0(t-t',\xi,\widehat{z}(t'))\,dt', \tag{4.4}
$$

and

$$
\partial_\xi^4\widehat{u}(\xi,t) = F_4(t,\xi,\widehat{u}_0) - \int_0^t F_4(t-t',\xi,\widehat{z}(t'))\,dt', \tag{4.5}
$$

where

$$
\widehat{z} = \frac{1}{2}\widehat{\partial_x u^2} = i\frac{\xi}{2}\widehat{u} * \widehat{u}.
$$

Next, we shall see that if $u_0 \in \dot{Z}_{4,4}$ all terms E_j, $j = 1, \dots, 10$ in (4.3) except

$$
E_5(t) = e^{-it|\xi|\xi}(-6it\delta\partial_\xi\widehat{u}_0(\xi)) = -6it\delta\partial_\xi\widehat{u}_0(0) = -6\,t\,\delta\int xu_0(x)dx, \tag{4.6}
$$

are in $L^2(\mathbb{R})$. Thus, we have

$$
\begin{cases}
\|E_1\|_2 &= \|12t^2\,e^{-it|\xi|\xi}\widehat{u}_0\|_2 \le c_t\,\|u_0\|_2,\\
\|E_2\|_2 &= \|48t^3\,|\xi|\xi e^{-it|\xi|\xi}\widehat{u}_0\|_2 \le c_t\,\|\partial_x^2 u_0\|_2,\\
\|E_3\|_2 &= \|16t^4\,\xi^4 e^{-it|\xi|\xi}\widehat{u}_0\|_2 \le c_t\,\|\partial_x^4 u_0\|_2,\\
\|E_4\|_2 &= \|48t^2\,\xi e^{-it|\xi|\xi}\partial_\xi\widehat{u}_0\|_2 \le c_t\,(\|u_0\|_2 + \|x\partial_x u_0\|_2),\\
\|E_6\|_2 &= \|24t^3\,|\xi|\xi^2 e^{-it|\xi|\xi}\partial_\xi\widehat{u}_0\|_2 \le c_t\,(\|x\partial_x^3 u_0\|_2 + \|\partial_x^2 u_0\|_2,\\
\|E_7\|_2 &= \|24t^2\,\xi^2 e^{-it|\xi|\xi}\partial_\xi^2\widehat{u}_0\|_2 \le c_t\,(\|u_0\|_2 + \|x^2\partial_x^2 u_0\|_2 + \|x\partial_x u_0\|_2),\\
\|E_8\|_2 &= \|12t\,\mathrm{sgn}(\xi)e^{-it|\xi|\xi}\partial_\xi^2\widehat{u}_0\|_2 \le c_t\,\|x^2 u_0\|_2,\\
\|E_9\|_2 &= \|8t\,|\xi|e^{-it|\xi|\xi}\partial_\xi^3\widehat{u}_0\|_2 \le c_t\,(\|x^3\partial_x u_0\|_2 + \|x^2 u_0\|_2),\\
\|E_{10}\|_2 &= \|e^{-it|\xi|\xi}\partial_\xi^4\widehat{u}_0\|_2 \le c_t\,\|x^4 u_0\|_2.
\end{cases}
\tag{4.7}
$$

Since $u_0 \in Z_{4,4} = H^4(\mathbb{R}) \cap L^2(|x|^8dx)$, by using interpolation it follows directly that all the terms on the left hand side of (4.7) are bounded.

Now we shall consider the integral term in (4.5)

$$
\Omega(t) \equiv \int_0^t F_4(t-t',\xi,\widehat{z}(t'))\,dt,
$$

with

$$
\widehat{z} = \frac{1}{2}\widehat{\partial_x u^2} = i\frac{\xi}{2}\widehat{u} * \widehat{u}.
$$

We are assuming that $u_0 \in \dot{Z}_{5,4}$, therefore from Theorem A we have that the corresponding solution $u(x,t)$ of (1.1) satisfies that

$$
u \in C(\mathbb{R} : \dot{Z}_{5,7/2-}).
$$

Thus,

$$\widehat{u} \in C(\mathbb{R} : Z_{7/2-,5}),$$

and hence

$$\widehat{u} * \widehat{u} \in C(\mathbb{R} : Z_{6,5}),$$

so we can conclude that

$$\xi\, \widehat{u} * \widehat{u} \in C(\mathbb{R} : Z_{4,4}). \tag{4.8}$$

Above we have seen that if $u_0 \in \dot{Z}_{4.4}$, then all the ten terms of $F_4(t, \xi, \widehat{u}_0)$ (see (4.3)) except E_5 (see (4.6)) are in L^2. The same argument, the fact that $u\partial_x u$ has mean value zero, (4.7), and (4.8) proves that

$$\Omega(t) = \int_0^t F_4(t-t', \xi, \widehat{z}(t'))dt' = \sum_{j=1}^{10} \int_0^t E_j(t-t', \xi, \widehat{z}(t'))dt'$$

with

$$\int_0^t E_j(t-t', \xi, \widehat{z}(t'))dt' \in C([-T,T] : L^2(\mathbb{R})),\ \ 1 \leq j \leq 10,\ \ j \neq 5,\ \ \forall\, t \in \mathbb{R}.$$

Therefore, for any $t \in \mathbb{R}$

$$\begin{aligned}
&\Omega(t) - \int_0^t E_5(t-t', \xi, \widehat{z}(t'))dt' \\
&= \Omega(t) + 6i \int_0^t (t-t')e^{-i(t-t')|\xi|\xi} \delta \partial_\xi (\frac{i\xi}{2} \widehat{u} * \widehat{u})(\xi, t')dt' \\
&= \Omega(t) + 6i\,\delta \int_0^t (t-t')\,\partial_\xi (\frac{i\xi}{2} \widehat{u} * \widehat{u})(0, t')dt \\
&\equiv \Omega(t) + B_5(t) \in L^2(\mathbb{R}).
\end{aligned} \tag{4.9}$$

We observe that

$$\begin{aligned}
&\partial_\xi (\frac{i\xi}{2} \widehat{u} * \widehat{u})(0, t') = -\widehat{ixu\partial_x} u(0,t') = -i \int_{-\infty}^{\infty} xu\partial_x u(x,t')dx \\
&= \frac{i}{2}\|u(t)\|_2^2 = \frac{i}{2}\|u_0\|_2^2 = i\,\frac{d}{dt} \int_{-\infty}^{\infty} xu_0(x)dx.
\end{aligned} \tag{4.10}$$

Using (4.10) and integration by parts it follows that

$$\begin{aligned}
B_5(t) &= 6i\,\delta \int_0^t (t-t')\,\partial_\xi (\frac{i\xi}{2} \widehat{u} * \widehat{u})(0, t')dt \\
&= 6i\,\delta \int_0^t (t-t')\,(i\,\frac{d}{dt} \int xu(x,t')dx)\,dt \\
&= -6\,\delta(\ (t-t') \int xu(x,t')dx|_{t'=0}^{t'=t}\ +\ \int_0^t (\int xu(x,t')dx)\,dt') \\
&= 6\,\delta(\,t \int xu_0(x)dx\ +\ \int_0^t (\int xu(x,t')dx)\,dt').
\end{aligned} \tag{4.11}$$

Collecting the above information we have that

$$\partial_\xi^4 \widehat{u}(\xi, t) = F_4(t, \xi, \widehat{u}_0) - \int_0^t F_4(t-t', \xi, \widehat{z}(t'))\,dt' - E_5(t) - B_5(t) \in L^2(\mathbb{R}).$$

But

$$\begin{aligned}
&E_5(t)+B_5(t)\\
&=-6\,t\,\delta\int xu_0(x)dx+6\,\delta\Big(t\int xu_0(x)dx+\int_0^t\Big(\int xu(x,t')dx\Big)\,dt'\Big)\\
&=6\,\delta\int_0^t\Big(\int xu(x,t')dx\Big)\,dt'=6\,\delta\int_0^t\Big(\int xu_0(x)dx+\frac{t'}{2}\,\|u_0\|_2^2\Big)dt\\
&=6\,\delta\,\Big(t\int xu_0(x)dx+\frac{t^2}{4}\|u_0\|_2^2\Big),
\end{aligned}$$

which vanishes only at

$$t^*=-\frac{4}{\|u_0\|_2^2}\int_{-\infty}^{\infty}x\,u_0(x)dx,$$

and at that time we have that

$$\partial_\xi^4 u(\xi,t^*)\in L^2(\mathbb{R}),$$

which yields the desired result.

ACKNOWLEDGMENT : This work was done while G. P. was visiting the Department of Mathematics at Universidad Autonoma de Madrid-Spain whose hospitality he gratefully acknowledges. G.P. was supported by NSF grant DMS-0800967.

References

[1] L. Abdelouhab, J. L. Bona, M. Felland and J.-C. Saut, *Nonlocal models for nonlinear dispersive waves*, Physica D. **40** (1989) 360–392.

[2] M. J. Ablowitz and A. S. Fokas, *The inverse scattering transform for the Benjamin-Ono equation, a pivot for multidimensional problems*, Stud. Appl. Math. **68** (1983) 1–10.

[3] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. **29** (1967) 559–592.

[4] N. Burq, and F. Planchon, *On the well-posedness of the Benjamin-Ono equation*, Math. Ann. **340** (2008) 497–542.

[5] A. P. Calderón, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A., **53** (1965) 1092–1099.

[6] R. Coifman and M. Wickerhauser, *The scattering transform for the Benjamin-Ono equation*, Inverse Problems **6** (1990) 825–860.

[7] L. Dawson, H. McGahagan and G. Ponce, *On the decay properties of solutions to a class of Schrödinger equations*, Proc. AMS. **136** (2008) 2081–2090.

[8] L. Escauriaza, C.E. Kenig, G. Ponce and L. Vega, *On uniqueness properties of solutions of the k-generalized KdV equations*, J. Funct. Anal. **244** (2007) 504–535.

[9] L. Escauriaza, C.E. Kenig, G. Ponce and L. Vega, *The Sharp Hardy Uncertainty Principle for Schrödinger Evolutions*, to appear in Duke Math. J.

[10] G. Fonseca and G. Ponce, *The IVP for the Benjamin-Ono equation in weighted Sobolev spaces*, J. Funct. Anal. **260** (2011) 436–459.

[11] A. A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, *Persistence Properties and Unique Continuation of solutions of the Camassa-Holm equation*, Comm. Math. Phys. **271** (2007) 511–522.

[12] A. D. Ionescu and C. E. Kenig, *Global well- posedness of the Benjamin-Ono equation on low-regularity spaces*, J. Amer. Math. Soc. **20**, 3 (2007) 753–798.

[13] R. J. Iorio, *On the Cauchy problem for the Benjamin-Ono equation*, Comm. P. D. E. **11** (1986) 1031–1081.

[14] R. J. Iorio, *Unique continuation principle for the Benjamin-Ono equation*, Diff. and Int. Eqs., **16** (2003) 1281–1291.

[15] C. E. Kenig, and K. D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Res. Letters **10** (2003) 879–895.

[16] C.E. Kenig, G. Ponce and L. Vega, *On the unique continuation of solutions to the generalized KdV equation*, Math. Res. Letters **10** (2003) 833–846.

[17] H. Koch and N. Tzvetkov, *On the local well- posedness of the Benjamin-Ono equation on $H^s(\mathbb{R})$*, Int. Math. Res. Not., **26** (2003) 1449-1464.

[18] H. Koch and N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation.*, Int. Math. Res. Not., **30** (2005) 1833–1847.

[19] L. Molinet and D.Pilot, *The Cauchy problem for the Benjamin-Ono equation in L^2 revisited*, to appear in Analysis and PDE.

[20] L. Molinet and F. Ribaud, *Well-posedness results for the Benjamin-Ono equation with arbitrary large initial data*, Int. Math. Res. Not., **70** (2004) 3757–3795.

[21] L. Molinet, J.C. Saut and N. Tzvetkov, *Ill- posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal. **33** (2001) 982-988.

[22] H. Ono, *Algebraic solitary waves on stratified fluids*, J. Phy. Soc. Japan **39** (1975) 1082–1091.

[23] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Diff. & Int. Eqs. **4** (1991) 527–542.

[24] J.-C. Saut, *Sur quelques généralisations de l' équations de Korteweg-de Vries*, J. Math. Pures Appl. **58** (1979) 21–61.

[25] T. Tao, *Global well-posedness of the Benjamin-Ono equation on H^1*, Journal Hyp. Diff. Eqs. **1** (2004) 27-49. Int. Math. Res. Not., Art. ID 95702 (2006) 1–44.

(G. Fonseca) Departamento de Matemáticas, Universidad Nacional de Colombia, Bogota, Colombia
E-mail address: gefonsecab@unal.edu.co

(F. Linares) Instituto de Matematica Pura e Aplicada, IMPA, Rio de Janeiro, Brazil
E-mail address: linares@impa.br

(G. Ponce) Department of Mathematics, University of California, Santa Barbara, CA 93106, USA.
E-mail address: ponce@math.ucsb.edu