THE IVP FOR THE DISPERSION GENERALIZED BENJAMIN-ONO EQUATION IN WEIGHTED SOBOLEV SPACES

GERMÁN FONSECA, FELIPE LINARES, AND GUSTAVO PONCE

Abstract. We study the initial value problem associated to the dispersion generalized Benjamin-Ono equation. Our aim is to establish well posedness results in weighted Sobolev spaces and to deduce from them some sharp unique continuation properties of solutions to this equation. In particular, we shall establish optimal decay rate for the solutions of this model.

Résumé. Nous étudions le problème de Cauchy associé à l'équation de Benjamin-Ono avec dispersion généralisée. Notre objectif est d'établir le caractère bien posé de cette équation dans des espaces de Sobolev avec poids et d'en déduire quelques propriétés de prolongement unique pour ses solutions. En particulier, nous établirons un taux de décroissance optimal pour les solutions de ce modèle.

1. Introduction

This work is concerned with the initial value problem (IVP) for the dispersion generalized Benjamin-Ono (DGBO) equation

$$\begin{cases}\partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \qquad t, x \in \mathbb{R}, \;\; 0 < a < 1,\\ u(x,0) = u_0(x),\end{cases} \tag{1.1}$$

where D^s denotes the homogeneous derivative of order $s \in \mathbb{R}$,

$$D^s = (-\Delta)^{s/2} \quad \text{so} \quad D^s f = c_s\big(|\xi|^s \widehat{f}\,\big)^{\vee}, \quad \text{with} \quad D^s = (\mathcal{H}\,\partial_x)^s \quad \text{if} \quad n = 1,$$

where $\mathcal{H}$ denotes the Hilbert transform,

$$\begin{aligned}\mathcal{H}f(x) &= \frac{1}{\pi}\text{v.p.}\big(\frac{1}{x} * f\big)(x)\\ &= \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int\limits_{|y|\geq\epsilon} \frac{f(x-y)}{y}\,dy = (-i\,\text{sgn}(\xi)\widehat{f}(\xi))^{\vee}(x).\end{aligned}$$

These equations model vorticity waves in the coastal zone, see [38] and references therein.

1991 *Mathematics Subject Classification.* Primary: 35B05. Secondary: 35B60.
Key words and phrases. Benjamin-Ono equation, weighted Sobolev spaces.
The second author was supported by CNPq and FAPERJ/Brasil.
The third author is supported by the NSF DMS-1101499.

When $a = 1$ the equation in (1.1) becomes the famous Korteweg-de Vries (KdV) equation

$$\partial_t u - \partial_x^3 u + u\partial_x u = 0, \qquad t, x \in \mathbb{R}, \tag{1.2}$$

and when $a = 0$ the equation in (1.1) agrees with the well known Benjamin-Ono (BO) equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \qquad t, x \in \mathbb{R}. \tag{1.3}$$

Both the KdV and the BO equations originally arise as models in one-dimensional waves propagation (see [34], [4], and [42]) and have widely been studied in many different contexts. They present several similarities: both possess infinite conserved quantities, define Hamiltonian systems, have multi soliton solutions and are completely integrable. The local well-posedness (LWP) and global well-posedness (GWP) of their associated IVP in the classical Sobolev spaces $H^s(\mathbb{R}),\ s \in \mathbb{R}$ have been extensively investigated.

In the case of the KdV equation this problem has been studied in [45], [5], [26], [29], [6], [30], [10], and finally [17] where global well-posedness was established for $s \geq -3/4$.

In the case of the BO equation the same well-posedness problem has been considered in [45], [1], [24], [44], [32], [27], [47], [36], [7], and [23] where global well-posedness was established for $s \geq 0$ (for further discussion we refer to [35]).

However, there are two remarkable differences between the existence theory for these two models. The first is the fact that one can give a local existence theory for the IVP associated to the KdV in $H^s(\mathbb{R})$ based only on the contraction principle. This can not be done in the case of the BO. This is a consequence of the lack of smoothness of the application data-solution in the BO setting established in [38]. There it was proved that this map is not locally C^2. Actually, in [33] it was proved that this map is not even locally uniformly continuous.

The second remarkable difference between these equations is concerned with the persistent property of the solutions (i.e. if the data $u_0 \in X$, a function space, then the corresponding solution $u(\cdot)$ describes a continuous curve in X, $u \in C([-T, T] : X),\ T > 0$) in weighted Sobolev spaces. In [26] it was shown that the KdV flow preserves the Schwartz class. However, it was first established by Iorio [24] and [25] that in general, polynomial type decay is not preserved by the BO flow. The results in [24], [25] were recently extended to fractional order weighted Sobolev spaces in [14]. In order to present these results, we introduce the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \ \ s, r \in \mathbb{R}, \tag{1.4}$$

and

$$\dot{Z}_{s,r} = \{f \in H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx) \,:\, \widehat{f}(0) = 0\}, \qquad s,\, r \in \mathbb{R}. \tag{1.5}$$

The well-posedness results for the IVP associated to the BO equation in weighted Sobolev spaces can be stated as:

Theorem A. ([14])

(i) *Let $s \geq 1$, $r \in [0,s]$, and $r < 5/2$. If $u_0 \in Z_{s,r}$, then the solution u of the IVP associated to the BO equation* (1.3) *satisfies that*

$$u \in C([0,\infty) : Z_{s,r}).$$

(ii) *For $s > 9/8$ ($s \geq 3/2$), $r \in [0,s]$, and $r < 5/2$ the IVP associated to the BO equation* (1.3) *is LWP (GWP resp.) in $Z_{s,r}$.*

(iii) *If $r \in [5/2, 7/2)$ and $r \leq s$, then the IVP* (1.3) *is GWP in $\dot{Z}_{s,r}$.*

Theorem B. ([14]) *Let $u \in C([0,T] : Z_{2,2})$ be a solution of the IVP* (1.3)*. If there exist two different times $t_1, t_2 \in [0,T]$ such that*

$$u(\cdot, t_j) \in Z_{5/2,5/2},\ j = 1,2,\ \textit{then}\ \widehat{u}_0(0) = 0,\ \ (\textit{so}\ u(\cdot,t) \in \dot{Z}_{5/2,5/2}). \tag{1.6}$$

Theorem C. ([14]) *Let $u \in C([0,T] : \dot{Z}_{3,3})$ be a solution of the IVP* (1.3)*. If there exist three different times $t_1, t_2, t_3 \in [0,T]$ such that*

$$u(\cdot, t_j) \in \dot{Z}_{7/2,7/2},\quad j = 1,2,3,\quad \textit{then}\quad u(x,t) \equiv 0. \tag{1.7}$$

We point out that Iorio's results correspond to the indexes $s \geq r = 2$ in Theorem A part (ii), $s \geq r = 3$ in Theorem A part (iii) and $s \geq r = 4$ in Theorem C.

Regarding the DGBO equation (1.1), we notice that for $a \in (0,1)$ the dispersive effect is stronger than the one for the BO equation but still too weak compared to that of the KdV equation. Indeed it was shown in [38] that for the IVP associated to the DGBO equation (1.1) the flow map data-solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ fails to be locally C^2 at the origin for any $T > 0$ and any $s \in \mathbb{R}$ as in the case of the BO equation. Therefore, so far local well-posedness in classical Sobolev spaces $H^s(\mathbb{R})$ for (1.1) cannot be obtained by an argument based only on the contraction principle. Local well-posedness in classical Sobolev spaces for (1.1) has been studied in [29], [19], [37], [18], and [20] where local well-posedness was established for $s \geq 0$.

Real solutions of the IVP (1.1) satisfy at least three conserved quantities:

$$\begin{aligned} I_1(u) &= \int_{-\infty}^{\infty} u(x,t)dx, \quad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx, \\ I_3(u) &= \int_{-\infty}^{\infty} (|D_x^{\frac{1+a}{2}} u|^2 + \frac{u^3}{6})(x,t)dx. \end{aligned} \tag{1.8}$$

In particular, we have that the local results in [20] extend globally in time.

Concerning the form of the traveling wave solution of (1.1) it is convenient to consider

$$v(x,t) = -u(x,-t),$$

where $u(x,t)$ satisfies equation (1.1). Thus,

$$\partial_t v - D^{1+a}\partial_x v + v\partial_x v = 0, \qquad t,x \in \mathbb{R}, \qquad 0 \leq a \leq 1, \tag{1.9}$$

Traveling wave solutions of (1.9) are solutions of the form

$$v(x,t) = c^{1+a}\,\phi_a(c(x - c^{1+a}\,t)), \qquad c>0,$$

where ϕ_a is called the ground state, which is an even, positive, decreasing (for $x>0$) function. In the case of the KdV equation ($a=1$ in (1.9)) one has that

$$\phi_1(x) = \frac{3}{2}\,\text{sech}^2\left(\frac{x}{2}\right),$$

whose uniqueness follows by elliptic theory.

In the case of the BO equation ($a=0$ in (1.9)) one has that

$$\phi_0(x) = \frac{4}{1+x^2}, \tag{1.10}$$

whose uniqueness (up to symmetry of the equation) was established in [2].

In the case $a \in (0,1)$ in (1.9) the existence of the ground state was established in [48] by variational arguments. Recently, uniqueness of the ground state for $a \in (0,1)$ was established in [16]. However, no explicit formula is know for ϕ_a, $a \in (0,1)$. In [28] the following upper bound for the decay of the ground state was deduced

$$\phi_a(x) \leq \frac{c_a}{(1+x^2)^{1+a/2}}, \qquad 0<a<1.$$

Thus, one has that for $a \in [0,1)$ the ground state has a very mild decay in comparison with that for the KdV equation $a=1$. Roughly speaking, this is a consequence of the non-smoothness of the symbol modeling the dispersive relation in (1.1) $\sigma_a(\xi) = |\xi|^{1+a}\,\xi$.

Our goal in this work is to extend the results in Theorems A-C for the DGBO equation (1.1), by proving persistent properties of solution of (1.1) in the weighted Sobolev spaces (1.4). This will lead us to obtain some optimal uniqueness properties of solutions of this equation as well as to establish what is the maximum rate of decay of a solution of (1.1).

In order to motivate our results we first recall the fact that for dispersive equations the decay of the data is preserved by the solution only if they have enough regularity. More precisely, persistence property of the solution $u = u(x,t)$ of the IVP (1.1) in the weighted Sobolev spaces $Z_{s,r}$ can only hold if $s \geq (1+a)r$. This can be seen from the fact that the linear part of the equation (1.1)

$$L = \partial_t + D^{1+a}\partial_x \quad \text{commutes with} \qquad \Gamma = x - (a+2)tD^{1+a}. \tag{1.11}$$

Hence, it is natural to consider well-posedness in the weighted Sobolev spaces $Z_{s,r}$, $s \geq (1+a)r$.

Let us state our main results:

Theorem 1.1. (a) *Let $a \in (0,1)$. If $u_0 \in Z_{s,r}$, then the solution u of the IVP (1.1) satisfies $u \in C([-T,T] : Z_{s,r})$ if either*

(i) $s \geq (1+a)$ *and* $r \in (0,1]$,

or

(ii) $s \geq 2(1+a)$ *and* $r \in (1,2]$,
or
(iii) $s \geq [(r+1)^-](1+a)$ *and* $2 < r < 5/2+a$, *with* $[\cdot]$ *denoting the integer part function.*

(b) *If* $u_0 \in \dot{Z}_{s,r}$, *then the solution* u *of the IVP* (1.1) *satisfies*

$$u \in C([-T,T] : \dot{Z}_{s,r}),$$

whenever

(iv) $s \geq [(r+1)^-](1+a)$ *and* $5/2+a < r < 7/2+a$.

Theorem 1.2. *Let* $a \in (0,1)$. *If* $u_0 \in Z_{s,r}$, $r \geq 2$ *and* $s \geq r(1+a)+(1-a)/2$. *Then there exists a unique solution* u *of the IVP* (1.1) *such that*

$$u \in C([-T,T] : Z_{s,r}) \tag{1.12}$$

with

$$D_x^l u \in L^\infty(\mathbb{R} : L^2(-T,T)), \qquad l \in [(1+a)/2, s+(1+a)/2]. \tag{1.13}$$

Theorem 1.3. *Let* $u \in C([-T,T] : Z_{s,(5/2+a)^-})$ *with*

$$T > 0 \quad \text{and} \quad s \geq (1+a)(5/2+a)+(1-a)/2$$

be a solution of the IVP (1.1). *If there exist two times* $t_1, t_2 \in [-T,T]$, $t_1 \neq t_2$, *such that*

$$u(\cdot,t_j) \in Z_{s,5/2+a}, \quad j = 1,2. \tag{1.14}$$

Then

$$\widehat{u}(0,t) = \int u(x,t)dx = \int u_0(x)\,dx = \widehat{u}_0(0) = 0 \quad \text{for all} \quad t \in [-T,T]. \tag{1.15}$$

Remarks.

a) *Theorem 1.3 shows that persistence in* $Z_{s,r}$ *with* $r = (5/2+a)^-$ *is the best possible for general initial data. In fact, it shows that for data* $u_0 \in Z_{s,r}$, $s \geq (1+a)r+(1-a)/2$, $r \geq 5/2+a$ *with* $\widehat{u}_0(0) \neq 0$ *the corresponding solution* $u = u(x,t)$ *verifies that*

$$|x|^{(5/2+a)^-} u \in L^\infty([0,T] : L^2(\mathbb{R})), \qquad T > 0,$$

but there does not exist a non-trivial solution u *corresponding to data* u_0 *with* $\widehat{u}_0(0) \neq 0$ *such that*

$$|x|^{5/2+a} u \in L^\infty([0,T'] : L^2(\mathbb{R})), \qquad \text{for some} \quad T' > 0.$$

b) *There is a gap between the lower values of* s *as a function of* r *given in the above theorems (*$s \geq s_r \equiv [(r+1)^-](1+a)$ *in Theorem 1.1 and* $s = r(1+a)+(1-a)/2$ *in Theorem 1.2) and that suggested by the commutative relation* (1.11) $s = r(1+a)$. *However, we observe that the commutative relation* (1.11) *requires that* $r \in \mathbb{Z}^+$, *so in this case the desired result follows by using the operator* Γ *in* (1.11). *In* [14] *for the case of the BO equation (*$a = 0$*) this difficulty was circumvented obtaining persistence property in* $Z_{s,r}$, $s \geq r \notin \mathbb{Z}^+$ *by* *using some special properties of the Hilbert transform. More precisely, by combining the extension of Calderón commutator estimate* [8] *established in* [11] *(see Lemma 2.1), the notion of A_p weight* [39] *and some related results in* [22] *and* [43]. *The same problem for the KdV ($a = 1$) (persistence property in $Z_{s,r}$, $s \geq 2r$, $r \notin \mathbb{Z}^+$) was recently considered in* [40].

c) *The proof of Theorem 1.2 follows by combining the contraction principle and the estimates in Propositions 2.3 and 2.7. The weight allows to pass from an L^1_x estimate to an L^2 weighted estimate for the maximal function. The extra regularity "$(1-a)/2$" required in s is to guarantee that the estimates for the weight can be completed. Roughly, from Proposition 2.7 one needs $r(1+a)$ derivatives to estimate $|x|^r u$ in the L^2-norm. Since the nonlinear term involves one derivative we have $1 + r(1+a)$ derivatives to control. Now the smoothing effect and the regularity of the data give us control of $s + (1+a)/2$ derivatives. So to close the estimate we need $s + (1+a)/2 \geq 1 + r(1+a)$, i.e. $s \geq r(1+a) + (1-a)/2$. For a similar argument (in fact more involved) we refer to* [40].

d) *The result in Theorem 1.1 for $s = 1 + a$ was established in* [9].

Theorem 1.4. *Let $u \in C([-T,T] : Z_{s,(7/2+a)^-})$ with*

$$T > 0 \quad \text{and} \quad s \geq (1+a)(7/2+a) + \frac{1-a}{2},$$

be a solution of the IVP (1.1). *If there exist three different times $t_1, t_2, t_3 \in [-T,T]$ such that*

$$u(\cdot, t_j) \in \dot{Z}_{s,7/2+a}, \quad j = 1,2,3. \tag{1.16}$$

Then

$$u \equiv 0.$$

Remarks.

a) *Theorem 1.4 shows that the decay $r = (7/2+a)^-$ is the largest possible. More precisely, Theorem 1.1 part (b) tells us that there are non trivial solutions $u = u(x,t)$ verifying*

$$|x|^{(7/2+a)^-} u \in L^\infty([0,T] : L^2(\mathbb{R})), \qquad T > 0,$$

and Theorem 1.4 guarantees that there does not exist a non-trivial solution such that

$$|x|^{7/2+a} u \in L^\infty([0,T'] : L^2(\mathbb{R})), \qquad \text{for some} \quad T' > 0.$$

b) *We shall prove this result in the most general case $s = (1+a)(7/2+a) + \frac{1-a}{2}$. Also, we will carry out the details in the case $a \in [1/2, 1)$. It will be clear from our argument how to extend the result to the case $a \in (0, 1/2)$.*

Theorem 1.5. *Let* $u \in C([-T,T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \quad \textit{and} \quad s \geq (1+a)(7/2+a) + (1-a)/2,$$

be a solution of the IVP (1.1)*. If there exist* $t_1, t_2 \in [-T,T]$*,* $t_1 \neq t_2$ *such that*

$$u(\cdot, t_j) \in \dot{Z}_{s,7/2+a}, \;\; j = 1,2,$$

and

$$\int xu(x,t_1)dx = 0 \quad \textit{or} \quad \int xu(x,t_2)dx = 0, \tag{1.17}$$

then

$$u \equiv 0.$$

Remark. *Theorem 1.5 tells us that the conditions of Theorem 1.4 can be reduced to two times provided the first momentum of the solution u vanishes at one of them.*

Theorem 1.6. *Let* $u \in C([-T,T] : Z_{s,(7/2+a)^-})$ *with*

$$T > 0 \quad \textit{and} \quad s \geq (1+a)(7/2+[1+2a]/2) + (1-a)/2$$

be a non-trivial solution of the IVP (1.1) *such that*

$$u_0 \in \dot{Z}_{s,\frac{7}{2}+\tilde{a}}, \;\; \tilde{a} = [1+2a]/2, \qquad \textit{and} \qquad \int_{-\infty}^{\infty} x\, u_0(x)\, dx \neq 0. \tag{1.18}$$

Then there exists $t^* \neq 0$ *with*

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{-\infty}^{\infty} x\, u_0(x)\, dx, \tag{1.19}$$

such that $u(t^*) \in \dot{Z}_{s,\frac{7}{2}+\tilde{a}}$.

Remarks.

a) *Notice that* $\tilde{a} > a$*, so Theorem 1.6 shows that the condition of Theorem 1.4 at two times is in general not sufficient to guarantee that* $u \equiv 0$*. So, in this regard Theorem 1.5 is optimal.*

b) *The results in Theorem 1.4 and Theorem 1.6 present an striking difference with other unique continuation properties deduced for other dispersive models. Using the information at two different times, uniqueness results have been established for the generalized KdV equation in* [12]*, for the semi-linear Schrödinger equation in* [13]*, and for the Camassa-Holm model in* [21]*. Theorem 1.6 affirms that the uniqueness condition with the weight* $|x|^{7/2+a}$ *does not hold at two different times but Theorem 1.4 guarantees that it does at three times. Similar result for the Benjamin-Ono equation (*$a = 0$ *in* (1.1)*) was obtained in* [15].

One can consider the IVP (1.1) with $a > 1$. In this case our results still hold, with the appropriate modification in the well-posedness in $H^s(\mathbb{R})$, if a is not an odd integer. In the case where a is an odd integer, one has solutions with exponential decay as in the case of the KdV equation ($a = 1$ in (1.1)).

Finally, we consider the generalization of the IVP (1.1) to higher nonlinearity

$$\begin{cases} \partial_t u + D^{1+a}\partial_x u + u^k \partial_x u = 0, & t, x \in \mathbb{R},\ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.20}$$

In this case our positive results, Theorems 1.1 – 1.3, still hold (with the appropriate modification in the well-posedness in $H^s(\mathbb{R})$). Our unique continuation results (Theorems 1.4–1.5) can be extended to the case where k in (1.20) is odd. In this case one has that the time evolution of the first momentum of the solution is given by the formula

$$\int_{-\infty}^{\infty} x\, u(x,t)\, dx = \int_{-\infty}^{\infty} x\, u_0(x)\, dx + \frac{1}{k+1}\int_0^t \int_{-\infty}^{\infty} u^{k+1}(x,t)\, dx.$$

Thus, it is an increasing function. Hence, defining $t^* \neq 0$ as the solution of the equation

$$\int_0^{t^*} \int_{-\infty}^{\infty} x\, u(x,t)\, dxdt = 0, \tag{1.21}$$

one sees that there is at most one solution of (1.21) but its existence it is not guaranteed. So the statements in Theorems 1.4–1.5 would have to be modified accordingly to this fact.

The rest of this paper is organized as follows: section 2 contains some preliminary estimates to be used in the coming sections. Section 3 contains the proof of Theorem 1.1. The proof of Theorem 1.2 will be omitted, see the remark after the statement of this theorem. Theorem 1.3, Theorem 1.4, Theorem 1.5, and Theorem 1.6 will be proven in sections 4, 5, 6, and 7 respectively.

2. Preliminary Estimates

We begin this section by introducing the notation needed in this work. We use $\|\cdot\|_{L^p}$ to denote the $L^p(\mathbb{R})$ norm. If necessary, we use subscript to inform which variable we are concerned with. The mixed norm $L^q_t L^r_x$ of $f = f(x,t)$ is defined as

$$\|f\|_{L^q_t L^r_x} = \left(\int \|f(\cdot,t)\|^q_{L^r_x} dt\right)^{1/q},$$

with the usual modifications when $q = \infty$ or $r = \infty$. The $L^r_x L^q_t$ norm is similarly defined.

We define the spatial Fourier transform of $f(x)$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

We shall also define J^s to be the Fourier multiplier with symbol $\langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$. Thus, the norm in the Sobolev space $H^s(\mathbb{R})$ is given by

$$\|f\|_{s,2} \equiv \|J^s f\|_{L^2_x} = \|\langle\xi\rangle^s \widehat{f}\|_{L^2_\xi}.$$

A function $\chi \in C_0^\infty$, supp $\chi \subseteq [-2,2]$ and $\chi \equiv 1$ in $(-1,1)$ will appear several times in our arguments.

For $a \in (0,1)$ fixed we introduce F_j's as being

$$F_j(t,\xi,\widehat{u}_0) = \partial_\xi^j (e^{-it|\xi|^{1+a}\xi} \widehat{u}_0(\xi)), \tag{2.1}$$

for $j = 0,1,2,3,4$. Thus

$$F_1(t,\xi,\widehat{u}_0) = -(2+a)it|\xi|^{1+a} e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi) + e^{-it|\xi|^{1+a}\xi}\partial_\xi\widehat{u}_0(\xi),$$

$$\begin{aligned}
F_2(t,\xi,\widehat{u}_0) = {} & e^{-it|\xi|^{1+a}\xi}(-it(2+a)(1+a)|\xi|^a \mathrm{sgn}(\xi)\widehat{u}_0(\xi) \\
& - (2+a)^2 t^2 |\xi|^{2(a+1)}\widehat{u}_0(\xi) \\
& - 2it(2+a)|\xi|^{1+a}\partial_\xi\widehat{u}_0(\xi) + \partial_\xi^2\widehat{u}_0(\xi)) \\
= {} & (B_1 + B_2 + B_3 + B_4)(t,\xi,\widehat{u}_0), \\
F_3(t,\xi,\widehat{u}_0) = {} & e^{-it|\xi|^{1+a}\xi}\big(- ita(1+a)(2+a)|\xi|^{a-1}\widehat{u}_0(\xi) \\
& - 3t^2(2+a)^2(1+a)|\xi|^{2a+1}\,\mathrm{sgn}(\xi)\widehat{u}_0(\xi) \\
& + it^3(2+a)^3|\xi|^{3(1+a)}\widehat{u}_0 \\
& - 3it(2+a)(1+a)\,|\xi|^a \mathrm{sgn}(\xi)\partial_\xi\widehat{u}_0(\xi) \\
& - 3t^2(2+a)^2|\xi|^{2(1+a)}\partial_\xi\widehat{u}_0(\xi) \\
& - 3it(2+a)|\xi|^{1+a}\partial_\xi^2\widehat{u}_0(\xi) + \partial_\xi^3\widehat{u}_0(\xi)\big) \\
= {} & (D_1 + D_2 + D_3 + D_4 + D_5 + D_7)(t,\xi,\widehat{u}_0),
\end{aligned}$$

$$\begin{aligned}
F_4(t,\xi,\widehat{u}_0) = {} & e^{-it|\xi|^{1+a}\xi}\big(- it(2+a)(1+a)a(a-1)|\xi|^{a-2}\,\mathrm{sgn}(\xi)\widehat{u}_0(\xi) \\
& - t^2(2+a)^2(1+a)(7a+3)|\xi|^{2a}\widehat{u}_0(\xi) \\
& + 6it^3(2+a)^3(1+a)|\xi|^{3a+2}\mathrm{sgn}(\xi)\widehat{u}_0(\xi) \\
& + t^4(2+a)^4|\xi|^{4(a+1)}(\xi)\widehat{u}_0(\xi) \\
& - 4ita(1+a)(2+a)|\xi|^{a-1}\partial_\xi\widehat{u}_0(\xi) \\
& - 12t^2(2+a)^2(1+a)|\xi|^{2a+1}\,\mathrm{sgn}(\xi)\partial_\xi\widehat{u}_0(\xi) \\
& + 4it^3(2+a)^3|\xi|^{3(1+a)}\partial_\xi\widehat{u}_0 - 6t^2(2+a)^2|\xi|^{2(1+a)}\partial_\xi^2\widehat{u}_0(\xi) \\
& - 6it(2+a)(1+a)\,|\xi|^a \mathrm{sgn}(\xi)\partial_\xi^2\widehat{u}_0(\xi) \\
& - 4it(2+a)|\xi|^{1+a}\partial_\xi^3\widehat{u}_0(\xi) + \partial_\xi^4\widehat{u}_0(\xi)\big) \\
= {} & (E_1 + \cdots + E_{11})(t,\xi,\widehat{u}_0).
\end{aligned}$$

The next two results will be essential in the analysis below.

The first one is an extension of the Calderón commutator theorem [8] found in [11].

Lemma 2.1. *Let $\mathcal{H}$ denote the Hilbert transform. Then for any $p \in (1,\infty)$ and any $l, m \in \mathbb{Z}^+ \cup \{0\}$ there exists $c = c(p;l;m) > 0$ such that*

$$\|\partial_x^l [\mathcal{H};\psi]\partial_x^m f\|_{L^p} \leq c\,\|\partial_x^{m+l}\psi\|_{L^\infty}\|f\|_{L^p}. \tag{2.2}$$

See Lemma 3.1 in [11].

Proposition 2.2. *Let $\alpha \in [0,1)$, $\beta \in (0,1)$ with $\alpha+\beta \in [0,1]$. Then for any $p, q \in (1,\infty)$ and for any $\delta > 1/q$ there exists $c = c(\alpha;\beta;p;q;\delta) > 0$ such that*

$$\|D_x^\alpha [D_x^\beta;\psi] D_x^{1-(\alpha+\beta)} f\|_{L^p} \leq c\,\|J^\delta \partial_x \psi\|_{L^q}\|f\|_{L^p}, \tag{2.3}$$

where $J := (1-\partial_x^2)^{1/2}$.

See Proposition 3.2 in [11].

Using the notation

$$W_a(t)f = \big(e^{-it|\xi|^{1+a}\xi}\,\widehat{f}\,\big)^{\vee} \tag{2.4}$$

we recall the following linear estimates:

Proposition 2.3. *(Smoothing Effects and Maximal Function)*

(1) *Homogeneous.*

$$\|D^{(1+a)/2} W_a(t) f\|_{L_x^\infty L_T^2} \leq c_a\,\|f\|_2. \tag{2.5}$$

(2) *Nonhomogeneous and Duality.*

$$\begin{aligned}&\|D^{s+a/2+1/2}\int_0^t W_a(t-t')F(t')dt'\|_{L_x^\infty L_T^2} + \|D^s\int_0^t W_a(t-t')F(t')dt'\|_{L_T^\infty L_x^2}\\ &\leq T^{a/2}\|D^{s-1/2+a/2}F\|_{L_x^{2/(2-a)}L_T^2}.\end{aligned} \tag{2.6}$$

(3) *Maximal function estimate*

$$\|W_a(t)f\|_{L_x^2 L_T^\infty} \leq c\,(1+T)^\rho\,\|f\|_{s,2} \tag{2.7}$$

where $\rho > 3/4$ and $s > (2+a)/4$.

Proof. For the proof of inequalities (2.5) and (2.7) see [29]. The inequality (2.6) follows by interpolation. □

Proposition 2.4.

(i) *Given $\phi \in L^\infty(\mathbb{R})$, with $\partial_x^\alpha \phi \in L^2(\mathbb{R})$ for $\alpha = 1, 2$, then for any $\theta \in (0,1)$*

$$\|[J^\theta;\phi]f\|_2 \leq c_{\theta,\phi}\,\|f\|_2. \tag{2.8}$$

(ii) *If* $\eta \in (0,1]$, *then*

$$\|J^\eta(fg) - fJ^\eta g\|_2 \le c\,\|\widehat{\partial_x f}\|_1\|g\|_2. \tag{2.9}$$

Proof. We first prove (2.8). Since

$$\begin{aligned}\big([J^\theta;\phi]f\big)^\wedge(\xi) &= (J^\theta(\phi f) - \phi J^\theta f)^\wedge(\xi)\\ &= \int \Big((1+\xi^2)^{\theta/2} - (1+\eta^2)^{\theta/2}\Big)\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)d\eta,\end{aligned}$$

the mean value theorem leads to

$$\big|\big([J^\theta;\phi]f\big)^\wedge(\xi)\big| \le c_\theta \int |\xi-\eta||\widehat{\phi}(\xi-\eta)||\widehat{f}(\eta)|\,d\eta = c_\theta\big(|\widehat{\partial_x\phi}| * |\widehat{f}\,|\big)(\xi).$$

Then by Young's inequality

$$\begin{aligned}\|[J^\theta;\phi]f\|_2 &\le c_\theta\||\widehat{\partial_x\phi}| * |\widehat{f}|\|_2 \le c_\theta\|\widehat{\partial_x\phi}\|_1\|\widehat{f}\,\|_2\\ &\le c_\theta\|\partial_x\phi\|_{1,2}\|f\|_2 \le c_{\theta,\phi}\|f\|_2.\end{aligned}$$

To show (2.9) we notice that

$$\begin{aligned}\|J^\eta(fg) - fJ^\eta g\|_2 &\le \Big\|\int |(1+|\xi|^2)^{\eta/2} - (1+|\zeta|^2)^{\eta/2}|\widehat{f}(\xi-\zeta)||\widehat{g}(\zeta)|d\zeta\Big\|_2\\ &\le \Big\|\int |\xi-\zeta||\widehat{f}(\xi-\zeta)||\widehat{g}(\zeta)|d\zeta\Big\|_2 \le \||\widehat{\partial_x f}| * |\widehat{g}|\,\|\\ &\le \|\widehat{\partial_x f}\|_1\|g\|_2.\end{aligned}$$

□

Proposition 2.5. *Given* $\phi \in L^\infty(\mathbb{R})$, *with* $\partial_x^\alpha\phi \in L^2(\mathbb{R})$ *for* $\alpha = 1, 2$, *then for any* $\theta \in (0,1)$

$$\|J^\theta(\phi f)\|_2 \le c_{\theta,\phi}\,\|J^\theta f\|_2. \tag{2.10}$$

Proof. We just need to write

$$J^\theta(\phi f) = [J^\theta, \phi]f + \phi\,J^\theta f$$

and use the hypotheses and Proposition 2.4 (2.8). □

We recall the following characterization of the $L^p_s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ spaces given in [46], (see [3] for the case $p = 2$).

Theorem 2.6 ([46])**.** *Let* $b \in (0,1)$ *and* $2n/(n+2b) < p < \infty$. *Then* $f \in L^p_b(\mathbb{R}^n)$ *if and only if*

$$(a)\quad f \in L^p(\mathbb{R}^n), \tag{2.11}$$

$$(b)\quad \mathcal{D}^b f(x) = \Big(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}}dy\Big)^{1/2} \in L^p(\mathbb{R}^n), \tag{2.12}$$

with

$$\|f\|_{b,p} \equiv \|(1-\Delta)^b f\|_p = \|J^s f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p. \tag{2.13}$$

For the proof of this theorem we refer the reader to [46]. One sees that from (2.11) for $p=2$ and $b\in(0,1)$ one has

$$\|\mathcal{D}^b(fg)\|_2 \le \|f\,\mathcal{D}^b g\|_2 + \|g\,\mathcal{D}^b f\|_2, \tag{2.14}$$

and

$$\|\mathcal{D}^b f\|_2 = c\,\|D^b f\|_2. \tag{2.15}$$

We shall use these estimates throughout in our arguments.

As an application of Theorem 2.6 we also have the following estimate:

Proposition 2.7. *Let $b\in(0,1)$. For any $t>0$*

$$\mathcal{D}^b(e^{-it|x|^{1+a}x}) \le c(|t|^{b/(2+a)} + |t|^b|x|^{(1+a)b}). \tag{2.16}$$

For the proof of Proposition 2.7 we refer to [41].

Also as consequence of the estimate (2.14) one has the following interpolation inequality.

Lemma 2.8. *Let α, $b>0$. Assume that $J^\alpha f=(1-\Delta)^{\alpha/2}f\in L^2(\mathbb{R})$ and $\langle x\rangle^b f=(1+|x|^2)^{b/2}f\in L^2(\mathbb{R})$. Then for any $\theta\in(0,1)$*

$$\|J^{\theta\alpha}(\langle x\rangle^{(1-\theta)b}f)\|_2 \le c\|\langle x\rangle^b f\|_2^{1-\theta}\,\|J^\alpha f\|_2^{\theta}. \tag{2.17}$$

Moreover, the inequality (2.17) is still valid with $\langle x\rangle_N^\theta$ in (3.2) instead of $\langle x\rangle$ with a constant c independent of N.

We refer to [14] for the proof of Lemma 2.8.

As a further direct consequence of Theorem 2.6 we deduce the following result. It will be useful in several of our arguments.

Proposition 2.9. *For any $\theta\in(0,1)$ and $\alpha>0$,*

$$\mathcal{D}^\theta\Big(|\xi|^\alpha\chi(\xi)\Big)(\eta) \sim \begin{cases} c\,|\eta|^{\alpha-\theta}+c_1, & \alpha\neq\theta, \quad |\eta|\ll 1,\\[2ex] c\,(-\ln|\eta|)^{\frac{1}{2}}, & \alpha=\theta, \quad |\eta|\ll 1,\\[2ex] \dfrac{c}{|\eta|^{1/2+\theta}}, & \qquad\qquad |\eta|\gg 1,\end{cases} \tag{2.18}$$

with $\mathcal{D}^\theta\Big(|\xi|^\alpha\chi(\xi)\Big)(\cdot)$ continuous in $\eta\in\mathbb{R}-\{0\}$.

In particular, one has that

$$\mathcal{D}^\theta\Big(|\xi|^\alpha\chi(\xi)\Big)\in L^2(\mathbb{R}) \quad \textit{if and only if} \quad \theta<\alpha+1/2.$$

Similar result holds for $\mathcal{D}^\theta\Big(|\xi|^\alpha sgn(\xi)\chi(\xi)\Big)(\eta)$.

Proof. We restrict ourselves to the case $\alpha\neq\theta$. First we consider the case $\mathcal{D}^\theta\Big(|\xi|^\alpha\chi(\xi)\Big)$. It is easy to see that for $\eta\neq0$, $\mathcal{D}^\theta\Big(|\xi|^\alpha\chi(\xi)\Big)(\eta)$ is continuous in $\epsilon<|\eta|<1/\epsilon$ for any $\epsilon>0$.

Let us consider $|\eta| < 1/2$. Without loss of generality we assume that $\eta \in (0, 1/2)$. Thus

$$\begin{aligned}\Big(\mathcal{D}^\theta\Big(|\xi|^\alpha\chi(\xi)\Big)(\eta)\Big)^2 &= \int \frac{\Big(|\xi+\eta|^\alpha\chi(\xi+\eta) - |\eta|^\alpha\chi(\eta)\Big)^2}{|\xi|^{1+2\theta}}\,d\xi \\ &\le c\int_{-\eta/2}^{\eta/2} \frac{(|\xi+\eta|^\alpha - |\eta|^\alpha)^2}{|\xi|^{1+2\theta}}\,d\xi + c\int_{\eta/2}^{2} \frac{(|\xi+\eta|^\alpha + |\eta|^\alpha)^2}{|\xi|^{1+2\theta}}\,d\xi \\ &\equiv A_1 + A_2.\end{aligned}$$

To bound A_1 we use that

$$|(\xi+\eta)^\alpha - \eta^\alpha| \le |\xi|\frac{c_\alpha}{\eta^{1-\alpha}} \quad \forall \xi: \;\; -\eta/2 < \xi < \eta/2.$$

Thus

$$A_1 \le c\int_{-\eta/2}^{\eta/2} \frac{|\xi|^2}{\eta^{2(1-\alpha)}|\xi|^{1+2\theta}}\,d\xi \le c\eta^{2(\alpha-\theta)}.$$

For A_2 we have that

$$|(\xi+\eta)^\alpha + \eta^\alpha| \le c\xi^\alpha \;\text{ for }\; \xi: \;\; \eta/2 \le \xi \le 2.$$

So

$$A_2 \le c\int_{\eta/2}^{2} \frac{\xi^{2\alpha}}{\xi^{1+2\theta}}\,d\xi \le c\,\eta^{2(\alpha-\theta)} + c_1, \quad \text{if} \quad \alpha \ne \theta,$$

where $c_1 = 0$ if $\theta > \alpha$.

Let us consider the case $|\eta| > 100$. Without loss of generality we assume $\eta > 100$. Then

$$\begin{aligned}\Big(\mathcal{D}^\theta(|\xi|^\alpha\chi(\xi))(\eta)\Big)^2 &= \int \frac{(|\xi+\eta|^\alpha\chi(\xi+\eta))^2}{|\xi|^{1+2\theta}}d\xi \\ &\le c\int_{-2-\eta}^{2-\eta} \frac{d\xi}{|\xi|^{1+2\theta}} \le \frac{c}{\eta^{1+2\theta}}.\end{aligned}$$

Finally we consider $\mathcal{D}^\theta(|\xi|^\alpha \mathrm{sgn}(\xi)\chi(\xi))(\eta)$. We notice that the previous computation for $|\eta| > 100$ is similar, so we just need to consider the case $|\eta| < 1$. Assuming without loss of generality that $0 < \eta < 1$.

Since

$$\mathcal{D}^\theta(|\xi|^\alpha \mathrm{sgn}(\xi)\chi(\xi))(\xi) =$$
$$\int_{-2-\eta}^{2-\eta} \frac{\Big(|\xi+\eta|^\alpha \mathrm{sgn}(\xi+\eta)\chi(\xi+\eta) - |\eta|^\alpha \mathrm{sgn}(\eta)\chi(\eta)\Big)^2}{|\xi|^{1+2\theta}}\,d\xi.$$

The bound for $\xi+\eta>0$ is similar to that given before, so we assume $\xi+\eta<0$ ($\xi<-\eta$) and consider

$$\int_{-2-\eta}^{-\eta}\frac{(|\xi+\eta|^{\alpha}+|\eta|^{\alpha})^2}{|\xi|^{1+2\theta}}\,d\xi=\int_{-2\eta}^{-\eta}\cdots+\int_{-2-\eta}^{-2\eta}\cdots=\tilde{A}_1+\tilde{A}_2.$$

A familiar argument shows that

$$\tilde{A}_1\le c\frac{\eta^{2\alpha}}{\eta^{1+2\theta}}\,\eta=\eta^{2(\alpha-\theta)},$$

and

$$\tilde{A}_2\le\int_{-2-\eta}^{-2\eta}\frac{|\xi|^{2\alpha}}{|\xi|^{1+2\theta}}\,d\xi=\eta^{2(\alpha-\theta)}+c,$$

with $c=0$ if $\theta>\alpha$.

□

3. Proof of Theorem 1.1

Proposition 3.1. *If $u_0\in H^{1+a}(\mathbb{R})$ and $|x|^{\theta}u_0\in L^2(\mathbb{R})$, $\theta\in(0,1)$, then the solution u of the IVP (1.1) satisfies*

$$u\in C([0,T]:H^{1+a}(\mathbb{R})\cap L^2(|x|^{2\theta})\equiv Z_{1+a,\theta}). \tag{3.1}$$

where T is given by the local theory.

Proof of Proposition 3.1. We use the differential equation and the local theory such that $u\in C([0,T]:H^{1+a}(\mathbb{R}))$ exists and is the limit of smooth solutions.

We define for $\theta\in(0,1)$

$$\langle x\rangle_N^{\theta}=\begin{cases}\langle x\rangle^{\theta}=(1+x^2)^{\theta/2}, & \text{if } |x|\le N,\\ (2N)^{\theta}, & \text{if } |x|\ge 3N,\end{cases} \tag{3.2}$$

with $\langle x\rangle_N^{\theta}$ smooth, even, nondecreasing for $|x|\ge 0$.

We multiply the equation in (1.1) by $\langle x\rangle_N^{2\theta}u$ and integrate in the x-variable to get

$$\frac{1}{2}\frac{d}{dt}\int(\langle x\rangle_N^{\theta}u)^2\,dx+\underbrace{\int\langle x\rangle_N^{\theta}D^{1+a}\partial_xu\langle x\rangle_N^{\theta}u\,dx}_{A_1}+\underbrace{\int\langle x\rangle_N^{2\theta}u^2\partial_xu\,dx}_{A_2}=0.$$

To estimate A_2 we integrate by parts to get

$$A_2=\frac{1}{3}\int\langle x\rangle_N^{2\theta}\,\partial_x(u^3)\,dx=-\frac{1}{3}\int\partial_x(\langle x\rangle_N^{2\theta})\,u^3\,dx. \tag{3.3}$$

We shall use that

$$\partial_x(\langle x\rangle_N^{2\theta})\le c_{\theta}\langle x\rangle_N^{2\theta-1}\le c_{\theta}\,\langle x\rangle_N^{\theta}, \tag{3.4}$$

since $\theta \in (0,1)$. Notice that c_θ is independent of N. Thus

$$A_2 \le \|\langle x\rangle_N^\theta\, u\|_2 \|u\|_2 \|u\|_\infty. \tag{3.5}$$

Now we turn to A_1. We write

$$\langle x\rangle_N^\theta\, D^{1+a}\partial_x u = D^a(\langle x\rangle_N^\theta\, D\partial_x u) - [D^a; \langle x\rangle_N^\theta]\, D\partial_x u \equiv B_1 + B_2. \tag{3.6}$$

Using Proposition 2.2

$$\|[D^a; \langle x\rangle_N^\theta]\, D^{1-a} f\|_2 \le c\, \|J^\delta \partial_x \langle x\rangle_N^\theta\|_q \|f\|_2, \tag{3.7}$$

with $\delta > 1/q$. Thus

$$\|B_2\|_2 = \|[D^a; \langle x\rangle_N^\theta]\, D\partial_x u\|_2 \le c_\theta\, \|D^a \partial_x u\|_2, \tag{3.8}$$

with c_θ independent of N since $\partial_x \langle x\rangle_N^\theta$ is bounded independent of N. Here we are assuming that $\theta < 1$ such that $J^\delta \partial_x \langle x\rangle_N^\theta \in L^q$ for appropriate values of δ, q with $\delta > 1/q$. When $\theta = 1$, $\|J^\delta \partial_x \langle x\rangle_N^1\|_q$ is not bounded uniformly on N by a constant and we cannot do this.

Also observe that the bound $\|D^a \partial_x u\|_2$ is natural from the fact that the operator $\Gamma = x - (2+a)tD^{1+a}$ which commutes with $\partial_t + D^{1+a}\partial_x$.

Hence it remains to consider B_1 in (3.6). We write

$$\begin{aligned} B_1 = D^a(\langle x\rangle_N^\theta D\partial_x u) &= D^a \partial_x(\langle x\rangle_N^\theta\, Du) - D^a((\partial_x \langle x\rangle_N^\theta) Du) \\ &\equiv C_1 + C_2, \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} \|C_2\|_2 &= \|D^a((\partial_x \langle x\rangle_N^\theta) Du)\|_2 \\ &\le \|[D^a; \partial_x \langle x\rangle_N^\theta] Du\|_2 + \|\partial_x \langle x\rangle_N^\theta\, D^{1+a} u\|_2 \\ &\underset{(3)}{\le} c\, \|J^\delta \partial_x^2 \langle x\rangle_N^\theta\|_q \|D^{1-a} u\|_2 + c\|D^{1+a} u\|_2 \\ &\le c\, \|J^{1+a} u\|_2, \quad c \text{ independent of } N, \end{aligned} \tag{3.10}$$

where in (3) we have used again (3.7) (Proposition 2.2).

To estimate C_1 in (3.9) we write

$$\begin{aligned} C_1 = D^a \partial_x(\langle x\rangle_N^\theta\, Du) &= D^a \partial_x D(\langle x\rangle_N^\theta\, u) - D^a \partial_x\, [D; \langle x\rangle_N^\theta] u \\ &\equiv K_1 + K_2. \end{aligned} \tag{3.11}$$

Since $D = \mathcal{H}\partial_x$ one has

$$\begin{aligned} [D; \langle x\rangle_N^\theta] f &= D(\langle x\rangle_N^\theta\, f) - \langle x\rangle_N^\theta\, Df \\ &= \mathcal{H}\partial_x(\langle x\rangle_N^\theta\, f) - \langle x\rangle_N^\theta\, \mathcal{H}\partial_x f \\ &= \mathcal{H}((\partial_x \langle x\rangle_N^\theta) f) - [\mathcal{H}; \langle x\rangle_N^\theta] \partial_x f. \end{aligned} \tag{3.12}$$

Therefore

$$K_2 = -D^a \partial_x \mathcal{H}((\partial_x \langle x\rangle_N^\theta)\, u) + D^a \partial_x [\mathcal{H}; \langle x\rangle_N^\theta] \partial_x u \equiv Q_1 + Q_2. \tag{3.13}$$

To bound Q_2 we use the commutator estimate in Proposition 2.1

$$\|\partial^j [\mathcal{H}; a] \partial^m f\|_2 \le c\, \|\partial^{j+m} a\|_\infty \|f\|_2, \tag{3.14}$$

and interpolation ($\|D^a f\|_2 \le \|f\|_2^{1-a}\|Df\|_2^a$) to get

$$\begin{aligned}\|Q_2\|_2 &= \|D^a\partial_x[\mathcal{H};\langle x\rangle_N^\theta]\partial_x u\|_2\\ &\le \|D\partial_x[\mathcal{H};\langle x\rangle_N^\theta]\partial_x u\|_2^a\|\partial_x[\mathcal{H};\langle x\rangle_N^\theta]\partial_x u\|_2^{1-a}\\ &\le \|\partial_x^2[\mathcal{H};\langle x\rangle_N^\theta]\partial_x u\|_2^a\,\|\partial_x[\mathcal{H};\langle x\rangle_N^\theta]\partial_x u\|_2^{1-a}\\ &\le c\left(\|\partial_x^3\langle x\rangle_N^\theta\|_\infty + \|\partial_x^2\langle x\rangle_N^\theta\|_\infty\right)\|u\|_2.\end{aligned} \tag{3.15}$$

Using previous arguments we also have

$$\begin{aligned}\|Q_1\|_2 &= \|\mathcal{H}D^a\partial_x((\partial_x\langle x\rangle_N^\theta)\,u)\|_2\\ &\le \|D^a((\partial_x^2\langle x\rangle_N^\theta)\,u)\|_2 + \|D^a((\partial_x\langle x\rangle_N^\theta)\,\partial_x u)\|_2\\ &\le \left(\|D^a\partial_x^2\langle x\rangle_N^\theta\|_2\|u\|_\infty + \|\partial_x^2\langle x\rangle_N^\theta\|_\infty\|D^a u\|_2\right)\\ &\quad + \|[D^a;\partial_x\langle x\rangle_N^\theta]\partial_x u\|_2 + \|(\partial_x\langle x\rangle_N^\theta)D^a\partial_x u\|_2\\ &\le c\,\|J^{1+a}u\|_2.\end{aligned} \tag{3.16}$$

Finally, we turn to the term K_1 in (3.11), Parseval's identity yields

$$\int D^a\partial_x D(\langle x\rangle_N^\theta\,u)\,\langle x\rangle_N^\theta\,u = \int \partial_x D^{(1+a)/2}(\langle x\rangle_N^\theta\,u)D^{(1+a)/2}(\langle x\rangle_N^\theta\,u) \equiv 0.$$

Since from the local existence theory [18] we know that

$$\sup_{[0,T]}\|u(t)\|_{1+a,2} = \sup_{[0,T]}\|u(t)\|_{H^{1+a}} \le c, \tag{3.17}$$

combining the above estimate and taking limit as $N\to\infty$ we obtain

$$\sup_{[0,T]}\|\langle x\rangle^\theta u(t)\|_2 \le \tilde{c}_\theta \quad \text{for all } \theta\in[0,1), \tag{3.18}$$

which yields the result.

We observe that the argument above shows that if in addition to $u_0 \in H^{1+a}(\mathbb{R})\cap L^2(|x|^{2\theta})$, $\theta\in(0,1)$, $u_0\in H^{1+a+\alpha}(\mathbb{R})$ with $D^\alpha u_0 \in L^2(|x|^{2\theta})$, for $\alpha>0$, then $D^\alpha u\in C([0,T]:H^{1+a}(\mathbb{R})\cap L^2(|x|^{2\theta}))$, $\alpha>0$.

□

3.1. **Case $s=1+a$, $r=1$.** Let $u_0\in H^{1+a}(\mathbb{R})\cap L^2(|x|^2\,dx)$. We observe that the persistence result in this case was already proved by Colliander, Kenig and Stafillani [9]. However, by convenience we present a different proof.

First notice that

$$\begin{aligned}x\,D^{1+a}\partial_x f &= (x\,D^{1+a}\partial_x f)^{\wedge\vee} = (i\partial_\xi\,(|\xi|^{1+a}\,i\xi\widehat{f}\,))^\vee\\ &= \left(-(2+a)|\xi|^{1+a}\widehat{f} + |\xi|^{1+a}i\xi i\partial_\xi\widehat{f}\,\right)^\vee\\ &= -(2+a)\,D_x^{1+a}f + D_x^{1+a}\partial_x(xf).\end{aligned} \tag{3.19}$$

Hence if u satisfies

$$\partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \tag{3.20}$$

then

$$\partial_t(xu) + D^{1+a}\partial_x(xu) - (2+a)D^{1+a}u + x\,u\partial_x u = 0. \tag{3.21}$$

In this case the standard energy estimate argument shows that

$$\frac{d}{dt}\|xu(t)\|_2^2 \le c_a\|D^{1+a}u(t)\|_2\|xu(t)\|_2 + \|u\|_\infty\|u(t)\|_2\|xu(t)\|_2. \tag{3.22}$$

Since

$$\sup_{[0,T]}\|u(t)\|_{1+a,2} \le c(a;T;\|u_0\|_{1+a}), \tag{3.23}$$

one has from (3.22) that

$$\sup_{[0,T]}\|xu(t)\|_{1+a,2} \le c(a;T;\|u_0\|_{1+a,2};\|xu_0\|_2). \tag{3.24}$$

Remark 3.2. *By taking derivatives D^α in the equation* (3.20) *and repeating the above argument we have that if in addition to $u_0 \in H^{1+a}(\mathbb{R})\cap L^2(|x|^2\,dx)$ one has*

$$D^\alpha u_0 \in H^{1+a}(\mathbb{R}),\ \ xD^\alpha u_0 \in L^2(\mathbb{R}),\ \ \textit{for some}\ \ \alpha > 0, \tag{3.25}$$

then

$$\sup_{[0,T]}\|xD^\alpha u(t)\|_2 \le c(T;a;\|u_0\|_{1+a+\alpha,2};\|xu_0\|_2;\|xD^\alpha u_0\|_2). \tag{3.26}$$

3.2. **Case** $s = 2(1+a)$**,** $r \in (1,2)$**.** Let $u_0 \in H^{2(1+a)}\cap L^2(|x|^{2r}\,dx)$, $r = 1+\theta$, $\theta \in (0,1)$.

Reapplying the method for the weight $\langle x\rangle_N^{2\theta}$, $\theta \in (0,1)$, multiplying the equation (3.21) by $\langle x\rangle_N^{2\theta}xu$ and integrating the result, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\big(\langle x\rangle_N^\theta xu\big)^2\,dx + \int\langle x\rangle_N^\theta D^{1+a}\partial_x(xu)\langle x\rangle_N^\theta xu\\ &-(2+a)\int\langle x\rangle_N^\theta D^{1+a}u\langle x\rangle_N^\theta xu + \int\langle x\rangle_N^\theta xu\partial_x\langle x\rangle_N^\theta xu = 0.\end{aligned} \tag{3.27}$$

From the previous analysis and Proposition 3.1, we only need to handle the last two terms in (3.27).

First we notice that

$$\int\langle x\rangle_N^\theta\,D_x^{1+a}u\,\langle x\rangle_N^\theta x\,u\,dx \le \|\langle x\rangle_N^\theta D^{1+a}u\|_2\|\langle x\rangle_N^\theta\,xu\|_2. \tag{3.28}$$

Next

$$\langle x\rangle_N^\theta D^{1+a}f = \underbrace{-[D^a;\langle x\rangle_N^\theta]D^{1-a}D^af}_{C_1} + \underbrace{D^a\langle x\rangle_N^\theta\,Df}_{C_2},$$

and by the commutator estimate

$$\|C_1\|_2 \le \|D^af\|_2 \quad \text{uniformly in } N \text{ for } \ \theta \in (0,1).$$

On the other hand,

$$C_2 = D^a\langle x\rangle_N^\theta\,\partial_x\mathcal{H}f = D^a\partial_x(\langle x\rangle_N^\theta\,\mathcal{H}f) - D^a(\partial_x\langle x\rangle_N^\theta\,\mathcal{H}f) \equiv K_1 + K_2,$$

where

$$\begin{aligned}\|K_2\|_2 &= \|D^a(\partial_x\langle x\rangle_N^\theta\,\mathcal{H}f)\|_2 \le \|J^a(\partial_x\langle x\rangle_N^\theta\,\mathcal{H}f)\|_2\\ &\le \|[J^a;\partial_x\langle x\rangle_N^\theta]\mathcal{H}f\|_2 + \|\partial_x\langle x\rangle_N^\theta\,J^a\mathcal{H}f\|_2\\ &\le \|f\|_2 + \|J^af\|_2 \le c\|J^af\|_2.\end{aligned}$$

So we consider K_1.

$$K_1 = D^a\partial_x(\langle x\rangle_N^\theta\,\mathcal{H}f) = D^a\mathcal{H}\partial_x(\langle x\rangle_N^\theta f) + D^a\partial_x[\langle x\rangle_N^\theta;\mathcal{H}]f = K_{1,1}+K_{1,2}.$$

For $K_{1,2}$ we write

$$\begin{aligned}\|K_{1,2}\|_2 &= \|D^a\partial_x[\langle x\rangle_N^\theta;\mathcal{H}]f\|_2\\ &\le \|D\partial_x[\langle x\rangle_N^\theta;\mathcal{H}]f\|_2^a\|\partial_x[\langle x\rangle_N^\theta;\mathcal{H}]f\|_2^{1-a}\\ &\le \|\partial_x^2\langle x\rangle_N^\theta\|_\infty^a\|f\|_2^a\|\partial_x\langle x\rangle_N^\theta\|_\infty^{1-a}\|f\|_2^{1-a} \le c_\theta\|f\|_2.\end{aligned}$$

Finally,

$$\begin{aligned}\|K_{1,1}\|_2 &= \|D^{1+a}(\langle x\rangle_N^\theta f)\|_2 \le \|J^{1+a}(\langle x\rangle_N^\theta f)\|_2\\ &\le \|J^{(1+a)(1+\theta)}f\|_2^{1/1+\theta}\|\langle x\rangle_N^{1+\theta}f\|_2^{\theta/1+\theta}\\ &\le \|J^{2(1+a)}f\| + \|\langle x\rangle_N^\theta\,xf\|_2 + \|f\|_2 + \|xf\|_2.\end{aligned}$$

which completes the estimate for (3.28) if $s \ge 2(1+a)$.

For the nonlinear term coming from (3.21) we have that

$$\int \langle x\rangle_N^\theta\,x\,u\partial_xu\langle x\rangle_N^\theta\,xu \le \|\langle x\rangle^\theta xu\|_2^2\|\partial_xu\|_\infty \le \|\langle x\rangle^\theta xu\|_2^2\|u\|_{2(1+a),2}.$$

This proves that if $u_0 \in H^{2(1+a)}(\mathbb{R})$ and $|x|^{1+\theta}\,u \in L^2(\mathbb{R})$, $\theta\in(0,1)$, then persistence holds in

$$H^{2(1+a)} \cap L^2(|x|^{2(1+\theta)}\,dx).$$

Remark 3.3. *The above argument also shows that if in addition to* $u_0 \in H^{2(1+a)}(\mathbb{R})\cap L^2(|x|^{2(1+\theta)}\,dx)$ *one has* $D^\alpha u_0 \in H^{2(1+a)}(\mathbb{R})\cap L^2(|x|^{2(1+\theta)}\,dx)$, $\alpha>0$, *then* $D^\alpha u \in C([0,T] : H^{2(1+a)}(\mathbb{R})\cap L^2(|x|^{2(1+\theta)}\,dx))$.

3.3. **Case** $s = 2(1+a)$**,** $r = 2$**.** We observe that an argument similar to that in [9] also gives the persistence of the solution to the IVP (1.1) in

$$u_0 \in H^{2(1+a)}(\mathbb{R}) \cap L^2(|x|^4\,dx). \tag{3.29}$$

In fact, using for

$$\begin{aligned}&x^2D^{1+a}\partial_xf = (x^2D^{1+a}\partial_xf)^{\wedge\vee} = (-\partial_\xi^2(|\xi|^{1+a}i\xi\widehat{f}))^\vee\\ &= (-(2+a)(1+a)|\xi|^a\,i\mathrm{sgn}(\xi)\,\widehat{f} - 2(2+a)|\xi|^{1+a}\widehat{xf} + |\xi|^{1+a}i\xi\widehat{(x^2f)})^\vee\\ &= -(2+a)(1+a)D^a\mathcal{H}f - 2(2+a)D^{1+a}(xf) + D^{1+a}\partial_x(x^2f).\end{aligned}$$

We get the equation for x^2u

$$\partial_t(x^2u)+D^{1+a}\partial_x(x^2u)-2(2+a)D^{1+a}(xu)-(1+a)(2+a)D^a\mathcal{H}u-x^2u\partial_xu = 0,$$

for which a familiar argument also shows that

$$\sup_{[0,T]} \|x^2 u(t)\|_2 \le c(T; a; \|x^2 u_0\|_2; \|u_0\|_{2(1+a),2}). \tag{3.30}$$

As before we notice that if in addition to (3.29) one has that

$$D^\alpha u_0 \in H^{2(1+a)}(\mathbb{R}) \cap L^2(|x|^4\, dx) \equiv Z_{2(1+a),2}, \quad \alpha > 0, \tag{3.31}$$

then

$$\sup_{[0,T]} \|x^2 D^\alpha u(t)\|_2 \le c(T; a; \|x^2 u_0\|_2; \|x^2 D^\alpha u_0\|_2; \|u_0\|_{2(1+a)+\alpha,2}). \tag{3.32}$$

3.4. **Case** $s = s_r \equiv [(r+1)^-](1+a)$**,** $r \in (2, 5/2+a)$**.** We observe that the equation for xu

$$\partial_t(xu) + D^{1+a}\partial_x(xu) - (2+a)D^{1+a}u + (xu)\partial_x u = 0, \tag{3.33}$$

and the previous argument for $|x|^l$, with $l \in (0,2)$ will provide the result if the contribution for the extra term in (3.33) $c_a\, D^{1+a}u$ can be handled, i.e. if for $l = 1+\theta$

$$\sup_{[0,T]} \||x|^{1+\theta} D^{1+a} u(t)\|_2 \le M. \tag{3.34}$$

We claim that if $\theta \in (0, a+1/2)$, $u_0 \in H^{s_r}(\mathbb{R}) \cap L^2(|x|^{2+\theta})$, $s = r[(r+1)^-](1+a)$, we obtain (3.34) and hence the desired result. From Remark 3.2 it will suffice to have

$$\||x|^{1+\theta} D^{1+a} u_0\|_2 \le c(\||x|^{2+\theta} u_0\|_2; \|D^{(1+a)(1+\theta)} u_0\|_2). \tag{3.35}$$

with c independent of N.

Proof of (3.35). Using the identity

$$x\, D^{1+a} f = D^{1+a}(xf) + (1+a) D^a \mathcal{H} f, \tag{3.36}$$

we have to control the L^2-norms of the terms

$$K_1 = |x|^\theta D^a \mathcal{H} u_0 \quad \text{and} \quad K_2 = |x|^\theta D^{1+a}(xu_0). \tag{3.37}$$

We can estimate K_1 as

$$\begin{aligned} \|K_1\|_2 \le & \|D_\xi^\theta(|\xi|^a \mathrm{sgn}(\xi)\chi(\xi)\widehat{u}_0(\xi))\|_2 \\ & + \|D_\xi^\theta(|\xi|^a \mathrm{sgn}(\xi)(1-\chi(\xi))\widehat{u}_0(\xi))\|_2 = K_{1,1} + K_{1,2}. \end{aligned} \tag{3.38}$$

where

$$K_{1,1} \le \|D_\xi^\theta(|\xi|^a \mathrm{sgn}(\xi)\chi(\xi))\widehat{u}_0(0))\|_2 + \|D_\xi^\theta(\underbrace{|\xi|^a \mathrm{sgn}(\xi)\chi(\xi)(\widehat{u}_0(\xi) - \widehat{u}_0(0))}_{L(\xi)})\|_2$$

$$= \tilde{K}_{1,1} + \|D_\xi^\theta L\|_2.$$

Next we have that

$$\|D_\xi^\theta L\|_2 \le \|L(\xi)\|_2^{1-\theta} \|\partial_\xi L(\xi)\|_2^\theta.$$

Then

$$\begin{aligned}\|\partial_\xi L\| &\le c\big(\|\partial_\xi \widehat{u}_0\|_\infty + \|D_\xi^\theta \widehat{u}_0\|_2\big)\\ &\le c\big(\|\widehat{xu_0}\|_\infty + \|\widehat{|x|^\theta u_0}\|_2\big)\\ &\le c\big(\|xu_0\|_{1,2} + \||x|^\theta u_0\|_2\big) \le c(\|\langle x\rangle u_0\|_2 + \|\langle x\rangle \partial_x u_0\|_2\big),\end{aligned}$$

and

$$\tilde{K}_{1,1} < \infty.$$

(by Proposition 2.9). □

Consider now $\|K_2\|_2$, we introduce the cutoff function χ to obtain

$$\begin{aligned}\|K_2\|_2 &= \||x|^\theta D^{1+a}(xu_0)\|_2\\ &= \|D_\xi^\theta(|\xi|^{1+a}(\widehat{xu_0}))\|_2\\ &\le \|D_\xi^\theta(|\xi|^{1+a}\chi(\xi)(\widehat{xu_0}))\|_2 + \|D_\xi^\theta(|\xi|^{1+a}(1-\chi(\xi))(\widehat{xu_0}))\|_2\\ &\le K_{2,1} + K_{2,2}.\end{aligned}$$

Using Stein's derivative, Leibniz rule (2.14) and Proposition 2.9, we estimate $K_{2,1}$ as

$$\begin{aligned}K_{2,1} &= \|D_\xi^\theta(|\xi|^{1+a}\chi(\xi)(\widehat{xu_0}))\|_2\\ &\le c\|\mathcal{D}_\xi^\theta(|\xi|^{1+a}\chi(\xi))\|_\infty\|\widehat{xu_0}\|_2 + \||\xi|^{1+a}\chi(\xi)\|_\infty\|\mathcal{D}_\xi^\theta(\widehat{xu_0})\|_2\\ &\le c_a\|xu_0\|_2 + c_a\|D_\xi^\theta(\widehat{xu_0})\|_2\\ &\le c_a\|\langle x\rangle^{1+\theta}u_0\|_2.\end{aligned}$$

On the other hand, we notice that $\varphi_1(\xi) = \frac{|\xi|^{1+a}(1-\chi(\xi))}{\langle\xi\rangle^{1+a}}$ and $\varphi_2(\xi) = \frac{\xi}{\langle\xi\rangle}$ satisfy the hypothesis in Proposition 2.5, and hence it follows that

$$\begin{aligned}K_{2,2} &= \|D_\xi^\theta(|\xi|^{1+a}(1-\chi(\xi))(\widehat{xu_0}))\|_2\\ &\le \|J_\xi^\theta\Big(\frac{|\xi|^{1+a}(1-\chi(\xi))}{\langle\xi\rangle^{1+a}}\langle\xi\rangle^{1+a}\widehat{xu_0}\Big)\|_2\\ &\le c\|J_\xi^\theta(\langle\xi\rangle^{1+a}\partial_\xi\widehat{u}_0)\|_2\\ &\le c\|J_\xi^\theta\partial_\xi(\langle\xi\rangle^{1+a}\widehat{u}_0)\|_2 + \|J_\xi^\theta(\partial_\xi(\langle\xi\rangle^{1+a})\widehat{u}_0)\|_2\\ &\le c\,\|J_\xi^{1+\theta}(\langle\xi\rangle^{1+a}\widehat{u}_0)\|_2 + \|J_\xi^\theta(\frac{\xi}{\langle\xi\rangle}\langle\xi\rangle^a\widehat{u}_0)\|_2\\ &\le c\,\|\langle x\rangle^{1+\theta}J^{1+a}u_0\|_2 + \|\langle x\rangle^\theta J^a u_0\|_2\\ &\le c_a\|\langle x\rangle^{2+\theta}u_0\|_2^{1-1/(2+\theta)}\|J^{(2+\theta)(1+a)}u_0\|_2^{1/(2+\theta)}\\ &\qquad + c_a\|\langle x\rangle^{2+\theta}u_0\|_2^{\theta/(2+\theta)}\|J^{(2+\theta)\frac{a}{2}}u_0\|_2^{2/(2+\theta)},\end{aligned}$$

where in the last inequality we have applied complex interpolation ($u_0 \in H^{s_r}(\mathbb{R}) \cap L^2(|x|^{2r}\,dx)$ with $r = 2+\theta \in (2, 5/2+a)$).

3.5. **Persistence property in $\dot{Z}_{s_r,r}$ with $s_r = [(r+1)^-](1+a)$ and $r \in [5/2+a, 7/2+a)$.** To simplify the exposition we assume $r \in [3, 7/2+a)$.

We have established persistence in

$$Z_{s_r,r} \quad \text{with} \quad r \in [2, 5/2+a), \tag{3.39}$$

for the equation (we are assuming $a \in (0,1/2)$ for simplicity)

$$\partial_t u + D^{1+a}\partial_x u + u\partial_x u = 0, \tag{3.40}$$

and that xu satisfies the equation

$$\partial_t(xu) + D^{1+a}\partial_x(xu) - (2+a)D^{1+a}u + xu\partial_x u = 0. \tag{3.41}$$

Thus if we prove that $u_0 \in \dot{Z}_{s_r,r}$ with $r \in [5/2+a, 7/2+a)$, then the solution u satisfies

$$|x|^\alpha D^{1+a}u \in L^2(\mathbb{R}) \qquad \text{for} \qquad \alpha = r-1 \in [2, 5/2+a), \tag{3.42}$$

the argument for proving the result in $Z_{s_r,r}$ as in (3.39) will provide the result.

Since $\widehat{u}(0,t) = \widehat{u}_0(0) = \int u_0(x)\,dx$ is preserved by the solution flow, it will suffice to show that if $u_0 \in \dot{Z}_{r(1+a),r}$, $r \in [5/2+a, 7/2+a)$, then $|x|^\alpha D^{1+a}u_0 \in L^2(\mathbb{R})$ for $\alpha = r-1$.

Since $\alpha \in [2, 5/2+a)$ with $a \in (0,1/2)$ write $\alpha = 2+\theta$, $\theta \in (0,1)$, and use that

$$x^2 D^{1+a}f = -(1+a)aD^{a-1}f - 2(1+a)D^a\mathcal{H}(xf) + D^{1+a}(x^2 f). \tag{3.43}$$

Thus

$$\begin{aligned} |x|^{2+\theta}D^{1+a}u_0 = & -(1+a)a|x|^\theta D^{a-1}u_0 - 2(1+a)D^a\mathcal{H}(xu_0) \\ & + |x|^\theta D^{1+a}(x^2u_0) \\ \equiv & \ G_1 + G_2 + G_3. \end{aligned} \tag{3.44}$$

First we write G_1 using that

$$\begin{aligned} \||x|^\theta D^{a-1}u_0\|_2 &= \|D^\theta_\xi(|\xi|^{a-1}\widehat{u}_0)\|_2 \\ &\leq \|D^\theta_\xi(|\xi|^{a-1}\chi(\xi)\widehat{u}_0)\|_2 + \|D^\theta_\xi(|\xi|^{a-1}(1-\chi(\xi))\widehat{u}_0)\|_2. \end{aligned}$$

Now using that $\widehat{u}_0(0) = 0$, the Taylor expansion allows to write

$$\widehat{u}_0(\xi) = \xi\partial_\xi\widehat{u}_0(0) + \int_0^\xi (\xi-\zeta)\partial_\xi^2\widehat{u}(\zeta)d\zeta. \tag{3.45}$$

So

$$\begin{aligned} |\xi|^{a-1}\chi(\xi)\widehat{u}_0(\xi) &= |\xi|^a\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(\xi) + \chi(\xi)|\xi|^{a-1}\int_0^\xi (\xi-\zeta)\partial_\xi^2\widehat{u}(\zeta)d\zeta \\ &\equiv |\xi|^a\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(\xi) + Q_1. \end{aligned}$$

and by Proposition 2.9

$$\begin{aligned} D^\theta_\xi\big(|\xi|^a\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(\xi)\big) \in L^2(\mathbb{R}) \quad & \text{if and only if} \quad \theta < a+1/2 \\ & \text{if and only if} \quad \alpha = 2+\theta < 5/2+a, \end{aligned}$$

which holds from our hypotheses, and

$$\|D^\theta_\xi Q_1\|_2 \le \|Q_1\|_2^{1-\theta}\|\partial_\xi Q_1\|_2^\theta.$$

$$\begin{aligned}\|Q_1\|_2 &\le \|\chi(\xi)|\xi|^{a+1}\|\partial_\xi^2\widehat{u}_0\|_{L^\infty_{\{|\xi|<1\}}}\|_2 \le \|\widehat{x^2u_0}\|_\infty \le \|x^2u_0\|_{1,2}\\ &\le \|\langle x\rangle^2u_0\|_2 + \|x^2\partial_x u_0\|_2 \le \|\langle x\rangle^2 u_0\|_2 + \|J(x^2\partial_x u_0)\|_2\\ &\le \|\langle x\rangle^2u_0\|_2 + \|J^3u_0\|_2^{1/3}\|\langle x\rangle^3u_0\|_2^{2/3},\end{aligned}$$

and

$$\begin{aligned}\|\partial_\xi Q_1\|_2 \le\ & \|\chi'(\xi)|\xi|^{a-1}\int_0^\xi(\xi-\zeta)\partial_\xi^2\widehat{u}_0(\zeta)\,d\zeta\|_2\\ &+\|\chi(\xi)(a-1)|\xi|^{a-2}\int_0^\xi(\xi-\zeta)\partial_\xi^2\widehat{u}_0(\zeta)\,d\zeta\|_2\\ &+\|\chi(\xi)|\xi|^{a-1}\int_0^\xi\partial_\xi^2\widehat{u}_0(\zeta)\,d\zeta\|_2\\ \le\ & \|\partial_\xi^2\widehat{u}_0\|_\infty + c_a\|\chi(\xi)|\xi|^a\|\partial_\xi^2\widehat{u}_0\|_\infty\|_2\\ \le\ & c_a\|\partial_\xi^2\widehat{u}_0\|_\infty \le c_a\|\widehat{x^2u_0}\|_\infty \le c_a\|x^2u_0\|_{1,2}\\ \le\ & c_a(\|\langle x\rangle^2u_0\|_2 + \|J^3u_0\|_2^{1/3}\|\langle x\rangle^3u_0\|_2^{2/3}).\end{aligned}$$

This provides the bound of G_1 in (3.44).

For G_2 we write

$$\begin{aligned}\||x|^\theta D^a\mathcal{H}(xu_0)\|_2 &= \|D_\xi^\theta(|\xi|^a\mathrm{sgn}(\xi)(\widehat{xu_0}))\|_2\\ &\le \|D_\xi^\theta(|\xi|^a\mathrm{sgn}(\xi)\chi(\xi)(\widehat{xu_0}))\|_2 + \|D_\xi^\theta(|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))(\widehat{xu_0}))\|_2\\ &\le \|\widehat{xu_0}\|_\infty + \|\widehat{xu_0}\|_2 + \|J_\xi^\theta(\langle\xi\rangle^a\widehat{xu_0})\|_2\\ &\le \|xu_0\|_{1,2} + \|xu_0\|_2 + \|J^{1+\theta}(\langle\xi\rangle^a\widehat{u}_0)\|_2 + \|J^\theta(\langle\xi\rangle^{a-1}\widehat{u}_0)\|_2\\ &\le \|\langle x\rangle u_0\|_2 + \|x\partial_x u\| + \|\langle x\rangle^{1+\theta}J^a u_0\|_2 + \|J^\theta u_0\|_2.\end{aligned}$$

To complete the estimate we use that

$$\|\langle x\rangle^{1+\theta}J^\theta u_0\|_2 \le \|\langle x\rangle^3u_0\|_2^{(1+\theta)/3}\|J^m u_0\|_2^{(2-\theta)/3},$$

holds if $m\Big(\dfrac{2-\theta}{3}\Big) = a < r(1+a)\Big(\dfrac{2-\theta}{3}\Big)$. Since $r\ge 3$ we just need $(1+a)(2-\theta) > a$, which holds since $\theta\in(0,1)$. This finishes the bound for G_2.

Finally, we consider G_3 in (3.44).

$$\begin{aligned}\||x|^\theta D^{1+a}(x^2u_0)\|_2 &= \|D_\xi^\theta(|\xi|^{1+a}\mathrm{sgn}(\xi)\partial_\xi^2\widehat{u}_0)\|_2\\ &\le \|D_\xi^\theta(|\xi|^{1+a}\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi^2\widehat{u}_0)\|_2 + \|D_\xi^\theta(|\xi|^{1+a}\mathrm{sgn}(\xi)(1-\chi(\xi))\partial_\xi^2\widehat{u}_0)\|_2\\ &= A_1 + A_2.\end{aligned}$$

The Leibniz rule and Stein derivatives give

$$\begin{aligned}A_1 &\leq \|D_\xi^\theta(|\xi|^{1+a}\mathrm{sgn}(\xi)\chi(\xi))\|_\infty\|\partial_\xi^2\widehat{u}_0\|_2 + \||\xi|^{1+a}\mathrm{sgn}(\xi)\chi(\xi)\|_\infty\|D_\xi^\theta\partial_\xi^2\widehat{u}_0\|_2\\ &\leq \|\langle x\rangle^{2+\theta}u_0\|_2,\end{aligned}$$

and

$$\begin{aligned}A_2 &\leq \|D_\xi^\theta\big(\partial_\xi^2(|\xi|^{1+a}\mathrm{sgn}(\xi)(1-\chi(\xi))\widehat{u}_0)\big)\|_2\\ &\quad+\|D_\xi^\theta\big(\partial_\xi(|\xi|^{1+a}\mathrm{sgn}(\xi)(1-\chi(\xi))\partial_\xi\widehat{u}_0\big)\|_2\\ &\quad+\|D_\xi^\theta\big(\partial_\xi^2(|\xi|^{1+a}\mathrm{sgn}(\xi)(1-\chi(\xi))\widehat{u}_0\big)\|_2\\ &\equiv K_1+K_2+K_3.\end{aligned}$$

Notice that

$$\partial_\xi(|\xi|^{1+a}\mathrm{sgn}(\xi)(1-\chi(\xi)))=(1+a)|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))+|\xi|^{1+a}\mathrm{sgn}(\xi)\chi'(\xi),$$

and

$$\begin{aligned}\partial_\xi^2(|\xi|^{1+a}\mathrm{sgn}(\xi)(1-\chi(\xi))) &= c\,|\xi|^{a-1}\mathrm{sgn}(\xi)(1-\chi(\xi))\\ &\quad+c|\xi|^a\mathrm{sgn}(\xi)\chi'(\xi)+|\xi|^{1+a}\mathrm{sgn}(\xi)\chi''(\xi).\end{aligned}$$

So to bound K_3 we just need to consider

$$\begin{aligned}\|D_\xi^\theta(|\xi|^{a-1}\mathrm{sgn}(\xi)(1-\chi(\xi))\widehat{u}_0\|_2 &\leq \|D_\xi^\theta(|\xi|^{a-1}\mathrm{sgn}(\xi)(1-\chi(\xi)))\|_\infty\|\widehat{u}_0\|_2\\ &\quad+\||\xi|^{a-1}\mathrm{sgn}(\xi)(1-\chi(\xi))\|_\infty\|D_\xi^\theta\widehat{u}_0\|_2\\ &\leq \|u_0\|_2+\||x|^\theta u_0\|_2.\end{aligned}$$

To bound K_2 we just need to consider

$$\begin{aligned}\|D_\xi^\theta(|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))\partial_\xi\widehat{u}_0\|_2 &\leq \|D_\xi^\theta\Big(\frac{|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\langle\xi\rangle^a\partial_\xi\widehat{u}_0\Big)\|_2\\ &\leq \|J_\xi^\theta\Big(\frac{|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\langle\xi\rangle^a\partial_\xi\widehat{u}_0\Big)\|_2\end{aligned}$$

Now notice that

$$\Phi(\xi)=\frac{|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\in L^\infty,\ \ \partial_\xi\Phi\cong\frac{1}{\langle x\rangle},\ \partial_\xi^2\Phi\cong\frac{1}{\langle x\rangle^2}\in L^2.$$

Thus Proposition 2.5 and interpolation (Lemma 2.8) yield

$$\begin{aligned}\|J_\xi^\theta\Big(\frac{|\xi|^a\mathrm{sgn}(\xi)(1-\chi(\xi))}{\langle\xi\rangle^a}\langle\xi\rangle^a\partial_\xi\widehat{u}_0\Big)\|_2 &\leq \|J_\xi^\theta\big(\langle\xi\rangle^a\partial_\xi\widehat{u}_0\big)\|_2\\ &\leq \|\partial_\xi J_\xi^\theta\big(\langle\xi\rangle^a\widehat{u}_0\big)\|_2+\|J_\xi^\theta\big(\langle\xi\rangle^{a-1}\widehat{u}_0\big)\|_2\\ &\leq \|J_\xi^{1+\theta}\big(\langle\xi\rangle^a\widehat{u}_0\big)\|_2+\|J_\xi^\theta\widehat{u}_0\|_2\\ &\leq \|\langle x\rangle^{1+\theta}J^au_0\|_2+\||x|^\theta u_0\|_2.\end{aligned}$$

Finally to bound K_3 we just need to consider

$$\begin{aligned}\|D_\xi^\theta\big(|\xi|^{a-1}\mathrm{sgn}(\xi)(1-\chi(\xi))\widehat{u}_0\big)\|_2 &\leq \|J_\xi^\theta(|\xi|^{a-1}\mathrm{sgn}(\xi)(1-\chi(\xi))\widehat{u}_0\big)\|_2\\ &\leq \|J_\xi^\theta\widehat{u}_0\|_2\leq\|\langle x\rangle^\theta u_0\|_2,\end{aligned}$$

where we have used Proposition 2.5.

4. Proof of Theorem 1.3

Without loss of generality we assume that $t_1 = 0 < t_2$.

We consider the case $0 < a < 1/2$ and hence $\frac{5}{2} + a = 2 + \alpha < 3$.

From the hypothesis we have that $u \in C([-T,T] : Z_{(1+a)(\frac{5}{2}+a)+\frac{1-a}{2},\frac{5}{2}+a-\epsilon})$, for some $0 < \epsilon <1$. Therefore

$$u\partial_x u \in C([-T,T] : Z_{(1+a)(\frac{5}{2}+a)-\frac{1+a}{2},4+2a-2\epsilon}),$$

and

$$u\partial_x u \in L^1([-T,T] : H^{s_0}(\mathbb{R})), \quad \text{with} \quad s_0 \in (0,(1+a)(\frac{5}{2}+a)).$$

The solution to the IVP (1.1) can be represented by Duhamel's formula

$$u(t) = W_a(t)u_0 - \int_0^t W_a(t-t')u(t')\partial_x u(t')dt', \tag{4.1}$$

or equivalently in Fourier space as

$$\widehat{u}(\xi,t) = e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi) - \frac{i}{2}\int_0^t e^{-i(t-t')|\xi|^{1+a}\xi}\xi\widehat{u^2}(\xi,t')dt'. \tag{4.2}$$

With the notation introduced in (2.1), we have

$$\begin{aligned}\partial_\xi^2\widehat{u}(\xi,t) &= F_2(t,\xi,\widehat{u}_0) - \frac{i}{2}\int_0^t F_2(t-t',\xi,\xi\widehat{u^2}(\xi,t'))\\ &= \sum_1^4 B_j(t,\xi,\widehat{u}_0) - \frac{i}{2}\int_0^t\sum_1^4 B_j(t-t',\xi,\xi\widehat{u^2})dt'.\end{aligned} \tag{4.3}$$

We claim that for any $t \in \mathbb{R}$, and any $j = 2,3,4$.

CLAIM 1. *Let $\alpha = \frac{1}{2} + a \in (\frac{1}{2},1)$ and $j = 2,3$. Then*

$$B_j(t,\xi,\widehat{u}_0) - \frac{i}{2}\int_0^t B_j(t-t',\xi,\xi\widehat{u^2})dt' \in H^\alpha(\mathbb{R}), \tag{4.4}$$

for all $t \in \mathbb{R}$.

If we assume (4.4), it follows that

$$\begin{aligned}&\partial_\xi^2\widehat{u}(\xi,t) \in H^\alpha(\mathbb{R}) \ \text{ if and only if}\\ &B_1(t,\xi,\widehat{u}_0)-\frac{i}{2}\int_0^t B_1(t-t',\xi,\xi\widehat{u^2})\,dt' \in H^\alpha(\mathbb{R}).\end{aligned} \tag{4.5}$$

We split now B_1 as

$$\begin{aligned}B_1(t,\xi,\widehat{u}_0) &= c_1 t|\xi|^a\text{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)\\ &= c_1 t|\xi|^a\text{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)(\chi(\xi)+(1-\chi(\xi)))\\ &= B_{1,1} + B_{1,2},\end{aligned} \tag{4.6}$$

where $c_1 = -i(2+a)(1+a)$. From the hypothesis, it is easy to check that for any $t \in \mathbb{R} - \{0\}$, $B_{1,2} \in H^1(\mathbb{R})$.

Next, we consider $B_{1,1}$

$$\begin{aligned} B_{1,1} &= c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)(e^{-it|\xi|^{1+a}\xi} - 1)\widehat{u}_0(\xi) + c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(\xi) \\ &= \tilde{B}^1_{1,1} + \tilde{B}^2_{1,1}. \end{aligned}$$

Once again, we can easily check that for any $t \in \mathbb{R} - \{0\}$, $\tilde{B}^1_{1,1} \in H^1(\mathbb{R})$.

We rewrite $\tilde{B}_{1,1}$ as:

$$\tilde{B}^2_{1,1} = c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)(\widehat{u}_0(\xi) - \widehat{u}_0(0)) + c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) \tag{4.7}$$

and notice that for any $t \in \mathbb{R} - \{0\}$, $c_1 t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)(\widehat{u}_0(\xi) - \widehat{u}_0(0)) \in H^1(\mathbb{R})$.

Now, we apply the above argument for the inhomogeneous term

$$\int_0^t B_1(t-t', \xi, \xi\widehat{u^2})dt',$$

to conclude that

$$\begin{aligned} &\big(B_1(t,\xi,\widehat{u}_0) - \frac{i}{2}\int_0^t B_1(t-t',\xi,\xi\widehat{u^2})dt'\big) - c_1\big(t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) \\ &- \frac{i}{2}\int_0^t (t-t')|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\xi\,\widehat{u^2}(0,t')\big) \in H^1(\mathbb{R}) \end{aligned} \tag{4.8}$$

and therefore from (4.5) and (4.8) we have that for any $t \in \mathbb{R} - \{0\}$

$$\partial_\xi^2 \widehat{u}(\xi,t) \in H^\alpha(\mathbb{R}) \quad \text{if and only if}$$

$$t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) - \frac{i}{2}\int_0^t (t-t')|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\xi\,\widehat{u^2}(0,t') \in H^\alpha(\mathbb{R}).$$

Finally, we observe that for any $t \in \mathbb{R} - \{0\}$

$$\frac{i}{2}\int_0^t (t-t')|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\xi\,\widehat{u^2}(0,t') \in H^1(\mathbb{R}),$$

and hence

$$\partial_\xi^2 \widehat{u}(\xi,t) \in H^\alpha(\mathbb{R}) \quad \text{if and only if} \quad t|\xi|^a \operatorname{sgn}(\xi)\chi(\xi)\widehat{u}_0(0) \in H^\alpha(\mathbb{R}) \tag{4.9}$$

and since $\alpha = \frac{1}{2} + a$, it follows from Proposition 2.9, that (4.9) holds at $t = t_2$ if and only if $\widehat{u}_0(0) = 0$.

In order to complete the proof we go back to Claim 1.

Proof of Claim 1: We have

$$B_2(t,\xi,\widehat{u}_0) = c_2 e^{-it|\xi|^{1+a}\xi} t^2 |\xi|^{2(a+1)}\widehat{u}_0(\xi) \tag{4.10}$$

and therefore

$$\|B_2(t,\cdot,\widehat{u}_0)\|_2 \le c_t \|D^{2(a+1)}u_0\|_2 \le c_t \|u_0\|_{2(a+1),2} \tag{4.11}$$

and from Proposition 2.7, Proposition 2.5 and Lemma 2.8

$$\begin{aligned}
&\|D^{\alpha}B_2(t,\cdot,\widehat{u}_0)\|_2\\
&\quad\le c_t(\|u_0\|_{2(a+1),2}+\||\xi|^{(2+\alpha)(2+a)}\widehat{u}_0\|_2+\|\mathcal{D}^{\alpha}_{\xi}(|\xi|^{2(a+1)}\widehat{u}_0)\|_2)\\
&\quad\le c_t(\|u_0\|_{(2+\alpha)(a+1),2}+\|\mathcal{D}^{\alpha}_{\xi}(\frac{|\xi|^{2(a+1)}}{\langle\xi\rangle^{2(1+a)}}\langle\xi\rangle^{2(1+a)}\widehat{u}_0)\|_2)\\
&\quad\le c_t(\|u_0\|_{(5/2+a)(a+1),2}+\|J^{\alpha}_{\xi}(\langle\xi\rangle^{2(1+a)}\widehat{u}_0)\|_2)\\
&\quad\le c_t(\|u_0\|_{(5/2+a)(a+1),2}+\|\langle x\rangle^{\alpha}J^{2(1+a)}u_0\|_2)\\
&\quad\le c_t(\|u_0\|_{(5/2+a)(a+1),2}+\|\langle x\rangle^{5/2+a}u_0\|_2^{\frac{\alpha}{2+\alpha}}\|J^{(5/2+a)(a+1)}u_0\|_2^{\frac{2}{2+\alpha}})
\end{aligned}\tag{4.12}$$

which are all finite since $u_0\in Z_{(5/2+a)(a+1),5/2+a}$.

5. Proof of Theorem 1.4

As it was remarked we carry out the details for $a\in[1/2,1)$. Thus, $7/2+a=4+\alpha$ with $\alpha=a-1/2\in[0,1/2)$.

We shall use the integral equation associated to the IVP (1.1)

$$u(t)=W_a(t)u_0-\int_0^t W_a(t-t')u(t')\partial_x u(t')dt',$$

which in Fourier space reads as

$$\widehat{u}(\xi,t)=e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)-\frac{i}{2}\int_0^t e^{-i(t-t')|\xi|^{1+a}\xi}\xi\widehat{u^2}(\xi,t')dt'.\tag{5.1}$$

With the notation introduced in (2.1), we have that

$$\begin{aligned}
\partial_{\xi}^4\widehat{u}(\xi,t)&=F_4(t,\xi,\widehat{u}_0)-\frac{i}{2}\int_0^t F_4(t-t',\xi,\xi\widehat{u^2}(\xi,t'))\\
&=\sum_1^{11}E_j(t,\xi,\widehat{u}_0)-\frac{i}{2}\int_0^t\sum_1^{11}E_j(t-t',\xi,\xi\widehat{u^2})dt'.
\end{aligned}\tag{5.2}$$

By hypothesis we have that

$$u\in C([-T,T]:\dot{Z}_{(1+a)(\frac{7}{2}+a)+\frac{1-a}{2},\frac{7}{2}+a-\epsilon}),\ \text{for some } 0<\epsilon<1.\tag{5.3}$$

Therefore

$$u\partial_x u\in C([-T,T]:Z_{(1+a)(\frac{7}{2}+a)-\frac{1+a}{2},6+2a-2\epsilon}),\tag{5.4}$$

and by using Proposition 2.3

$$u\partial_x u\in L^1([-T,T]:H^{s_0}(\mathbb{R})),\qquad s_0\in(0,(1+a)(7/2+a)).\tag{5.5}$$

In Fourier space these last two properties are

$$\widehat{u}\in C([-T,T]:Z_{\frac{7}{2}+a-\epsilon,(1+a)(\frac{7}{2}+a)+\frac{1-a}{2}}),\tag{5.6}$$

and

$$\xi\, \widehat{u} * \widehat{u} \in C([-T,T] : Z_{6+2a-2\epsilon,(1+a)(\frac{7}{2}+a)-\frac{1+a}{2}}). \tag{5.7}$$

Also for $j = 1, 2, 3$ one has that

$$u(\cdot, t_j) \in \dot{Z}_{(7/2+a)(1+a),7/2+a} \quad \text{and so} \quad \widehat{u}(\cdot, t_j) \in \dot{Z}_{7/2+a,(7/2+a)(1+a)}. \tag{5.8}$$

We observe that from the equation in (1.1) it follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu(x,t)dx = \frac{1}{2}\,\|u(t)\|_2^2 = \frac{1}{2}\,\|u_0\|_2^2, \tag{5.9}$$

and hence

$$\int_{-\infty}^{\infty} xu(x,t)dx = \int_{-\infty}^{\infty} xu_0(x)dx + \frac{t}{2}\,\|u_0\|_2^2 \tag{5.10}$$

so the first momentum of a non-null solution is a strictly increasing function of t. If we prove that there exist $\tilde{t}_1 \in (t_1, t_2)$ and $\tilde{t}_2 \in (t_2, t_3)$ such that for $j = 1, 2$

$$\int_{-\infty}^{\infty} xu(x,\tilde{t}_j)dx = 0, \tag{5.11}$$

it will follow that $\|u_0\|_2 = 0$, and therefore $u \equiv 0$.

So we just need to show that using the hypothesis in (5.8) and (5.9) for $j = 1, 2$ there exists $\tilde{t}_1 \in (t_1, t_2)$ such that

$$\int_{-\infty}^{\infty} xu(x,\tilde{t}_1)dx = 0. \tag{5.12}$$

Without loss of generality we assume that $t_1 = 0 < t_2 < t_3$. Then, going back to equation (5.2) we observe that

$$\begin{aligned} E_1 = E_1(t,\xi,\widehat{u}_0) &= c_1 t|\xi|^{a-2}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi) \\ &= c_1 t|\xi|^{a-2}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\widehat{u}_0(\xi)(\chi(\xi) + 1 - \chi(\xi)) \\ &= E_{1,1} + E_{1,2}, \end{aligned} \tag{5.13}$$

with $E_{1,2} \in H^1(\mathbb{R})$ for any $t \in \mathbb{R}$. On the other hand, by using Taylor's formula and the fact that $\widehat{u}_0(0) = 0$, we obtain

$$\widehat{u}_0(\xi) = \xi\partial_\xi\widehat{u}_0(0) + \int_0^\xi (\xi-\theta)\partial_\xi^2\widehat{u}(\theta)d\theta \equiv \xi\partial_\xi\widehat{u}_0(0) + R_2(\xi). \tag{5.14}$$

Therefore we can write $E_{1,1}$ as

$$\begin{aligned} E_{1,1}(t,\xi,\widehat{u}_0) &= c_1 t|\xi|^{a-2}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\big(\xi\partial_\xi\widehat{u}_0(0) + R_2(\xi)\big) \\ &= c_1 t|\xi|^{a-1}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0) + \tilde{R}_2(\xi,t). \end{aligned} \tag{5.15}$$

Let us see that for any $t \in \mathbb{R}$, $\tilde{R}_2(\xi,t) \in H^1(\mathbb{R})$. Thus

$$\begin{aligned} \|\tilde{R}_2(\cdot,t)\|_2 &\le c_t\||\xi|^{a-2}\chi(\xi)|\xi|^2\|\partial_\xi^2\widehat{u}_0\|_\infty\|_2 \\ &\le c_t\|\widehat{x^2u_0}\|_\infty \ \le c_t\|x^2u_0\|_{1,2} \\ &\le c_t(\|\langle x\rangle^2u_0\|_2 + \|\langle x\rangle^2\partial_x u_0\|_{1,2}). \end{aligned}$$

Since $a \in (1/2, 1)$ (so $a-1 > -1/2$)

$$\begin{aligned}\|\partial_\xi \tilde{R}_2(.,t)\|_2 \le c_t(&\||\xi|^{a-3}\chi(\xi)|\xi|^2\|\partial_\xi^2\widehat{u}_0\|_\infty\|_2\\ &+\||\xi|^{a-2}\delta(0)\chi(\xi)|\xi|^2\|\partial_\xi^2\widehat{u}_0\|_\infty\|_2\\ &+\||\xi|^{2a+1}\chi(\xi)|\|\partial_\xi^2\widehat{u}_0\|_\infty\|_2\\ &+\||\xi|^{a-2}\chi(\xi)|\xi|\|\partial_\xi^2\widehat{u}_0\|_\infty\|_2)\\ \le c_t(&\|\langle x\rangle^2 u_0\|_2 + \|\langle x\rangle^2\partial_x u_0\|_{1,2}).\end{aligned} \tag{5.16}$$

Next we observe that

$$\begin{aligned}&t|\xi|^{a-1}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0)\\ &= t|\xi|^{a-1}\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0)\big(1+(e^{-it|\xi|^{1+a}\xi}-1)\big)\\ &= t|\xi|^{a-1}\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0) + Q_2(t,\xi)\end{aligned} \tag{5.17}$$

with

$$Q_2(t,\cdot) \in H^1(\mathbb{R}), \tag{5.18}$$

for all $t \in \mathbb{R}$, which follows by combining the estimate $|e^{i\tau}-1| \le |\tau|$ and that $a \in (1/2, 1)$.

Gathering the information from (5.13) to (5.18) we can conclude

$$E_1 - c_1 t|\xi|^{a-1}\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0) \in H^1(\mathbb{R}), \tag{5.19}$$

for all $t \in \mathbb{R}$, with $c_1 = -(2+a)(1+a)a(a-1)i$.

Combining the above argument and (5.4) we also conclude that

$$\int_0^t E_1(t-t',\xi,\xi\widehat{u^2}(\xi,t'))dt' - c_1\int_0^t (t-t')|\xi|^{a-1}\chi(\xi)\partial_\xi(\xi\widehat{u^2})(0,t'))dt' \in H^1(\mathbb{R})$$

for all $t \in \mathbb{R}$.

Now we shall rewrite the terms E_5's in (2.1) as

$$\begin{aligned}E_5 = E_5(t,\xi,\widehat{u}_0) &= c_5 t|\xi|^{a-1}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\partial_\xi\widehat{u}_0(\xi)\\ &= c_1 t|\xi|^{a-2}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\partial_\xi\widehat{u}_0(\xi)(\chi(\xi)+1-\chi(\xi))\\ &= E_{5,1}+E_{5,2}.\end{aligned} \tag{5.20}$$

with $E_{5,2} \in H^1(\mathbb{R})$ for any $t \in \mathbb{R}$. In fact

$$\begin{aligned}\|E_{5,2}\|_{1,2} &\le c_t(\|D^{2a}(xu_0)\|_2 + \|\langle x\rangle^2 u_0\|_2)\\ &\le c_t(\|xu_0\|_{2,2} + \|\langle x\rangle^2 u_0\|_2)\\ &\le c_t(\|\partial_x u_0\|_2 + \|\langle x\rangle\partial_x^2 u_0\|_2 + \|\langle x\rangle^2 u_0\|_2).\end{aligned} \tag{5.21}$$

Also

$$\begin{aligned}E_{5,1} = &c_5 t|\xi|^{a-1}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0)\\ &+ c_5 t|\xi|^{a-1}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)(\partial_\xi\widehat{u}_0(\xi)-\partial_\xi\widehat{u}_0(0)).\end{aligned} \tag{5.22}$$

An argument similar to that one in (5.15)-(5.16) shows that

$$c_5 t|\xi|^{a-1}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)(\partial_\xi\widehat{u}_0(\xi)-\partial_\xi\widehat{u}_0(0)) \in H^1(\mathbb{R}), \tag{5.23}$$

for all $t \in \mathbb{R}$. Now we consider

$$\begin{aligned} c_5 t|\xi|^{a-1}\mathrm{sgn}(\xi)e^{-it|\xi|^{1+a}\xi}\chi(\xi)\partial_\xi\widehat{u}_0(0) &= c_5\, t|\xi|^{a-1}\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0)\\ &+ c_5\, t|\xi|^{a-1}\mathrm{sgn}(\xi)(e^{-it|\xi|^{1+a}\xi}-1)\chi(\xi)\partial_\xi\widehat{u}_0(0). \end{aligned} \tag{5.24}$$

The argument in (5.17) and (5.18) shows that

$$t|\xi|^{a-1}\mathrm{sgn}(\xi)(e^{-it|\xi|^{1+a}\xi}-1)\chi(\xi)\partial_\xi\widehat{u}_0(0) \in H^1(\mathbb{R}), \tag{5.25}$$

for all $t \in \mathbb{R}$. Hence gathering the information from (5.20) to (5.24)

$$E_5(t,\xi,\widehat{u}_0) - c_5 t|\xi|^{a-1}\mathrm{sgn}(\xi)\chi(\xi)\partial_\xi\widehat{u}_0(0) \in H^1(\mathbb{R}) \tag{5.26}$$

for all $t \in \mathbb{R}$, with $c_5 = -4i(2+a)(1+a)a$.

The above argument and (5.4) show that

$$\int_0^t E_5(t-t',\xi,\xi\widehat{u^2}(\xi,t')) - c_5\int_0^t (t-t')|\xi|^{a-1}\chi(\xi)\partial_\xi(\xi\widehat{u^2})(0,t')) \in H^1(\mathbb{R}),$$

for all $t \in \mathbb{R}$. We claim that for all $t \in \mathbb{R}$,

$$E_2(t,\cdot,\widehat{u}_0),\ E_3(t,\cdot,\widehat{u}_0) \in H^1(\mathbb{R}). \tag{5.27}$$

It suffices to consider E_3. So

$$\|E_3\|_2 \le c_t\|u_0\|_{3a+2,2}, \tag{5.28}$$

and

$$\begin{aligned} \|\partial_\xi E_3\|_2 &\le c_t(\|\langle\xi\rangle^{4a+3}\widehat{u}_0\|_2 + \|\langle\xi\rangle^{3a+2}\partial_\xi\widehat{u}_0\|_2)\\ &\le c_t(\|\langle\xi\rangle^{4a+3}\widehat{u}_0\|_2 + \|\partial_\xi(\langle\xi\rangle^{3a+2}\widehat{u}_0)\|_2)\\ &\le c_t(\|u_0\|_{4a+3,2} + \|J_\xi(\langle\xi\rangle^{3a+2}\widehat{u}_0)\|_2)\\ &\le c_t(\|u_0\|_{4a+3,2} + \|J_\xi^4\widehat{u}_0\|_2^{\frac14}\|(\langle\xi\rangle^{4a+\frac83}\widehat{u}_0)\|_2^{\frac34})\\ &\le c_t(\|u_0\|_{4a+3,2} + \|\langle x\rangle^4 u_0\|_2^{\frac14}\|J^{4a+\frac83}u_0\|_2^{\frac34}). \end{aligned} \tag{5.29}$$

Since $4a+\frac83 \le (\frac72+a)(1+a)$, and $4 \le \frac72 + a$, the claim is proved.

A similar argument and (5.5) shows that for $j = 2, 3$

$$\int_0^t E_j(t-t',\xi,\xi\widehat{u^2}(\xi,t'))dt' \in H^1(\mathbb{R}), \tag{5.30}$$

for all $t \in \mathbb{R}$. Let us see now that for $j = 6, 9$,

$$E_j(t,\cdot,\widehat{u}_0) \in H^1(\mathbb{R}), \tag{5.31}$$

for all $t \in \mathbb{R}$. In both cases the proof is similar so we just consider the case $j = 9$. So

$$\begin{aligned} \|E_9\|_2 &\le c_t\|\langle\xi\rangle^a\partial_\xi^2\widehat{u}_0\|_2\\ &\le c_t(\|\langle\xi\rangle^a J_\xi^2\widehat{u}_0\|_2 + \|\langle\xi\rangle^a\widehat{u}_0\|_2)\\ &\le c_t(\|J^a\langle x\rangle^2 u_0\|_2 + \|J^a u_0\|_2), \end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\|\partial_\xi E_9\|_2 &\leq c_t(\||\xi|^{a-1}\partial_\xi^2\widehat{u}_0\|_2+\|\langle\xi\rangle^{1+2a}\partial_\xi^2\widehat{u}_0\|_2\|\langle\xi\rangle^a\partial_\xi^3\widehat{u}_0\|_2)\\ &\leq c_t(\|\partial_\xi^2\widehat{u}_0\|_\infty+\|\langle\xi\rangle^{1+2a}J_\xi^2\widehat{u}_0\|_2\|J_\xi^3\langle\xi\rangle^a\widehat{u}_0\|_2+\|\langle\xi\rangle^{1+2a}\widehat{u}_0\|_2)\\ &\leq c_t(\|x^2u_0\|_{1,2}+\|J^{2a+1}\langle x\rangle^2u_0\|_2+\|\langle x\rangle^3J^au_0\|_2+\|u_0\|_{2a+1,2})\\ &\leq c_t(\|x^2u_0\|_2+\|J^{2a+1}\langle x\rangle^2u_0\|_2+\|\langle x\rangle^3J^au_0\|_2+\|u_0\|_{2a+1,2})),\end{aligned}\tag{5.33}$$

which can be bounded by interpolation as well.

Also a similar argument and (5.5) shows again that for $j=6,9$

$$\int_0^t E_j(t-t',\xi,\xi\widehat{u^2}(\xi,t'))dt'\in H^1(\mathbb{R}),\tag{5.34}$$

for all $t\in\mathbb{R}$. By hypotheses (5.3), (5.5) and (5.8) we have

CLAIM 2. *Let $\alpha=a-\frac{1}{2}\in(0,\frac{1}{2})$ and $j=4,7,8$ and 11. Then*

$$E_j(t-t',\xi,\partial_\xi(\xi\widehat{u^2})),\ \int_0^t E_j(t-t',\xi\widehat{u^2})(\xi,t')dt'\in H^\alpha(\mathbb{R}),\tag{5.35}$$

for all $t\in\mathbb{R}$.

Proof of Claim 2: Due to the form of the terms, by interpolation it suffices to consider the cases $j=4$ and $j=11$. Thus,

$$\|E_4\|_2\leq c_t\||\xi|^{4(1+a)}\widehat{u}_0\|_2\leq c_t\|u_0\|_{4(1+a),2},\tag{5.36}$$

and

$$\|E_{11}\|_2\leq c_t\|\partial_\xi^4\widehat{u}_0\|_2\leq c_t\|\langle\xi\rangle^4u_0\|_2,\tag{5.37}$$

and hence both quantities are finite. Now

$$\begin{aligned}\|D_\xi^\alpha E_4\|_2 &\leq c_t(\||\xi|^{1+a}\widehat{u}_0\|_2+\||\xi|^{(4+\alpha)(1+a)}\widehat{u}_0\|_2+\|\mathcal{D}_\xi^\alpha(|\xi|^{4(1+a)}\widehat{u}_0)\|_2)\\ &\leq c_t(\|\langle\xi\rangle^{(4+\alpha)(1+a)}\widehat{u}_0\|_2+\|\mathcal{D}_\xi^\alpha(|\xi|^{4(1+a)}\widehat{u}_0)\|_2)\\ &\leq c_t(\|u_0\|_{(4+\alpha)(1+a),2}+\|\mathcal{D}_\xi^\alpha(|\xi|^{4(1+a)}\widehat{u}_0)\|_2),\end{aligned}$$

but

$$\begin{aligned}\|\mathcal{D}_\xi^\alpha(|\xi|^{4(1+a)}\widehat{u}_0)\|_2 &\leq\|\mathcal{D}_\xi^\alpha(\frac{|\xi|^{4(a+1)}}{\langle\xi\rangle^{4(1+a)}}\langle\xi\rangle^{4(1+a)}\widehat{u}_0)\|_2\\ &\leq c(\|\langle\xi\rangle^{4(1+a)}\widehat{u}_0\|_2+\|\mathcal{D}_\xi^\alpha(\langle\xi\rangle^{4(1+a)}\widehat{u}_0)\|_2)\\ &\leq c(\|J^{4(1+a)}u_0\|_2+\|J_\xi^\alpha(\langle\xi\rangle^{4(1+a)}u_0)\|_2)\\ &\leq c(\|u_0\|_{4(1+a),2}+\|\langle x\rangle^\alpha J^{4(1+a)}u_0\|_2)\\ &\leq c(\|u_0\|_{4(1+a),2}+\|\langle x\rangle^{4+\alpha}u_0\|_2^{\frac{\alpha}{4+\alpha}}\|J^{(4+\alpha)(a+1)}u_0\|_2^{\frac{4}{4+\alpha}}),\end{aligned}\tag{5.38}$$

therefore

$$\|D_\xi^\alpha E_4\|_2\leq c(\|u_0\|_{4(1+a),2}+\|\langle x\rangle^{4+\alpha}u_0\|_2),\tag{5.39}$$

which is finite by the hypothesis at $t_1 = 0$. Also

$$\begin{aligned}
\|D_\xi^\alpha E_{11}\|_2 &\le c_t(\|\partial_\xi^4 \widehat{u}_0\|_2 + \||\xi|^{\alpha(1+a)}\partial_\xi^4\widehat{u}_0\|_2 + \|\mathcal{D}_\xi^\alpha(\partial_\xi^4\widehat{u}_0)\|_2)\\
&\le c_t(\|\langle x\rangle^4 u_0\|_2 + \|\langle \xi\rangle^{\alpha(1+a)}\partial_\xi^4\widehat{u}_0\|_2 + \||x|^{4+\alpha}u_0\|_2)\\
&\le c_t(\|J^{\alpha(1+a)}(x^4u_0)\|_2 + \|\langle x\rangle^{4+\alpha}u_0\|_2)\\
&\le c_t(\|J^{\alpha(1+a)}(\langle x\rangle^4u_0)\|_2 + \|J^{\alpha(1+a)}(\langle x\rangle^2u_0)\|_2\\
&\quad + \|J^{\alpha(1+a)}u_0\|_2 + \|\langle x\rangle^{4+\alpha}u_0\|_2)\\
&\le c_t(\|\langle x\rangle^{4+\alpha}u_0\|_2^{\frac{4}{4+\alpha}}\|J^{(4+\alpha)(a+1)}u_0\|_2^{\frac{\alpha}{4+\alpha}}\\
&\quad + \|\langle x\rangle^4u_0\|_2^{\frac12}\|J^{2\alpha(a+1)}u_0\|_2^{\frac12}\\
&\quad + \|J^{\alpha(1+a)}u_0\|_2 + \|\langle x\rangle^{4+\alpha}u_0\|_2)\\
&\le c_t(\|u_0\|_{(4+\alpha)(1+a),2} + \|\langle x\rangle^{4+\alpha}u_0\|_2).
\end{aligned} \tag{5.40}$$

Combining (5.36)–(5.40) and (5.5) it follows that

$$\int_0^t E_j(t-t',\xi,\xi\widehat{u^2})(\xi,t')dt' \in H^\alpha(\mathbb{R}), \tag{5.41}$$

for all $t\in\mathbb{R}$ for $j=4,11$, and consequently for $j=7,8,9$ as well.

Summing up, we can conclude that

$$\begin{aligned}
&D_\xi^\alpha\partial_\xi^4\widehat{u}(\cdot,t)\in L^2(\mathbb{R}) \quad \text{if and only if}\\
&D_\xi^\alpha\Big(t|\xi|^{1-a}\chi(\xi)\partial_\xi\widehat{u}_0(0) - \int_0^t(t-t')|\xi|^{1-a}\chi(\xi)\partial_\xi\widehat{u^2}(0,t')dt'\Big)\in L^2(\mathbb{R}),
\end{aligned} \tag{5.42}$$

for any fixed $t\in\mathbb{R}$. We observe that

$$\partial_\xi\widehat{u}_0(0) = \widehat{-ixu_0}(0) = -i\int xu_0(x)\,dx, \tag{5.43}$$

and by (5.9)

$$\begin{aligned}
\partial_\xi(i\frac{\xi}{2}\widehat{u}^2)(0,t') &= -\widehat{ixu\partial_x}u(0,t') = -i\int xu\partial_x u(x,t')\,dx,\\
&= \frac{i}{2}\|u(t')\|_2^2 = \frac{i}{2}\|u(0)\|_2^2 = i\frac{d}{dt}\int xu(x,t)dx,
\end{aligned} \tag{5.44}$$

and by integration by parts

$$
\begin{aligned}
&t_2\partial_\xi\widehat{u}_0(0)-\frac{i}{2}\int_0^{t_2}(t_2-t')\partial_\xi(\xi\widehat{u}^2)(0,t')dt'\\
&=-it_2\int xu_0\,dx-i\int_0^{t_2}(t_2-t')\frac{d}{dt'}\int xu(x,t')dx\,dt'\\
&=-it_2\int xu_0(x)dx-i(t_2-t')\int xu(x,t')dx\Big|_{t'=0}^{t'=t_2}\\
&\quad -i\int_0^{t_2}\int xu(x,t')dxdt'\\
&=-i\int_0^{t_2}\int xu(x,t')dx\,dt'.
\end{aligned}
\tag{5.45}
$$

Thus from our hypothesis at $t=t_2$, it follows that

$$
D_\xi^\alpha(\chi(\xi)|\xi|^{a-1})\int_0^{t_2}\int xu(x,t')dx\,dt'\in L^2(\mathbb{R}),
\tag{5.46}
$$

but we recall that $\alpha=a-1/2$ and from Proposition 2.9

$$
D_\xi^\alpha(\chi(\xi)|\xi|^{a-1})\approx\mathcal{D}_\xi^\alpha(\chi(\xi)|\xi|^{\alpha-1/2})\notin L^2(\mathbb{R})\text{ if }\alpha\in(0,1).
$$

Therefore for (5.46) to hold is necessary that

$$
\int_0^{t_2}\int xu(x,t')dx\,dt'=0,
\tag{5.47}
$$

and since $F(t)=\int xu(x,t)dx$ is a continuous function, there exists $\tilde{t}_1\in(0,t_2)$ where $F(t)$ must vanish and this completes the proof.

6. Proof of Theorem 1.5

Without loss of generality we can assume that

$$
t_1=0\qquad\text{and}\qquad\int xu_0(x)dx=0.
\tag{6.1}
$$

Thus in this case, combining (5.43), (5.45)-(5.47) and (5.10), we have for $t_2\neq 0$ that

$$
\begin{aligned}
&\mathcal{D}_\xi^\alpha\partial_\xi^4\widehat{u}(\cdot,t)\in L^2(\mathbb{R}),\quad\text{if and only if}\\
&\int_0^{t_2}\int xu(x,t')dx\,dt'=0,\quad\text{if and only if}\\
&\int_0^{t_2}\frac{1}{2}t'\|u_0\|_2^2dt'=\frac{t_2^2}{4}\|u_0\|_2^2=0,\quad\text{if and only if}\\
&\|u_0\|_2^2=0\ \text{ if and only if }\ u_0=0.
\end{aligned}
\tag{6.2}
$$

7. Proof of Theorem 1.6

We shall consider only the case $a \in [1/2, 1)$, so that $\tilde{a} = 1$.

From the argument of the proof in Theorem 1.4, with $\alpha = 1/2$ in (5.36)–(5.40) and (5.5), we can conclude from our hypothesis $s \geq (\frac{7}{2}+a)(1+a)+\frac{1-a}{2}$, that for $t \neq 0$

$$D_\xi^{1/2}\partial_\xi^4\widehat{u}(\cdot,t) \in L^2(\mathbb{R}), \quad \text{if and only if}$$

$$D_\xi^{1/2}\Big(t|\xi|^{a-1}\chi(\xi)\partial_\xi\widehat{u}_0(0) - \int_0^t (t-t')|\xi|^{a-1}\chi(\xi)\partial_\xi(\xi\widehat{u^2}(0,t))dt'\Big) \in L^2(\mathbb{R}),$$

if and only if

$$D_\xi^{1/2}(\chi(\xi)|\xi|^{a-1})\int_0^t \int xu(x,t')dx\,dt' \in L^2(\mathbb{R}),$$

if and only if

$$\begin{aligned}\int_0^t \int xu(x,t')dx\,dt' &= \int_0^t (\int xu_0(x)\,dx + \frac{1}{2}t'\|u_0\|_2^2)dt'\\ &= t(\int xu_0(x)\,dx + \frac{1}{4}t\|u_0\|_2^2) = 0.\end{aligned}$$

References

[1] L. Abdelouhab, J. L. Bona, M. Felland and J.-C. Saut, *Nonlocal models for nonlinear dispersive waves*, Physica D. **40** (1989) 360–392.

[2] C. Amick and J. Toland, *Uniqueness and related analytic properties for the Benjamin-Ono equation – a nonlinear Neumann problem in the plane*, Acta Math. **167** (1991) 107–126.

[3] N. Aronszajn and K. T. Smith, *Theory of Bessel potentials. I.* Ann. Inst. Fourier (Grenoble) **11** (1961) 385–475.

[4] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. **29** (1967) 559–592.

[5] J. L. Bona and R. Smith, *The initial value problem for the Korteweg-de Vries equation*, Roy. Soc. London **Ser A 278** (1978) 555–601.

[6] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II* , Geom. Funct. Anal. **3** (1993) 107–156, 209–262.

[7] N. Burq and F. Planchon, *On the well-posedness of the Benjamin-Ono equation*, Math. Ann. **340** (2008) 497–542.

[8] A. P. Calderón, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A., **53** (1965) 1092–1099.

[9] J. Colliander, C. E. Kenig and G. Stafillani, *Local well-posedness for dispersion generalized Benjamin-Ono equations*, Diff. and Int. Eqs **16**, 12 (2002) 1441–1471.

[10] J. Colliander, M. Keel, G. Stafillani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$ equations*, J. Amer. Math. Soc. **16**, 3 (2003) 705–749.

[11] L. Dawson, H. McGahagan and G. Ponce, *On the decay properties of solutions to a class of Schrödinger equations*, Proc. AMS. **136** (2008) 2081–2090.

[12] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, *On uniqueness properties of solutions of the k-generalized KdV equations*, J. Funct. Anal. **244** (2007) 504–535.

[13] L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, *The Sharp Hardy Uncertainty Principle for Schrödinger Evolutions*, Duke Math. J. **155** (2010) 163–187.

[14] G. Fonseca and G. Ponce, *The I.V.P for the Benjamin-Ono equation in weighted Sobolev spaces*, J. Funct. Anal. **260** (2011) 436–459.

[15] G. Fonseca, F. Linares, and G. Ponce, *The I.V.P for the Benjamin-Ono equation in weighted Sobolev spaces II*, preprint.

[16] R. L. Frank and E. Lenzmann, *Uniqueness and non-degeneracy of ground states for $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ in $\mathbb{R}$*, preprint.

[17] Z. Guo, *Global well-posedness of the Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$*, J. Math. Pures Appl. **91** (2009) 583–597.

[18] Z. Guo, *Local Well-posedness for dispersion generalized Benjamin-Ono equations in Sobolev spaces*, preprint.

[19] S. Herr, *Well-posedness for equations of Benjamin-Ono type*, Illinois J. Math. **51** (2007) 951–976.

[20] S. Herr, A. Ionescu, C. E. Kenig, and H. Koch *A para-differential renormalization technique for nonlinear dispersive equations* Comm. P. D. E. **35** (2010) 1827–1875.

[21] A. A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, *Persistence Properties and Unique Continuation of solutions of the Camassa-Holm equation*, Comm. Math. Phys.**271** (2007) 511–522.

[22] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. AMS. **176** (1973) 227–251.

[23] A. D. Ionescu and C. E. Kenig, *Global well-posedness of the Benjamin-Ono equation on low-regularity spaces*, J. Amer. Math. Soc. **20**, 3 (2007) 753–798.

[24] R. J. Iorio, *On the Cauchy problem for the Benjamin-Ono equation*, Comm. P. D. E. **11** (1986) 1031–1081.

[25] R. J. Iorio, *Unique continuation principle for the Benjamin-Ono equation*, Diff. and Int. Eqs. **16** (2003) 1281–1291.

[26] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Mathematics Supplementary Studies, Studies in Applied Math. **8** (1983) 93–128.

[27] C. E. Kenig, and K. D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Res. Letters **10** (2003) 879–895.

[28] C. E. Kenig, Y. Martel and L. Robbiano, *Local well-posedness and blow up in the energy space for a class of L^2 critical dispersion generalized Benjamin-Ono equations*, preprint.

[29] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem Korteweg-de Vries equation* , J. Amer. Math. Soc **4** (1991) 323–347.

[30] C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation* , J. Amer. Math. Soc **9** (1996) 573–603.

[31] C. E. Kenig, G. Ponce and L. Vega, *On the unique continuation of solutions to the generalized KdV equation*, Math. Res. Letters **10** (2003) 833–846.

[32] H. Koch and N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation on* $H^s(\mathbb{R})$, Int. Math. Res. Not., **26** (2003) 1449–1464.

[33] H. Koch and N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation*, Int. Math. Res. Not., **30** (2005), 1833–1847.

[34] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. 5 **39** (1895), 22–443.

[35] L. Molinet and D. Pilod, *The Cauchy problem for the Benjamin-Ono equation in* L^2*revisited*, to appear in Analysis and PDE.

[36] L. Molinet and F. Ribaud, *Well-posedness results for the Benjamin-Ono equation with arbitrary large initial data*, Int. Math. Res. Not. **70** (2004), 3757–3795.

[37] L. Molinet and F. Ribaud, *On global well-posedness for a class of nonlocal dispersive wave equations*, Discrete Contin. Dyn. Syst. **15** (2006), no. 2, 657–668.

[38] L. Molinet, J.-C. Saut and N. Tzvetkov, *Ill- posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal. **33** (2001) 982–988.

[39] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. AMS. **165** (1972) 207–226.

[40] J. Nahas, *A decay property of solutions to the k-generalized KdV equation*, pre-print.

[41] J. Nahas and G. Ponce, *On the persistent properties of solutions to semi-linear Schrödinger equation*, Comm. P.D.E. **34** (2009) 1–20.

[42] H. Ono, *Algebraic solitary waves on stratified fluids*, J. Phy. Soc. Japan **39** (1975) 1082–1091.

[43] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical* A_p *characteristic*, Amer. J. Math. **129** (2007) 1355–1375.

[44] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Diff. & Int. Eqs. **4** (1991) 527–542.

[45] J.-C. Saut, *Sur quelques généralisations de l' équations de Korteweg-de Vries*, J. Math. Pures Appl. **58** (1979) 21–61.

[46] E. M. Stein, *The Characterization of Functions Arising as Potentials*, Bull. Amer. Math. Soc. **67** (1961) 102–104.

[47] T. Tao, *Global well-posedness of the Benjamin-Ono equation on* H^1, Journal Hyp. Diff. Eqs. **1** (2004) 27-49. Int. Math. Res. Not., Art. ID 95702 (2006) 1–44.

[48] M. I. Weinstein, *Solitary waves of nonlinear dispersive evolution equations with critical power nonlinearities*, J. Diff. Eqs **69** (1987) 192–203.

(G. Fonseca) Departamento de Matemáticas, Universidad Nacional de Colombia, Bogota, Colombia
E-mail address: gefonsecab@unal.edu.co

(F. Linares) IMPA, Rio de Janeiro, Brazil
E-mail address: linares@impa.br

(G. Ponce) Department of Mathematics, University of California, Santa Barbara, CA 93106, USA.
E-mail address: ponce@math.ucsb.edu