DECAY PROPERTIES FOR SOLUTIONS OF FIFTH ORDER NONLINEAR DISPERSIVE EQUATIONS

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ABSTRACT. We consider the initial value problem associated to a large class of fifth order nonlinear dispersive equations. This class includes several models arising in the study of different physical phenomena. Our aim is to establish special (space) decay properties of solutions to these systems. These properties complement previous unique continuation results and in some case, show that they are optimal. These decay estimates reflect the "parabolic character" of these dispersive models in exponential weighted spaces. This principle was first obtained by T. Kato in solutions of the KdV equation.

1. INTRODUCTION

In this work we shall study decay and uniqueness properties of solutions to a class of higher order dispersive models. More precisely, we shall be concerned with one space dimensional (1D) dispersive models in which the dispersive relation is described by the fifth order operator ∂_x^5. Roughly, the general form of the class of equations to be considered here is

I1
$$\partial_t u - \partial_x^5 u + P(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = 0 \tag{1.1}$$

where $P(\cdot)$ is a polynomial without constant or linear term, i.e.

I2
$$P = P(x_1, x_2, x_3, x_4) = \sum_{2 \le |\alpha| \le N} a_\alpha x^\alpha, \quad N \in \mathbb{Z}^+, \quad N \ge 2, \quad a_\alpha \in \mathbb{R}. \tag{1.2}$$

In this class one finds a large set of models arising in both mathematical and physical settings. Thus, the case

I2b
$$P(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = 10u\partial_x^3 u + 20\partial_x u \partial_x^2 u - 30u^2 \partial_x u \tag{1.3}$$

corresponds to the third equation in the KdV hierarchy, where

I3
$$\partial_t u + \partial_x u = 0 \tag{1.4}$$

and the KdV

I4
$$\partial_t u + \partial_x^3 u + u\partial_x u = 0 \tag{1.5}$$

are the first and second ones respectively in the hierarchy the jth being

I4b
$$\partial_t u + (-1)^j \partial_x^{2j+1} u + Q_j(u, \dots, \partial_x^{2j-1} u) = 0, \quad j \in \mathbb{Z}^+, \tag{1.6}$$

with $Q_j(\cdot)$ an appropriate polynomial (see [6]).

Further examples of integrable models of the equation in (1.1)-(1.2) were deduced in [14] and [25] which also arise in the study of higher order models of water waves.

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In [1] the cases

I5
$$P_1 = c_1\, u\partial_x u, \quad P_2 = u\partial_x^3 u + 2\partial_x u \partial_x^2 u \tag{1.7}$$

were proposed as models describing the interaction between long and short waves.

In [20] the example of (1.1)-(1.2) with

I6
$$P = (u+u^2)\partial_x u + (1+u)(\partial_x u\partial_x^2 u + u\partial_x^3 u) \tag{1.8}$$

was deduced in the study of the motion of a lattice of anharmonic oscillators (by simplicity the values of the coefficients in (1.8) have been taken equal to one). In [11] the equations (1.1)-(1.2) with $P = P_1$ as in (1.7) were proposed as a model for magneto-acoustic waves at the critical angle in cold plasma.

Other cases of the equations (1.1)-(1.2) have been studied in [21], [15], ...

The well-posedness of the initial value problem (IVP) and the periodic boundary value problem (PBVP) associated to the equation (1.1) have been extensively studied (in the well-posedness of these problems in a function space X one includes existence, uniqueness of a solution $u \in C([0,T] : X) \cap \dots$ with $T = T(\|u_0\|_X) > 0$, $u(\cdot,0) = u_0$, and the map $u_0 \mapsto u$ being locally continuous).

In [24] Saut proved the existence of solutions corresponding to smooth data for the IVP for the whole KdV hierarchy sequence of equations in (1.6).

In [26] Schwarz considered the PBVP for the KdV hierarchy (1.6) establishing existence and uniqueness in $H^s(\mathbb{T})$ for $s \geq 3j-1$.

In [23] Ponce showed that the IVP for (1.1)-(1.2) with (1.2) as in (1.3) (i.e. the third equation in the KdV hierarchy) is globally well posed in $H^s(\mathbb{R})$ for $s \geq 4$.

In [17] Kenig, Ponce and Vega established the local well-posedness in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^m)$ of the IVP for the sequence of equations in (1.6) for any polynomial $Q_j(u, \dots, \partial_x^{2j} u)$, for $s \geq jm$ and $s \geq s_0(j)$.

The latter work motivated several further studies concerning the minimal regularity required in the Sobolev scale to guarantee that the IVP associated to (1.1)-(1.2) is locally well-posed in $H^s(\mathbb{R})$. These results heavily depend on the structure of the nonlinearity $P(\cdot)$ in (1.2) considered. In [22] Pilod showed that the IVP associated to the equation (1.1)-(1.2) with a nonlinearity involving a quadratic term depending on $\partial_x^3 u$ (as in (1.3)) cannot be solved by an argument based on the contraction principle. This is not the case when $P(\cdot)$ in (1.2) has the form $P = P(u, \partial_x u, \partial_x^2 u)$, see [17]. This is quite different to the case for the KdV equation (1.5) for which the well posedness can be established via contraction principle, see [19] for details and references.

Concerning the model (1.1) with P as in (1.3) in [18] Kwon obtained local well posedness in $H^s(\mathbb{R})$ with $s > 5/2$. For the same problem Kenig and Pilod [16] and Guo, Kwak and Kwon [7] simultaneously established local and global results in the energy space, i.e. $H^s(\mathbb{R})$ with $s \geq 2$.

For other well posedness results concerning the IVP associated to the equation (1.1) with different P in (1.2) see [2], [13], [3], [8] and references therein.

Special uniqueness properties of solutions to the IVP associated to the equation (1.1)-(1.2) were studied by Dawson [4]. It was established in [4] that if $u_1, u_2 \in C([0,T] : H^6(\mathbb{R}) \cap L^2(|x|^3\, dx))$, $T > 1$, are two solutions of (1.1)-(1.2) such that

I8
$$(u_1 - u_2)(\cdot, 0),\ (u_1 - u_2)(\cdot, 1) \in L^2\big(e^{x_+^{4/3+\varepsilon}}\, dx\big) \tag{1.9}$$

for some $\varepsilon > 0$, then $u_1 \equiv u_2$. Moreover, in the case where in (1.2) one has

$$P = P(u, \partial_x u) = \sum_{2 \leq \alpha_1 + \alpha_2 \leq N} a_{\alpha_1,\alpha_2}\, u^{\alpha_1} (\partial_x u)^{\alpha_2}, \qquad N \in \mathbb{Z}^+$$

the exponent $4/3$ can be replaced by $5/4$.

In fact one should expect the general result in [4] to hold with $5/4$ in (1.9) instead of $4/3$ for all $P(\cdot)$ in (1.2). However the argument of proof in [4] follows that given in [5] for the KdV equation. More precisely, it was established in [5] that there exists $a_0 > 0$ such that if $u_1, u_2 \in C([0,T] : H^4(\mathbb{R}) \cap L^2(|x|^2\,dx))$, $T > 1$, are solutions of the IVP associated to the KdV equation (1.5) with

I9
$$(u_1 - u_2)(\cdot, 0),\ (u_1 - u_2)(\cdot, 1) \in L^2\left(e^{a_0 x_+^{3/2}}\,dx\right) \tag{1.10}$$

then $u_1 \equiv u_2$. (Although, the statements in [5] and [4] contain stronger hypotheses than the ones described in (1.9) and (1.10) respectively, these can be deduced by interpolation between (1.9) and (1.10) and the corresponding inequalities following by the assumptions on the class of solutions considered).

The value of the exponents above are dictated by the following decay estimate concerning the fundamental solution of the associated linear problem

I10
$$\begin{cases} \partial_t v + \partial_x^{2j+1} v = 0, \\ v(x,0) = v_0(x). \end{cases} \tag{1.11}$$

In [27] it was shown that

I11
$$v(x,t) = \frac{c_j}{t^{1/(2j+1)}} \int_{-\infty}^{\infty} K_j\Big(\frac{x - x'}{t^{1/(2j+1)}}\Big) v_0(x')\,dx' \tag{1.12}$$

with $K_j(\cdot)$ satisfying

I12
$$|K_j(x)| \le \frac{c}{(1 + x_-)^{(2j-1)/4j}}\, e^{-c_j x_+^{(2j+1)/2j}} \tag{1.13}$$

with $x_+ = \max\{x, 0\}$, $x_- = -\min\{x,\ 0\}$. Thus

I13a
$$K_j\Big(\frac{x}{t^{1/(2j+1)}}\Big) \sim e^{-c_j\left(\frac{x^{(2j+1)}}{t}\right)^{1/2j}}. \tag{1.14}$$

For this reason and the result in [5] one should expect that (1.9) holds with $5/4$ instead of $4/3$ for a large class of polynomials in (1.2) including that in (1.3). The obstruction appears in [4] when the Carleman estimate deduced in [5] is used in this higher order setting.

In [9] we proved that the result in [5] commented above is optimal. More precisely, the following results were established in [9].

(i) If $u_0 \in L^2(\mathbb{R}) \cap L^2(e^{a_0 x_+^{3/2}}\,dx)$, $a_0 > 0$, then, for any $T > 0$, the solution of the IVP for the KdV equation (1.5) satisfies

I13
$$\sup_{t\in[0,T]} \int_{-\infty}^{\infty} e^{a(t) x_+^{3/2}} |u(x,t)|^2\,dx \le c^* = c^*(a_0;\, \|u_0\|_2;\, \|e^{\frac{1}{2} a_0 x_+^{3/2}} u_0\|_2; T), \tag{1.15}$$

with

I14
$$a(t) = \frac{a_0}{(1 + 27 a_0^2 t/4)^{1/2}}. \tag{1.16}$$

(ii) If $u_1, u_2 \in C([0,\infty) : H^1(\mathbb{R}) \cap L^2(|x|\,dx))$ are solutions of the IVP for the KdV equation (1.5) such that

I15
$$\int_{-\infty}^{\infty} e^{a_0 x_+^{3/2}} |u_1(x,0) - u_2(x,0)|^2\,dx < \infty, \tag{1.17}$$

then, for any $T > 0$,

I16
$$\sup_{[0,T]} \int_{-\infty}^{\infty} e^{a(t) x_+^{3/2}} |u_1(x,t) - u_2(x,t)|^2\,dx \le c^* \tag{1.18}$$

with

$$c^* = c^*(a_0; \|u_1(\cdot,0)\|_{1,2}; \|u_2(\cdot,0)\|_{1,2}; \||x|^{1/2} u_1(x,0)\|_2;$$
$$\|(u_1-u_2)(\cdot,0)\|_{1,2}; \|e^{\frac{1}{2}a_0 x_+^{3/2}} (u_1-u_2)(\cdot,0)\|_2; T)$$

with $a(t)$ as in (1.16).

In order to simplify the exposition we state our main result first for the case of the IVP

I17
$$\begin{cases} \partial_t u - \partial_x^5 u + b_1 u\, \partial_x^3 u + b_2 \partial_x u\, \partial_x^2 u + b_3 u^2 \partial_x u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.19}$$

with b_1, b_2, $b_3 \in \mathbb{R}$ arbitrary constants.

theorem1 **Theorem 1.1.** *Let a_0 be a positive constant. For any given data*

I18
$$u_0 \in H^3(\mathbb{R}) \cap L^2(e^{a_0 x_+^{5/4}}\,dx). \tag{1.20}$$

The unique solution $u(\cdot)$ of the IVP (1.19) *provided in* [16]

$$u \in C([0,T] : H^3(\mathbb{R})) \cap \dots$$

satisfies

I19
$$\sup_{[0,T]} \int_{-\infty}^{\infty} e^{a(t) x_+^{5/4}} |u(x,t)|^2\,dx \le c^* = c^*(a_0; \|u_0\|_{3,2}; \|e^{a_0 x_+^{5/4}} u_0\|_2; T) \tag{1.21}$$

with

I20
$$a(t) = \frac{a_0}{\sqrt[4]{1 + k\, a_0^4 t}} \qquad \textit{with} \qquad k = 11\,\frac{5^5}{4^5}. \tag{1.22}$$

theorem2 **Theorem 1.2.** *Let a_0 be a positive constant. Let u_1, u_2 be solutions of the IVP* (1.19) *such that*

$$u_1 \in C([0,T] : H^8(\mathbb{R}) \cap L^2(|x|^4\,dx)),$$
$$u_2 \in C([0,T] : H^4(\mathbb{R})) \cap L^2(|x|^2\,dx)).$$

If

I21
$$\Lambda \equiv \int_{-\infty}^{\infty} e^{a_0 x_+^{5/4}} |u_1(x,0) - u_2(x,0)|^2\,dx = \int_{-\infty}^{\infty} e^{a_0 x_+^{5/4}} |u_{01}(x) - u_{02}(x)|^2\,dx < \infty, \tag{1.23}$$

then, for $0 < \varepsilon \ll 1$

I22
$$\sup_{[0,T]} \int_{-\infty}^{\infty} e^{a(t) x_+^{5/4}} |u_1(x,t) - u_2(x,t)|^2\,dx \le c^{**} \tag{1.24}$$

where $c^{**} = c^{**}(a_0; \|u_{01}\|_{8,2}; \|u_{02}\|_{4,2}; \|x^2u_{01}\|_2; \|x\,u_{02}\|_2; \Lambda; \varepsilon; T)$ *and*

$$a(t) = \frac{a_0}{\sqrt[4]{1+k\,a_0^4 t}} \qquad \text{with} \qquad k = k(\varepsilon) = \frac{5^5}{4^5}\Big(\frac{3}{2} + \frac{25}{4(5-\varepsilon)}\Big).$$

Remarks.

(i) *Our method of proof is based on weighted energy estimates for which one needs that*

I22a (1.25)
$$\partial_x^3 u \in L^1([-T,T] : L^\infty(\mathbb{R})).$$

By using Strichartz estimates it was shown in [16] *that* (1.25) *holds for solutions corresponding to datum* $u_0 \in H^s(\mathbb{R})$ *with* $s > 9/4$, *(see section 2.5 in* [16]*). However, to obtain some interpolation inequalities needed in the proof and to simplify the exposition we shall assume that* $u_0 \in H^3(\mathbb{R})$.

(ii) *In the case when the local solutions extend to global ones, for example for the case of the model described in* (1.3) *for which the solutions satisfy infinitely many conservation laws, the result in Theorem 1.1 holds in any time interval* $[0,T]$.

(iii) *In the statement of Theorem 1.2 and Theorem 1.3 below we did not intend to optimize the hypothesis on the regularity and decay of the data.*

Our next results generalize those in Theorems 1.1 and 1.2 to the following class of polynomials:

I22b (1.26)
$$P(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = Q_0(u, \partial_x u, \partial_x^2 u)\,\partial_x^3 u + Q_1(u, \partial_x u, \partial_x^2 u)$$

with

$$Q_0(x_1,x_2,x_3) = \sum_{1\le|\alpha|\le N} a_\alpha x^\alpha, \qquad N \in \mathbb{Z}^+,\ N \ge 1,\ a_\alpha \in \mathbb{R},$$

and

$$Q_1(x_1,x_2,x_3) = \sum_{2\le|\alpha|\le M} b_\alpha x^\alpha, \qquad M \in \mathbb{Z}^+,\ M \ge 2,\ b_\alpha \in \mathbb{R}.$$

Notice that all the nonlinearities in the models previously discussed belong to this class. For further discussion on the form of the polynomial $P(\cdot)$ in (1.1)-(1.2) see remark (iv) after the statements of Theorem 1.3 and Theorem 1.4.

theorem3 **Theorem 1.3.** *Let* a_0 *be a positive constant. For any given data*

I23 (1.27)
$$u_0 \in H^{10}(\mathbb{R}) \cap L^2(e^{a_0 x_+^{5/4}}\,dx).$$

The unique solution $u(\cdot)$ *of the IVP associated to the equation* (1.1) *with* $P(\cdot)$ *as in* (1.26)

I24 (1.28)
$$u \in C([0,T] : H^{10}(\mathbb{R})) \cap \dots$$

satisfies

I25 (1.29)
$$\sup_{[0,T]} \int_{-\infty}^{\infty} e^{a(t)x_+^{5/4}} |u(x,t)|^2\,dx \le c^* = c^*(a_0; \|u_0\|_{10,2}; \|\langle x\rangle^4 u_0\|_2; \|e^{a_0 x_+^{5/4}} u_0\|_2; T)$$

with

I26 (1.30)
$$a(t) = \frac{a_0}{\sqrt[4]{1+k\,a_0^4 t}} \qquad \text{with} \qquad k = 11\,\frac{5^5}{4^5}.$$

theorem4 **Theorem 1.4.** *Let a_0 be a positive constant. Let u_1, u_2 be solutions of the IVP associated to the equation* (1.1) *with $P(\cdot)$ as in* (1.26) *in the class*

$$u_1, u_2 \in C([0,T] : H^{10}(\mathbb{R}) \cap L^2(\langle x\rangle^4\, dx)) \cap \dots$$

If

I27
$$\Lambda \equiv \int_{-\infty}^{\infty} e^{a_0 x_+^{5/4}} |u_1(x,0) - u_2(x,0)|^2\, dx = \int_{-\infty}^{\infty} e^{a_0 x_+^{5/4}} |u_{01}(x) - u_{02}(x)|^2\, dx < \infty, \tag{1.31}$$

then for $0 < \varepsilon \ll 1$

I28
$$\sup_{[0,T]} \int_{-\infty}^{\infty} e^{a(t) x_+^{5/4}} |u_1(x,t) - u_2(x,t)|^2\, dx \le c^{**} \tag{1.32}$$

where $c^{**} = c^{**}(a_0;\ \sum_{j=1}^{2} \big(\|u_{0j}\|_{10,2} + \|\langle x\rangle^4 u_{0j}\|_2;\ \Lambda; \varepsilon; T)$ and

$$a(t) = \frac{a_0}{\sqrt[4]{1 + k\, a_0^4 t}} \qquad \text{with} \qquad k = k(\varepsilon) = \frac{5^5}{4^5}\Big(\frac{3}{2} + \frac{25}{4(5-\varepsilon)}\Big).$$

Remarks.

(i) *The hypotheses on the regularity of the initial data (and consequently on the solutions) in Theorems 1.3 and 1.4 are not sharp. They depend on the structure of the polynomial under consideration. However as written they apply to any polynomial $P(\cdot)$ in the class described in* (1.26).

(ii) *Theorem 1.1 and Theorem 1.3 tell us that solutions of the equation* (1.1) *with $P(\cdot)$ as in* (1.2) *decay accordingly to that of the fundamental solution $K_2(\cdot)$ described in* (1.14) *with $j = 2$.*

(iii) *Theorem 1.2 and Theorem 1.4 suggest that the results in* [4] *described in* (1.9) *should hold with $5/4$ instead of $4/3$ in the exponent for any polynomial in* (1.2). *However this remains as an open problem.*

(iv) *We recall that in* [17] *a local existence theory in weighted Sobolev spaces ($H^s(\mathbb{R}) \cap L^2(\langle x\rangle^{2j}\, dx)$ with $s > 4j$, s large enough) for the IVP associated to the equation* (1.1) *with a general polynomial $P(\cdot)$ as in* (1.2) *was established. This involves the use of a gauge transformation which transforms the equation* (1.1) *into an equivalent system. So one can ask if our argument presented here in Theorem 1.3 and Theorem 1.4 extends to this general case. In this case, however, this general result requires (decay and regularity) hypotheses involving the data (Theorem 1.3) or the solutions (Theorem 1.4) as well as some some of their derivatives. Also in this case the constant function $a(t)$ described in* (1.22) *may be smaller (i.e. weaker decay).*

(v) *Concerning the existence of the solutions u_1, u_2 in the class described in Theorem 1.2 and Theorem 1.4 we recall the result in* [17] . *The fact that the operator $\Gamma = x + 5t\partial_x^4$ commutes with $L = \partial_t - \partial_x^5$, and the "identity"*

$$|x|^{\alpha} W(t) u_0 = W(t)|x|^{\alpha} u_0 + W(t)\{\Phi_{a,t,\alpha}(\widehat{u}_0)(\xi)\}^{\vee}(x)$$

which holds for $\alpha \in (0,1)$, where

$$\|\{\Phi_{t,\alpha}(\widehat{u}_0)(\xi)\}^{\vee}\|_2 \le c(1 + |t|)(\|u_0\|_2 + \|D^{4\alpha} u_0\|_2),$$

and $W(t)$ denotes the unitary group associated to the linear equation in (1.1) *(see* [10]*), imply in particular that in order to control x^{α} decay in the L^2 norm one needs at least to have $D_x^{4\alpha}$ derivatives in L^2. Thus combining these ideas the result of existence of the solutions u_1, u_2 in the class described follows.*

In the proofs of Theorem 1.1 and Theorem 1.3 we need an intermediate decay result concerning the solutions of the IVP. More precisely, in [12] T. Kato showed that H^2-solutions u of the generalized KdV defined in the time interval $[0,T]$

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \qquad k \in \mathbb{Z}^+,$$

with data $u_0 \in L^2(e^{\beta x}\,dx)$, $\beta > 0$ satisfy

I29

$$e^{\beta x} u \in C([0,T] : L^2(e^{\beta x}\,dx)) \cap C((0,T) : H^{\infty}(\mathbb{R})). \tag{1.33}$$

Roughly in the linear case this follows from his observation that if u is solution of

$$\partial_t u + \partial_x^3 u = 0$$

and $v(x,t) = e^{\beta x} u(x,t)$, then

$$\partial_t v + (\partial_x - \beta)^3 v = 0.$$

So our next result, which will be used in the proofs of Theorems 1.1 and 1.3 extends Kato's result to solutions of the IVP (1.1) with $P(\cdot)$ as in (1.26).

theorem5

Theorem 1.5. *Let $u \in C([0,T] : H^6(\mathbb{R}))$ be a solution of the IVP for the equation* (1.1) *with P as in* (1.26)*, corresponding to data $u_0 \in H^6(\mathbb{R}) \cap L^2(e^{\beta x}\,dx)$, $\beta > 0$. Then*

$$e^{\beta x} u \in C([0,T] : L^2(\mathbb{R})) \cap C((0,T) : H^{\infty}(\mathbb{R})),$$

and

$$\|e^{\beta x} u(t)\|_2 \le c\, \|e^{\beta x} u_0\|_2, \quad t \in [0,T].$$

We recall that although this is a subclass of the previously considered in Theorems 1.1-1.4, this class includes all models previously discussed. The restriction appears in our wish to use Kato's approach. In fact by using the idea developed in the proof of Theorems 1.1-1.4 one can extend the result in Theorem 1.5 to the whole class in (1.2).

Also the hypotheses in Theorem 1.5, $u_0 \in H^6(\mathbb{R})$ can be significantly lower once a particular form of the polynomial P in (1.26) is considered.

The paper is organized as follows. The construction of the weights to put forward the theory will be given in Section 1. The proofs of Theorem 1.1 and Theorem 1.2 will be presented in Section 3. In Section 4, Theorem 1.3 and Theorem 1.4 will be proven. Finally, the proof of the extension of Kato's result will be detailed in Section 5.

2. Construction of weights

Consider the equation

A1

$$\partial_t u - \partial_x^5 u = F(x,t), \qquad t \ge 0, \; x \in \mathbb{R}. \tag{2.1}$$

Formally, we perform (weighted) energy estimates in the equation (2.1), i.e., we multiply (2.1) by $u\phi_N$, with $\phi_N = \phi_N(x,t)$ and $N \in \mathbb{Z}^+$, and integrate the result in the space variable. Thus after several integration by parts one gets

A2

$$\begin{aligned}\frac{d}{dt}\int u^2 \phi_N\,dx - \int u^2\,\partial_t \phi_N\,dx + 5\int (\partial_x^2 u)^2\,\partial_x \phi_N\,dx \\ - 5\int (\partial_x u)^2\,\partial_x^3 \phi_N\,dx + \int u^2 \partial_x^5 \phi_N\,dx = 2\int F u \phi_N\,dx.\end{aligned} \tag{2.2}$$

Using that

A3 (2.3)
$$-5\int(\partial_x u)^2\partial_x^3\phi_N\,dx = 5\int u\,\partial_x^2 u\,\partial_x^3\phi_N\,dx - \frac{5}{2}\int u^2\partial_x^5\phi_N\,dx.$$

From the Cauchy-Schwarz and Young inequalities we have that for any $\varepsilon\in[0,1]$

A4 (2.4)
$$\begin{aligned}5\Big|\int u\,\partial_x^2 u\,\partial_x^3\phi_N\,dx\Big| &\le 5\big(\int(\partial_x^2 u)^2\partial_x\phi_N\,dx\big)^{1/2}\big(\int u^2\frac{(\partial_x^3\phi_N)^2}{\partial_x\phi_N}\,dx\big)^{1/2}\\ &\le (5-\varepsilon)\int(\partial_x^2 u)^2\partial_x\phi_N\,dx + \frac{25}{4(5-\varepsilon)}\int u^2\frac{(\partial_x^3\phi_N)^2}{\partial_x\phi_N}\,dx\end{aligned}$$

(we remark that the integral above are taken on the set where $\partial_x^3\phi_N$ does not vanish. We will show that $\partial_x\phi_N$ does not vanish in the support of $\partial_x^3\phi_N$). Then, from (2.2)-(2.4) it follows that for any $\varepsilon\in[0,1]$

A5 (2.5)
$$\begin{aligned}&\frac{d}{dt}\int u^2\phi_N\,dx - \int u^2\,\partial_t\phi_N\,dx + \varepsilon\int(\partial_x^2 u)^2\partial_x\phi_N\,dx\\ &-\frac{3}{2}\int u^2\partial_x^5\phi_N\,dx - \frac{25}{4(5-\varepsilon)}\int u^2\frac{(\partial_x^3\phi_N)^2}{\partial_x\phi_N}\,dx\\ &\le 2\int Fu\phi_N\,dx,\end{aligned}$$

i.e. for $\varepsilon\in[0,1]$

A6 (2.6)
$$\begin{aligned}&\frac{d}{dt}\int u^2\phi_N\,dx + \varepsilon\int(\partial_x^2 u)^2\partial_x\phi_N\,dx\\ &\le \int u^2\Big(\partial_t\phi_N + \frac{3}{2}\partial_x^5\phi_N + \frac{25}{4(5-\varepsilon)}\,\frac{(\partial_x^3\phi_N)^2}{\partial_x\phi_N}\Big)\,dx\\ &\quad + 2\int uF\phi_N\,dx.\end{aligned}$$

We shall use the inequality (2.6) with $0<\varepsilon\ll 1$. Then in order to simplify the proof we shall carry the details in the case $\varepsilon = 0$ and remark that all the estimates involving the coefficient $25/4(5-\varepsilon)$ are strict inequalities which also proves their extension to $\varepsilon>0$ with $\varepsilon\ll 1$.

We shall construct a sequence of weights $\{\phi_N\}_{N=1}^{\infty}$ which will be a key ingredient in the proof of our main theorems.

theoremA **Theorem 2.1.** *Given $a_0>0$ and $\varepsilon\in[0,1]$, $\varepsilon\ll 1$, there exists a sequence $\{\phi_{\varepsilon,N}\}_{N=1}^{\infty}\equiv\{\phi_N\}_{N=1}^{\infty}$ of functions with*

A7 (2.7)
$$\phi_N:\mathbb{R}\times[0,\infty)\to\mathbb{R}$$

satisfying for any $N\in\mathbb{Z}^+$

(i) $\phi_N\in C^4(\mathbb{R}\times[0,\infty))$ *with* $\partial_x^5\phi_N\phi(\cdot,t)$ *having a jump discontinuity at $x=N$.*
(ii) $\phi_N(x,t)>0$ *for all* $(x,t)\in\mathbb{R}\times[0,\infty)$.
(iii) $\partial_x\phi_N(x,t)\ge 0$ *for all* $(x,t)\in\mathbb{R}\times[0,\infty)$.
(iv) *There exist constants $c_N=c(N)>0$ and $c_0=c_0(a_0)>0$ such that*

A8 (2.8)
$$\phi_N(x,t)\le c_N\,c_0\,\langle x_+\rangle^4$$

with

A9 (2.9)
$$x_+=\max\{0;x\},\qquad \langle x\rangle=(1+x^2)^{1/2}.$$

(v) *For $T>0$ there is $N_0 \in \mathbb{Z}^+$ such that*

A9b (2.10)
$$\phi_N(x,0) \le e^{a_0 x_+^{5/4}} \quad \text{if} \quad N \ge N_0.$$

Also

$$\lim_{N\uparrow\infty} \phi_N(x,t) = e^{a(t)x_+^{5/4}}$$

for any $t>0$ and $x \in (-\infty,0)\cap(1,\infty)$ where

$$a(t) = \frac{a_0}{\sqrt[4]{1+k a_0^4 t}} \qquad \text{with} \quad k = k(\varepsilon) = \frac{5^5}{4^5}\Big(\frac{3}{2}+\frac{25}{4(5-\varepsilon)}\Big).$$

(vi) *There exists a constant $c_0 = c_0(a_0) > 0$ such that for any $\varepsilon \in [0,1]$, $\varepsilon \ll 1$,*

A10 (2.11)
$$\partial_t \phi_N + \frac{3}{2}\partial_x^5 \phi_N + \frac{25}{4(5-\varepsilon)}\,\frac{(\partial_x^3\phi_N)^2}{\partial_x \phi_N} \le c_0\,\phi_N$$

for any $(x,t) \in \mathbb{R}\times[0,\infty)$.

(vii) *There exist constants $c_j = c_j(j;a_0) > 0$, $j=1,2,\dots,5$ such that*

A11 (2.12)
$$|\partial_x^j \phi_N(x,t)| \le c_j\,\langle x\rangle^{j/4}\,\phi_N(x,t)$$

for any $(x,t) \in \mathbb{R}\times[0,\infty)$.

Proof of Theorem (2.1)

Given $a_0 > 0$, for $N \in Z^+$ we define

A12 (2.13)
$$\phi_N(x,t) = \begin{cases} e^{a(t)\varphi(x)}, & -\infty < x \le 1, \\ e^{a(t)x^{5/4}}, & 1 \le x \le N, \\ P_N(x,t), & x \ge N, \end{cases}$$

where

A13 (2.14)
$$a(t) = \frac{a_0}{\sqrt[4]{1+4k a_0^4 t}} \le a_0, \quad t \ge 0,$$

a_0 being the initial parameter and $k = k(\varepsilon) > 1$ is a constant whose precise value will be deduced below,

A14 (2.15)
$$\varphi(x) = (1-\eta(x))x_+^5 + \eta(x)x^{5/4}, \quad x_+ = \max\{x;0\}$$

for $x \in (-\infty,1]$ where $\eta \in C^\infty(\mathbb{R})$, $\eta' \ge 0$ and

A15 (2.16)
$$\eta(x) = \begin{cases} 0, & x \le 1/2, \\ 1, & x \ge 3/4, \end{cases}$$

(i.e. for each $x \in [0,1]$ $\varphi(x)$ is a convex combination of x^5 and $x^{5/4}$) and $P_N(x,t)$ is a polynomial of order 4 in x which matches the value of $e^{a(t)x^{5/4}}$ and its derivatives up to order 4 at $x=N$:

A16
$$\begin{aligned}
&P_N(x,t)=\\
&\Big\{1+\frac{5}{4}aN^{1/4}(x-N)+\frac{5}{4^2}\big(5a^2N^{2/4}+aN^{-3/4}\big)\frac{(x-N)^2}{2}\\
&\quad+\frac{5}{4^3}\big(25a^3N^{3/4}+15a^2N^{-2/4}-3aN^{-7/4}\big)\frac{(x-N)^3}{3!}\\
&\quad+\frac{5}{4^4}\big(125a^4N+150a^3N^{-1/4}-45a^2N^{-6/4}+21aN^{-11/4}\big)\frac{(x-N)^4}{4!}\Big\}\,e^{aN^{5/4}},
\end{aligned} \tag{2.17}$$

with $a=a(t)$ as in (2.14).

Thus to prove (2.8)-(2.12) (i)-(vii) we consider the intervals $(-\infty,0]$, $[0,1]$, $[1,N]$ and $[N,\infty)$.

The interval $(-\infty,0]$: In this case

$$\phi_N(x,t)=e^{a(t)\cdot 0}=1$$

which clearly satisfies (2.8) (i)-(vii).

The interval $[0,1]$: In this case

$$\phi_N(x,t)=e^{a(t)\varphi(x)}$$

with

$$\varphi(x)=(1-\eta(x))x^5+\eta(x)x^{5/4}\geq 0,\quad x\in[0,1]$$

with η as in (2.16). Since in this interval $x^{5/4}\geq x^5$ it follows that

A17
$$\begin{aligned}
\varphi'(x)&=(1-\eta(x))5x^4+\eta(x)\frac{5}{4}x^{1/4}+\eta'(x)(x^{5/4}-x^5)\\
&\geq(1-\eta(x))5x^4+\eta(x)\frac{5}{4}x^{1/4}\geq 0,
\end{aligned} \tag{2.18}$$

and there exist $c_j>0,\ j=0,1,\dots,5$ such that

A18
$$\varphi^{(j)}(x)\leq c_j,\quad x\in[0,1]. \tag{2.19}$$

Since

A19
$$a'(t)\leq 0\quad\text{one has}\quad a(t)\leq a_0\quad\text{for}\quad t\geq 0 \tag{2.20}$$

and we can conclude that

A20
$$\partial_x^j\phi_N(x,t)\leq c(j;a_0)\,\phi_N(x,t),\quad x\in[0,1],\ t\geq 0. \tag{2.21}$$

Also

A21
$$\partial_t\phi_N(x,t)=a'(t)\,\varphi(x)\,\phi_N(x,t)\leq 0. \tag{2.22}$$

Next we want to show that in this interval there exists $c_0=c_0(a_0)>0$ such that

A22
$$\frac{(\partial_x^3\phi_N(x,t))^2}{\partial_x\phi_N(x,t)}\leq c_0\,\phi_N(x,t), \tag{2.23}$$

i.e.

A23
$$(\partial_x^3\phi_N(x,t))^2\leq c_0\,\phi_N(x,t)\,\partial_x\phi_N(x,t). \tag{2.24}$$

Since

$$\begin{aligned}
&\partial_x\phi_N = a\varphi'\phi_N,\\
&\partial_x^2\phi_N = (a\varphi^{(2)} + (a\varphi')^2)\phi_N,\\
&\partial_x^3\phi_N = (a\varphi^{(3)} + 3a^2\varphi^{(2)}\varphi' + (a\varphi')^3)\phi_N,
\end{aligned}$$

one has that for $x \sim 0$ $(x \geq 0)$

$$\begin{aligned}
&\partial_x\phi_N \sim a5x^4\phi_N,\\
&\partial_x^2\phi_N \sim (a20x^3 + a^2 25x^8)\phi_N,\\
&\partial_x^3\phi_N \sim (a60x^2 + 3a^2 100x^7 + a^3 125x^{12})\phi_N,
\end{aligned}$$

Hence for $x \sim 0$ $(x \geq 0)$

$$\left(\partial_x^3\phi_N\right)^2 \leq c\,(a + a^3)^2\, x^4\, \phi_N^2$$

and

$$\phi_N\,\partial_x\phi_N \geq 5ax^4\,\phi_N^2.$$

Using (2.20) (i.e. $a(t) \leq a_0$ for $t \geq 0$) it follows that there exists $\delta > 0$ and a universal constant $c > 0$ such that

A24 (2.25) $$\left(\partial_x^3\phi_N\right)^2 \leq c\,(a_0 + a_0^5)\,\phi_N\,\partial_x\phi_N \quad \text{for} \quad x \in [0,\delta),\ t \geq 0.$$

In the interval $[\delta, 1]$ is easy to see that (2.25) still holds (with a possible large $c > 0$).

Combining the above estimates we see that (2.8) (i)-(vii) hold in this interval.

The interval $[1,N]$: In this region

A25 (2.26) $$\phi_N(x,t) = e^{a(t)\,x^{5/4}}, \qquad x \in [1,N],\ t \geq 0.$$

We calculate

A26 (2.27)
$$\begin{aligned}
\partial_x\phi_N &= \frac{5}{4}\,ax^{1/4}\phi_N > 0,\\
\partial_x^2\phi_N &= \frac{5}{4^2}\,(5a^2x^{2/4} + ax^{-3/4})\phi_N,\\
\partial_x^3\phi_N &= \frac{5}{4^3}\,(25a^3x^{3/4} + 15a^2x^{-2/4} - 3ax^{-7/4})\phi_N,\\
\partial_x^4\phi_N &= \frac{5}{4^4}\,(125a^4x + 150a^3x^{-1/4} - 45a^2x^{-6/4} + 21ax^{-11/4})\,\phi_N,\\
\partial_x^5\phi_N &= \frac{5}{4^5}\,(625a^5x^{5/4} + 1250a^4 - 375a^3x^{-5/4} + 375a^2x^{-10/4} - 231ax^{-15/4})\phi_N.
\end{aligned}$$

Hence $\phi_N,\ \partial_x\phi_N > 0$ and

A27 (2.28) $$\frac{\left(\partial_x^3\phi_N\right)^2}{\partial_x\phi_N} = \frac{5}{4^5}\Big(625a^5x^{5/4} + 750a^4 + 75a^3x^{-5/4} - 90a^2x^{-10/4} + 9ax^{-15/4}\Big)\,\phi_N.$$

Hence

A28 (2.29)
$$\begin{aligned}
&\partial_t\phi_N + \frac{3}{2}\partial_x^5\phi_N + \frac{5}{4}\,\frac{\left(\partial_x^3\phi_N\right)^2}{\partial_x\phi_N} =\\
&= \Big\{a'\,x^{5/4} + k\,a^5x^{5/4} + c_4\,a^4 + c_3\,a^3x^{-5/4} + c_2\,a^2x^{-10/4} + c_1\,ax^{-15/4}\Big\}\,\phi_N
\end{aligned}$$

with

A29

$$
(2.30)\qquad
\begin{aligned}
&\text{(a)}\quad k=\frac{5^5}{4^5}\Big(\frac{3}{2}+\frac{5}{4}\Big)>1\\
&\text{(b)}\quad c_4=\frac{5}{4^5}\Big(\frac{3}{2}1250+\frac{5}{4}750\Big)>0\\
&\text{(c)}\quad c_3=\frac{5}{4^5}\Big(\frac{3}{2}(-375)+\frac{5}{4}75\Big)<0\\
&\text{(d)}\quad c_2=\frac{5}{4^5}\Big(\frac{3}{2}(375)+\frac{5}{4}(-90)\Big)>0\\
&\text{(e)}\quad c_1=\frac{5}{4^5}\Big(\frac{3}{2}(-231)+\frac{5}{4}9\Big)<0.
\end{aligned}
$$

Notice that if we change the coefficient $5/4$ in (2.29) by $25/4(5-\varepsilon)$, $\varepsilon\in[0,1]$, $\varepsilon\ll 1$, the factor $5/4$ in (2.30) (a)-(e) changes in a similar manner, and the value of k in (2.30) (a) will increase to

A30

$$
(2.31)\qquad k(\varepsilon)=\frac{5^5}{4^5}\Big(\frac{3}{2}+\frac{25}{4(5-\varepsilon)}\Big)>1
$$

and c_1,c_2,c_3,c_4 remain with the same sign, uniformly bounded in $\varepsilon\in[0,1]$, $\varepsilon\ll 1$, and as we shall see below, the exact values of c_j's, $j=1,2,3,4$, are not relevant in the discussion below.

Next we solve the equation

A31

$$
(2.32)\qquad a'(t)=-k\,a^5(t)
$$

which eliminates the terms with power $5/4$ on the right hand side of (2.29). Thus

A32

$$
(2.33)\qquad a(t)=\frac{a_0}{\sqrt[4]{1+4\,k\,a_0^4\,t}}.
$$

Therefore to show that

A33

$$
(2.34)\qquad \partial_t\phi_N+\frac{3}{2}\partial_x^5\phi_N+\frac{5}{4}\frac{\big(\partial_x^3\phi_N\big)^2}{\partial_x\phi_N}\le c_0\,\phi_N
$$

with $c_0=c_0(a_0)>0$ from (2.30) it suffices to see that for $x\ge 1$.

A34

$$
(2.35)\qquad c_4a^4+c_3a^3x^{-5/4}+c_2a^2x^{-10/4}+c_1ax^{-15/4}\le c_0.
$$

Since $a(t)=a\le a_0$, $c_1,c_3\le 0$, and $x\ge 1$ one just needs to take c_0 such that

$$
c_4a_0^4+c_2a_0^2\le c_0.
$$

Next, from (2.27)

A35

$$
(2.36)\qquad
\begin{aligned}
&\partial_x\phi_N=\frac{5}{4}ax^{1/4}\phi_N\le c\,a_0\langle x\rangle^{1/4}\phi_N,\\
&\partial_x^2\phi_N\le c\,(a_0^2+a_0)\,\langle x\rangle^{1/2}\phi_N,\\
&\quad\vdots\qquad\qquad\vdots\\
&\partial_x^5\phi_N\le c\,(a_0^5+a_0)\,\langle x\rangle^{5/4}\phi_N.
\end{aligned}
$$

Finally we remark that

A36

$$
(2.37)\qquad \phi_N(x,t)=e^{a(t)x^{5/4}}\le e^{a_0N^{5/4}}\quad\text{for}\quad t\ge 0,\ \ x\in[1,N]
$$

which completes the proof of (2.8) (i)-(vi) in this interval.

<u>The interval $[N,\infty)$</u>: In this region

A37
$$
\begin{aligned}
\phi_N(x,t) = \ & P_N(x,t) = \Big\{1+\frac{5}{4}aN^{1/4}(x-N)\\
&+\frac{5}{4^2}\Big(\frac{5}{2}a^2N^{2/4}+\frac{1}{2}aN^{-3/4}\Big)(x-N)^2\\
&+\frac{5}{4^3}\Big(\frac{25}{6}a^3N^{3/4}+\frac{15}{6}a^2N^{-2/4}-\frac{3}{6}aN^{-7/4}\Big)(x-N)^3\\
&+\frac{5}{4^4}\Big(\frac{125}{24}a^4N^{4/4}+\frac{150}{24}a^3N^{-1/4}-\frac{45}{24}a^2N^{-6/4}+\frac{21}{24}aN^{-11/4}\Big)(x-N)^4\Big\}\,e^{aN^{5/4}},
\end{aligned}
\tag{2.38}
$$

with $a=a(t)$ as in (2.33).

First we shall show that the negative coefficients of $(x-N)^3$, i.e. $-15aN^{-7/4}/384$ and of $(x-N)^4$, i.e. $-225a^2N^{-6/4}/6144$ can be controlled by the other ones. More precisely, we shall see that there exists a universal constant $c>0$ such that

A34b
$$
\begin{aligned}
&\frac{5}{4^4}\Big(\frac{150}{24}a^3N^{-1/4}-\frac{45}{24}a^2N^{-6/4}+\frac{21}{24}aN^{-11/4}\Big)(x-N)^4\\
&-\frac{5}{4^3}\frac{3}{6}aN^{-7/4}(x-N)^3+\frac{5}{4^2}\frac{1}{2}aN^{-3/4}(x-N)^2\equiv R_N(x,t)\\
&\geq c\Big\{(a^3N^{-1/4}+a^2N^{-6/4}+aN^{-11/4})(x-N)^4\\
&\qquad+aN^{-7/4}(x-N)^3+aN^{-3/4}(x-N)^2\Big\},
\end{aligned}
\tag{2.39}
$$

A35b
$$
\begin{aligned}
\partial_x R_N(x,t)\geq c\Big\{&(a^3N^{-1/4}+a^2N^{-6/4}+aN^{-11/4})(x-N)^3\\
&+aN^{-7/4}(x-N)^2+aN^{-3/4}(x-N)\Big\},
\end{aligned}
\tag{2.40}
$$

and

A36b
$$
\begin{aligned}
\frac{\partial_t R_N(x,t)}{a'(t)}\geq c\Big\{&(a^2N^{-1/4}+aN^{-6/4}+N^{-11/4})(x-N)^4\\
&+N^{-7/4}(x-N)^3+N^{-3/4}(x-N)^2\Big\}.
\end{aligned}
\tag{2.41}
$$

Once (2.39)-(2.41) have been established it follows that there exists $c>0$ such that for $x\geq N$

A37b
$$
\begin{aligned}
P_N(x,t)\geq\ & c\Big\{1+aN^{1/4}(x-N)+(a^2N^{2/4}+aN^{-3/4})\frac{(x-N)^2}{2!}\\
&+(a^3N^{3/4}+a^2N^{-2/4}+aN^{-7/4})\frac{(x-N)^3}{3!}\\
&+(a^4N^{4/4}+a^3N^{-1/4}+a^2N^{-6/4}+aN^{-11/4})\frac{(x-N)^4}{4!}\Big\}\,e^{aN^{5/4}}\\
&\geq c\,e^{aN^{5/4}}>0,
\end{aligned}
\tag{2.42}
$$

(which proves (2.8) (ii) in this interval).

From (2.38)-(2.40) it can also be seen that

A40a-A38
$$
\begin{aligned}
\partial_x P_N(x,t)\geq\Big\{&\frac{5}{4}aN^{1/4}+\frac{5^2}{4^2}a^2N^{2/4}(x-N)\\
&+\frac{5^3}{4^3}\frac{a^3N^{3/4}}{2}(x-N)^2+\frac{5^4}{4^4}\frac{a^4N^{4/4}}{6}(x-N)^3\Big\}\,e^{aN^{5/4}}\geq 0
\end{aligned}
\tag{2.43}
$$

(which proves (2.8) (iii) in this interval) and

A39
$$\partial_t P_N(x,t) = a'(t)\, S_N(x,t)\, e^{aN^{5/4}} + a'(t)\, N^{5/4}\, P_N(x,t), \tag{2.44}$$

where

A40
$$\begin{aligned} S_N(x,t) \geq c\Big\{ & N^{1/4}(x-N) + (aN^{2/4} + N^{-3/4})\,\frac{(x-N)^2}{2!} \\ & + (a^2N^{3/4} + aN^{-2/4} + aN^{-7/4})\,\frac{(x-N)^3}{3!} \\ & + (a^3N^{4/4} + a^2N^{-1/4} + aN^{-6/4} + aN^{-11/4})\,\frac{(x-N)^4}{4!}\Big\} \geq 0. \end{aligned} \tag{2.45}$$

The proof of (2.39) and (2.40) are similar, so we restrict ourselves to present the details of that for (2.39).

First we observe that for any $\alpha > 0$

A41
$$a^2N^{-6/4} = \alpha a^{3/2} N^{-1/8}\,\frac{a^{1/2}N^{-11/8}}{\alpha} \leq \frac{1}{2}\Big(\alpha^2 a^3 N^{-1/4} + \frac{aN^{-11/4}}{\alpha^2}\Big). \tag{2.46}$$

Thus taking $\alpha = \sqrt{6}$ it follows that

A42
$$5(45+1)a^2N^{-6/4} \leq 5(149\,a^3N^{-1/4} + 4\,aN^{-11/4}). \tag{2.47}$$

Hence

A43
$$\begin{aligned} & \frac{5}{4^4 24}\big(150a^3N^{-1/4} - 45\,a^2N^{-6/4} + 21aN^{-11/4}\big) \\ & \qquad \geq \frac{5}{4^4 24}\big(a^3N^{-1/4} + a^2N^{-6/4} + 17aN^{-11/4}\big). \end{aligned} \tag{2.48}$$

This takes care of the term with coefficient $-45\cdot 5a^2N^{-6/4}/(4^4\cdot 24)$ in $(x-N)^4$, see (2.42). To handle the term $-15aN^{-7/4}(x-N)^3/(4^3\cdot 6)$ we write

A44
$$\begin{aligned} N^{-7/4}(x-N)^3 &= \alpha N^{-3/8}\,(x-N)\,\frac{N^{-11/8}(x-N)^2}{\alpha} \\ &\leq \frac{1}{2}\Big(\alpha^2 N^{-3/4}(x-N)^2 + \frac{N^{-11/4}(x-N)^4}{\alpha^2}\Big) \end{aligned} \tag{2.49}$$

for any $\alpha > 0$, and so taking $\alpha = \sqrt{6}$ one has that

A45
$$\begin{aligned} \frac{15+1}{4^3\cdot 6}aN^{-7/4}(x-N)^3 &= \frac{1}{24}aN^{-7/4}(x-N)^3 \\ &\leq \frac{1}{8}aN^{-3/4}(x-N)^2 + \frac{80}{4^4\cdot 24}aN^{-11/4}(x-N)^4. \end{aligned} \tag{2.50}$$

Collecting the above estimates we obtain (2.39).

Next we shall show that there exists $c_0 = c_0(a_0) > 0$ (independent of N) such that if $x \geq N$

A46
$$\partial_t\phi_N + \frac{3}{2}\partial_x^5\phi_N + \frac{25}{4(5-\varepsilon)}\,\frac{(\partial_x^3\phi_N)^2}{\partial_x\phi_N} \leq c_0\,\phi_N, \tag{2.51}$$

which in this region reduces to

A47
$$\partial_t P_N\,\partial_x P_N + \frac{25}{4(5-\varepsilon)}\,(\partial_x^3 P_N)^2 \leq c_0\,P_N\,\partial_x P_N, \tag{2.52}$$

for any $\varepsilon \in [0,1]$, $\varepsilon \ll 1$. As we have done before we first consider the case $\varepsilon = 0$.

Thus we have

A48

$$\begin{aligned}\frac{5}{4}\left(\partial_x^3 P_N\right)^2 = \frac{5}{4}\Big\{ & \Big(\frac{5^3}{4^3}a^3N^{3/4} + \frac{75}{4^3}a^2N^{-2/4} - \frac{15}{4^3}aN^{-7/4}\Big) \\ & + \Big(\frac{5^4}{4^4}a^4N^{4/4} + \frac{750}{4^4}a^3N^{-1/4} - \frac{225}{4^4}a^2N^{-6/4} + \frac{105}{4^4}aN^{-11/4}\Big)(x-N)\Big\}^2 e^{2aN^{5/4}},\end{aligned} \tag{2.53}$$

since, $a'(t) < 0$, by (2.44) and (2.42)

A49

$$\partial_t P_N \le a'(t)N^{5/4}P_N \le a'(t)N^{5/4}\, e^{aN^{5/4}}. \tag{2.54}$$

Thus, by (2.43)

A50

$$\begin{aligned}&\partial_t P_N \partial_x P_N \le a'(t)\,N^{5/4} \times \\ &\qquad \Big\{\frac{5}{4}aN^{1/4} + \frac{5^2}{4^2}a^2N^{2/4}(x-N) + \frac{5^3}{4^3}a^3\frac{N^{3/4}}{2}(x-N)^2\Big\}\, e^{2aN^{5/4}}.\end{aligned} \tag{2.55}$$

First we shall use $\partial_t P_N\, \partial_x P_N$ to control the terms in (2.53) involving the highest power in N. (Notice that we only handle the positive terms in (2.53)).

Thus using (2.30)–(2.32) it follows that

A51

$$\begin{aligned}&\frac{5}{4}\frac{5^6}{4^6}a^6N^{6/4} + a'(t)N^{5/4}\frac{5}{4}aN^{1/4} = aN^{6/4}\frac{5}{4}\Big(\frac{5^6}{4^6}a^5 + a'(t)\Big) \\ &= aN^{6/4}\frac{5}{4}\Big(\frac{5^6}{4^6}a^5 - ka^5\Big) = a^6N^{6/4}\Big(\frac{5}{4}\frac{5^6}{4^6} - \frac{5^6}{4^6}\Big(\frac{3}{2} + \frac{25}{4\cdot 5}\Big)\Big) < 0\end{aligned} \tag{2.56}$$

Notice that the last inequality above still holds with $25/4(5-\varepsilon)$, $\varepsilon \in [0,1]$, $\varepsilon \ll 1$, instead of $25/4\cdot 5$ and $5/4$.

Also by (2.30)–(2.32)

A52

$$\begin{aligned}&\frac{5}{4}2\Big(\frac{5^3}{4^3}a^3N^{3/4}\Big)\Big(\frac{5^4}{4^4}a^4N^{4/4}\Big)(x-N) + a'(t)N^{5/4}\frac{5^2}{4^2}a^2N^{2/4}(x-N) \\ &= a^2N^{7/4}\frac{5^2}{4^2}(x-N)\Big(2\frac{5^6}{4^6}a^5 + a'(t)\Big) \\ &= a^2N^{7/4}\frac{5^2}{4^2}(x-N)\Big(2\frac{5^6}{4^6}a^5 - ka^5\Big) \\ &= a^7N^{7/4}\frac{5^2}{4^2}(x-N)\Big(2\frac{5^6}{4^6} - \frac{5^5}{4^5}\Big(\frac{3}{2} + \frac{25}{20}\Big)\Big) \le 0\end{aligned} \tag{2.57}$$

(where the remark after (2.56) also applies), and again by (2.30)–(2.32)

A53

$$\begin{aligned}&\frac{5}{4}\frac{5^8}{4^8}a^8N^2(x-N)^2 + a'(t)N^{5/4}\frac{5^3}{4^3}\frac{a^3N^{3/4}}{2}(x-N)^2 \\ &= \frac{5^3}{4^3}a^3N^2(x-N)^2\Big(\frac{5^6}{4^6}a^5 + \frac{1}{2}a'(t)\Big) \\ &= \frac{5^3}{4^3}a^3N^2(x-N)^2\Big(\frac{5^6}{4^6}a^5 - \frac{1}{2}ka^5\Big) \\ &= \frac{5^3}{4^3}a^8N^2(x-N)^2\Big(\frac{5^6}{4^6} - \frac{1}{2}\frac{5^5}{4^5}\Big(\frac{3}{2} + \frac{25}{20}\Big)\Big) \le 0\end{aligned} \tag{2.58}$$

(where the remark after (2.56) also applies).

We bound the remaining terms in (2.53) by $c_0\partial_t P_N \partial_x P_N$. For that we use the fact that

$$e^{aN^{5/4}}\partial_x P_N \le P_N\, \partial_x P_N, \quad x \ge N.$$

Thus, from (2.43)

A54
$$\frac{5}{4}2\frac{5^4}{4^4}a^4N^{4/4}\frac{750}{4^4}a^3N^{-1/4}(x-N)^2e^{2aN^{5/4}} \leq c_0\frac{5^3}{4^3}a^3\frac{N^{3/4}}{2}(x-N)^2e^{2aN^{5/4}} \\ \leq c_0\,\partial_xP_Ne^{aN^{5/4}} \leq c_0\,P_N\partial_xP_N, \tag{2.59}$$

by taking

A55
$$c_0 > ca_0^4, \quad c \quad \text{universal constant.} \tag{2.60}$$

Notice that with this choice of c_0 (2.59) holds even when the factor $5/4$ in the left hand side is replaced by $25/4(5-\varepsilon)$, $\varepsilon \in [0,1]$, $\varepsilon \ll 1$. Also, from (2.43)

A56
$$2\frac{5}{4}\Big(\frac{5^4}{4^4}a^4N^{4/4}\frac{75}{4^3}a^2N^{-2/4} + \frac{750}{4^4}a^3N^{-1/4}\frac{5^3}{4^3}a^3N^{3/4}\Big)(x-N)e^{2aN^{5/4}} \\ \leq c_0\frac{5^2}{4^2}a^2N^{2/4}\,(x-N)\,e^{2aN^{5/4}} \leq c_0\,\partial_xP_N\,e^{aN^{5/4}} \leq c_0\,P_N\partial_xP_N \tag{2.61}$$

by taking c_0 as in (2.60) (where the remark after (2.60) also applies). Also

A57
$$2\frac{5}{4}\frac{5^3}{4^3}a^3N^{3/4}\frac{75}{4^3}a^2N^{-2/4}e^{2aN^{5/4}} \leq c_0\frac{5}{4}aN^{1/4}e^{2aN^{5/4}} \leq c_0\,P_N\partial_xP_N \tag{2.62}$$

by taking c_0 as in (2.60) (and the remark after (2.60) also applies).

This handles all the terms in (2.53) having a positive coefficient and a positive power of N. The reminder ones having positive coefficients can be bounded by

$$c_0\,\frac{5}{4}aN^{1/4}.$$

Combining the above estimates with (2.39)–(2.45) completes the proof of (2.52).

Finally (2.42) yields (2.12) in this region $x \geq N$.

To finish the proof we need to prove (v) in the region $[N,\infty)$. We use (2.27) with $t=0$ and observe that the negative terms in the expression for $\frac{d^5}{dx^5}e^{a_0x^{5/4}}$ can be absorbed by the positive terms for $x \geq N$ and N sufficiently large. More precisely,

$$1250a_0^4 > 2\cdot 375a_0^3x^{-5/4} \quad \text{and} \quad 375a_0^2N^{-10/4} > 2\cdot 231a_0x^{-15/4}$$

if $x^{5/4} > c/a_0$, where c is an absolute constant. To have this for $x \geq N$, it sufficies to take N in such a way that $N^{5/4} > c/a_0$. This is, $N > c^{4/5}a_0^{-4/5} \equiv N_0$.

In this way,

$$\frac{d^5}{dx^5}(e^{a_0x^{5/4}} - P_N(x,0)) = \frac{d^5}{dx^5}e^{a_0x^{5/4}} \geq 0$$

for $x > N > N_0$. Since $e^{a_0x^{5/4}}$ and $P_N(x,0)$ coincide at $x=N$ up to the fourth derivative, we conclude that $e^{a_0x^{5/4}} \geq P_N(x,0)$ for $x \geq N \geq N_0$, which proves (v) in this region.

Thus we have completed the proof of (2.7) (i)-(vii), (2.8)–(2.12).

corollaryA **Corollary 2.2.** *There exists $\widetilde{c}_0 = \widetilde{c}_0(a_0;T) > 0$ such that for any $N \in \mathbb{Z}^+$ sufficiently large, $x \in \mathbb{R}$, $t \in [0,T]$*

A58
$$\phi_N(x,t) \leq \widetilde{c}_0\left(1 + \langle x\rangle\,\partial_x\phi_N(x,t)\right). \tag{2.63}$$

The proof follows from the construction of the weight ϕ_N.

3. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1

Using the result in Theorem 1.5 (and the remark afterwards) we have that our solution u of the IVP (1.19) satisfies

3.2
$$u \in C([0,T];H^3(\mathbb{R})\cap L^2(e^{\beta x}dx)) \quad \text{for any } \beta>0. \tag{3.1}$$

Therefore, by interpolation one has that

3.2b
$$\partial_x^j u \in C([0,T];H^{3-j}(\mathbb{R})\cap L^2(e^{(4-j)\beta x/4}dx)) \quad j=0,1,2,3. \tag{3.2}$$

In particular $u\in C([0,T];L^2(\langle x\rangle^k dx))$, for any k. Suppose first that u is sufficiently regular, say $u\in C([0,T];H^5(\mathbb{R}))$. Then we can perform energy estimates for u using the weights $\{\phi_N\}$ (since $\phi_N\le c\langle x\rangle^4$). Thus, we multiply the equation in (1.19) by $u\phi_N$ and integrate by parts in the space variable to obtain

3.1
$$\int \partial_t u u\phi_N - \int \partial_x^5 u u\phi_N + b_1\int u\partial_x^3 u u\phi_N + b_2\int \partial_x u\partial_x^2 u u\phi_N + b_3\int u^2\partial_x u u\phi_N = 0, \tag{3.3}$$

and applying (2.6) and (2.11) we have

3a
$$\begin{aligned} 2\Big(\int \partial_t u u\phi_N\,dx - \int \partial_x^5 u u\phi_N\,dx\Big) &\ge \frac{d}{dt}\int u^2\phi_N\,dx + \varepsilon\int(\partial_x^2 u)^2\partial_x\phi_N\,dx \\ &\quad - \int u^2\Big(\partial_t\phi_N + \frac{3}{2}\partial_x^5\phi_N + \frac{25}{4(5-\varepsilon)}\frac{(\partial_x^3\phi_N)^2}{\partial_x\phi_N}\Big)\,dx \\ &\ge \frac{d}{dt}\int u^2\phi_N\,dx + \varepsilon\int(\partial_x^2 u)^2\partial_x\phi_N\,dx - c_0\int u^2\phi_N\,dx, \end{aligned} \tag{3.4}$$

with $\varepsilon\in[0,1]$, $\varepsilon << 1$, and $c_0=c_0(a_0)$. In the proof of Theorem 1.1 we will only use (3.4) with $\varepsilon=0$.

Now we shall handle the third, fourth and fifth terms on the right hand of (3.3). Thus we write

3.6
$$\int u\partial_x^3 u\,u\phi_N\,dx \le c\,\|\partial_x^3 u\|_\infty \int u^2\phi_N\,dx, \tag{3.5}$$

by integration by parts

3.7
$$\begin{aligned} \int \partial_x u\partial_x^2 u\,u\phi_N\,dx &= -\frac{1}{2}\int \partial_x^3 u\,u^2\phi_N\,dx - \frac{1}{2}\int \partial_x^2 u u^2\partial_x\phi_N\,dx \\ &\equiv E_1+E_2. \end{aligned} \tag{3.6}$$

We recall hat the ten $\|\partial_x^3 u(t)\|_\infty$ is integrable in the time interval $[-T,T]$, (see remark (i) after the statement of Theorem 1.2).

The bound for E_1 is similar to that in (3.5). To control E_2 we recall that (see (2.12))

3.8
$$0\le \partial_x\phi_N(x,t)\le c_1\,\langle x\rangle^{1/4}\phi_N(x,t)\le c(1+e^x)\,\phi_N(x,t) \tag{3.7}$$

so

$$E_2 \le c\left(\|e^x\,\partial_x^2 u\|_\infty + \|\partial_x^2 u\|_\infty\right)\int u^2\phi_N\,dx.$$

Notice that by combining Sobolev embedding and (3.2) one has that $\int_0^T \|e^x\partial_x^2 u\|_\infty(t)\,dt$ is finite.

Finally for the fifth term in (3.3) we have that

3.9
$$\int u^2\partial_x u\,u\phi_N\,dx \le \|u\,\partial_x u\|_\infty \int u^2\phi_N\,dx. \tag{3.8}$$

Collecting the above information, from (3.4) we can conclude that for any $N \in \mathbb{Z}^+$

$$\frac{d}{dt}\int u^2(x,t)\phi_N(x,t)\,dx \le M(t)\int u^2(x,t)\phi_N(x,t)\,dx$$

with $M(t) \in L^\infty([0,T])$, where $M(\cdot)$ depends on a_0; $\|e^x u_0\|_2$; $\|u_0\|_{3,2}$. Hence, from property (v) in (2.10), and Gronwall's Lemma we see that for $t \in [0,T]$

G

$$\begin{aligned}\int u^2(x,t)\phi_N(x,t)\,dx &\le c\Big(\int u_0^2(x)\phi_N(x,0)\,dx\Big)e^{\int_0^T M(t')\,dt'}\\ &\le c(a_0, \|e^{\frac12 a_0^2 x_+^{5/4}} u_0\|_2, \|u_0\|_{3,2}, T)\int u_0^2(x)e^{a_0 x_+^{5/4}}\,dx.\end{aligned} \tag{3.9}$$

Now, we will establish (3.9) for our less regular solution $u \in C([0,T];H^3(\mathbb{R}))$. To do that, we consider the IVP (1.19) with regularized initial data $u_{0,\delta} := \rho_\delta * u(\cdot+\delta,0)$, where $\delta > 0$, $\rho_\delta = \frac{1}{\delta}\rho(\frac{\cdot}{\delta})$, $\rho \in C^\infty(\mathbb{R})$ is supported in (-1,1), and $\int \rho = 1$. Since

3c

$$u_{0,\delta} \to u_0 \;\text{ in }\; H^3(\mathbb{R}) \;\text{ as }\; \delta \to 0, \tag{3.10}$$

by the well-posedness result in [16] for the IVP (1.19) in $H^3(\mathbb{R})$, the corresponding solutions u_δ satisfy $u_\delta(t) \to u(t)$ in $H^3(\mathbb{R})$ uniformly for $t \in [0,T]$ as $\delta \to 0$. In particular, by Sobolev embeddings, for fixed t

3d

$$u_\delta(x,t) \to u(x,t) \quad \text{for all } x \in \mathbb{R} \quad \text{as } \delta \to 0. \tag{3.11}$$

Also, it can be proved (see Theorem 1.1 in [9]) that

3e

$$\|e^{\frac12 a_0 x_+^{5/4}} u_{0,\delta}\|_2 \le \|e^{\frac12 a_0 x_+^{5/4}} u_0\|_2. \tag{3.12}$$

Since u_δ is sufficiently regular we have (3.9) with u_δ and $u_{0,\delta}$ instead of u and u_0. In this way, for t fixed, using (3.10)-(3.12), and applying Fatou's Lemma we see that

$$\int u^2(x,t)\phi_N(x,t)\,dx \le C(a_0, \|e^{\frac12 a_0 x_+^{5/4}} u_0\|_2, \|u_0\|_{3,2}, T)\int u_0^2(x)e^{a_0 x_+^{5/4}}\,dx \tag{3.13}$$

Now, we make $N \to \infty$ and apply property (v) in Theorem 2.1 and Fatou's Lemma again to obtain

$$\sup_{t\in[0,T]}\int u^2(x,t)e^{a(t)x_+^{5/4}}\,dx \le c*,$$

which is the desired result.

<u>Proof of Theorem 1.2</u>

We consider the equation for the difference of the two solutions

3.10

$$w(x,t) = (u_1 - u_2)(x,t) \tag{3.14}$$

that is,

3.11

$$\begin{aligned}\partial_t w - \partial_x^5 w = &-b_1(u_1\partial_x^3 w + \partial_x^3 u_2 w) - b_2(\partial_x u_1 \partial_x^2 w + \partial_x^2 u_2 \partial_x w)\\ &- b_3(\partial_x u_2(u_1+u_2)w + u_1^2\partial_x w).\end{aligned} \tag{3.15}$$

We follow the argument given in the proof of Theorem 1.1 with $\varepsilon \in [0,1]$, $\varepsilon \ll 1$. Hence we multiply (3.15) by $w\phi_N$ and integrate in the variable x and use that

3.12

$$\int u_1\partial_x^3 w\, w\phi_N\,dx = \frac12\int u_1\phi_N\partial_x^3(w^2)\,dx - 3\int u_1\phi_N\partial_x w\partial_x^2 w\,dx \equiv F_1 + F_2 \tag{3.16}$$

where

3.13 (3.17)
$$F_1 = -\frac{1}{2}\int \partial_x^3(u_1\phi_N)w^2\,dx.$$

Then using (2.12) it follows that

3.14 (3.18)
$$|F_1| \le \sum_{j=0}^{3} \|\langle x\rangle^{j/4}\partial_x^{3-j}u_1\|_\infty \int w^2\phi_N\,dx$$

and after some integration by parts

3.15 (3.19)
$$\begin{aligned} F_2 = -\frac{3}{2}\int u_1\phi_N\partial_x(\partial_x w)^2\,dx &= -\frac{3}{2}\int \partial_x(u_1\phi_N)w\partial_x^2 w\,dx + \frac{3}{4}\int \partial_x^3(u_1\phi_N)w^2\,dx \\ &\equiv F_2^1 + F_2^2. \end{aligned}$$

We observe that the same bound for F_1 given in (3.18) applies to F_2^2. For F_2^1 we write

3.16 (3.20)
$$F_2^1 = -\frac{3}{2}\int \partial_x u_1\phi_N w\partial_x^2 w\,dx - \frac{3}{2}\int u_1\partial_x\phi_N w\partial_x^2 w\,dx \equiv F_2^{1,1} + F_2^{1,2},$$

with

3.17 (3.21)
$$\begin{aligned} |F_2^{1,2}| &\le \frac{\varepsilon}{4}\int (\partial_x^2 w)^2\partial_x\phi_N\,dx + \frac{4}{\varepsilon}\int u_1^2 w^2\partial_x\phi_N\,dx \\ &\le \frac{\varepsilon}{4}\int (\partial_x^2 w)^2\partial_x\phi_N\,dx + \frac{c}{\varepsilon}\|u_1\langle x\rangle^{1/8}\|_\infty^2\int w^2\phi_N\,dx \end{aligned}$$

using (2.12) and

$$\begin{aligned} |F_2^{1,1}| &\le c_0\int |\partial_x u_1(1+\langle x\rangle\partial_x\phi_N)w\partial_x^2 w|\,dx \\ &= c_0\int |\partial_x u_1 w\partial_x^2 w|\,dx + c_0\int |\partial_x u_1\langle x\rangle w\partial_x^2 w|\partial_x\phi_N\,dx \\ &\le c_0\int |\partial_x u_1 w\partial_x^2 w|\,dx + \frac{\varepsilon}{4}\int (\partial_x^2 w)^2\partial_x\phi_N\,dx + \frac{c_0'}{\varepsilon}\int |\partial_x u_1|^2\langle x\rangle^2 w^2\partial_x\phi_N\,dx \\ &\le c_0\int |\partial_x u_1 w\partial_x^2 w|\,dx + \frac{\varepsilon}{4}\int (\partial_x^2 w)^2\partial_x\phi_N\,dx + \frac{c}{\varepsilon}\|\langle x\rangle^{1+1/8}\partial_x u_1\|_\infty^2\int w^2\phi_N\,dx \end{aligned}$$

by using Corollary 2.2 (2.63) and (2.12).

Directly one has that

$$\int \partial_x^3 u_2 w^2\phi_N\,dx \le \|\partial_x^3 u_2\|_\infty\int w^2\phi_N\,dx.$$

The estimate for the term

$$\int \partial_x u_1\partial_x^2 w w\phi_N\,dx$$

is similar to that given above for $F_2^{1,1}$.

Similarly, we have that

$$\int \partial_x^2 u_2\partial_x w w\phi_N\,dx = -\frac{1}{2}\int \partial_x(\partial_x^2 u_2\phi_N)w^2\,dx = -\frac{1}{2}\int \partial_x^3 u_2 w^2\phi_N\,dx - \frac{1}{2}\int \partial_x^2 u_2\partial_x\phi_N w^2\,dx$$

with

$$\Big|\int \partial_x^2 u_2\partial_x\phi_N w^2\,dx\Big| \le \int |\partial_x^2 u_2|\langle x\rangle^{1/4}\phi_N w^2\,dx \le \|\langle x\rangle^{1/4}\partial_x^2 u_2\|_\infty\int w^2\phi_N\,dx$$

and

$$\Big|\int \partial_x^3 u_2 w^2\phi_N\,dx\Big| \le \|\partial_x^3 u_2\|_\infty\int w^2\phi_N\,dx.$$

Finally, the terms

$$\int \partial_x u_2(u_1+u_2)w^2\phi_n\,dx+\int u_1^2\partial_x ww\phi_N\,dx$$

can be handled analogously.

Thus combining the inequalities

$$\int \partial_t ww\phi_N\,dx-\int \partial_x^5 ww\phi_N\,dx\geq\ 2\frac{d}{dt}\int w^2\phi_N\,dx+\varepsilon\int(\partial_x^2w)^2\partial_x\phi_N\,dx-c_0\int w^2\,\phi_N\,dx$$

(see (3.4)),

$$\int|\partial_x u_2 w\partial_x^2 w|\,dx\leq\|\partial_x u_2\|_\infty\|w\|_2\|\partial_x^2 w\|_2\equiv L(t),$$

and the above estimates we have that

$$\frac{d}{dt}\int w^2(x,t)\phi_N(x,t)\,dx\leq M(t)\int w^2(x,t)\phi_N(x,t)\,dx+L(t)$$

where

$$\begin{aligned}M(t)=c(\varepsilon)\Big(&\sum_{j=0}^{3}\|\langle x\rangle^{j/4}\partial_x^{3-j}u_1\|_\infty+\|\langle x\rangle^{1+1/8}\partial_x u_1\|_\infty\\&+\|\partial_x^3u_2\|_\infty+\|\langle x\rangle^{1/4}\partial_x^2u_2\|_\infty+\|\partial_x u_2\|_\infty(\|u_1\|_\infty+\|u_2\|_\infty)\Big)\end{aligned}$$

with $M,L\in L^\infty([0,T])$. Therefore

$$\sup_{[0,T]}\int w^2(x,t)\phi_N(x,t)\,dx\leq c\Big(\int w^2(x,0)\phi_N(x,0)\,dx+\int_0^T L(t)\,dt\Big)\,e^{\int_0^T M(t)\,dt}.$$

which basically yields the desired result.

4. Proofs of Theorem 1.3 and Theorem 1.4

<u>Proof of Theorem 1.3</u>

To simplify the exposition and illustrate the argument of proof we restrict ourselves to consider the most difficult case $P(u,\partial_x u,\partial_x^2u,\partial_x^3u)=\partial_x^2u\partial_x^3u$. Thus we have the equation

$$\partial_t u-\partial_x^5u+a\,\partial_x^2u\partial_x^3u=0,\qquad a\in\mathbb{R}.$$

Now we follow the argument given in the proof of Theorem 1.1. Then we need to consider the term

$$I=\int\partial_x^2u\partial_x^3uu\phi_N\,dx.$$

By integration by parts it follows that

$$\begin{aligned}I&=\frac{1}{20}\int\partial_x^5(u\,u)u\phi_N\,dx-\frac{1}{10}\int u\partial_x^5uu\phi_N\,dx-\frac{1}{2}\int\partial_x u\partial_x^4uu\phi_N\,dx\\&=I_1+I_2+I_3.\end{aligned}$$

Hence

$$I_1=-\frac{1}{20}\int\partial_x^5(u\phi_N)\,u^2\,dx$$

Thus

4.1 $$|I_1|\leq c\sum_{j=0}^{5}\|\langle x\rangle^{j/4}\partial_x^{5-j}u\|_\infty\int u^2\phi_N\,dx.\tag{4.1}$$

Similarly,

$$I_2 \le c\|\partial_x^5 u\|_\infty \int u^2 \phi_N \, dx,$$

and after integration by parts

$$I_3 \le \left(\|\partial_x^5 u\|_\infty + \|\langle x\rangle^{1/4}\partial_x^4 u\|_\infty\right) \int u^2 \phi_N \, dx.$$

Therefore one has that

4.2
$$|I| \le c \sum_{j=0}^{5} \|\langle x\rangle^{j/4}\partial_x^{5-j} u\|_\infty \int u^2 \phi_N \, dx. \tag{4.2}$$

Now using Theorem 1.5 and interpolation one has that for $j = 0, 1, \dots, 6$

$$\sup_{[0,T]} \|e^{(6-j)\beta x}\partial_x^j u(t)\|_2 \le c,$$

which combined with (4.2) and Sobolev embedding yields the desired result.

<u>Proof of Theorem 1.4</u>

As in the proof of Theorem 1.3 we shall consider the most significant form of the polynomial $P(\cdot)$ in (1.26), $P(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = \partial_x^2 u \partial_x^3 u$. Thus we consider the equation

$$\partial_t u - \partial_x^5 u + a\,\partial_x^2 u \partial_x^3 u = 0, \qquad a \in \mathbb{R}.$$

Hence $w = u_1 - u_2$ satisfies

$$\partial_t w - \partial_x^5 w + a\,\partial_x^2 u_1 \partial_x^3 w + a\,\partial_x^3 u_2 \partial_x^2 w = 0.$$

Following the argument given in the proof of Theorem 1.2 we shall estimate

$$E_1 = \int \partial_x^2 u_1 \partial_x^3 w w \phi_N \, dx$$

and

$$E_2 = \int \partial_x^3 u_2 \partial_x^2 w w \phi_N \, dx.$$

More precisely, we have

4.3
$$\frac{d}{dt}\int w^2 \phi_N \, dx + \varepsilon \int (\partial_x^2 w)^2 \partial_x \phi_N \, dx \le c_0 \int w^2 \phi_n \, dx + E_1 + E_2. \tag{4.3}$$

To bound E_2 we use Corollary 2.2 and (2.11)

4.4
$$\begin{aligned} |E_2| &\le \tilde{c}_0 \int \partial_x^3 u_2 \partial_x^2 w w (1 + \langle x\rangle \partial_x \phi_N) \, dx \\ &\le \tilde{c}_0 \int \partial_x^3 u_2 \, \partial_x^2 (u_1 - u_2)(u_1 - u_2) \, dx \\ &\quad + c_{\varepsilon'} \int (\partial_x^3 u_2)^2 \langle x\rangle^2 w^2 \partial_x \phi_N \, dx + \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N \, dx \\ &\le M(t) + c_{\varepsilon'} \int (\partial_x^3 u_2)^2 \langle x\rangle^{2+1/4} w^2 \phi_N \, dx + \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N \, dx \\ &\le M(t) + c_{\varepsilon'} \|\langle x\rangle^{1+1/8}\partial_x^3 u_2\|_\infty^2 \int w^2 \phi_N \, dx + \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N \, dx, \end{aligned} \tag{4.4}$$

where $0 < \varepsilon' \ll \varepsilon$.

To control E_1 we write

$$
\begin{aligned}
E_1 &= -\int \partial_x^3 u_1 \partial_x^2 w w \phi_N\, dx - \int \partial_x^2 u_1 \partial_x^2 w \partial_x w \phi_N - \int \partial_x^2 u_1 \partial_x^2 w w \partial_x \phi_N\, dx \\
&= E_1^1 + E_1^2 + E_1^3.
\end{aligned}
$$

The bound for E_1^1 is similar to the one deduced above for E_2. For E_1^3 we write

$$
|E_1^3| \le \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N\, dx + c_{\varepsilon'} \int (\partial_x u_1)^2 w^2 \partial_x \phi_N\, dx,
$$

hence a bound similar to that obtained for $|E_2|$ applies.

Finally, to estimate E_1^2 we write

$$
\begin{aligned}
E_1^2 &= \int \partial_x^2 u_1 \partial_x^2 w \partial_x w \phi_N\, dx = \frac{1}{2}\int \partial_x^3 u_1 \partial_x w \partial_x w \phi_N\, dx + \frac{1}{2}\int \partial_x^2 u_1 \partial_x w \partial_x w \partial_x \phi_N\, dx \\
&= \frac{1}{4}\int w^2 \Big[\partial_x(\partial_x^4 u_1 \phi_N) + 2\partial_x(\partial_x^3 u_1 \partial_x \phi_N) + \partial_x(\partial_x^2 u_1 \partial_x^2 \phi_N)\Big]\, dx \\
&\quad - \frac{1}{2}\int \partial_x^3 u_1 \partial_x^2 w w \phi_N\, dx - \frac{1}{2}\int \partial_x^2 u_1 w \partial_x^2 w \partial_x \phi_N\, dx \\
&= E_1^{2,1} + E_1^{2,2} + E_1^{2,3}.
\end{aligned}
$$

Thus from (2.12) one has that

$$
|E_1^{2,1}| \le \sum_{j=0}^{5} \|\langle x\rangle^{j/4} \partial_x^{5-j} u_1\|_\infty \int w^2 \phi_N\, dx
$$

and also

$$
\begin{aligned}
|E_1^{2,3}| &\le \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N\, dx + c_{\varepsilon'} \int (\partial_x^2 u_1)^2 w^2 \partial_x \phi_N\, dx \\
&\le \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N\, dx + c_{\varepsilon'} \int (\partial_x^2 u_1)^2 \langle x\rangle^{1/4} w^2 \phi_N\, dx.
\end{aligned}
$$

Finally an argument similar to that given in (4.4) shows that

$$
|E_1^{2,2}| \le M(t) + c_{\varepsilon'} \int (\partial_x^3 u_1)^2 \langle x\rangle^{2+1/4} w^2 \phi_N\, dx + \varepsilon' \int (\partial_x^2 w)^2 \partial_x \phi_N\, dx.
$$

Inserting these estimates in (4.3) one gets the desired result.

5. Proof of Theorem 1.5

Proof of Theorem 1.5

We shall follow Kato's approach in [12] and define for $\beta > 0$

5.1 (5.1)
$$
\varphi_\delta(x) = \frac{e^{\beta x}}{1+\delta e^{\beta x}} \quad \text{for} \quad \delta \in (0,1), \quad \delta \ll 1.
$$

Thus one has

5.2 (5.2)
$$
\varphi_\delta \in L^\infty(\mathbb{R}) \quad \text{and} \quad \|\varphi_\delta\|_\infty = \frac{1}{\delta}.
$$

5.3 (5.3)
$$
0 \le \partial_x \varphi_\delta(x) = \frac{\beta\, e^{\beta x}}{(1+\delta e^{\beta x})^2} \le \beta\, \varphi_\delta(x),
$$

5.4 (5.4)
$$
\partial_x^2 \varphi_\delta(x) = \frac{\beta^2\, e^{\beta x}(1-\delta e^{\beta x})}{(1+\delta e^{\beta x})^3},
$$

then

5.5
$$|\partial_x^2 \varphi_\delta(x)| \le \beta^2 \frac{e^{\beta x}}{(1+\delta e^{\beta x})^2}. \tag{5.5}$$

5.6
$$\partial_x^3 \varphi_\delta(x) = \frac{\beta^3 e^{\beta x}(1 - 4\delta e^{\beta x} + \delta^2 e^{2\beta x})}{(1+\delta e^{\beta x})^4}, \tag{5.6}$$

so

5.7
$$|\partial_x^3 \varphi_\delta(x)| \le 2\beta^3 \frac{e^{\beta x}}{(1+\delta e^{\beta x})^2}. \tag{5.7}$$

and

5.8
$$|\partial_x^j \varphi_\delta(x)| \le c_j \beta^j \frac{e^{\beta x}}{(1+\delta e^{\beta x})^2}, \quad j = 1,2,3,4,5. \tag{5.8}$$

Also we have that

5.9
$$0 \le \frac{(\partial_x^3 \varphi_\delta)^2}{\partial_x \varphi_\delta} \le 4\beta^5 \frac{e^{\beta x}}{(1+\delta e^{\beta x})^2}. \tag{5.9}$$

Therefore

5.10
$$\frac{3}{2}|\partial_x^5 \varphi_\delta(x)| + \frac{25}{4(5-\varepsilon)} \frac{(\partial_x^3 \varphi_\delta)^2}{\partial_x \varphi_\delta} \le c_0 \beta^5 \frac{e^{\beta x}}{(1+\delta e^{\beta x})^2} \le c_0 \beta^5 \varphi_\delta(x). \tag{5.10}$$

Moreover

5.11
$$\varphi_\delta(x) \le \varphi_{\delta'}(x) \quad x \in \mathbb{R} \quad \text{if} \quad 0 < \delta' < \delta \tag{5.11}$$

and

5.12
$$\lim_{\delta \downarrow 0} \varphi_\delta(x) = e^{\beta x}. \tag{5.12}$$

As in Theorem 1.3 and Theorem 1.4 we shall consider the most relevant case in (1.26).

5.13
$$P(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = a \partial_x^2 u \partial_x^3 u, \quad a \in \mathbb{R}, \tag{5.13}$$

to get the equation

5.14
$$\partial_t u - \partial_x^5 u + a \partial_x^2 u \partial_x^3 u = 0. \tag{5.14}$$

We employ an argument similar to that exposed in (3.5). Indeed, we multiply equation (5.14) by $u\varphi_\delta$ and integrate by parts. Then we use the Cauchy-Schwarz and Young inequalities and the property (5.10), to obtain the estimate

5.14b
$$\begin{aligned}
&2\Big(\int \partial_t u\, u \varphi_\delta \, dx - \int \partial_x^5 u \varphi_\delta \, dx\Big) \\
&= \frac{d}{dt} \int u^2 \varphi_\delta \, dx + 5 \int (\partial_x^2 u)^2 \partial_x \varphi_\delta \, dx + 5 \int u \partial_x^2 u \partial_x^3 \varphi_\delta \, dx - \frac{3}{2} \int u^2 \partial_x^5 \varphi_\delta \, dx \\
&\ge \frac{d}{dt} \int u^2 \varphi_\delta \, dx + \varepsilon \int (\partial_x^2 u)^2 \partial_x \varphi_\delta \, dx - \int u^2 \Big(\frac{3}{2}|\partial_x^5 \varphi_\delta| + \frac{25}{4(5-\varepsilon)} \frac{(\partial_x^3 \varphi_\delta)^2}{\partial_x \varphi_\delta}\Big) dx \\
&\ge \frac{d}{dt} \int u^2 \varphi_\delta \, dx + \varepsilon \int (\partial_x^2 u)^2 \partial_x \varphi_\delta \, dx - c_0 \beta^5 \int u^2 \varphi_\delta \, dx
\end{aligned} \tag{5.15}$$

with $\varepsilon \in [0,1)$, $\varepsilon \ll 1$ and $c_0 > 0$. With this estimate we deduced that

5.15 (5.16)
$$\begin{aligned}\frac{d}{dt}\int u^2\varphi_\delta(x)\,dx + \varepsilon\int(\partial_x^2 u)^2\partial_x\varphi_\delta(x)\,dx\\ \le c_0\beta^5\int u^2\varphi_\delta(x)\,dx + |a\int\partial_x^2 u\partial_x^3 uu\varphi_\delta(x)\,dx|.\end{aligned}\tag{5.16}$$

Next we estimate the last term of (5.16). We integrate by parts and write

$$\begin{aligned}\int\partial_x^2 u\partial_x^3 uu\varphi_\delta(x)\,dx &= \frac{1}{20}\int\partial_x^5(u^2)u\varphi_\delta(x)\,dx - \frac{1}{10}\int u\partial_x^5 uu\varphi_\delta(x)\,dx - \frac{1}{2}\int\partial_x u\partial_x^4 uu\varphi_\delta(x)\,dx\\ &= E_1 + E_2 + E_3.\end{aligned}$$

Thus one has

$$E_1 = -\frac{1}{20}\int u^2\partial_x^5(u\varphi_\delta(x))\,dx.$$

Therefore by (5.1)-(5.8)

$$|E_1| \le c\sum_{j=0}^{5}\beta^j\|\partial_x^{5-j}u(t)\|_\infty\int u^2\varphi_\delta(x)\,dx.$$

Also

$$|E_3| \le c\,(\|\partial_x^5 u\|_\infty + \beta\|\partial_x^4 u\|_\infty)\int u^2\varphi_\delta(x)\,dx$$

and

$$|E_2| \le \|\partial_x^5 u\|_\infty\int u^2\varphi_\delta(x)\,dx.$$

Inserting these estimates in (5.16) it follows that

$$\frac{d}{dt}\int u^2\varphi_\delta(x)\,dx \le c_0\big(\beta^5 + \sum_{j=0}^{5}\beta^j\|\partial_x^{5-j}u(t)\|_\infty\big)\int u^2\varphi_\delta(x)\,dx$$

which implies that

5.16 (5.17)
$$\begin{aligned}\sup_{[0,T]}\int u(x,t)\varphi_\delta(x)\,dx &\le \int u_0(x)\varphi_\delta(x)\,dx\,e^{\int_0^T N(t)\,dt}\\ &\le \int u_0(x)\varphi_0(x)\,dx\,e^{\int_0^T N(t)\,dt}\end{aligned}\tag{5.17}$$

with

$$N(t) = c_0\big(\beta^5 + \sum_{j=0}^{5}\beta^j\|\partial_x^{5-j}u(t)\|_\infty\big).$$

Since the right hand side of (5.17) is independent of δ taking $\delta \downarrow 0$ we obtain the desired result.

We shall notice that in the argument above we assumed the solution sufficiently smooth to perform the integration by parts, otherwise we consider the IVP associated to the equation (5.14) with regularized initial data as was done in the proof of Theorem 1.1.

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