HARDY UNCERTAINTY PRINCIPLE, CONVEXITY AND PARABOLIC EVOLUTIONS

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Abstract. We give a new proof of the L^2 version of Hardy's uncertainty principle based on calculus and on its dynamical version for the heat equation. The reasonings rely on new log-convexity properties and the derivation of optimal Gaussian decay bounds for solutions to the heat equation with Gaussian decay at a future time. We extend the result to heat equations with lower order variable coefficient.

1. Introduction

In this paper we continue the study in [18, 6, 8, 9, 10, 11] related to the Hardy uncertainty principle and its relation to unique continuation properties for some evolutions.

One of our motivations came from a well known result due to G. H. Hardy ([14], [21, pp. 131]), which concerns the decay of a function f and its Fourier transform,

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

If $f(x) = O(e^{-|x|^2/\beta^2})$, $\widehat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $1/\alpha\beta > 1/4$, then $f \equiv 0$. Also, if $1/\alpha\beta = 1/4$, f is a constant multiple of $e^{-|x|^2/\beta^2}$.

As far as we know, the known proofs for this result and its variants - before the one in [18, 6, 9, 10, 11] - use complex analysis (the Phragmén-Lindelöf principle). There has also been considerable interest in a better understanding of this result and on extensions of it to other settings: [3], [15], [20], [1] and [2].

The result can be rewritten in terms of the free solution of the Schrödinger equation

$$i\partial_t u + \triangle u = 0, \text{ in } \mathbb{R}^n \times (0, +\infty),$$

with initial data f,

$$u(x,t) = (4\pi i t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} f(y)\, dy = (2\pi i t)^{-\frac{n}{2}}\, e^{\frac{i|x|^2}{4t}}\, \widehat{e^{\frac{i|\cdot|^2}{4t}} f}\left(\frac{x}{2t}\right),$$

in the following way:

If $u(x,0) = O(e^{-|x|^2/\beta^2})$, $u(x,T) = O(e^{-|x|^2/\alpha^2})$ and $T/\alpha\beta > 1/4$, then $u \equiv 0$. Also, if $T/\alpha\beta = 1/4$, u has as initial data a constant multiple of $e^{-\left(1/\beta^2 + i/4T\right)|y|^2}$.

The corresponding results in terms of L^2-norms, established in [4], are the following:

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If $e^{|x|^2/\beta^2} f$, $e^{4|\xi|^2/\alpha^2} \widehat{f}$ *are in* $L^2(\mathbb{R}^n)$ *and* $1/\alpha\beta \geq 1/4$, *then* $f \equiv 0$.

If $e^{|x|^2/\beta^2} u(x,0)$, $e^{|x|^2/\alpha^2} u(x,T)$ *are in* $L^2(\mathbb{R}^n)$ *and* $T/\alpha\beta \geq 1/4$, *then* $u \equiv 0$.

In [10] we proved a uniqueness result in this direction for variable coefficients Schrödinger evolutions

$$\partial_t u = i\left(\triangle u + V(x,t)u\right), \text{ in } \mathbb{R}^n \times [0,T]. \tag{1.1}$$

with bounded potentials V verifying, $V(x,t) = V_1(x) + V_2(x,t)$, with V_1 real-valued and

$$\sup_{[0,T]} \|e^{T^2|x|^2/(\alpha t + \beta(T-t))^2} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$$

or

$$\lim_{R \to +\infty} \int_0^T \|V(t)\|_{L^\infty(\mathbb{R}^n \setminus B_R)}\, dt = 0.$$

More precisely, we showed that the only solution u to (1.1) in $C([0,T], L^2(\mathbb{R}^n))$, which verifies

$$\|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

is the zero solution, when $T/\alpha\beta > 1/4$. When $T/\alpha\beta = 1/4$, we found a complex valued potential potential V with

$$|V(x,t)| \lesssim \frac{1}{1+|x|^2}, \text{ in } \mathbb{R}^n \times [0,T]$$

and a nonzero smooth solution u in $C^\infty([0,T], \mathcal{S}(\mathbb{R}^n))$ of (1.1) with

$$\|e^{|x|^2/\beta^2} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/\alpha^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty.$$

Thus, we established in [10] that the optimal version of Hardy's Uncertainty Principle in terms of L^2-norms holds for solutions to (1.1) holds when $T/\alpha\beta > 1/4$ for many general bounded potentials, while it can fail for some complex-valued potentials in the end-point case, $T/\alpha\beta = 1/4$. Finally, in [11] we showed that the reasonings in [18, 6, 8, 9, 10, 11] provide the first proof (up to the end-point case) that we know of Hardy's uncertainty principle for the Fourier transform without the use of holomorphic functions.

The Hardy uncertainty principle also has a dynamical version associated to the heat equation,

$$\partial_t u - \Delta u = 0, \text{ in } \mathbb{R}^n \times (0, +\infty),$$

with initial data f,

$$u(x,t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\, dy, \ \widehat{u}(\xi,t) = e^{-t|\xi|^2} \widehat{f}(\xi), \ x, \ \xi \in \mathbb{R}^n, \ t > 0.$$

In particular, its L^∞ and L^2 versions yield the following statements:

If $u(0)$ *is a finite measure in* $\mathbb{R}^n$, $u(x,T) = O(e^{-|x|^2/\delta^2})$ *and* $\delta < \sqrt{4T}$, *then* $f \equiv 0$. *Also, if* $\delta = \sqrt{4T}$, *then* $u(0)$ *is a multiple of the Dirac delta function.*

If $u(0)$ *is in* $L^2(\mathbb{R}^n)$, $\|e^{|x|^2/\delta^2} u(T)\|_{L^2(\mathbb{R}^n)}$ *is finite and* $\delta \leq \sqrt{4T}$, *then* $u \equiv 0$.

In [9, Theorem 4] we proved that a dynamical L^2-version of Hardy uncertainty principle holds for solutions u in $C([0,T], L^2(\mathbb{R}^n)) \cap L^2([0,T], H^1(\mathbb{R}^n))$ to

$$\partial_t u = \Delta u + V(x,t)u, \text{ in } \mathbb{R}^n \times [0,T], \tag{1.2}$$

when V is any bounded complex potential in $\mathbb{R}^n \times [0,T]$ and $\delta < \sqrt{T}$. Here, we find the optimal interior Gaussian decay over $[0,1]$ for solutions to (1.2) with

$$\|e^{|x|^2/\delta^2} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty,$$

when $\delta > \sqrt{4T}$ and derive from it the full dynamical L^2 version of the Hardy uncertainty principle for solutions to (1.2), reaching the end-point case, $\delta = \sqrt{4T}$.

Theorem 1. *Assume that u in $C([0,T], L^2(\mathbb{R}^n)) \cap L^2([0,T], H^1(\mathbb{R}^n))$ verifies (1.2) with V in $L^\infty(\mathbb{R}^n \times [0,T])$. Assume that*

$$\|e^{T|x|^2/4(T^2+R^2)} u(T)\|_{L^2(\mathbb{R}^n)} < +\infty \tag{1.3}$$

for some $R > 0$. Then, there is a universal constant N such that

$$\begin{aligned} &\sup_{[0,T]} \|e^{t|x|^2/4(t^2+R^2)} u(t)\|_{L^2(\mathbb{R}^n)} \\ &\le e^{N\left(1+T^2\|V\|^2_{L^\infty(\mathbb{R}^n\times[0,T])}\right)} \left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{T|x|^2/4(T^2+R^2)} u(T)\|_{L^2(\mathbb{R}^n)}\right]. \end{aligned} \tag{1.4}$$

Moreover, u must be identically zero when $\|e^{|x|^2/4T} u(T)\|_{L^2(\mathbb{R}^n)}$ is finite.

Theorem 1 is optimal because

$$u_R(x,t) = (t - iR)^{-\frac{n}{2}} e^{-|x|^2/4(t-iR)} = (t - iR)^{-\frac{n}{2}} e^{-(t+iR)|x|^2/4(t^2+R^2)}, \tag{1.5}$$

is a solution to the heat equation and for each fixed $t > 0$, $t/4(t^2+R^2)$ is decreasing in the R-variable for $R > 0$. Also, observe that $t/4(t^2+R^2)$ attains its maximum value in the interior of $[0,T]$, when $R \neq T$,

Notice that the finiteness condition on condition on $\|e^{|x|^2/4T} u(T)\|_{L^2(\mathbb{R}^n)}$ is independent of the size of the potential or the dimension and that we do not assume any regularity or strong decay of the potentials.

This improvement of our results in [9, Theorem 4] on the relation between Hardy uncertainty principle and its dynamical version for parabolic evolutions comes from a better understanding of the solutions to (1.2) which have Gaussian decay and of the adaptation to the parabolic context of the same kind of log-convexity arguments that we used in [10] to derive the dynamical version of the Hardy uncertainty principle for Schrödinger evolutions.

We have not tried to extend the results in Theorems 1 to parabolic evolutions with nonzero drift terms

$$\partial_t u = \Delta u + W(x,t)\cdot\nabla u + V(x,t)u. \tag{1.6}$$

We expect that similar methods will yield analogue results for solutions to (1.6) (See [5] for initial results following the approach initiated in [18] and [6] for the case of Schr̈odiger evolutions).

In what follows, N denotes a universal constant depending at most on the dimension, $N_{a,\xi,\dots}$ a constant depending on the parameters $a, \xi, \dots$ In section 2 we give three Lemmas which are necessary for our proof in section 3 of Theorem 1.

2. A few Lemmas

In the sequel

$$(f,g) = \int_{\mathbb{R}^n} f\overline{g}\,dx\ ,\ \|f\|^2 = (f,f) \text{ and } \|V\|_\infty = \|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}.$$

In Lemma 1, $\mathcal{S}$ and $\mathcal{A}$ denote respectively a symmetric and skew-symmetric bounded linear operators on $\mathcal{S}(\mathbb{R}^n)$. Both are allowed to depend smoothly on the time-variable, $\mathcal{S}_t = \partial_t \mathcal{S}$ and $[\mathcal{S}, \mathcal{A}]$ is the space commutator of $\mathcal{S}$ and $\mathcal{A}$. The reader can find a proof of Lemma 1 in [10, Lemma 2].

Lemma 1. *Let $\mathcal{S}$ and $\mathcal{A}$ be as above, f lie in $C^\infty([c,d], \mathcal{S}(\mathbb{R}^n))$ and $\gamma : [c,d] \longrightarrow (0, +\infty)$ be a smooth function such that*

$$(\gamma\, \mathcal{S}_t f(t) + \gamma\, [\mathcal{S}, \mathcal{A}]\, f(t) + \dot{\gamma}\, \mathcal{S} f(t), f(t)) \geq 0, \ \textit{when } c \leq t \leq d.$$

Then, if $H(t) = \|f(t)\|^2$ and $\epsilon > 0$

$$H(t) + \epsilon \leq (H(c) + \epsilon)^{\theta(t)}\, (H(d) + \epsilon)^{1-\theta(t)}\, e^{M_\epsilon(t) + 2N_\epsilon(t)}, \ \textit{when } c \leq t \leq d,$$

where M_ϵ verifies

$$\partial_t\left(\gamma\, \partial_t M_\epsilon\right) = -\gamma\, \frac{\|\partial_t f - \mathcal{S} f - \mathcal{A} f\|^2}{H + \epsilon}\,, \ \textit{in } [c,d], \quad M_\epsilon(c) = M_\epsilon(d) = 0,$$

$$N_\epsilon = \int_c^d \left| Re\, \frac{(\partial_s f(s) - \mathcal{S} f(s) - \mathcal{A} f(s), f(s))}{H(s) + \epsilon} \right| ds$$

and

$$\theta(t) = \frac{\int_t^d \frac{ds}{\gamma}}{\int_c^d \frac{ds}{\gamma}}\,.$$

A calculation (see formulae (2.12), (2.13) and (2.14) in [9] with $\gamma = 1$) shows that given smooth functions $a : [0,1] \longrightarrow [0, +\infty)$, $b : [0,1] \longrightarrow \mathbb{R}$ and $T : [0,1] \longrightarrow \mathbb{R}$, and ξ in $\mathbb{R}^n$

$$e^{a(t)|x|^2 + b(t) x\cdot\xi - T(t)|\xi|^2}\left(\partial_t - \triangle\right) e^{-a(t)|x|^2 - b(t)x\cdot\xi + T(t)|\xi|^2} = \partial_t - \mathcal{S} - \mathcal{A},$$

where $\mathcal{S}$ and $\mathcal{A}$ are the symmetric and skew-symmetric linear bounded operators on $\mathcal{S}(\mathbb{R}^n)$ given by

$$\mathcal{S} = \Delta + \left(a' + 4a^2\right)|x|^2 + (b' + 4ab)\, x\cdot\xi + \left(b^2 - T'\right)|\xi|^2, \tag{2.1}$$

$$\mathcal{A} = -2\left(2ax + b\,\xi\right)\cdot\nabla - 2na. \tag{2.2}$$

and

$$\begin{aligned}\mathcal{S}_t + [\mathcal{S}, \mathcal{A}] = -8a\,\Delta + \left(a'' + 16aa' + 32a^3\right)|x|^2 \\ + \left(b'' + 8ab' + 8a'b + 32a^2 b\right) x\cdot\xi + \left(8ab^2 + 4bb' - T''\right)|\xi|^2.\end{aligned} \tag{2.3}$$

In Lemma 2 we make choices of a, b and T which make non-negative the self-adjoint operator

$$e^{8A}\left(\mathcal{S}_t + [\mathcal{S}, \mathcal{A}]\right) + \left(e^{8A}\right)' \mathcal{S},$$

where A denotes an anti-derivative of a in $[0,1]$ with $A(1) = 0$, .

Lemma 2. *Let $a : [0,1] \longrightarrow \mathbb{R}$ be a smooth function verifying*

$$\left(e^{8A} a\right)'' \geq 0, \ \textit{in } [0,1], \tag{2.4}$$

and let b and T be the solutions to

$$\begin{cases} \left(e^{8A} b\right)'' = 2\left(e^{8A} a\right)'', \ \textit{in } [0,1], \\ b(0) = b(1) = 0, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \left(e^{8A}T'\right)' = 2\left(e^{8A}b^2\right)' - \left(e^{8A}a\right)'', \ \textit{in } [0,1], \\ T(0) = T(1) = 0. \end{cases} \tag{2.6}$$

Then,

$$\left(e^{8A}\mathcal{S}_t f + e^{8A}\left[\mathcal{S}, \mathcal{A}\right] f + \left(e^{8A}\right)' \mathcal{S} f, f\right) \geq 0, \ \textit{when } f \in \mathcal{S}(\mathbb{R}^n) \ \textit{and } 0 \leq t \leq 1.$$

Proof. From (2.1), (2.3), the identities

$$\begin{aligned} \left(e^{8A}a\right)'' &= e^{8A}\left(a'' + 24aa' + 64a^3\right). \\ \left(e^{8A}b\right)'' &= e^{8A}\left(b'' + 16ab' + 8a'b + 64a^2b\right). \\ \left(e^{8A}b^2\right)' &= e^{8A}\left(8ab^2 + 2bb'\right), \end{aligned}$$

and the definitions of b and T, we have

$$\begin{aligned} &e^{8A}\left(\mathcal{S}_t + [\mathcal{S}, \mathcal{A}]\right) + \left(e^{8A}\right)' \mathcal{S} \\ &\quad = \left(e^{8A}a\right)'' |x|^2 + \left(e^{8A}b\right)'' x \cdot \xi + \left(2\left(e^{8A}b^2\right)' - \left(e^{8A}T'\right)'\right)|\xi|^2 \\ &\quad = \left(e^{8A}a\right)''\left(|x|^2 + 2x\cdot\xi + |\xi|^2\right) = \left(e^{8A}a\right)'' |x+\xi|^2. \end{aligned}$$

The later and (2.4) implies Lemma 2. □

In the next Lemma we assume that u in $C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ verifies (1.2) in $\mathbb{R}^n \times (0,1]$ and

$$\|e^{|x|^2/\delta^2}u(1)\| < +\infty.$$

Lemma 3. *Let $a : [0,1] \longrightarrow [0,+\infty)$ be a smooth function with $a(0) = 0$, $a(1) = 1/\delta^2$, $\left(e^{8A}a\right)'' > 0$ in $[0,1]$ and*

$$\sup_{[0,1]} \|e^{(a(t)-\epsilon)|x|^2}u(t)\| < +\infty, \ \textit{when } 0 < \epsilon \leq 1.$$

Then, there is a universal constant N such that for b and T as in (2.5) and (2.6),

$$\|e^{a(t)|x|^2 + b(t)x\cdot\xi - T(t)|\xi|^2}u(t)\| \leq e^{N\left(1+\|V\|_\infty^2\right)}\left(\|u(0)\| + \|e^{|x|^2/\delta^2}u(1)\|\right),$$

when ξ is in $\mathbb{R}^n$ and $0 \leq t \leq 1$.

Proof. For ξ in $\mathbb{R}^n$ and $\epsilon > 0$, set

$$f_\epsilon(x,t) = e^{a_\epsilon|x|^2 + b_\epsilon x\cdot\xi - T_\epsilon|\xi|^2}u(x,t),$$

with $a_\epsilon = a - \epsilon$, $A_\epsilon = A + \epsilon(1-t)$, and with b_ϵ and T_ϵ as in Lemma 2 but with a and A replaced by a_ϵ and A_ϵ respectively. The local Schauder estimates for solutions to (1.2) show that

$$\begin{aligned} &r|\nabla u(x,t)| + r^2\left(\fint_{B_r(x)\times(t-r^2,t]} |\partial_s u|^p + |D^2u|^p \, dyds\right)^{\frac{1}{p}} \\ &\qquad \leq N_p\left(1 + r^2\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}\right)\fint_{B_{2r}(x)\times(t-4r^2,t]} |u| \, dyds \end{aligned}$$

for $1 < p < \infty$, $0 < r \le \sqrt{t}/2$, $0 < t \le 1$. Thus, f_ϵ is in $W_2^{2,1}(\mathbb{R}^n \times [\varrho, 1])$ and verifies

$$\begin{aligned} &\sup_{[0,1]} \|f_\epsilon(t)\| \le N_{a,\epsilon,\xi} \sup_{[0,1]} \|e^{\left(a-\frac{\epsilon}{2}\right)|x|^2} u(t)\|, \\ &\sup_{[\varrho,1]} \|\nabla f_\epsilon(t)\| \le N_{a,\epsilon,\xi,\varrho} \sup_{[0,1]} \|e^{\left(a-\frac{\epsilon}{2}\right)|x|^2} u(t)\| \end{aligned} \tag{2.7}$$

for $0 < \varrho \le \frac{1}{2}$ and

$$\partial_t f_\epsilon - \mathcal{S}_\epsilon f_\epsilon - \mathcal{A}_\epsilon f_\epsilon = V(x,t) f_\epsilon, \text{ in } \mathbb{R}^n \times (0,1], \tag{2.8}$$

where $\mathcal{S}_\epsilon$ and $\mathcal{A}_\epsilon$ are the operators defined in (2.1) and (2.2) with a, A, b and T replaced by a_ϵ, A_ϵ, b_ϵ and T_ϵ respectively. Also, (2.7), the equation (2.8) verified by f_ϵ and [22, Lemma 1.2] show that f_ϵ is in $C((0,1], L^2(\mathbb{R}^n))$.

Extend f_ϵ as zero outside $\mathbb{R}^n \times [0,1]$ and let θ in $C^\infty(\mathbb{R}^{n+1})$ be a mollifier supported in the unit ball of $\mathbb{R}^{n+1}$. For $0 < \rho \le \frac{1}{4}$, set $f_{\epsilon,\rho} = f_\epsilon * \theta_\rho$ and

$$\theta_\rho^{x,t}(y,s) = \rho^{-n-1} \theta\left(\tfrac{x-y}{\rho}, \tfrac{t-s}{\rho}\right).$$

Then, $f_{\epsilon,\rho}$ is in $C^\infty([0,1], \mathcal{S}(\mathbb{R}^n))$ and for x in $\mathbb{R}^n$ and $\rho \le t \le 1-\rho$,

$$\begin{aligned} &\left(\partial_t f_{\epsilon,\rho} - \mathcal{S}_\epsilon f_{\epsilon,\rho} - \mathcal{A}_\epsilon f_{\epsilon,\rho}\right)(x,t) = (V f_\epsilon) * \theta_\rho(x,t) \\ &+ \int f_\epsilon \left(q_\epsilon(y,s,\xi) - q_\epsilon(x,t,\xi)\right) \theta_\rho^{x,t} \, dyds \\ &+ \int \nabla_y f_\epsilon \cdot \left[(a_\epsilon(t)x + 2b_\epsilon(t)\xi) - (a_\epsilon(s)y + 2b_\epsilon(s)\xi)\right] \theta_\rho^{x,t} \, dyds, \end{aligned} \tag{2.9}$$

with

$$\begin{aligned} q_\epsilon(x,t,\xi) = &\left(a_\epsilon'(t) + 4a_\epsilon^2(t)\right)|x|^2 \\ &+ \left(b_\epsilon'(t) + 4a_\epsilon(t)b_\epsilon(t)\right) x\cdot\xi + \left(b_\epsilon^2(t) - T_\epsilon'(t)\right)|\xi|^2 - 2na_\epsilon(t). \end{aligned}$$

The last identity gives,

$$\left(\partial_t f_{\epsilon,\rho} - \mathcal{S}_\epsilon f_{\epsilon,\rho} - \mathcal{A}_\epsilon f_{\epsilon,\rho}\right)(x,t) = (V f_\epsilon) * \theta_\rho(x,t) + A_{\epsilon,\rho}(x,t),$$

in $\mathbb{R}^n \times [\rho, 1-\rho]$, where $A_{\epsilon,\rho}$ denotes the sum of the second and third integrals in the right hand side of (2.9). Moreover, from (2.7) there is $N_{a,\epsilon,\xi,\varrho}$ such that for $0 < \varrho < \frac{1}{2}$ and $0 < \rho \le \varrho$,

$$\sup_{[\varrho,1-\varrho]} \|A_{\epsilon,\rho}(t)\|_{L^2(\mathbb{R}^n)} \le \rho N_{a,\epsilon,\xi,\varrho} \sup_{[-1,1]} \|e^{\left(a(t)-\frac{\epsilon}{2}\right)|x|^2} u(t)\|.$$

Also, $\left(e^{8A_\epsilon} a_\epsilon\right)'' > 0$ in $[0,1]$, when $0 < \epsilon \le \epsilon_a$, and from Lemma 2 we can apply to $\mathcal{S}_\epsilon$, $\mathcal{A}_\epsilon$ and $f_{\epsilon,\rho}$, the conclusions of Lemma 1 with $[c,d] = [\varrho, 1-\varrho]$, $\gamma = e^{8A_\epsilon}$ and $H_{\epsilon,\rho}(t) = \|f_{\epsilon,\rho}(t)\|^2$. Thus,

$$H_{\epsilon,\rho}(t) \le \left(H_{\epsilon,\rho}(\varrho) + H_{\epsilon,\rho}(1-\varrho) + 2\epsilon\right) e^{M_{\epsilon,\rho}(t) + 2N_{\epsilon,\rho}}, \text{ when } \varrho \le t \le 1-\varrho, \tag{2.10}$$

where $M_{\epsilon,\rho}$ verifies

$$\begin{cases} \partial_t \left(e^{8A_\epsilon} \partial_t M_{\epsilon,\rho}\right) = -e^{8A_\epsilon} \frac{\|\partial_t f_{\epsilon,\rho} - \mathcal{S}_\epsilon f_{\epsilon,\rho} - \mathcal{A}_\epsilon f_{\epsilon,\rho}\|^2}{H_{\epsilon,\rho} + \epsilon}, \text{ in } [\varrho, 1-\varrho], \\ M_{\epsilon,\rho}(\varrho) = M_{\epsilon,\rho}(1-\varrho) = 0, \end{cases}$$

and

$$N_{\epsilon,\rho} = \int_\varrho^{1-\varrho} \frac{\|\partial_s f_{\epsilon,\rho}(s) - \mathcal{S}_\epsilon f_{\epsilon,\rho}(s) - \mathcal{A}_\epsilon f_{\epsilon,\rho}(s)\|}{\sqrt{H_{\epsilon,\rho}(s) + \epsilon}} \, ds.$$

We can now pass to the limit in (2.10), when ρ tends to zero and derive that for $H_\epsilon(t)=\|f_\epsilon(t)\|^2$, $0<\varrho\le\frac12$ and $0<\epsilon\le\epsilon_a$, we have

$$H_\epsilon(t)\le\left[H_\epsilon(\varrho)+H_\epsilon(1-\varrho)+2\epsilon\right]e^{M_\epsilon(t)+2\|V\|_\infty},\ \text{in }[\varrho,1-\varrho],\tag{2.11}$$

with

$$\begin{cases}\partial_t\left(e^{8A_\epsilon}\partial_t M_\epsilon\right)=-\,e^{8A_\epsilon}\,\frac{\|\partial_t f_\epsilon-\mathcal{S}_\epsilon f_\epsilon-\mathcal{A}_\epsilon f_\epsilon\|^2}{H_\epsilon}\,,\ \text{in }[0,1].\\ M_\epsilon(0)=M_\epsilon(1)=0,\end{cases}\tag{2.12}$$

By writing an explicit formula for the solution to (2.12), it follows from the monotonicity of A; i.e. $A'\ge0$ in $[0,1]$ and (2.8) that

$$M_\epsilon(t)\le N\left(1+\|V\|_\infty^2\right).$$

Also, there is $N_a>0$ such that $|b_\epsilon'|+|T_\epsilon'|\le N_a$, when $0<\epsilon\le\epsilon_a$. The later, the continuity of f_ϵ in $C((0,1],L^2(\mathbb{R}^n))$ and the fact that $a(0)=b_\epsilon(0)=T_\epsilon(0)=0$ show, that for each fixed $\xi\in\mathbb{R}^n$ and all $0<\epsilon<\epsilon_a$, there is ϱ_ϵ with $\lim_{\epsilon\to0^+}\varrho_\epsilon=0$ such that $H_\epsilon(1-\varrho_\epsilon)\le H_\epsilon(1)+\epsilon$ and $H_\epsilon(\varrho_\epsilon)\le\sup_{[0,1]}\|u(t)\|$. Thus, after taking $\varrho=\varrho_\epsilon$ in (2.11), we get

$$\|e^{a_\epsilon(t)|x|^2+b_\epsilon(t)x\cdot\xi-T_\epsilon(t)|\xi|^2}u(t)\|\le e^{N\left(1+\|V\|_\infty^2\right)}\left[\sup_{[0,1]}\|u(t)\|+\|e^{|x|^2/\delta^2}u(1)\|+3\epsilon\right],$$

for $\varrho_\epsilon\le t\le1-\varrho_\epsilon$. Then, let $\epsilon\to0^+$ and recall the L^2 energy inequality verified by solutions to (1.2). $\square$

3. Proof of Theorem 1

Proof. By scaling it suffices to prove Theorem 1 when $T=1$. Assume first that u in $C([0,1],L^2(\mathbb{R}^n))\cap L^2([0,1],H^1(\mathbb{R}^n))$ verifies (1.2) in $\mathbb{R}^n\times(0,1]$ and

$$\|e^{|x|^2/\delta^2}u(1)\|<+\infty$$

for some $\delta>2$. Following [9, Theorem 4], for $\alpha=1$ and $\beta=1+\frac{2}{\delta}$, define

$$\widetilde u(x,t)=\left(\tfrac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{\frac n2}u\left(\tfrac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t},\tfrac{\beta t}{\alpha(1-t)+\beta t}\right)e^{\frac{(\alpha-\beta)|x|^2}{4(\alpha(1-t)+\beta t)}}.$$

Then, $\widetilde u$ is in $C([0,1],L^2(\mathbb{R}^n))\cap L^2([0,1],H^1(\mathbb{R}^n))$ and from [9, Lemma 5] with $A+iB=1$

$$\partial_t\widetilde u=\Delta\widetilde u+\widetilde V(x,t)\widetilde u,\ \text{in }\mathbb{R}^n\times(0,1],$$

with

$$\widetilde V(x,t)=\tfrac{\alpha\beta}{(\alpha(1-t)+\beta t)^2}V\left(\tfrac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t},\tfrac{\beta t}{\alpha(1-t)+\beta t}\right).$$

Also, for $\gamma=\frac{1}{2\delta}$

$$\|e^{\gamma|x|^2}\widetilde u(0)\|=\|u(0)\|\ \text{and}\ \|e^{\gamma|x|^2}\widetilde u(1)\|=\|e^{|x|^2/\delta^2}u(1)\|.$$

From the log-convexity property of $\|e^{\gamma|x|^2}\widetilde u(t)\|$ established in [9, Lemma 3], we know that

$$\sup_{[0,1]}\|e^{\gamma|x|^2}\widetilde u(t)\|\le e^{N\left(1+\|\widetilde V\|^2_{L^\infty(\mathbb{R}^n\times[0,1])}\right)}\left(\|e^{\gamma|x|^2}\widetilde u(0)\|+\|e^{\gamma|x|^2}\widetilde u(1)\|\right).\tag{3.1}$$

The last claim in [9, Lemma 5] shows that with $s=\frac{\beta t}{\alpha(1-t)+\beta t}$,

$$\|e^{\gamma|x|^2}\widetilde u(t)\|=\|e^{\left[\frac{\gamma\alpha\beta}{(\alpha s+\beta(1-s))^2}+\frac{\alpha-\beta}{4(\alpha s+\beta(1-s))}\right]|y|^2}u(s)\|,\ \text{for }0\le t\le1.\tag{3.2}$$

From (3.1) and (3.2), we find that

$$\sup_{[0,1]} \|e^{\frac{t|x|^2}{(\delta+2-2t)^2}} u(t)\| \le e^{N\left(1+\|V\|_\infty^2\right)} \left[\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\|\right]. \tag{3.3}$$

We then begin an inductive procedure where at the kth step we have constructed k smooth functions, $a_j : [0,1] \longrightarrow [0,+\infty)$ verifying

$$0 < a_1 < a_2 < \cdots < a_k < \cdots \le \frac{1}{\delta^2 - 4}\,, \text{ in } (0,1), \tag{3.4}$$

$$a_j(0) = 0,\ a_j(1) = 1/\delta^2,\ \left(e^{8A_j} a_j\right)'' > 0, \text{ in } [0,1], \tag{3.5}$$

$$\sup_{[0,1]} \|e^{a_j(t)|x|^2} u(t)\| \le e^{N\left(1+\|V\|_\infty^2\right)} \left[\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\|\right], \tag{3.6}$$

when $j = 1, \dots, k$, with $A_j' = a_j$, $A_j(1) = 0$. The case $k = 1$ follows from (3.3) with $a_1(t) = t/\left(\delta + 2 - 2t\right)^2$. Assume now that $a_1, \dots, a_k$ have been constructed and let b_k and T_k be the functions defined in Lemma 3 for $a = a_k$. Then,

$$\begin{aligned} &\|e^{a_k(t)|x|^2 + b_k(t) x\cdot\xi - T_k(t)|\xi|^2} u(t)\|^2 \\ &\qquad\le e^{2N\left(1+\|V\|_\infty^2\right)} \left(\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\|\right)^2, \end{aligned} \tag{3.7}$$

for $0 \le t \le 1$ and all $\xi \in \mathbb{R}^n$. Observe that (3.7) and the existence of the solutions u_R defined in (1.5) imply that $T_k > 0$ in $(0,1)$, when $\delta > 2$. Otherwise, (3.7) implies that $u_R \equiv 0$, when $2\sqrt{1+R^2} < \delta$.

For $\epsilon > 0$, multiply (3.7) by $e^{-2\epsilon T_k(t)|\xi|^2}$ and integrate the new inequality with respect to ξ in $\mathbb{R}^n$. It gives,

$$\sup_{[0,1]} \|e^{a_{k+1}^\epsilon(t)|x|^2} u(t)\| \le \left(1 + \tfrac{1}{\epsilon}\right)^{\frac{n}{4}} e^{N\left(1+\|V\|_\infty^2\right)} \left(\|u(0)\| + \|e^{|x|^2/\delta^2} u(1)\|\right),$$

with

$$a_{k+1}^\epsilon = a_k + \frac{b_k^2}{4\left(1+\epsilon\right) T_k}\,.$$

On the other hand, $e^{8A_k} b_k$ is strictly convex and $b_k < 0$ in $[0,1]$,

$$b_k(t) = 2\left(a_k(t) - t e^{-8A_k(t)} \delta^{-2}\right) \tag{3.8}$$

and

$$T_k(t) = 2\int_0^t b_k^2(s)\,ds - a_k(t) - 8\int_0^t a_k^2(s)\,ds - \alpha_k \int_0^t e^{-8A_k(s)}\,ds,$$

with

$$\alpha_k = \left(2\int_0^1 b_k^2(s)\,ds - \frac{1}{\delta^2} - 8\int_0^1 a_k^2(s)\,ds\right)\left(\int_0^1 e^{-8A_k(s)}\,ds\right)^{-1}.$$

The last two formulae and (3.4) show that there is $N_\delta \ge 1$, independent of $k \ge 1$, such that

$$T_k(t) \le 2\left(\int_0^t b_k^2(s)\,ds + N_\delta\right) \text{ and } N_\delta + \frac{b_k}{2} \ge 1, \text{ in } [0,1]. \tag{3.9}$$

Also, $\left(\left(a_k' + 4a_k^2\right) e^{16A_k}\right)' = e^{8A_k}\left(e^{8A_k} a_k\right)''$, $\left(a_k' + 4a_k^2\right) e^{16A_k}$ is non decreasing in $[0,1]$ and

$$a_k' + 4a_k^2 \geq 0 \text{ in } [0,1]. \tag{3.10}$$

Set then,

$$a_{k+1}(t) = a_k(t) + \frac{b_k^2(t)}{8\left(\int_0^t b_k^2(s)\,ds + N_\delta\right)}. \tag{3.11}$$

We have, $a_k < a_{k+1}$ in $(0,1)$, $a_{k+1}(0) = 0$, $a_{k+1}(1) = \frac{1}{\delta^2}$,

$$A_{k+1} = A_k + \frac{1}{8}\log\left(\int_0^t b_k^2(s)\,ds + N_\delta\right) - \frac{1}{8}\log\left(\int_0^1 b_k^2(s)\,ds + N_\delta\right), \tag{3.12}$$

and

$$\sup_{[0,1]} \|e^{(a_{k+1}(t)-\epsilon)|x|^2} u(t)\| < +\infty, \text{ for all } \epsilon > 0.$$

The identity $\left(e^{8A}\right)''' = 8\left(e^{8A}a\right)''$ and (3.12) show that $\left(e^{8A_{k+1}}a_{k+1}\right)''$ is a positive multiple of

$$\begin{aligned}\left(e^{8A_k}\left(\int_0^t b_k^2(s)\,ds + N_\delta\right)\right)''' &= \left(e^{8A_k}\right)'''\left(\int_0^t b_k^2(s)\,ds + N_\delta\right)\\ &+ 3\left(e^{8A_k}\right)'' b_k^2 + 6\left(e^{8A_k}\right)' b_k b_k' + 2e^{8A_k}\left(b_k'' b_k + b_k'^2\right)\end{aligned}$$

The equation verified by b_k shows that the last sum is equal to

$$\begin{aligned}&\left(e^{8A_k}\right)'''\left(\int_0^t b_k^2(s)\,ds + N_\delta + \frac{b_k}{2}\right)\\ &\qquad + 8\left(a_k' + 8a_k^2\right)e^{8A_k} b_k^2 + 2e^{8A_k} b_k'^2 + 16 e^{8A_k} a_k b_k b_k'.\end{aligned}$$

From (3.9) and (3.10), the above sum is bounded from below by

$$\left(e^{8A_k}\right)''' + 2e^{8A_k}\left(4a_k b_k + b_k'\right)^2 > 0, \text{ in } [0,1].$$

The later and Lemma 3 show that (3.6) holds up to $j = k+1$. Finally, because (3.10) holds with k replaced by $k+1$,

$$-\left(\frac{1}{a_{k+1}}\right)' + 4 \geq 0, \text{ in } (0,1],$$

and the integration of this identity over $[t,1]$ shows that $a_{k+1}(t) \leq \frac{1}{\delta^2 - 4}$ in $(0,1)$.

Thus, there exists $a(t) = \lim_{k\to+\infty} a_k(t)$ and from (3.11), $\lim_{k\to+\infty} b_k(t) = 0$. This and (3.8) show that

$$ae^{8A} = t\delta^{-2}, \text{in } [0,1]. \tag{3.13}$$

Write $a(1) = 1/\delta^2$ as $1/4\left(1+R^2\right)$, for some $R > 0$. Then, $a(t) = t/4\left(t^2 + R^2\right)$ follows from the integration of (3.13) and (1.4) from (3.6) after letting $j \to +\infty$. Finally, when $\delta = 2$, we have

$$\sup_{[0,1]} \|e^{t|x|^2/4\left(t^2+R^2\right)} u(t)\| \leq e^{N\left(1+\|V\|_\infty^2\right)}\left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/4}u(1)\|\right],$$

for all $R > 0$. Letting $R \to 0^+$, we get

$$\sup_{[0,1]} \|e^{|x|^2/4t} u(t)\| \leq e^{N\left(1+\|V\|_\infty^2\right)}\left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{|x|^2/4}u(1)\|\right],$$

and it implies, $u \equiv 0$. □

Remark 1. Theorem 1 holds when (1.3) and (1.4) are replaced respectively by

$$\|e^{Tx_1^2/4(T^2+R^2)}u(T)\|_{L^2(\mathbb{R}^n)} < +\infty$$

and

$$\sup_{[0,T]} \|e^{tx_1^2/4(t^2+R^2)}u(t)\|_{L^2(\mathbb{R}^n)}$$
$$\le e^{N\left(1+T^2\|V\|^2_{L^\infty(\mathbb{R}^n\times[0,T])}\right)} \left[\|u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{Tx_1^2/4(T^2+R^2)}u(T)\|_{L^2(\mathbb{R}^n)}\right].$$

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