MATH 201-A FALL 2013 FIRST HOMEWORK

Problem 1 Consider the function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q} \cap [0,1], \\ \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \ \ (p,\, q \in \mathbb{Z}^+ \text{ relative primes}). \end{cases} \tag{1}$$

Where is f discontinuous? Using the definition prove that f is Riemann integrable on $[0, 1]$

Problem 2 Consider the function $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q} \cap [0,1], \\ 1, & \text{if } x = \dfrac{p}{q}, \quad p, q \in \mathbb{Z}^+. \end{cases} \tag{1a}$$

Where is f discontinuous? Using the definition prove that f is not Riemann integrable on $[0,1]$.

Definition 1. *A set* $E \subset \mathbb{R}$ *is said to be of "measure zero" if given* $\epsilon > 0$ *there is a countable collection of intervals* $\{I_j\}_{j\in\mathbb{Z}^+}$ *which covers* E,

$$E \subset \bigcup_{j\in\mathbb{Z}^+} I_j \quad \text{such that} \quad \sum_{j=1}^{\infty} |I_j| = \sum_{j=1}^{\infty} \text{length of } I_j < \epsilon.$$

Thus : (i) Prove that every countable set of $\mathbb{R}$ is a set of measure zero. and (ii) the countable union of sets of measure zero has measure zero.

Problem 3 Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Prove that E is uncountable set of measure zero.

Definition 2. *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function,* $b - a < \infty$. *For* $x \in [a,b]$ *and* $\eta > 0$ *define*

$$\Omega(f, x, \eta) = \sup\{|f(x_1) - f(x_2)| \ : \ x_1,\ x_2 \in (x - \eta, x + \eta) \cap [a,b]\},$$

and the oscillation of f at a point $x \in [a,b]$.

$$\omega_f(x) = \lim_{\eta \to 0^+} \Omega(f, x, \eta) = \inf_{\eta > 0} \Omega(f, x, \eta).$$

Problem 4 Prove that $\omega_f(x)$ is defined for any $x \in [a,b]$.

Problem 5 Prove that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

Problem 6 Prove that for any $\mu > 0$ the set $A_\mu = \{x \in [a,b] \, : \, \omega_f(x) \geq \mu\}$ is compact.

Problem 7 Prove that the set of discontinuities of f can be written as

$$D_f = \bigcup_{j \in \mathbb{Z}^+} A_j = \{x \in [a,b] \ : \ \omega_f(x) \geq \frac{1}{j}\}.$$

Problem 8 Prove that if for some $\epsilon > 0$, $\omega_f(x) < \epsilon$ for any $x \in [a, b]$, then there exists a $\eta > 0$ such that

$$\Omega(f, x, \eta) < \epsilon, \qquad \text{for any } x \in [a, b].$$

Hint : Use the compactness of $[a, b]$.

At this point we are ready to prove the Lebesgue criterion for Riemann integrability.

Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function and* $b-a < \infty$*. Then* f *is Riemann integrable on* $[a, b]$ *if and only if the set of discontinuities of* f, D_f *is a set of measure zero.*

Problem 9 Prove theorem 1.

Hint : Assume that f is RI and that D_f does not have measure zero. Prove that for some $j_0 \in \mathbb{Z}^+$

$$A_{j_0} = \{x \in [a, b] \ : \ \omega_f(x) \geq \frac{1}{j_0}\}$$

is not a set of measure zero. Take a partition P of $[a, b]$, observe that the sub-intervals of this partition cover A_{j_0}. Evaluate the difference between its upper and lower sum to get a contradiction.

Assume that D_f has measure zero. Then for any j the set A_j is compact and has measure zero. For j large one can cover A_j by a countable collection of intervals whose sum of their lengths is arbitrary small, say less than $\epsilon/2$. Show that one can expand these intervals to obtain a new collection of OPEN intervals covering A_j and whose sum of their lengths is less than ϵ. Use the compactness of A_j to construct an appropriate partition, which allows to conclude the proof.

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We saw in class that if B is a open set of $\mathbb{R}$, then B is the union of an at most countable collection of disjoint open intervals.

Let B be a open bounded set of $\mathbb{R}$. Define χ_B $\chi_B(x) = 1$, if $x \in B$, and $\chi_B(x) = 0$, if $x \notin B$.

$$\int \chi_B(x)dx = ?$$

Problem 10 Give an example of B open and bounded such that χ_B is not Riemann integrable. Moreover, χ_B is not Riemann integrable even after any modification on a set of measure zero.