

THE IVP FOR THE BENJAMIN-ONO EQUATION IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. We study the initial value problem associated to the Benjamin-Ono equation. The aim is to establish persistence properties of the solution flow in the weighted Sobolev spaces $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r} dx)$, $s \in \mathbb{R}$, $s \geq 1$ and $s \geq r$. We also prove some unique continuation properties of the solution flow in these spaces. In particular, these continuation principles demonstrate that our persistence properties are sharp.

1. INTRODUCTION

This work is concerned with the initial value problem (IVP) for the Benjamin-Ono (BO) equation

$$(1.1) \quad \begin{cases} \partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0, & t, x \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases}$$

where \mathcal{H} denotes the Hilbert transform

$$(1.2) \quad \begin{aligned} \mathcal{H}f(x) &= \frac{1}{\pi} \text{p.v.} \left(\frac{1}{x} * f \right)(x) \\ &= \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy = -i (\text{sgn}(\xi) \widehat{f}(\xi))^\vee(x). \end{aligned}$$

The BO equation was deduced by Benjamin [3] and Ono [29] as a model for long internal gravity waves in deep stratified fluids. It was also shown that it is a completely integrable system (see [2], [7] and references therein).

Several works have been devoted to the problem of finding the minimal regularity, measured in the Sobolev scale $H^s(\mathbb{R}) = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R})$, which guarantees that the IVP (1.1) is locally or globally wellposed (LWP and GWP, resp.), i.e. existence and uniqueness hold in a space embedded in $C([0, T] : H^s(\mathbb{R}))$, with the map data-solution from $H^s(\mathbb{R})$ to $C([0, T] : H^s(\mathbb{R}))$ being locally continuous. Let us recall them : in [32] $s > 3$ was proven, in [1] and [17] $s > 3/2$, in [31] $s \geq 3/2$, in [24] $s > 5/4$, in [21] $s > 9/8$, in [35] $s \geq 1$, in [5] $s > 1/4$, and finally in [16] $s \geq 0$ was established.

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Real valued solutions of the IVP (1.1) satisfy infinitely many conservation laws (time invariant quantities), the first three are the following:

$$(1.3) \quad \begin{aligned} I_1(u) &= \int_{-\infty}^{\infty} u(x, t) dx, & I_2(u) &= \int_{-\infty}^{\infty} u^2(x, t) dx, \\ I_3(u) &= \int_{-\infty}^{\infty} (|D_x^{1/2} u|^2 - \frac{u^3}{3})(x, t) dx, \end{aligned}$$

where $D_x = \mathcal{H} \partial_x$.

Roughly, for $k \geq 2$ the k -conservation law I_k provides an *a priori* estimate of the L^2 -norm of the derivatives of order $(k-2)/2$ of the solution, i.e. $\|D_x^{(k-2)/2} u(t)\|_2$. This allows one to deduce GWP from LWP results.

For existence of solutions with non-decaying at infinity initial data we refer to [19] and [12].

In the BO equation the dispersive effect is described by a non-local operator and is significantly weaker than that exhibited by the Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0.$$

Indeed, it was proven in [26] that for any $s \in \mathbb{R}$ the map data-solution from $H^s(\mathbb{R})$ to $C([0, T] : H^s(\mathbb{R}))$ is not locally C^2 , and in [25] that it is not locally uniformly continuous. This implies that no LWP results can be obtained by an argument based only on a contraction method. This is certainly not the case of the KdV (see [23]).

Our interest here is to study real valued solutions of the IVP (1.1) in weighted Sobolev spaces

$$(1.4) \quad Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r} dx), \quad s, r \in \mathbb{R},$$

and decay properties of solutions of the equation (1.1). In this direction R. Iorio [17] proved the following results:

Theorem A. ([17]) (i) The IVP (1.1) is GWP in $Z_{2,2}$.

(ii) If $\widehat{u}_0(0) = 0$, then the IVP (1.1) is GWP in $\dot{Z}_{3,3}$.

(iii) If $u(x, t)$ is a solution of the IVP (1.1) such that $u \in C([0, T] : Z_{4,4})$ for arbitrary $T > 0$, then $u(x, t) \equiv 0$.

Above we have introduced the notation

$$(1.5) \quad \dot{Z}_{s,r} = \{f \in H^s(\mathbb{R}) \cap L^2(|x|^{2r} dx) : \widehat{f}(0) = 0\}, \quad s, r \in \mathbb{R}.$$

Notice that the conservation law I_1 in (1.3) tells us that the property $\widehat{u}_0(0) = 0$ is preserved by the solution flow.

We observe that the linear part of the equation in (1.1) $L = \partial_t + \mathcal{H} \partial_x^2$ commutes with the operator $\Gamma = x - 2t \mathcal{H} \partial_x$, i.e.

$$[L; \Gamma] = L\Gamma - \Gamma L = 0.$$

In fact, one can deduce (see [17]) that for a solution $v(x, t)$ of the associated linear problem

$$(1.6) \quad v(x, t) = U(t)v_0(x) = e^{-it\mathcal{H}\partial_x^2} v_0(x) = (e^{-it\xi|\xi|} \widehat{v}_0)^\vee(x),$$

to satisfy that $v(\cdot, t) \in L^2(|x|^{2k} dx)$, $t \in [0, T]$, one needs $v_0 \in Z_{k,k}$, $k \in \mathbb{Z}^+$ for $k = 1, 2$ and

$$\int_{-\infty}^{\infty} x^j v_0(x) dx = 0, \quad j = 0, 1, \dots, k-3, \quad \text{if } k \geq 3.$$

Also one notices that the traveling wave $\phi_c(x+t)$, $c > 0$ for the BO equation

$$\phi(x) = \frac{-4}{1+x^2}, \quad \phi_c(x+t) = c\phi(c(x+ct)),$$

has very mild decay at infinity. In this case, the traveling wave is negative and travels to the left. To get a positive traveling wave moving to the right one needs to consider the equation

$$(1.7) \quad \partial_t v - \mathcal{H}\partial_x^2 v + v\partial_x v = 0, \quad t, x \in \mathbb{R},$$

and observes that if $u(x, t)$ is a solution of (1.1) then

$$v(x, t) = -u(x, -t),$$

satisfies equation (1.7). In particular, (1.7) has the traveling wave solution

$$v(x, t) = \psi_c(x-t) = c\psi(c(x-ct)), \quad c > 0 \quad \text{with} \quad \psi(x) = -\phi(x).$$

In [18] R. Iorio strengthened his unique continuation result in $Z_{4,4}$ found in [17] (Theorem A, part (iii)) by proving:

Theorem B. ([18]) *Let $u \in C([0, T] : H^2(\mathbb{R}))$ be a solution of the IVP (1.1). If there exist three different times $t_1, t_2, t_3 \in [0, T]$ such that*

$$(1.8) \quad u(\cdot, t_j) \in Z_{4,4}, \quad j = 1, 2, 3, \quad \text{then} \quad u(x, t) \equiv 0.$$

Our goal in this work is to extend the results in Theorem A and Theorem B from integer values to the continuum optimal range of indices (s, r) . Our main results are the following:

Theorem 1. (i) *Let $s \geq 1$, $r \in [0, s]$, and $r < 5/2$. If $u_0 \in Z_{s,r}$, then the solution $u(x, t)$ of the IVP (1.1) satisfies that $u \in C([0, \infty) : Z_{s,r})$.*

(ii) *For $s > 9/8$ ($s \geq 3/2$), $r \in [0, s]$, and $r < 5/2$ the IVP (1.1) is LWP (GWP resp.) in $Z_{s,r}$.*

(iii) *If $r \in [5/2, 7/2)$ and $r \leq s$, then the IVP (1.1) is GWP in $\dot{Z}_{s,r}$.*

Theorem 2. *Let $u \in C([0, T] : Z_{2,2})$ be a solution of the IVP (1.1). If there exist two different times $t_1, t_2 \in [0, T]$ such that*

$$(1.9) \quad u(\cdot, t_j) \in Z_{5/2, 5/2}, \quad j = 1, 2, \quad \text{then} \quad \widehat{u}_0(0) = 0, \quad (\text{so } u(\cdot, t) \in \dot{Z}_{5/2, 5/2}).$$

Theorem 3. *Let $u \in C([0, T] : \dot{Z}_{3,3})$ be a solution of the IVP (1.1). If there exist three different times $t_1, t_2, t_3 \in [0, T]$ such that*

$$(1.10) \quad u(\cdot, t_j) \in Z_{7/2, 7/2}, \quad j = 1, 2, 3, \quad \text{then} \quad u(x, t) \equiv 0.$$

Remarks : (a) Theorem 2 shows that the condition $\widehat{u}_0(0) = 0$ is necessary to have persistence property of the solution in $Z_{s, 5/2}$, with $s \geq 5/2$, so in that regard Theorem 1 parts (i)-(ii) are sharp. Theorem 3 affirms that there is an upper limit of the spacial L^2 -decay rate of the solution (i.e. $|x|^{7/2}u(\cdot, t) \notin L^\infty([0, T] : L^2(\mathbb{R}))$),

for any $T > 0$) regardless of the decay and regularity of the non-zero initial data u_0 . In particular, Theorem 3 shows that Theorem 1 part (iii) is sharp.

(b) In part (ii) of Theorem 1 we shall use that in that case the solution $u(x, t)$ satisfies

$$\partial_x u \in L^1([0, T] : L^\infty(\mathbb{R})),$$

(see [21], [24], and [31]) to establish that the map data-solution is locally continuous from $Z_{s,r}$ into $C([0, T] : Z_{s,r})$.

(c) The condition in Theorem 3 involving three times seems to be technical and may be reduced to two different times as that in Theorem 2. We recall that unique continuation principles for the nonlinear Schrödinger equation and the generalized Korteweg-de Vries equation have been established in [10] and [11] resp. under assumptions on the solutions at two different times. Following the idea in [18] one finds from the equation (1.1) that

$$(1.11) \quad \frac{d}{dt} \int_{-\infty}^{\infty} xu(x, t) dx = \frac{1}{2} \|u(t)\|_2^2 = \frac{1}{2} \|u_0\|_2^2,$$

so the first momentum of a non-null solution of the BO equation is strictly increasing. On the other hand, using the integral equation version of the BO equation from the hypotheses one can deduce that the first momentum must vanish somewhere in the time intervals (t_1, t_2) and (t_2, t_3) . This implies that $u(x, t) \equiv 0$.

(d) We recall that if for a solution $u \in C([0, T] : H^s(\mathbb{R}))$ of (1.1) one has that $\exists t_0 \in [0, T]$ such that $u(x, t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0, T] : H^{s'}(\mathbb{R}))$. So we shall mainly consider the most interesting case $s = r$ in (1.4).

(e) Consider the IVP for generalized Benjamin-Ono (gBO) equation

$$(1.12) \quad \begin{cases} \partial_t u + \mathcal{H} \partial_x^2 u \pm u^k \partial_x u = 0, & t, x \in \mathbb{R}, k \in \mathbb{Z}^+, \\ u(x, 0) = u_0(x), \end{cases}$$

with u_0 a real valued function. In this case the best LWP available results are : for $k = 2$, $s \geq 1/2$ (see [22]), for $k = 3$, $s > 1/34$ (see [36]), and for $k \geq 4$, $s \geq 1/2 - 1/k$ (see [36]). So for any power $k = 1, 2, \dots$ with focussing (+) or defocussing (-) non-linearity the IVP (1.12) is LWP in $H^1(\mathbb{R})$. So the local results in Theorems 1 and Theorem 2 and their proofs extend to the IVP (1.12) with possible different values $s = s(k)$ for the minimal regularity required. This is also the case for Theorem 3 when the power k in (1.12) is odd in the focusing and defocusing regime.

(f) In [20] the number $7/2$ was mentioned as a possible threshold in the spaces (1.4).

The proof of Theorem 1 is based on weighted energy estimates and involves several inequalities for the Hilbert transform \mathcal{H} . Among them we shall use the A_p condition introduced in [27], (see Definition 1). It was proven in [15] that this is a necessary and sufficient condition for the Hilbert transform \mathcal{H} to be bounded in $L^p(w(x)dx)$ (see [15],), i.e. $w \in A_p$, $1 < p < \infty$ if and only if

$$(1.13) \quad \left(\int_{-\infty}^{\infty} |\mathcal{H}f|^p w(x) dx \right)^{1/p} \leq c^* \left(\int_{-\infty}^{\infty} |f|^p w(x) dx \right)^{1/p},$$

(see Theorem 4).

In order to justify some of our arguments in the proofs we need some further continuity properties of the Hilbert transform. More precisely, our proof requires the constant c^* in (1.13) to depend only on $c(w)$ the constant describing the A_p condition (see (2.2)) and on p . In [30] precise bounds for the constant c^* in (2.3)

were given which are sharp in the case $p = 2$ and sufficient for our purpose (see Theorem 5).

It will be essential in our arguments that some commutator operators involving the Hilbert transform \mathcal{H} are of “order zero”. More precisely, we shall use the following estimate: $\forall p \in (1, \infty)$, $l, m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \geq 1$ $\exists c = c(p; l; m) > 0$ such that

$$(1.14) \quad \|\partial_x^l [\mathcal{H}; a] \partial_x^m f\|_p \leq c \|\partial_x^{l+m} a\|_\infty \|f\|_p.$$

In the case $l + m = 1$, (1.14) is Calderón’s first commutator estimate [6]. In the case $l + m \geq 2$, (1.14) was proved in [8].

The rest of this paper is organized as follows: section 2 contains some preliminary estimates to be utilized in the coming sections. Theorem 1 will be proven in section 3. Finally, the proofs of Theorem 2 and Theorem 3 will be given in sections 4 and 5, respectively.

2. PRELIMINARY ESTIMATES

We shall use the following notations:

$$(2.1) \quad \|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|,$$

$$\|f\|_{s,2} = \|(1 - \partial_x^2)^{s/2} f\|_2, \quad s \in \mathbb{R}.$$

Let us first recall the definition of the A_p condition. We shall restrict here to the cases $p \in (1, \infty)$ and the 1-dimensional case \mathbb{R} (see [27]).

Definition 1. A non-negative function $w \in L^1_{loc}(\mathbb{R})$ satisfies the A_p inequality with $1 < p < \infty$ if

$$(2.2) \quad \sup_Q \text{interval} \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} = c(w) < \infty,$$

where $1/p + 1/p' = 1$.

Theorem 4. ([15]) *The condition (2.2) is necessary and sufficient for the boundedness of the Hilbert transform \mathcal{H} in $L^p(w(x)dx)$, i.e.*

$$(2.3) \quad \left(\int_{-\infty}^{\infty} |\mathcal{H}f|^p w(x) dx \right)^{1/p} \leq c^* \left(\int_{-\infty}^{\infty} |f|^p w(x) dx \right)^{1/p}.$$

In the case $p = 2$, a previous characterization of w in (2.3) was found in [14] (for further references and comments we refer to [9], [13], and [34]). However, even though we will be mainly concerned with the case $p = 2$, the characterization (2.3) will be the one used in our proof. In particular, one has that in \mathbb{R}

$$(2.4) \quad |x|^\alpha \in A_p \Leftrightarrow \alpha \in (-1, p-1).$$

In order to justify some of the arguments in the proof of Theorem 1 we need some further continuity properties of the Hilbert transform. More precisely, our proof requires the constant c^* in (2.3) to depend only on $c(w)$ in (2.2) and on p (in fact, this is only needed for the case $p = 2$).

Theorem 5. ([30]) *For $p \in [2, \infty)$ the inequality (2.3) holds with $c^* \leq c(p)c(w)$, with $c(p)$ depending only on p and $c(w)$ as in (2.2). Moreover, for $p = 2$ this estimate is sharp.*

Next, we define the truncated weights $w_N(x)$ using the notation $\langle x \rangle = (1+x^2)^{1/2}$ as

$$(2.5) \quad w_N(x) = \begin{cases} \langle x \rangle & \text{if } |x| \leq N, \\ 2N & \text{if } |x| \geq 3N, \end{cases}$$

$w_N(x)$ are smooth and non-decreasing in $|x|$ with $w'_N(x) \leq 1$ for all $x \geq 0$.

Proposition 1. *For any $\theta \in (-1, 1)$ and any $N \in \mathbb{Z}^+$, $w_N^\theta(x)$ satisfies the A_2 inequality (2.2). Moreover, the Hilbert transform \mathcal{H} is bounded in $L^2(w_N^\theta(x)dx)$ with a constant depending on θ but independent of $N \in \mathbb{Z}^+$.*

The proof of Proposition 1 follows by combining the fact that for a fixed $\theta \in (-1, 1)$ the family of weights $w_N^\theta(x)$, $N \in \mathbb{Z}^+$ satisfies the A_2 inequality in (2.2) with a constant c independent of N , and Theorem 5.

Next, we have the following generalization of Calderón commutator estimates [6] found in [8] and already commented in the introduction:

Theorem 6. *For any $p \in (1, \infty)$ and $l, m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \geq 1$ there exists $c = c(p; l; m) > 0$ such that*

$$(2.6) \quad \|\partial_x^l [\mathcal{H}; a] \partial_x^m f\|_p \leq c \|\partial_x^{l+m} a\|_\infty \|f\|_p.$$

We shall also use the pointwise identities

$$[\mathcal{H}; x] \partial_x f = [\mathcal{H}; x^2] \partial_x^2 f = 0,$$

and more generally

$$[\mathcal{H}; x] f = 0 \quad \text{if and only if} \quad \int f dx = 0.$$

We recall the following characterization of the $L_s^p(\mathbb{R}^n) = (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n)$ spaces given in [33].

Theorem 7. *Let $b \in (0, 1)$ and $2n/(n + 2b) < p < \infty$. Then $f \in L_b^p(\mathbb{R}^n)$ if and only if*

$$(2.7) \quad \begin{aligned} (a) \quad & f \in L^p(\mathbb{R}^n), \\ (b) \quad & \mathcal{D}^b f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2b}} dy \right)^{1/2} \in L^p(\mathbb{R}^n), \end{aligned}$$

with

$$(2.8) \quad \|f\|_{b,p} \equiv \|(1 - \Delta)^{b/2} f\|_p = \|J^b f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p.$$

Above we have used the notation: for $s \in \mathbb{R}$

$$D^s = (-\Delta)^{s/2} \quad \text{with} \quad D^s = (\mathcal{H} \partial_x)^s, \quad \text{if } n = 1.$$

For the proof of this theorem we refer the reader to [33]. One sees that from (2.7) for $p = 2$ and $b \in (0, 1)$ one has

$$(2.9) \quad \|\mathcal{D}^b(fg)\|_2 \leq \|f \mathcal{D}^b g\|_2 + \|g \mathcal{D}^b f\|_2.$$

We shall use this estimate in the proof of Theorem 3. As applications of Theorem 7 we have the following estimate:

Proposition 2. *Let $b \in (0, 1)$. For any $t > 0$*

$$(2.10) \quad \mathcal{D}^b(e^{-itx|x|}) \leq c(|t|^{b/2} + |t|^b|x|^b).$$

For the proof of Proposition 2 we refer to [28].

As a further direct consequence of Theorem 7 we deduce the following result to be used in the proof of Theorem 3.

Proposition 3. *Let $p \in (1, \infty)$. If $f \in L^p(\mathbb{R})$ such that there exists $x_0 \in \mathbb{R}$ for which $f(x_0^+)$, $f(x_0^-)$ are defined and $f(x_0^+) \neq f(x_0^-)$, then for any $\delta > 0$, $\mathcal{D}^{1/p} f \notin L^p_{loc}(B(x_0, \delta))$ and consequently $f \notin L^p_{1/p}(\mathbb{R})$.*

Also as consequence of the estimate (2.9) one has the following interpolation inequality.

Lemma 1. *Let $a, b > 0$. Assume that $J^a f = (1 - \Delta)^{a/2} f \in L^2(\mathbb{R})$ and $\langle x \rangle^b f = (1 + |x|^2)^{b/2} f \in L^2(\mathbb{R})$. Then for any $\theta \in (0, 1)$*

$$(2.11) \quad \|J^{\theta a}(\langle x \rangle^{(1-\theta)b} f)\|_2 \leq c \|\langle x \rangle^b f\|_2^{1-\theta} \|J^a f\|_2^\theta.$$

Moreover, the inequality (2.11) is still valid with $w_N(x)$ in (2.5) instead of $\langle x \rangle$ with a constant c independent of N .

Proof. It will suffice to consider the case : $a = 1 + \alpha$, $\alpha \in (0, 1)$. We denote by $\rho(x)$ a function equal to $\langle x \rangle$ or equal to $w_N(x)$ as in (2.5) and consider the function

$$F(z) = e^{(z^2-1)} \int_{-\infty}^{\infty} J^{az}(\rho^{b(1-z)} f(x)) \overline{g(x)} dx$$

with $g \in L^2(\mathbb{R}^n)$ with $\|g\|_2 = 1$, which is continuous in $\{z = \eta + iy : 0 \leq \eta \leq 1\}$ and analytic in its interior. Moreover,

$$|F(0 + iy)| \leq e^{-(y^2+1)} \|\rho^b f\|_2,$$

and since $|\rho'/\rho| + |\rho''/\rho| \leq c$ (independent of N) combining (2.7) and (2.9) one has

$$\begin{aligned} |F(1 + iy)| &\leq e^{-y^2} \|J^a(\rho^{iby} f)\|_2 \leq e^{-y^2} (\|\rho^{iby} f\|_2 + \|D^\alpha \partial_x(\rho^{iby} f)\|_2) \\ &\leq e^{-y^2} (\|f\|_2 + \|D^\alpha(\rho^{iby} \partial_x f)\|_2 + |by| \|D^\alpha(\rho^{iby-1} \rho' f)\|_2) \\ &\leq e^{-y^2} (\|f\|_2 + \|D^\alpha(\rho^{iby} \partial_x f)\|_2 + |by| \|D^\alpha(\rho^{iby-1} \rho' f)\|_2) \\ &\leq e^{-y^2} (\|f\|_2 + \|D^\alpha(\rho^{iby}) \partial_x f\|_2 + \|(\rho^{iby}) D^\alpha \partial_x f\|_2 \\ &\quad + |by| \|D^\alpha(\rho^{iby-1} \rho') f\|_2 + |by| \|(\rho^{iby-1} \rho') D^\alpha f\|_2) \\ &\leq c_\alpha e^{-y^2} (1 + |yb|^2) (\|f\|_2 + \|D^\alpha f\|_2 + \|\partial_x f\|_2 + \|D^\alpha \partial_x f\|_2) \\ &\leq c_\alpha e^{-y^2} (1 + |yb|^2) \|J^{1+\alpha} f\|_2 = c_\alpha e^{-y^2} (1 + |yb|^2) \|J^a f\|_2, \end{aligned}$$

using that for $\alpha \in (0, 1)$

$$\|D^\alpha h\|_\infty \leq c_\alpha (\|h\|_\infty + \|\partial_x h\|_\infty).$$

Therefore, the three lines theorem yields the desired result. \square

We shall also employ the following simple estimate.

Proposition 4. *If $f \in L^2(\mathbb{R})$ and $\phi \in H^1(\mathbb{R})$, then*

$$(2.12) \quad \|[D^{1/2}; \phi]f\|_2 \leq c \|\phi\|_{1,2} \|f\|_2.$$

Finally, to complete this section we recall the result obtained in [31] concerning regularity properties of the solutions of the IVP (1.1) with data $u_0 \in H^s(\mathbb{R})$, $s \geq 3/2$. This will be used in the proof of Theorem 3.

Theorem 8. *For any $u_0 \in H^s(\mathbb{R})$ with $s \geq 3/2$ the IVP (1.1) has a unique global solution $u \in C([0, T] : H^s(\mathbb{R}))$ such that for any $T > 0$*

$$J^{s+1/2}u \in l_k^\infty(L^2([k, k+1] \times [0, T])), \quad Ju \in l_k^2(L^\infty([k, k+1] \times [0, T]))$$

and

$$J^{s-3/2}\partial_x u \in L^4([0, T] : L^\infty(\mathbb{R})).$$

3. PROOF OF THEOREM 1

We consider several cases :

Case 1: $s = 1$ and $r = \theta \in (0, 1]$. Part (i) in Theorem 1:

We multiply the differential equation by $w_N^{2\theta}u$ (see (2.5)) with $0 < \theta \leq 1$ and integrate on \mathbb{R} to obtain

$$(3.1) \quad \frac{1}{2} \frac{d}{dt} \int (w_N^\theta u)^2 dx + \int w_N^\theta \mathcal{H} \partial_x^2 u w_N^\theta u dx + \int w_N^{2\theta} u^2 \partial_x u dx = 0.$$

To handle the second term on the left hand side (l.h.s.) of (3.1) we write

$$\begin{aligned} w_N^\theta \mathcal{H} \partial_x^2 u &= [w_N^\theta; \mathcal{H}] \partial_x^2 u + \mathcal{H}(w_N^\theta \partial_x^2 u) \\ &= A_1 + \mathcal{H} \partial_x^2 (w_N^\theta u) - 2\mathcal{H}(\partial_x w_N^\theta \partial_x u) - \mathcal{H}(\partial_x^2 w_N^\theta u) \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

We observe that by Theorem 6 and our assumption on $\theta \in (0, 1]$ the terms A_1, A_4 are bounded by the L^2 -norm of the solution u and A_3 is bounded by the H^1 -norm of the solution with constants independent of N , thus they are bounded uniformly on $N \in \mathbb{Z}^+$ by

$$M_1 = \sup_{t \in [0, T]} \|u(t)\|_{1,2}.$$

We insert the term A_2 in (3.1) and use integration by parts, to get that

$$\int \mathcal{H} \partial_x^2 (w_N^\theta u) w_N^\theta u dx = 0.$$

Finally, using integration by parts, we bound the nonlinear term (the third term on the l.h.s.) in (3.1) as

$$(3.2) \quad \left| \int w_N^{2\theta} u^2 \partial_x u dx \right| \leq c \|u\|_\infty \|u\|_2 \|w_N^\theta u\|_2 \leq c \|u\|_{1,2}^2 \|w_N^\theta u\|_2.$$

Inserting this information in (3.1) we get

$$\frac{d}{dt} \|w_N^\theta u(t)\|_2 \leq cM, \quad \text{with } c \text{ independent of } N,$$

which tells us that

$$\sup_{t \in [0, T]} \|w_N^\theta u(t)\|_2 \leq c \|\langle x \rangle^\theta u_0\|_2 e^{TM}, \quad \text{with } c \text{ independent of } N,$$

which yields the result $u \in L^\infty([0, T] : L^2(|x|^{2\theta}))$ for any $T > 0$.

To see that $u \in C([0, T] : L^2(|x|^{2\theta}))$ one considers the sequence

$$(w_N^\theta u)_{N \in \mathbb{Z}^+} \subseteq C([0, T] : L^2(\mathbb{R})),$$

and reapply the above argument to find that it is a Cauchy sequence.

Finally, we point out that the use of the differential equation in (1.1) can be justified by the locally continuous dependence of the solution upon the data from $H^s(\mathbb{R})$ to $C([0, T] : H^s(\mathbb{R}))$.

Case 2: $s \in (1, 2]$ and $r = s$. Part (i) in Theorem 1.

We multiply the differential equation by $w_N^{2+2\theta} u$ (see (2.5)) with $0 \leq \theta \leq 1$ and integrate on \mathbb{R} to obtain

$$(3.3) \quad \frac{1}{2} \frac{d}{dt} \int (w_N^{1+\theta} u)^2 dx + \int w_N^{1+\theta} \mathcal{H} \partial_x^2 u w_N^{1+\theta} u dx + \int w_N^{2+2\theta} u^2 \partial_x u dx = 0.$$

To control the second term on the l.h.s. of (3.3) we write

$$\begin{aligned} w_N^{1+\theta} \mathcal{H} \partial_x^2 u &= [w_N^{1+\theta}; \mathcal{H}] \partial_x^2 u + \mathcal{H}(w_N^{1+\theta} \partial_x^2 u) \\ &= B_1 + \mathcal{H} \partial_x^2 (w_N^{1+\theta} u) - 2\mathcal{H}(\partial_x w_N^{1+\theta} \partial_x u) - \mathcal{H}(\partial_x^2 w_N^{1+\theta} u) \\ &= B_1 + B_2 + B_3 + B_4. \end{aligned}$$

We observe that by Theorem 6 and our assumption $\theta \in (0, 1]$ the terms B_1, B_4 are bounded by the L^2 -norm of the solution. Inserting the term B_2 on the (3.3) and using integration by parts one finds that its contribution is null. So it remains to control $B_3 = -2\mathcal{H}(\partial_x w_N^{1+\theta} \partial_x u)$. Since

$$|\partial_x w_N^{1+\theta}| = |(1+\theta)w_N^\theta \partial_x w_N| \leq c w_N^\theta, \quad c \text{ independent of } N,$$

one has

$$(3.4) \quad \|B_3\|_2 \leq c \|w_N^\theta \partial_x u\|_2 \leq c \|\partial_x (w_N^\theta u)\|_2 + c \|\partial_x w_N^\theta u\|_2 \leq c \|\partial_x (w_N^\theta u)\|_2 + c \|u\|_2.$$

Then by the interpolation inequality in (2.11) it follows that

$$(3.5) \quad \|\partial_x (w_N^\theta u)\|_2 \leq \|J(w_N^\theta u)\|_2 \leq c \|w_N^{1+\theta} u\|_2^{\theta/(1+\theta)} \|J^{1+\theta} u\|_2^{1/(1+\theta)},$$

with a constant c independent of N . So by Young's inequality in (3.5) and (3.4) the term B_3 is appropriately bounded. Finally, for the last term on the l.h.s. of (3.3) we write

$$(3.6) \quad \left| \int w_N^{2+2\theta} u^2 \partial_x u dx \right| \leq c \|u\|_\infty \|w_N^{1+\theta} u\|_2^2 \leq c \|u\|_{1,2} \|w_N^{1+\theta} u\|_2^2,$$

with c independent of N .

So inserting the above information in (3.3) we obtain the result.

Case 3: $s \in (9/8, 2]$ and $r = s$. Part (ii) in Theorem 1.

In this case it remains to establish the continuous dependence of the solution $C([0, T] : Z_{s,r})$ upon the data in $Z_{s,r}$. We are considering the most interesting case $s = r \in (9/8, 2]$. Suppose that $u, v \in C([0, T] : Z_{s,s})$ are two solutions of the BO equation in (1.1) corresponding to data u_0, v_0 respectively. Hence,

$$(3.7) \quad \partial_t(u-v) + \mathcal{H} \partial_x(u-v) + \partial_x u(u-v) + v \partial_x(u-v) = 0.$$

We will reapply the argument used in the previous case. However, we notice that the nonlinear term in (3.7) is different than that in (3.3). So we recall the result in [21] which affirms that for $s > 9/8$

$$(3.8) \quad \partial_x u, \partial_x v \in L^1([0, T] : L_x^\infty(\mathbb{R})),$$

and use integration by parts to obtain that

$$(3.9) \quad \begin{aligned} & \left| \int w_N^{2+2\theta} (\partial_x u (u-v)^2 + v(u-v) \partial_x (u-v)) dx \right| \\ & \leq c(\|\partial_x u(t)\|_\infty + \|\partial_x v(t)\|_\infty + \|v(t)\|_\infty) \|w_N^{1+\theta} (u-v)\|_2^2. \end{aligned}$$

Hence, combining the argument in the previous section, the estimates (3.9) and (3.8), and the continuous dependence of the solution in $C([0, T] : H^s(\mathbb{R}))$ upon the data in $H^s(\mathbb{R})$ the desired result follows.

Case 4: $s = r \in (2, 5/2)$. Part (ii) in Theorem 1.

We recall that from the previous cases we know the result for $s \geq r \in (0, 2]$. Also we shall write $r = 2 + \theta$, $\theta \in (0, 1/2)$, and we multiply the differential equation by $x^2 w_N^{2+2\theta} u$ (see (2.5)) and integrate on \mathbb{R} to obtain

$$(3.10) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (w_N^{1+\theta} x u)^2 dx \\ & + \int w_N^{1+\theta} x \mathcal{H} \partial_x^2 u w_N^{1+\theta} x u dx + \int x^2 w_N^{2+2\theta} u^2 \partial_x u dx = 0. \end{aligned}$$

From our previous proofs it is clear that we just need to handle the second term on the l.h.s. of (3.10). First we write the identity

$$(3.11) \quad x \mathcal{H} \partial_x^2 u = \mathcal{H}(x \partial_x^2 u) = \mathcal{H}(\partial_x^2(xu)) - 2\mathcal{H} \partial_x u = E_1 + E_2.$$

To bound the contribution of the term E_2 inserted in (3.10) we shall use that w_N^θ with $\theta \in (0, 1/2)$ satisfies the A_2 inequality uniformly in N (see Proposition 1) so

$$(3.12) \quad \begin{aligned} \|w_N^{1+\theta} E_2\|_2 &= 2 \|w_N^{1+\theta} \mathcal{H} \partial_x u\|_2 \leq c \|w_N^\theta \mathcal{H} \partial_x u\|_2 + c \|w_N^\theta x \mathcal{H} \partial_x u\|_2 \\ &\leq c \|w_N^\theta \partial_x u\|_2 + c \|w_N^\theta \mathcal{H}(x \partial_x u)\|_2 \\ &\leq c \|w_N^\theta \partial_x u\|_2 + c \|w_N^\theta x \partial_x u\|_2 = F_1 + F_2. \end{aligned}$$

Now using complex interpolation one gets (see Lemma 1)

$$(3.13) \quad \begin{aligned} \|w_N^\theta \partial_x u\|_2 &\leq \|\partial_x(w_N^\theta u)\|_2 + \|\partial_x w_N^\theta u\|_2 \\ &\leq \|\partial_x(w_N^\theta u)\|_2 + c \|u\|_2 \leq c \|J(w_N^\theta u)\|_2 + c \|u\|_2 \\ &\leq c \|J^2 u\|_2^{1/2} \|\langle x \rangle^{2\theta} u\|_2^{1/2} + c \|u\|_2, \end{aligned}$$

which has been bounded in the previous cases. So it remains to bound the term

$$(3.14) \quad F_2 = \|w_N^\theta x \partial_x u\|_2,$$

which will be considered later.

Inserting the term E_1 in (3.11) into (3.10) one obtains the term

$$(3.15) \quad G_1 = \int w_N^{1+\theta} \mathcal{H} \partial_x^2(xu) w_N^{1+\theta} x u dx.$$

As before we write

$$(3.16) \quad \begin{aligned} w_N^{1+\theta} \mathcal{H} \partial_x^2(xu) &= -[\mathcal{H}; w_N^{1+\theta}] \partial_x^2(xu) + \mathcal{H}(w_N^{1+\theta} \partial_x^2(xu)) \\ &= K_1 + \mathcal{H}(\partial_x^2(w_N^{1+\theta} xu)) - 2\mathcal{H}(\partial_x w_N^{1+\theta} \partial_x(xu)) - \mathcal{H}(\partial_x^2 w_N^{1+\theta}(xu)) \\ &= K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Thus, by Theorem 6 and the results in the previous cases the contribution of K_1, K_4 in (3.15) is bounded. Also inserting the term K_2 in (3.15) one has by

integration by parts that its contribution is null. So in (3.16) it only remains to consider the contribution from K_3 in (3.15). But using that

$$(3.17) \quad \begin{aligned} \|K_3\|_2 &= \|\mathcal{H}(\partial_x w_N^{1+\theta} \partial_x(xu))\|_2 = \|\partial_x w_N^{1+\theta} \partial_x(xu)\|_2 \\ &\leq \|\partial_x w_N^{1+\theta} u\|_2 + \|\partial_x w_N^{1+\theta} x \partial_x u\|_2 \\ &\leq c(\|w_N^\theta u\|_2 + \|w_N^\theta x \partial_x u\|_2) = R_1 + R_2, \end{aligned}$$

since R_1 was previously bounded, it remains to estimate R_2 which is equal to the term F_2 in (3.14). To estimate this term we use the BO equation in (1.1) to obtain the new equation

$$(3.18) \quad \partial_t(x \partial_x u) + \mathcal{H} \partial_x^2(x \partial_x u) - 2\mathcal{H} \partial_x^2 u + x \partial_x(u \partial_x u) = 0.$$

The differential equation (3.18) multiplied by $w_N^{2\theta} x \partial_x u$ leads to the identity

$$(3.19) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int (w_N^\theta x \partial_x u)^2 dx + \int w_N^\theta \mathcal{H} \partial_x^2(x \partial_x u) w_N^\theta(x \partial_x u) dx \\ - 2 \int w_N^\theta \mathcal{H} \partial_x^2 u w_N^\theta x \partial_x u dx + \int w_N^\theta x \partial_x(u \partial_x u) w_N^\theta x \partial_x u dx = 0. \end{aligned}$$

Sobolev inequality and integration by parts lead to

$$(3.20) \quad \begin{aligned} \left| \int w_N^\theta x \partial_x(u \partial_x u) w_N^\theta x \partial_x u dx \right| \\ \leq c \|u\|_{2,2} \|w_N^\theta x \partial_x u\|_2 (\|w_N^\theta x \partial_x u\|_2 + \|w_N^\theta u\|_2), \end{aligned}$$

and since

$$(3.21) \quad \begin{aligned} w_N^\theta \mathcal{H} \partial_x^2(x \partial_x u) &= -[\mathcal{H}; w_N^\theta] \partial_x^2(x \partial_x u) + \mathcal{H}(w_N^\theta \partial_x^2(x \partial_x u)) \\ &= V_1 + \mathcal{H} \partial_x^2(w_N^\theta x \partial_x u) - 2\mathcal{H}(\partial_x w_N^\theta \partial_x(x \partial_x u)) - \mathcal{H}(\partial_x^2(w_N^\theta) x \partial_x u) \\ &= V_1 + V_2 + V_3 + V_4, \end{aligned}$$

Theorem 6, the previous results, and interpolation allow to bound the L^2 -norm of the terms V_1 and V_4 . As before by integration by parts the contribution of the term V_2 in (3.19) is null. So it just remains to consider the term V_3 in (3.21). In fact,

$$V_3 = -2\mathcal{H}(\partial_x w_N^\theta \partial_x u) - 2\mathcal{H}(\partial_x w_N^\theta (x \partial_x^2 u)) = V_{3,1} + V_{3,2},$$

so one just needs to handle the term $V_{3,2}$. Using that

$$|\partial_x w_N^\theta x| \leq c w_N^\theta, \quad c \text{ independent of } N,$$

it suffices to consider

$$(3.22) \quad \|w_N^\theta \partial_x^2 u\|_2 \leq c \|J^2(w_N^\theta u)\|_2 + c \|u\|_{1,2} + c \|w_N^\theta u\|_2,$$

with c independent on N . So we just need to consider the first term on the r.h.s. of the inequality (3.22). Using interpolation it follows that

$$(3.23) \quad \|J^2(w_N^\theta u)\|_2 \leq c \|J^{2+\theta} u\|_2^{2/(2+\theta)} \|w_N^{2+\theta} u\|_s^{\theta/(2+\theta)}.$$

We notice that the first term on the r.h.s. of (3.23) is bounded and the second one is bounded by the one we were estimating in (3.10). Therefore, (3.10) and (3.19) yield closed differential inequalities for $\|x w_N^{1+\theta} u\|_2$ and $\|w_N^\theta x \partial_x u\|_2$, and consequently the desired result.

Case 5: $s = r \in [5/2, 7/2)$. Part (iii) in Theorem 1.

First, by differentiating the BO equation in (1.1) one gets

$$\partial_t(\partial_x u) + \mathcal{H}\partial_x^2(\partial_x u) + u\partial_x(\partial_x u) + \partial_x u \partial_x u = 0,$$

so by reapplying the argument in the previous cases it follows that

$$(3.24) \quad \sup_{t \in [0, T]} \|\langle x \rangle^{s-1} \partial_x u(t)\|_2 \leq M,$$

with M depending on $\|u_0\|_{s,2}$, $\|\langle x \rangle^s u_0\|_2$, and T .

Next, we multiply the BO equation in (1.1) by $x^2 w_N^{\tilde{\theta}}$ with $\tilde{\theta} \in [1/2, 3/2)$ to get

$$(3.25) \quad \partial_t x^2 w_N^{\tilde{\theta}} u + x^2 w_N^{\tilde{\theta}} \mathcal{H} \partial_x^2 u + x^2 w_N^{\tilde{\theta}} u \partial_x u = 0,$$

so a familiar argument leads to

$$(3.26) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (x^2 w_N^{\tilde{\theta}} u)^2 dx + \int x^2 w_N^{\tilde{\theta}} \mathcal{H} \partial_x^2 u x^2 w_N^{\tilde{\theta}} u dx \\ & + \int x^2 w_N^{\tilde{\theta}} u \partial_x u x^2 w_N^{\tilde{\theta}} u dx = 0. \end{aligned}$$

Using the identity

$$x^2 \mathcal{H} \partial_x^2 u = \mathcal{H} \partial_x^2 (x^2 u) + 4\mathcal{H} \partial_x (xu) + \mathcal{H} u,$$

the linear dispersive part of (3.25) (the second term on the l.h.s. of (3.25)) can be written

$$(3.27) \quad \begin{aligned} w_N^{\tilde{\theta}} x^2 \mathcal{H} \partial_x^2 u &= w_N^{\tilde{\theta}} \mathcal{H} \partial_x^2 (x^2 u) + 4w_N^{\tilde{\theta}} \mathcal{H} \partial_x (xu) + w_N^{\tilde{\theta}} \mathcal{H} u \\ &= Q_1 + Q_2 + Q_3. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} u_0(x) dx = \int_{-\infty}^{\infty} u(x, t) dx = 0, \quad \text{then} \quad H(xu) = xHu,$$

for $\tilde{\theta} \in [1/2, 1]$ one has

$$\|Q_3\|_2 = \|w_N^{\tilde{\theta}} \mathcal{H} u\|_2 \leq \|(1 + |x|) \mathcal{H} u\|_2 \leq \|u\| + \|xu\|_2,$$

and for $\tilde{\theta} \in (1, 3/2)$ using Proposition 1

$$\begin{aligned} \|Q_3\|_2 &= \|w_N^{\tilde{\theta}} \mathcal{H} u\|_2 \leq \|(1 + |x|) w_N^{\tilde{\theta}-1} \mathcal{H} u\|_2 \\ &\leq \|w_N^{\tilde{\theta}-1} u\| + \|w_N^{\tilde{\theta}-1} x u\|_2, \end{aligned}$$

so in both cases by the previous results Q_3 in (3.27) is bounded in L^2 .

To control Q_2 we first consider the case $\tilde{\theta} \in [1/2, 1]$ and use Calderón commutator theorem to get

$$\begin{aligned} \|Q_2\|_2 &= 4 \|w_N^{\tilde{\theta}} \mathcal{H} \partial_x (xu)\|_2 \\ &\leq c(\|[\mathcal{H}; w_N^{\tilde{\theta}}] \partial_x (xu)\|_2 + \|\mathcal{H}(w_N^{\tilde{\theta}} \partial_x (xu))\|_2) \\ &\leq c(\|xu\|_2 + \|w_N^{\tilde{\theta}} x \partial_x u\|_2 + \|w_N^{\tilde{\theta}} u\|_2). \end{aligned}$$

Thus, in the case $\tilde{\theta} \in [1/2, 1]$, (3.24) provides the appropriate bound on the L^2 -norm of Q_2

For the case $\tilde{\theta} = 1 + \theta$, $\theta \in (0, 1/2)$ we combine Proposition 1 and the hypothesis on the mean value of u_0 to deduce that

$$\begin{aligned} \|Q_2\|_2 &= 4\|w_N^{\tilde{\theta}} \mathcal{H} \partial_x(xu)\|_2 \leq c(\|w_N^{\tilde{\theta}} x \mathcal{H} \partial_x(xu)\|_2 + \|w_N^{\tilde{\theta}} \mathcal{H} \partial_x(xu)\|_2) \\ &\leq c(\|w_N^{\tilde{\theta}} \mathcal{H}(x \partial_x(xu))\|_2 + \|w_N^{\tilde{\theta}} \partial_x(xu)\|_2) \\ &\leq c(\|w_N^{\tilde{\theta}} x \partial_x(xu)\|_2 + \|w_N^{\tilde{\theta}} \partial_x(xu)\|_2). \end{aligned}$$

Hence, (3.24) yields the appropriate bound on the L^2 -norm of Q_2 .

Finally, we turn to the contribution of the term Q_1 when inserted in (3.27). Thus, we write

$$\begin{aligned} w_N^{\tilde{\theta}} \mathcal{H} \partial_x^2(x^2 u) &= -[\mathcal{H}; w_N^{\tilde{\theta}}] \partial_x^2(x^2 u) + \mathcal{H}(w_N^{\tilde{\theta}} \partial_x^2(x^2 u)) \\ &= V_1 + \mathcal{H}(\partial_x^2(w_N^{\tilde{\theta}} x^2 u)) - 2\mathcal{H}(\partial_x w_N^{\tilde{\theta}} \partial_x(x^2 u)) - \mathcal{H}(\partial_x^2 w_N^{\tilde{\theta}}(x^2 u)) \\ &= V_1 + V_2 + V_3 + V_4. \end{aligned}$$

From the previous cases it follows that the L^2 -norm of the terms V_1, V_4 are bounded. By integration by parts, the contribution of the term V_2 is null. So it just remains to consider $V_3 = -2\mathcal{H}(\partial_x w_N^{\tilde{\theta}} \partial_x(x^2 u))$ in L^2 , but

$$\partial_x w_N^{\tilde{\theta}} \partial_x(x^2 u) = \partial_x w_N^{\tilde{\theta}}(x^2 \partial_x u + 2xu) = V_{2,1} + V_{2,2}.$$

Since $\tilde{\theta} \in (1/2, 3/2]$

$$\|V_{2,2}\|_2 \leq c\|\langle x \rangle^2 u\|_2$$

which has been found to be bounded in the previous cases. Now since

$$|\partial_x w_N^{\tilde{\theta}} x^2| \leq \langle x \rangle^{1+\tilde{\theta}},$$

it follows that

$$\|V_{2,1}\|_2 \leq \|\langle x \rangle^{1+\tilde{\theta}} \partial_x u\|_2,$$

so (3.24) gives the bound. Gathering the above information one completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

Without loss of generality we assume that $t_1 = 0 < t_2$.

Since $u(t_1) \in Z_{\frac{5}{2}, \frac{5}{2}}$, we have that $u \in C([0, T] : H^{2+1/2} \cap L^2(|x|^{5^-} dx))$.

Let us denote by $U(t)u_0 = (e^{-it|\xi|\xi} \widehat{u_0})^\vee$ the solution of the IVP for the linear equation associated to the BO equation with datum u_0 . Therefore, the solution to the IVP (1.1) can be represented by Duhamel's formula

$$(4.1) \quad u(t) = U(t)u_0 - \int_0^t U(t-t')u(t')\partial_x u(t')dt'.$$

From Plancherel's equality we have that for any t , $|x|^{2+1/2}U(t)u_0 \in L^2(\mathbb{R})$ if and only if $D_\xi^{1/2}\partial_\xi^2(e^{-it|\xi|\xi}\widehat{u_0}) \in L^2(\mathbb{R})$ and since

$$(4.2) \quad \partial_\xi^2(e^{-it|\xi|\xi}\widehat{u_0}) = -e^{-it|\xi|\xi}(4t^2\xi^2\widehat{u_0} + 2it\text{sgn}(\xi)\widehat{u_0} + 4it|\xi|\partial_\xi\widehat{u_0} - \partial_\xi^2\widehat{u_0}),$$

we show that with the hypothesis on the initial data, all terms in Duhamel's formula for our solution u except the one involving $\text{sgn}(\xi)$, arising from the linear part in (4.2), have the appropriate decay at a later time. The argument in our proof requires localizing near the origin in Fourier frequencies by a function $\chi \in C_0^\infty$, $\text{supp } \chi \subseteq (-\epsilon, \epsilon)$ and $\chi = 1$ on $(-\epsilon/2, \epsilon/2)$

Let us start with the computation for the linear part in (4.1) by introducing a commutator as follows

$$(4.3) \quad \begin{aligned} \chi D_\xi^{1/2} \partial_\xi^2 (e^{-it|\xi|\xi} \widehat{u}_0) &= [\chi; D_\xi^{1/2}] \partial_\xi^2 (e^{-it|\xi|\xi} \widehat{u}_0) + D_\xi^{1/2} (\chi \partial_\xi^2 (e^{-it|\xi|\xi} \widehat{u}_0)) \\ &= A + B. \end{aligned}$$

From Proposition 4 and identity (4.2) we have that

$$(4.4) \quad \begin{aligned} \|A\|_2 &= \|[\chi; D_\xi^{1/2}] \partial_\xi^2 (e^{-it|\xi|\xi} \widehat{u}_0)\|_2 \\ &\leq c \|\partial_\xi^2 (e^{-it|\xi|\xi} \widehat{u}_0)\|_2 \\ &\leq c (t^2 \|\xi^2 \widehat{u}_0\|_2 + t \|\operatorname{sgn}(\xi) \widehat{u}_0\|_2 + t \|\xi |\partial_\xi \widehat{u}_0\|_2 + \|\partial_\xi^2 \widehat{u}_0\|_2) \\ &\leq c (t^2 \|\partial_2^2 u_0\|_2 + t \|u_0\|_2 + t \|\partial_x(xu_0)\|_2 + \|x^2 u_0\|_2), \end{aligned}$$

which are all finite since $u_0 \in Z_{2,2}$.

On the other hand,

$$(4.5) \quad \begin{aligned} B &= D_\xi^{1/2} (\chi \partial_\xi^2 (e^{-it|\xi|\xi} \widehat{u}_0)) \\ &= 4D_\xi^{1/2} (\chi e^{-it|\xi|\xi} t^2 \xi^2 \widehat{u}_0) + 2iD_\xi^{1/2} (\chi e^{-it|\xi|\xi} t \operatorname{sgn}(\xi) \widehat{u}_0) \\ &\quad + 4iD_\xi^{1/2} (\chi e^{-it|\xi|\xi} t |\xi| \widehat{u}_0) - D_\xi^{1/2} (\chi e^{-it|\xi|\xi} \partial_\xi^2 \widehat{u}_0) \\ &= B_1 + B_2 + B_3 + B_4. \end{aligned}$$

Next, we shall estimate B_4 in $L^2(\mathbb{R})$. From Theorem 7, Proposition 2, and the fractional product rule type inequality (2.10) we get that

$$(4.6) \quad \begin{aligned} \|B_4\|_2 &\leq c (\|\chi e^{-it|\xi|\xi} \partial_\xi^2 \widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2} (\chi e^{-it|\xi|\xi} \partial_\xi^2 \widehat{u}_0)\|_2) \\ &\leq c (\|\partial_\xi^2 \widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2} (e^{-it|\xi|\xi}) \chi \partial_\xi^2 \widehat{u}_0\|_2 + \|e^{-it|\xi|\xi} \mathcal{D}_\xi^{1/2} (\chi \partial_\xi^2 \widehat{u}_0)\|_2) \\ &\leq c (\|x^2 u_0\|_2 + \|(t^{1/4} + t^{1/2} |\xi|^{1/2}) \chi \partial_\xi^2 \widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2} (\chi \partial_\xi^2 \widehat{u}_0)\|_2) \\ &\leq c(T) (\|x^2 u_0\|_2 + \|\mathcal{D}_\xi^{1/2} (\chi)\|_2 \|\partial_\xi^2 \widehat{u}_0\|_\infty + \|\chi\|_\infty \|\mathcal{D}_\xi^{1/2} (\partial_\xi^2 \widehat{u}_0)\|_2) \\ &\leq c(T) \|\langle x \rangle^{2+1/2} u_0\|_2. \end{aligned}$$

Estimates for B_1 and B_3 in $L^2(\mathbb{R})$ are easily obtained in a similar manner involving lower decay and regularity of the initial data. On the other hand for the analysis of B_2 we introduce $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\tilde{\chi} \equiv 1$ on $\operatorname{supp}(\chi)$. Then we can express this term as

$$(4.7) \quad \begin{aligned} D_\xi^{1/2} (\chi e^{-it|\xi|\xi} t \operatorname{sgn}(\xi) \widehat{u}_0) &= t D_\xi^{1/2} (e^{-it|\xi|\xi} \tilde{\chi} \chi \operatorname{sgn}(\xi) \widehat{u}_0) \\ &= t ([e^{-it|\xi|\xi} \tilde{\chi}, D_\xi^{1/2}] \chi \operatorname{sgn}(\xi) \widehat{u}_0 + e^{-it|\xi|\xi} \tilde{\chi} D_\xi^{1/2} (\chi \operatorname{sgn}(\xi) \widehat{u}_0)) \\ &= t(S_1 + S_2). \end{aligned}$$

Proposition 4 can be applied to bound S_1 in $L^2(\mathbb{R})$ as

$$(4.8) \quad \begin{aligned} \|S_1\|_2 &\leq \|[e^{-it|\xi|\xi} \tilde{\chi}, D_\xi^{1/2}] \chi \operatorname{sgn}(\xi) \widehat{u}_0\|_2 \\ &\leq c \|\chi \operatorname{sgn}(\xi) \widehat{u}_0\|_2 \\ &\leq c \|u_0\|_2. \end{aligned}$$

Therefore, once we show that the integral part in Duhamel's formula (4.1) lies in $L^2(|x|^5 dx)$, we will be able to conclude that

$$S_2, \tilde{\chi} D_\xi^{1/2} \chi \operatorname{sgn}(\xi) \widehat{u}_0, D_\xi^{1/2} (\tilde{\chi} \chi \operatorname{sgn}(\xi) \widehat{u}_0) \in L^2(\mathbb{R}),$$

then from Proposition 3 it will follow that $\widehat{u}_0(0) = 0$, and from the conservation law I_1 in (1.3), this would necessarily imply that $\widehat{u}_0(0) = \int u(x, t) dx = 0$.

As we just mentioned above, in order to complete the proof, we consider the integral part in Duhamel's formula. We localize again with the help of $\chi \in C_0^\infty(\mathbb{R})$ so that the integral in equation (4.1) after weights and a commutator reads now in Fourier space as

$$(4.9) \quad \begin{aligned} & \int_0^t ([\chi; D_\xi^{1/2}] (e^{-i(t-t')|\xi|\xi} (4(t-t')^2 \xi^2 \widehat{z} \\ & + 2i(t-t') \operatorname{sgn}(\xi) \widehat{z} + 4i(t-t') |\xi| \partial_\xi \widehat{z} - \partial_\xi^2 \widehat{z})) \\ & + D_\xi^{1/2} (\chi (e^{-i(t-t')|\xi|\xi} (4(t-t')^2 \xi^2 \widehat{z} \\ & + 2i(t-t') \operatorname{sgn}(\xi) \widehat{z} + 4i(t-t') |\xi| \partial_\xi \widehat{z} - \partial_\xi^2 \widehat{z}))) dt' \\ & = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 \end{aligned}$$

where $\widehat{z} = \frac{1}{2} \widehat{\partial_x u^2} = i \frac{\xi}{2} \widehat{u} * \widehat{u}$.

We limit our attention to the terms in (4.9) involving the highest order derivatives of u , i.e. \mathcal{A}_1 and \mathcal{B}_1 , and remark that the others can be treated in a similar way by using that the function \widehat{z} vanishes at $\xi = 0$.

Combining Proposition 4, Holder's inequality and Theorem 8 one has that

$$(4.10) \quad \begin{aligned} \|\mathcal{A}_1\|_{L_T^\infty L_x^2} &\leq c \|(t-t')^2 \xi^2 \widehat{u} * \widehat{u}\|_{L_T^1 L_x^2} \\ &\leq c(T) \|\partial_x^3(uu)\|_{L_T^2 L_x^2} \\ &\leq c(T) (\|u \partial_x^3 u\|_{L_T^2 L_x^2} + \|\partial_x u \partial_x^2 u\|_{L_T^2 L_x^2}) \\ &\leq c(T) (\|u\|_{l_k^2 L_T^\infty L_x^\infty(Q_k^T)} \|\partial_x^3 u\|_{l_k^\infty L_T^2 L_x^2(Q_k^T)} \\ &\quad + \|\partial_x u\|_{L_T^\infty L_x^\infty} \|\partial_x^2 u\|_{L_T^\infty L_x^2}) \\ &\leq c(T) (\|u\|_{l_k^2 L_T^\infty L_x^\infty(Q_k^T)} \|\partial_x^3 u\|_{l_k^\infty L_T^2 L_x^2(Q_k^T)} + \|u\|_{L_T^\infty H^2}^2), \end{aligned}$$

where $Q_k^T = [k, k+1] \times [0, T]$.

For \mathcal{B}_1 we obtain from Theorem 7

$$(4.11) \quad \begin{aligned} \|\mathcal{B}_1\|_{L_T^\infty L_x^2} &\leq c \int_0^T \|D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi \xi^2 \widehat{u} * \widehat{u})\|_2 dt \\ &\leq c (\|e^{-i(t-t')|\xi|\xi} \chi \xi^2 \widehat{u} * \widehat{u}\|_{L_T^1 L_x^2} + \|D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi \xi^2 \widehat{u} * \widehat{u})\|_{L_T^1 L_x^2}) \\ &= \mathcal{Y}_1 + \mathcal{Y}_2. \end{aligned}$$

These terms can be handled as follows

$$(4.12) \quad \mathcal{Y}_1 \leq c \|\widehat{u} * \widehat{u}\|_{L_T^1 L_x^2} \leq c \| \|u\|_\infty \|u\|_2 \|L_T^1\| \leq cT \sup_{[0, T]} \|u(t)\|_{1,2}^2,$$

and using Proposition 2, (2.9), (2.10), and (4.12)

$$\begin{aligned}
(4.13) \quad \mathcal{Y}_2 &= \|\mathcal{D}_\xi^{1/2}(e^{-i(t-t')|\xi|\xi}\chi\xi^2\xi\widehat{u} * \widehat{u})\|_{L_T^1 L_x^2} \\
&\leq c\|\mathcal{D}_\xi^{1/2}(e^{-i(t-t')|\xi|\xi}\chi\xi^2\xi\widehat{u} * \widehat{u})\|_{L_T^1 L_x^2} + c\|\mathcal{D}_\xi^{1/2}(\chi\xi^2\xi\widehat{u} * \widehat{u})\|_{L_T^1 L_x^2} \\
&\leq c\|(t^{1/2} + t^{1/2}|\xi|^{1/2})\chi\xi^2\xi\widehat{u} * \widehat{u}\|_{L_T^1 L_x^2} + c\|\mathcal{D}_\xi^{1/2}(\chi\xi^3)\|_\infty \|\widehat{u} * \widehat{u}\|_2 \| \cdot \|_{L_T^1} \\
&\quad + c\|\chi\xi^3\|_\infty \|\mathcal{D}_\xi^{1/2}(\widehat{u} * \widehat{u})\|_2 \| \cdot \|_{L_T^1} \\
&\leq c(T)\|\widehat{u} * \widehat{u}\|_{L_T^1 L_x^2} + c\|\mathcal{D}_\xi^{1/2}(\widehat{u} * \widehat{u})\|_{L_T^1 L_x^2} \\
&\leq c(T) \sup_{[0,T]} \|u(t)\|_{1,2}^2 + c(T)\| |x|^{1/2} u \|_{L_T^\infty L_x^2} \sup_{[0,T]} \|u(t)\|_{1,2}.
\end{aligned}$$

Hence the terms in (4.9) are all bounded, so by applying the argument after inequality (4.8) we complete the proof.

5. PROOF OF THEOREM 3

From the previous results and the hypothesis we have that for any $\epsilon > 0$

$$u \in C([0, T] : \dot{Z}_{7/2, 7/2-\epsilon}) \quad \text{and} \quad u(\cdot, t_j) \in H^{7/2}(\mathbb{R}), \quad j = 1, 2, 3.$$

Hence,

$$\widehat{u} \in C([0, T] : H^{7/2-\epsilon}(\mathbb{R}) \cap L^2(|\xi|^7 d\xi)) \quad \text{and} \quad \widehat{u}(\cdot, t_j) \in L^2(|\xi|^7 d\xi), \quad j = 1, 2, 3.$$

for any $\epsilon > 0$. Thus, in particular it follows that

$$(5.1) \quad \widehat{u} * \widehat{u} \in C([0, T] : H^6(\mathbb{R}) \cap L^2(|\xi|^7 d\xi)).$$

Let us assume that $t_1 = 0 < t_2 < t_3$. An explicit computation shows that

$$\begin{aligned}
(5.2) \quad F(t, \xi, \widehat{u}_0) &= \partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0) \\
&= e^{-it|\xi|\xi}(8it^3\xi^3\widehat{u}_0 - 12t^2\xi\widehat{u}_0 - 12t^2\xi^2\partial_\xi\widehat{u}_0 \\
&\quad - 6it\operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0 - 6it|\xi|\partial_\xi^2\widehat{u}_0 - 2it\delta\widehat{u}_0 + \partial_\xi^3\widehat{u}_0),
\end{aligned}$$

where we observe that since the initial data u_0 has zero mean value the term involving the Dirac function in (5.2) vanishes. Hence in order to prove our theorem, via Plancherel's Theorem and Duhamel's formula (4.1), it is enough to show that the assumption that

$$(5.3) \quad D_\xi^{1/2}F(t, \xi, \widehat{u}_0) - \int_0^t D_\xi^{1/2}F(t-t', \xi, \widehat{z}(t')) dt',$$

lies in $L^2(\mathbb{R})$ for times $t_1 = 0 < t_2 < t_3$, where $\widehat{z} = \frac{1}{2}\widehat{\partial_x u^2} = i\frac{\xi}{2}\widehat{u} * \widehat{u}$, leads to a contradiction. Let us show that the first term in equation (5.3) which arises from the linear part in Duhamel's formula persists in L^2

We proceed as in the proof of Theorem 2 and localize one more time by introducing $\chi \in C_0^\infty$, $\operatorname{supp} \chi \subseteq (-\epsilon, \epsilon)$ and $\chi = 1$ on $(-\epsilon/2, \epsilon/2)$ so that

$$\begin{aligned}
(5.4) \quad \chi D_\xi^{1/2}\partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0) &= [\chi; D_\xi^{1/2}]\partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0) + D_\xi^{1/2}(\chi\partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0)) \\
&= \widetilde{A} + \widetilde{B}.
\end{aligned}$$

As for the first term, \tilde{A} , from Proposition 4, this is bounded in $L^2(\mathbb{R})$ by $\|\partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0)\|_2$, which is finite as can easily be observed from its explicit representation in (5.2), the assumption on the initial data u_0 , and the quite similar computation already performed in (4.4), therefore we omit the details.

On the other hand, for \tilde{B} , we notice that

$$\begin{aligned}
(5.5) \quad \tilde{B} &= D_\xi^{1/2}(\chi\partial_\xi^3(e^{-it|\xi|\xi}\widehat{u}_0)) \\
&= 8iD_\xi^{1/2}(\chi e^{-it|\xi|\xi}t^3|\xi|^3\widehat{u}_0) - 12D_\xi^{1/2}(\chi e^{-it|\xi|\xi}t^2\xi\widehat{u}_0) \\
&\quad - 12D_\xi^{1/2}(\chi e^{-it|\xi|\xi}t^2\xi^2\partial_\xi\widehat{u}_0) - 6iD_\xi^{1/2}(\chi e^{-it|\xi|\xi}t\operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0) \\
&\quad - 6iD_\xi^{1/2}(\chi e^{-it|\xi|\xi}t|\xi|\partial_\xi^2\widehat{u}_0) + D_\xi^{1/2}(\chi e^{-it|\xi|\xi}\partial_\xi^3\widehat{u}_0) \\
&= \tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3 + \tilde{B}_4 + \tilde{B}_5 + \tilde{B}_7.
\end{aligned}$$

Notice that from the remark made after the identity (5.2) \tilde{B}_6 does not appear, and that \tilde{B}_1 and \tilde{B}_7 are the terms involving the highest regularity and decay of the initial data. Therefore we show in detail their L^2 estimates along with the argument to exploit a nice cancellation property of \tilde{B}_4 , and a term arising in the integral part in Duhamel's formula (4.1).

For \tilde{B}_1 we obtain from Theorem 7, fractional product rule type estimate (2.9), (2.10), and Holder's inequality that

$$\begin{aligned}
(5.6) \quad \|\tilde{B}_1\|_2 &\leq c(\|\chi e^{-it|\xi|\xi}t^3\xi^3\widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\chi e^{-it|\xi|\xi}t^3|\xi|^3\widehat{u}_0)\|_2) \\
&\leq ct^3(\|u_0\|_2 + \|\mathcal{D}_\xi^{1/2}(e^{-it|\xi|\xi})\chi|\xi|^3\widehat{u}_0\|_2 + \|e^{-it|\xi|\xi}\mathcal{D}_\xi^{1/2}(\chi|\xi|^3\widehat{u}_0)\|_2) \\
&\leq ct^3(\|u_0\|_2 + \|(t^{1/4} + t^{1/2}|\xi|^{1/2})\chi|\xi|^3\widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\chi|\xi|^3\widehat{u}_0)\|_2) \\
&\leq c(T)(\|u_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\chi\xi^3)\|_\infty\|\widehat{u}_0\|_2 + \|\chi\xi^3\|_\infty\|\mathcal{D}_\xi^{1/2}\widehat{u}_0\|_2) \\
&\leq c(T)(\|u_0\|_2 + \|x|^{1/2}u_0\|_2),
\end{aligned}$$

and similarly

$$\begin{aligned}
(5.7) \quad \|\tilde{B}_7\|_2 &\leq c(\|\chi e^{-it|\xi|\xi}\partial_\xi^3\widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\chi e^{-it|\xi|\xi}\partial_\xi^3\widehat{u}_0)\|_2) \\
&\leq c(\|\partial_\xi^3\widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2}(e^{-it|\xi|\xi})\chi\partial_\xi^3\widehat{u}_0\|_2 + \|e^{-it|\xi|\xi}\mathcal{D}_\xi^{1/2}(\chi\partial_\xi^3\widehat{u}_0)\|_2) \\
&\leq c(\|x^3u_0\|_2 + \|(t^{1/4} + t^{1/2}|\xi|^{1/2})\chi\partial_\xi^3\widehat{u}_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\chi\partial_\xi^3\widehat{u}_0)\|_2) \\
&\leq c(t)(\|x^3u_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\chi)\|_\infty\|\partial_\xi^3\widehat{u}_0\|_2 + \|\chi\|_{L^\infty}\|\mathcal{D}_\xi^{1/2}(\partial_\xi^3\widehat{u}_0)\|_2) \\
&\leq c(T)(\|x^3u_0\|_2 + \|\mathcal{D}_\xi^{1/2}(\partial_\xi^3\widehat{u}_0)\|_2) \\
&\leq c(T)\|\langle x \rangle^{3+1/2}u_0\|_2.
\end{aligned}$$

Now, let us go over the integral part that can be written in Fourier space and with the help of a commutator as

$$\begin{aligned}
(5.8) \quad & \int_0^t ([\chi; D_\xi^{1/2}](e^{-i(t-t')|\xi|\xi}(8i(t-t')^3\xi^3\widehat{z} - 12(t-t')^2\xi\widehat{z}) \\
& - 12(t-t')^2\xi^2\partial_\xi\widehat{z} - 6i(t-t')|\xi|\partial_\xi^2\widehat{z} - 2i(t-t')\delta\widehat{z} + \partial_\xi^3\widehat{z})) \\
& + D_\xi^{1/2}(\chi(8i(t-t')^3\xi^3\widehat{z} - 12(t-t')^2\xi\widehat{z} \\
& - 12(t-t')^2\xi^2\partial_\xi\widehat{z} - 6i(t-t')|\xi|\partial_\xi^2\widehat{z} - 2i(t-t')\delta\widehat{z} + \partial_\xi^3\widehat{z})) dt \\
& = \widetilde{\mathcal{A}}_1 + \widetilde{\mathcal{A}}_2 + \widetilde{\mathcal{A}}_3 + \widetilde{\mathcal{A}}_5 + \widetilde{\mathcal{A}}_6 + \widetilde{\mathcal{A}}_7 + \widetilde{\mathcal{B}}_1 + \widetilde{\mathcal{B}}_2 + \widetilde{\mathcal{B}}_3 + \widetilde{\mathcal{B}}_5 + \widetilde{\mathcal{B}}_6 + \widetilde{\mathcal{B}}_7 + \widetilde{\mathcal{C}},
\end{aligned}$$

where

$$(5.9) \quad \widetilde{\mathcal{C}} = -6i \int_0^t D_\xi^{1/2}(e^{-i(t-t')|\xi|\xi}\chi(t-t')\operatorname{sgn}(\xi)\partial_\xi\widehat{z}) dt',$$

and $\widehat{z} = \frac{1}{2}\widehat{\partial_x u^2} = i\frac{\xi}{2}\widehat{u} * \widehat{u}$.

Notice that $\widetilde{\mathcal{A}}_6, \widetilde{\mathcal{B}}_6$ vanish since $u\partial_x u$ has zero mean value and for $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \widetilde{\mathcal{A}}_3, \widetilde{\mathcal{A}}_5, \widetilde{\mathcal{A}}_7, \widetilde{\mathcal{B}}_1, \widetilde{\mathcal{B}}_2, \widetilde{\mathcal{B}}_3, \widetilde{\mathcal{B}}_5$ and $\widetilde{\mathcal{B}}_7$ the estimates in $L^2(\mathbb{R})$ are essentially the same for their counterparts in equation (4.9), in the proof of Theorem 2, so we omit the details of their estimates.

Therefore from the assumption that $u_0, u(t_2) \in \dot{Z}_{\frac{7}{2}, \frac{7}{2}}$, equation (5.8), and the estimates above, we conclude that

$$\begin{aligned}
(5.10) \quad R &= -6iD_\xi^{1/2}(e^{-it|\xi|\xi}\chi t \operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0) - \widetilde{\mathcal{C}} \\
&= -6iD_\xi^{1/2}(e^{-it|\xi|\xi}\chi t \operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0) \\
&\quad + 6i \int_0^t D_\xi^{1/2}(e^{-i(t-t')|\xi|\xi}\chi(t-t')\operatorname{sgn}(\xi)\partial_\xi(\frac{i\xi}{2}\widehat{u} * \widehat{u})) dt',
\end{aligned}$$

is a function in $L^2(\mathbb{R})$ at time $t = t_2$. But

$$\begin{aligned}
(5.11) \quad R &= 6i \int_0^t (D_\xi^{1/2}(e^{-i(t-t')|\xi|\xi}\chi(t-t')\operatorname{sgn}(\xi) \\
&\quad (\partial_\xi(\frac{i\xi}{2}\widehat{u} * \widehat{u}) - \partial_\xi(\frac{i\xi}{2}\widehat{u} * \widehat{u})(0))) dt' \\
&\quad - 6iD_\xi^{1/2}(e^{-it|\xi|\xi}\chi t \operatorname{sgn}(\xi)(\partial_\xi\widehat{u}_0 - \partial_\xi\widehat{u}_0(0))) \\
&\quad - 6iD_\xi^{1/2}(e^{-it|\xi|\xi}\chi t \operatorname{sgn}(\xi)\partial_\xi\widehat{u}_0(0)) \\
&\quad + 6i \int_0^t D_\xi^{1/2}(e^{-i(t-t')|\xi|\xi}\chi(t-t')\operatorname{sgn}(\xi)(\partial_\xi(\frac{i\xi}{2}\widehat{u} * \widehat{u}(0))) dt' \\
&= R_1 + R_2 + R_3 + R_4.
\end{aligned}$$

We shall show that R_1 and R_2 are $L^2(\mathbb{R})$ functions. This will imply that $(R_3 + R_4)(t_2)$ is also an $L^2(\mathbb{R})$ function.

For R_1 we observe that from (5.1)

$$e^{-i(t-t')|\xi|\xi}\chi(\xi)\operatorname{sgn}(\xi)(\partial_\xi(\frac{i\xi}{2}\widehat{u} * \widehat{u})(\xi, t') - \partial_\xi(\frac{i\xi}{2}\widehat{u} * \widehat{u})(0, t'))$$

is a Lipschitz function with compact support in the ξ variable. Therefore, using Theorem 7 one sees that $R_1(t) \in L^2(\mathbb{R})$. A similar argument shows that $R_2(t) \in L^6(\mathbb{R})$. Therefore, we have that $(R_3 + R_4)(t_2) \in L^2(\mathbb{R})$.

On the other hand

$$\partial_\xi \left(\frac{i\xi}{2} \widehat{u} * \widehat{u} \right) (0) = -i \widehat{xu \partial_x u} (0) = -i \int xu \partial_x u dx = \frac{i}{2} \|u\|_2^2,$$

and from the Benjamin-Ono equation we have

$$(5.12) \quad \frac{d}{dt} \int x u dx + \int x \partial_x^2 \mathcal{H} u dx + \int x u \partial_x u dx = 0,$$

which implies that

$$(5.13) \quad \frac{d}{dt} \int x u dx = - \int x u \partial_x u dx = \frac{1}{2} \|u_0\|_2^2,$$

and hence

$$\partial_\xi \left(\frac{i\xi}{2} \widehat{u} * \widehat{u} \right) (0) = i \frac{d}{dt} \int x u dx.$$

Substituting this into R_4 gives us after integration by parts

$$(5.14) \quad \begin{aligned} R_4 &= -6 \int_0^t D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi(t-t') \operatorname{sgn}(\xi) \left(\frac{d}{dt'} \int x u dx \right)) dt' \\ &= -6 D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi(t-t') \operatorname{sgn}(\xi) \int x u dx \Big|_{t'=0}^{t'=t}) \\ &\quad + 6 \int_0^t D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi(t-t') (i|\xi|\xi(t-t') - 1) \operatorname{sgn}(\xi) \left(\int x u dx \right)) dt' \\ &= 6 D_\xi^{1/2} (e^{-it|\xi|\xi} \chi t \operatorname{sgn}(\xi) \int x u_0(x) dx) \\ &\quad + 6i \int_0^t D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi(t-t') |\xi|\xi \operatorname{sgn}(\xi) \left(\int x u dx \right)) dt' \\ &\quad - 6 \int_0^t D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi \operatorname{sgn}(\xi) \left(\int x u dx \right)) dt'. \end{aligned}$$

We observe that the second term after the last equality in (5.14) belongs to $L^2(\mathbb{R})$ and the first cancels out with R_3 since

$$(5.15) \quad \partial_\xi \widehat{u}_0(0) = -i \widehat{xu_0}(0) = -i \int x u_0(x) dx,$$

and therefore

$$(5.16) \quad R_3 = -6 D_\xi^{1/2} (e^{-it|\xi|\xi} \chi t \operatorname{sgn}(\xi) \int x u_0(x) dx).$$

So the argument above implies that

$$(5.17) \quad -6 \int_0^t D_\xi^{1/2} (e^{-i(t-t')|\xi|\xi} \chi \operatorname{sgn}(\xi) \left(\int x u(x, t') dx \right)) dt'$$

is in $L^2(\mathbb{R})$ at time $t = t_2$, and from Theorem 7 this is equivalent to have that

$$(5.18) \quad D_\xi^{1/2} (\chi(\xi) \operatorname{sgn}(\xi) \int_0^{t_2} e^{-i(t_1-t')|\xi|\xi} \int x u(x, t') dx dt') \in L^2(\mathbb{R}),$$

which from Proposition 3 (choosing the support $(-\epsilon, \epsilon)$ of χ sufficiently small) implies that $\int_0^{t_2} \left(\int x u(x, t') dx \right) dt' = 0$ and consequently $\int x u(x, t) dx$ must be zero at some time in $(0, t_2)$. We reapply the same argument to conclude that $\int x u(x, t) dx$ is again zero at some other time in (t_2, t_3) . Finally, the identity (1.11) completes the proof of the theorem.

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