THE NON-LINEAR SCHRÖDINGER EQUATION WITH A PERIODIC δ–INTERACTION

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Abstract. We study the existence and stability of space-periodic standing waves for the space-periodic cubic nonlinear Schrödinger equation with a point defect determined by a space-periodic Dirac distribution at the origin. This equation admits a smooth curve of positive space-periodic solutions with a profile given by the Jacobi elliptic function of dnoidal type. Via a perturbation method and continuation argument, we prove that in the case of an attractive defect the standing wave solutions are stable in $H^1_{per}([-\pi,\pi])$ with respect to perturbations which have the same space-periodic as the wave itself. In the case of a repulsive defect, the standing wave solutions are stable in the subspace of even functions of $H^1_{per}([-\pi,\pi])$ and unstable in $H^1_{per}([-\pi,\pi])$ with respect to perturbations which have the same space-periodic as the wave itself.

1. Introduction

Consider the semi-linear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u \pm |u|^p u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function and $0 < p < \infty$. This is a canonical dispersive equation which arises as a model in several physical situations, see for example [48], [14], and references therein.

The mathematical study of the NLS (the local well posedness of its initial value problem (IVP) and its periodic boundary value problem (PBVP) under minimal regularity assumptions on the data, the long time behavior of their solutions, blow up and scattering results, etc) has attracted a great deal of attention and is a very active research area (see [16], [10], [49], and [38]).

in [51] it was established that the 1-dimensional cubic case of (1.1) (i.e. NLS with $(n,p) = (1,2)$) is completely integrable. Thus, using the inverse scattering theory it can be solved in the line $\mathbb{R}$ (IVP) and in the circle $\mathbb{T}$ (PBVP) (see [1], [39] and references therein).

Special solutions of the NLS equation (1.1) have been widely considered in analytic, numerical and experimental works. In particular, in the focussing case (+ in (1.1)) one has the "standing waves" solutions

$$u_s(x,t) = e^{i\omega t}\phi(x), \qquad \omega > 0, \tag{1.2}$$

or their generalization "travelling waves" solutions

$$u_{tw}(x,t) = e^{i\omega t}\, e^{i(c\cdot x - |c|^2 t)}\phi(x - 2ct), \qquad \omega > 0, \;\; c \in \mathbb{R}^n, \tag{1.3}$$

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with $\phi = \phi_{\omega,p}$ being the unique positive, radially symmetric solution (ground state) of the nonlinear elliptic problem

$$-\Delta\phi + \omega\phi(x) - \phi^{p+1}(x) = 0, \quad x \in \mathbb{R}^n, \tag{1.4}$$

satisfying the boundary condition $\phi(x) \to 0$ as $|x| \to \infty$. In the one dimensional case, $n = 1$, ϕ is given by the explicit formula (modulo translation)

$$\phi(x) = \phi_{\omega,p}(x) = \Big[\frac{(p+2)\omega}{2} sech^2\Big(\frac{p\sqrt{\omega}}{2}x\Big)\Big]^{\frac{1}{p}}. \tag{1.5}$$

The stability and instability properties of the standing waves have been extensively studied. A crucial role in the stability analysis is played by the symmetries of the NLS equation in $\mathbb{R}^n$. The most important ones for this purpose are :

(1) *phase invariance*: $u(x,t) \to e^{i\theta}u(x,t),\ \theta \in \mathbb{R}$;
(2) *translation invariance*: $u(x,t) \to u(x+y,t),\ y \in \mathbb{R}^n$;
(3) *Galilean invariance*: $u(x,t) \to e^{i(v\cdot x - |v|^2 t)}u(x-2vt,t),\ v \in \mathbb{R}^n$.
(4) *Scaling invariance*: $u(x,t) \to \lambda^{2/p}u(\lambda x, \lambda^2 t),\ \lambda \in \mathbb{R}$.

So, if one considers the orbit generated by the solution $\phi = \phi_{\omega,p}$ of (1.4) and the phase-invariance symmetries above, namely,

$$\Theta(\phi_{\omega,p}) = \{e^{i\theta}\phi_{\omega,p}(\cdot + y) : \theta \in [0,2\pi), y \in \mathbb{R}^n\}, \tag{1.6}$$

is known that in the one dimensional case, $n = 1$, $\Theta(\phi_{\omega,p})$ is stable in $H^1(\mathbb{R})$ by the flow of the NLS equation provided that $p < 4$ and unstable for $p \geqq 4$ (for details and results in higher dimensions see Cazenave&Lions [17], Cazenave [16], Weinstein [50]). This means that for $p < 4$, if u_0 is close to $\Theta(\phi_{\omega,p})$ in $H^1(\mathbb{R}^n)$, then the corresponding solution of (1.1) $u(t)$ with initial data u_0 remains close to the orbit $\Theta(\phi_{\omega,p})$ for each $t \in \mathbb{R}$.

From now on we shall restrict our attention to the one dimensional focussing NLS

$$i\partial_t u + \partial_x^2 u + |u|^p u = 0, \qquad p > 0. \tag{1.7}$$

In contrast to the standing waves solutions in the line commented above relatively less is known about the existence and stability of periodic standing wave solutions, i.e., ϕ in (1.2) being a periodic solution of the equation (1.4).

A partial *spectral stability* analysis was carried out by Rowlands [45] for the case $p = 2$ with respect to long-wave disturbances, who showed that space-periodic waves with real-valued profile are unstable. Similar results were also obtained for certain NLS-type equations with spatially periodic potentials by Bronski&Rapti [12]. The first results concerning the *nonlinear stability* of space-periodic standing waves are due to Angulo [5]. In [5] it was established the existence of a smooth family of *dnoidal waves* for the cubic NLS equation ($p = 2$ in (1.7)) of the form

$$\omega \in \Big(\frac{\pi^2}{2L^2}, +\infty\Big) \to \phi_{\omega,0} \in H^\infty_{per}([-L,L]), \tag{1.8}$$

where the profile of $\phi_{\omega,0}$ is given by the Jacobian elliptic function called *dnoidal*, *dn*, by the formula

$$\phi_{\omega,0}(\xi) = \eta_1 dn\Big(\frac{\eta_1}{\sqrt{2}}\xi; k\Big), \tag{1.9}$$

with $\eta_1 \in (\sqrt{\omega}, \sqrt{2\omega})$ and the modulus $k \in (0,1)$ depending smoothly on ω. Angulo showed that for every $\omega > \frac{\pi^2}{2L^2}$ the $2L$-periodic wave $\phi_{\omega,0}$ is *orbitally stable* with respect to perturbations which have the same period as the wave itself, and *nonlinearly unstable* with respect to perturbations which have two times the period ($4L$) as the wave itself. Indeed, the same analysis used to obtain the instability result provides the *nonlinear instability* of the dnoidal wave by perturbations which have j-times ($j > 2$) the period as the wave itself (for further details see also [5] and [6]).

In [24]-[25] Gallay&Haragus have shown the stability of space-periodic traveling waves described in (1.3) for the cubic NLS equation by allowing the profile ϕ being complex-valued. In the case $p = 4$ in (1.7), Angulo&Natali [9] have shown the existence of a family of periodic waves of the form described in (1.2) for which there is a unique (threshold) value of the phase-velocity ω which separates the two global scenarios: stability and instability.

In the past years, the following nonlinear Schrödinger model (NLS-δ henceforth) in the line

$$i\partial_t u + \partial_x^2 u + Z\delta(x)u + |u|^p u = 0, \tag{1.10}$$

where δ is the Dirac distribution at the origin, namely, $\langle \delta, v\rangle = v(0)$ for $v \in H^1(\mathbb{R})$, and $Z \in \mathbb{R}$, it has received a good attention by mathematicians and physicists. The equation (1.10), $Z \neq 0$ has been considered in a variety of physical models with a point defect, for instance, in nonlinear optics and Bose-Einstein condensates. Indeed, the Dirac distribution is used to model an impurity, or defect, localized at the origin, and it is is described by the following boundary problem (see Caudrelier&Mintchev&Ragoucy [18])

$$\begin{cases} i\partial_t u(x,t) + \partial_x^2 u(x,t) = -\left|u(x,t)\right|^p u(x,t), \quad x \neq 0 \\ \lim\limits_{x\to 0^+} [u(x,t) - u(-x,t)] = 0, \\ \lim\limits_{x\to 0^+} [\partial_x u(x,t) - \partial_x u(-x,t)] = -Zu(0,t) \\ \lim\limits_{x\to \pm\infty} u(x,t) = 0, \end{cases} \tag{1.11}$$

hence $u(x,t)$ must be solution of the non-linear Schrödinger equation on $\mathbb{R}^-$ and $\mathbb{R}^+$, continuous at $x = 0$ and satisfy a "jump condition" at the origin and it also vanish at infinity. Also the NLS-δ equation (1.10) can be viewed as a prototype model for the interaction of a wide soliton with a highly localized potential. In nonlinear optics, this models a soliton propagating in a medium with a point defect or the interaction of a wide soliton with a much narrower one in a bimodal fiber, see [26], [47], [15], [42], [41], [2], [11], [20], [46], and the reference therein.

Equation (1.10) in the line with $p = 2$ has been considered by several authors. In a series of papers [29], [30], [31], and [32] the phenomenon of soliton scattering by the effect of the defect was comprehensibly studied. In particular, in [31] for the equation (1.10) with $p = 2$ and data

$$u(x,0) = e^{icx} sech(x - x_0), \quad x_0 << -1, \tag{1.12}$$

it was shown that for the $|Z| << 1$ the corresponding solution, the traveling wave for $t > |x_0|/c$ remains intact. The case $Z > 0$ and $|c| >> 1$ was examinated in [29], [30] where it was proven how the defect separate the soliton into two parts: one part is transmitted past the defect, the other one is reflected at the defect. The case $Z < 0$ and $|c| >> 1$ was considered in [19].

The existence of standing wave solutions of the equation (1.10) requires that the profile $\phi = \phi_{\omega,Z,p}$ satisfy the semi-linear elliptic equation

$$\Big(-\frac{d^2}{dx^2} - Z\delta(x)\Big)\phi + \omega\phi - |\phi|^p\phi = 0. \tag{1.13}$$

In Fukuizumi&Jeanjean [22] (see also [26]) it was deduced the formula for the unique positive even solution of (1.13), modulo rotations :

$$\phi_{\omega,Z,p}(x) = \Big[\frac{(p+2)\omega}{2} sech^2\Big(\frac{p\sqrt{\omega}}{2}|x| + \tanh^{-1}\Big(\frac{Z}{2\sqrt{\omega}}\Big)\Big)\Big]^{\frac{1}{p}}, \quad x \in \mathbb{R}, \tag{1.14}$$

if $\omega > Z^2/4$. This solution is constructed from the known solution (1.5) in the case $Z = 0$ on each side of the defect pasted together at $x = 0$ to satisfy the conditions of continuity and the jump condition in the first derivative at $x = 0$, $\phi'(0+) - \phi'(0-) = -Z\phi(0)$ determined by (1.11). So ϕ belongs to the domain of the formal expression $-\partial_x^2 - Z\delta$ (see Section 3 below or [3])

$$\{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : u'(0+) - u'(0-) = -Zu(0)\}.$$

Notice that there is no nontrivial solution of (1.13) in $H^1(\mathbb{R})$ when $\omega \leqq Z^2/4$.

The basic symmetry associated to equation (1.10) is the phase-invariance since the translation invariance of the solutions is not hold due to the defect. Thus, the notion of stability and instability will be based only on this symmetry and is formulated as follows:

Definition 1.1. *For $\eta > 0$, let ϕ be a solution of* (1.13) *and define*

$$U_\eta(\phi) = \Big\{v \in X : \inf_{\theta\in\mathbb{R}} \|v - e^{i\theta}\phi\|_X < \eta\Big\}.$$

The standing wave $e^{i\omega t}\phi$ is (orbitally) stable in X if for any $\epsilon > 0$ there exists $\eta > 0$ such that for any $u_0 \in U_\eta(\phi)$, the solution $u(t)$ of (1.10) *with $u(0) = u_0$ satisfies $u(t) \in U_\epsilon(\phi)$ for all $t \in \mathbb{R}$. Otherwise, $e^{i\omega t}\phi$ is said to be (orbitally) unstable in X.*

Gathering the information in [22], [23], [26], [37], and [43] in the cases of $X = H^1(\mathbb{R})$ or $X = H^1_{rad}(\mathbb{R})$, one can summarize the known results on the stability and instability of standing waves associated to the solitary wave-peak in (1.14) as follows:

- Let $Z > 0$ and $\omega > Z^2/4$.
  - (a) If $0 < p \leqq 4$, the standing wave $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/4, +\infty)$.
  - (b) If $p > 4$, there exists a unique $\omega_1 > Z^2/4$ such that $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/4, \omega_1)$, and unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1, +\infty)$.
- Let $Z < 0$ and $\omega > Z^2/4$.
  - (a) If $0 < p \leqq 2$, the standing wave $e^{i\omega t}\phi_{\omega,Z,p}$ is stable in $H^1_{rad}(\mathbb{R})$ for any $\omega \in (Z^2/4, +\infty)$.
  - (b) If $0 < p \leqq 2$, the standing wave $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/4, +\infty)$.
  - (c) If $2 < p < 4$, there exists a $\omega_2 > Z^2/4$ such that $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/4, \omega_2)$, and stable in $H^1_{rad}(\mathbb{R})$ for any $\omega \in (\omega_2, +\infty)$.
  - (d) If $2 < p < 4$, the standing wave $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_2, +\infty)$, where ω_2 is that in item (c) above, and unstable in $H^1_{rad}(\mathbb{R})$ for $\omega = \omega_2$.
  - (e) if $p \geqq 4$, then the standing wave $e^{i\omega t}\phi_{\omega,Z,p}$ is unstable in $H^1(\mathbb{R})$.

In this paper, we study the existence and nonlinear stability in $H^1_{\rm per}([-\pi, \pi])$ of space-periodic standing waves solutions of (1.10) in the case $p = 2$ and $Z \neq 0$, namely, for the NLS-δ model

$$i\partial_t u + \partial_x^2 u + Z\delta(x)u + |u|^2 u = 0, \tag{1.15}$$

where $\delta = \delta_0$ represents the periodic Dirac distribution at the origin, namely, $\langle \delta, v\rangle = v(0)$ for $v \in H^1_{\rm per}([-\pi, \pi])$ and $u(t) \in H^1_{\rm per}([-\pi, \pi])$, $t \in \mathbb{R}$, satisfying the boundary conditions

$$\begin{cases} \lim\limits_{x\to 0^+} [u(x, t) - u(-x, t)] = 0, \\ \lim\limits_{x\to 0^+} [\partial_x u(x, t) - \partial_x u(-x, t)] = -Zu(0, t). \end{cases}$$

We note that by the periodic boundary conditions, the periodic Dirac distribution at the origin δ_0 can be changed by any periodic Dirac distribution $\delta_{2j\pi}$ centered in the point $2j\pi$, $j \in \mathbb{Z}$. Here, we show the existence of a branch of periodic solutions, $\omega \to \varphi_{\omega,Z}$, for the semi-linear elliptic equation

$$\Big(-\frac{d^2}{dx^2} - Z\delta(x)\Big)\varphi_{\omega,Z} + \omega\varphi_{\omega,Z} = \varphi^3_{\omega,Z}, \tag{1.16}$$

where $\varphi_{\omega,Z} > 0$ is a periodic real-valued function with prescribe period 2π and where ω will belong to a determined interval in $\mathbb{R}$ with $\omega > Z^2/4$. Our solutions $\varphi = \varphi_{\omega,Z} : \mathbb{R} \to \mathbb{R}$ (see (1.23) and Figure 3 below) satisfy the following boundary conditions:

$$\begin{aligned} &(1)\ \varphi_{\omega,Z}(x+2\pi) = \varphi_{\omega,Z}(x), \quad \text{for all } x \in \mathbb{R}. \\ &(2)\ \varphi_{\omega,Z} \in C^2(\mathbb{R} - \{2n\pi : n \in \mathbb{Z}\}) \cap C(\mathbb{R}), \\ &(3)\ -\varphi''_{\omega,Z}(x) + \omega\varphi_{\omega,Z}(x) = \varphi^3_{\omega,Z}(x) \quad \text{for } x \neq \pm 2n\pi,\ n \in \mathbb{N}. \\ &(4)\ \varphi'_{\omega,Z}(0+) - \varphi'_{\omega,Z}(0-) = -Z\varphi_{\omega,Z}(0). \end{aligned} \tag{1.17}$$

The notation $\varphi'_{\omega,Z}(0\pm)$ in (1.17) is defined as $\varphi'_{\omega,Z}(0\pm) = \lim_{\epsilon\downarrow 0}\varphi'_{\omega,Z}(\pm\epsilon)$. From the periodicity of the function $\varphi_{\omega,Z}$ one also has that $\varphi'_{\omega,Z}(\pm 2n\pi+) - \varphi'_{\omega,Z}(\pm 2n\pi-) = -Z\varphi_{\omega,Z}(2n\pi)$, for $n \in \mathbb{N}$. We recall that if $\varphi_{\omega,Z}$ is a solution of (1.16) then $\varphi_{\omega,Z}(\cdot + y)$, $y \in \mathbb{R}$, is not necessarily a solution of (1.16). Hence, our stability study for the "*periodic-peaks*" $\varphi_{\omega,Z}$ will be for the orbit generated by this solution and defined in the form

$$\Omega_{\varphi_{\omega,Z}} = \{e^{i\theta}\varphi_{\omega,Z} : \theta \in [0, 2\pi]\}. \tag{1.18}$$

From equation (1.16) arises naturally the condition that our solutions $\varphi_{\omega,Z}$ need to belong to the domain of the formal expression

$$-\Delta_{-Z} \equiv -\frac{d^2}{dx^2} - Z\delta. \tag{1.19}$$

So, we shall develop a precise formulation for this *periodic* point interaction, also called a periodic δ-interaction at the origin. We present a detailed study of the model of quantum mechanics (1.19) with a potential supported on a δ and in the framework of periodic functions. In our study of the "solvability" of this model we will describe their resolvents explicitly in terms of the interactions strengths, Z, and the location of the source, $x = 0$. We start by establishing the definition of all the self-adjoint extensions of the operator $A^0 = -\frac{d^2}{dx^2}$ with domain

$$D(A^0) = \{\psi \in D(A) : \delta(\psi) \equiv \psi(0) = 0\}, \tag{1.20}$$

which is a densely defined symmetric operator on $L^2_{per}([-\pi, \pi])$ with deficiency indices $(1, 1)$. Here A represents the self-adjoint operator $-\frac{d^2}{dx^2}$ on $L^2_{per}([-\pi, \pi])$ with the natural domain $D(A) = H^2_{per}([-\pi, \pi])$. Using the von Neumann theory of self-adjoint extensions for symmetric operators we can parametrize all the self-adjoint extensions of A^0 with the help of Z. Indeed, for $Z \in [-\infty, \infty)$ we have

$$\begin{cases} -\Delta_{-Z} = -\dfrac{d^2}{dx^2} \\ D(-\Delta_{-Z}) = \{\zeta \in H^1_{\text{per}}([-\pi, \pi]) \cap H^2((0, 2\pi)) : \zeta'(0+) - \zeta'(0-) = -Z\zeta(0)\}, \qquad Z \neq -\infty, \end{cases} \tag{1.21}$$

the case $Z = -\infty$ is discussed in Theorem 3.1 below. Then for $\zeta \in D(-\Delta_{-Z})$ we have

$$-\Delta_{-Z}\zeta(x) = -\zeta''(x), \qquad \text{for} \quad x \neq 2n\pi,\ \ n \in \mathbb{Z}.$$

From (1.21) we have the following observations. For $\zeta \in D(-\Delta_{-Z})$ we have by periodicity $\zeta|_{(-2\pi,0)} \in H^2((-2\pi, 0))$, then in particular $\zeta|_{(-\pi,\pi)} \in H^2((-\pi, \pi) - \{0\})$. Moreover, since $\zeta|_{(0,2\pi)} \in H^2((0, 2\pi))$, $\zeta|_{(0,2\pi)} \in C^1([0, 2\pi])$, so that $\zeta(0), \zeta'(0+)$ and $\zeta'(0-)(= \zeta'(2\pi-))$ are well defined. These definitions and observations are not only important to determine solutions for equation in (1.16) but also for our nonlinear stability theory.

In Section 4, by using the theory of elliptic integral, the theory of Jacobi elliptic functions and the implicit function theorem we will find a smooth branch of positive, even, periodic-peak solutions of (1.16), $\omega \to \phi_{\omega,Z} \in H^n_{per}([-\pi, \pi])$, $n = 1, 2, 3, \cdots$, such that $\phi_{\omega,Z} \in D(-\Delta_{-Z})$ (therefore we obtain the conditions in (1.17)) and satisfying

$$\lim_{Z\to 0^+} \phi_{\omega,Z} = \phi_{\omega,0} \tag{1.22}$$

where $\phi_{\omega,0}$ is the dnoidal traveling wave defined in (1.9). For $Z>0$, the profile of $\phi_{\omega,Z}$ obtained is based in the Jacobian elliptic function *dnoidal* and determined by the pattern

$$\phi_{\omega,Z}(\xi)=\eta_{1,Z}dn\Big(\frac{\eta_{1,Z}}{\sqrt{2}}|\xi|+a;k\Big),\qquad \xi\in[-\pi,\pi] \tag{1.23}$$

where $\eta_{1,Z}$ and the modulus k depend smoothly of ω and Z. The shift value a is also a smooth function of ω and Z satisfies that $\lim_{Z\to 0^+}a(\omega,Z)=0$. See Figure 3 below for a general profile of $\phi_{\omega,Z}$.

Similarly, we obtain for $Z<0$ a smooth branch of positive, even, periodic-peak solutions of (1.16), $\omega\to\zeta_{\omega,Z}\in H^n_{per}([-\pi,\pi])$, such that $\zeta_{\omega,Z}\in D(-\Delta_{-Z})$ and satisfying

$$\lim_{Z\to 0^-}\zeta_{\omega,Z}=\phi_{\omega,0} \tag{1.24}$$

where $\phi_{\omega,0}$ is the dnoidal wave defined in (1.9). The profile of $\zeta_{\omega,Z}$ is determined by the pattern

$$\zeta_{\omega,Z}(\xi)=\eta_{1,Z}dn\Big(\frac{\eta_{1,Z}}{\sqrt{2}}|\xi|-a;k\Big),\qquad \xi\in[-\pi,\pi]. \tag{1.25}$$

See Figure 4 below for a general profile of $\zeta_{\omega,Z}$. We note that the periodic-peak $\phi_{\omega,Z}$ and $\zeta_{\omega,Z}$ "converge" to the solitary wave-peak $\phi_{\omega,Z,2}$ in (1.14) when we consider $\eta_{1,Z}\to\sqrt{2\omega}$. We refer the reader to Section 4 for the precise details on this convergence.

Our approach for the stability theory of the periodic-peak family

$$\varphi_{\omega,Z}=\begin{cases}\phi_{\omega,Z}, & Z>0,\\ \zeta_{\omega,Z}, & Z<0,\end{cases} \tag{1.26}$$

with $\phi_{\omega,Z}$ and $\zeta_{\omega,Z}$ given in (1.23)-(1.24) and ω large (by technical reasons), it will be based in the general framework developed by Grillakis&Shatah&Strauss [27], [28], for a Hamiltonian system which is invariant under a one-parameter unitary group of operators. This theory requires the following informations :

- The *Cauchy problem*: The initial value problem associated to the NLS-δ equation is well-posedness in $H^1_{per}([-\pi,\pi])$.
- The *spectral condition*:
  (a) The self-adjoint operator $\mathcal{L}_{2,Z}$

$$\mathcal{L}_{2,Z}\zeta=-\frac{d^2}{dx^2}\zeta+\omega\zeta-\varphi^2_{\omega,Z}\zeta \tag{1.27}$$

  with domain $\mathcal{D}=D(-\Delta_{-Z})$ given in (1.21), is a nonnegative operator with the eigenvalue zero being simple and with eigenfunction $\varphi_{\omega,Z}$ in (1.26).
  (b) The self-adjoint operator $\mathcal{L}_{1,Z}$

$$\mathcal{L}_{1,Z}\zeta=-\frac{d^2}{dx^2}\zeta+\omega\zeta-3\varphi^2_{\omega,Z}\zeta \tag{1.28}$$

  with domain $\mathcal{D}=D(-\Delta_{-Z})$ given in (1.21), has a trivial kernel for all $Z\in\mathbb{R}-\{0\}$.
  (c) The number of negative eigenvalues of the operator $\mathcal{L}_{1,Z}$.
- The *slope condition*: The sign of $\partial_\omega\int_{-\pi}^{\pi}\varphi^2_{\omega,Z}(\xi)d\xi$.

The local well-posedness of the Cauchy problem for (1.15) in $H^1_{per}([-\pi,\pi])$ is an consequence from Theorem 3.7.1 in [16] and from the theory spectral established in Section 3 below for the operator $-\partial_x^2-Z\delta$ for $Z\neq 0$ (see Proposition 3.1 below). The global existence of solutions **in** $H^1_{per}([-\pi,\pi])$ is an immediate consequence of the following conserved quantities for (1.15): the energy and the charge, respectively,

$$\begin{aligned} E(v(\cdot,0)) = E(v(\cdot,t)) &= \frac{1}{2}\int_{-\pi}^{\pi} |\partial_x v(x,t)|^2\, dx - \frac{Z}{2}|v(0,t)|^2 - \frac{1}{4}\int_{-\pi}^{\pi} |v(x,t)|^4\, dx, \\ Q(v(\cdot,0)) = Q(v(\cdot,t)) &= \frac{1}{2}\int_{-\pi}^{\pi} |v(x,t)|^2\, dx. \end{aligned} \tag{1.29}$$

Now, since the profile $\varphi_{\omega,Z}$ is positive and $\mathcal{L}_{2,Z}\varphi_{\omega,Z} = 0$, we obtain from classical theory for self-adjoint operators that $\mathcal{L}_{2,Z} \geqq 0$ (see Proposition 5.1 below). That the kernel of $\mathcal{L}_{1,Z}$ is trivial is a very delicate point in our theory (see Theorem 5.1 below). Here we need to use the specific spectral structure associated to the periodic eigenvalue problem in $L^2_{\rm per}([0, 2K])$ for the Lamé's equation,

$$\begin{cases} \Phi''(x) + [\lambda - 6k^2 sn^2(x; k)]\Phi(x) = 0, \quad x \in (0, 2K) \\ \Phi(0) = \Phi(2K(k)), \;\; \Phi'(0) = \Phi'(2K(k)), \;\; k \in (0, 1) \end{cases} \tag{1.30}$$

with sn being the Jacobian elliptic function called snoidal and K the complete elliptic integral of first type.

Lastly, to count the number of negative eigenvalues of linear operators is in general a delicate issue in any stability theory. In the case of the self-adjoint operator $\mathcal{L}_{1,Z}$ our strategy is based in two basic facts. The first one is that in the case $Z = 0$, the spectrum of the self-adjoint operator $\mathcal{L}_0 \equiv \mathcal{L}_{1,0}$

$$\mathcal{L}_0\zeta = -\frac{d^2}{dx^2}\zeta + \omega\zeta - 3\phi^2_{\omega,0}\zeta \tag{1.31}$$

with general domain $H^2_{per}([0, 2L])$ and $\omega > \pi^2/2L^2$, has already been described in [5] and in [8]: there is only one negative eigenvalue which is simple, zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\phi_{\omega,0}$. The rest of the spectrum is positive and discrete. The second one is that for Z small, $\mathcal{L}_{1,Z}$ can be considered as a *real-holomorphic perturbation* of $\mathcal{L}_0$. So, we have that the spectrum of $\mathcal{L}_{1,Z}$ depends holomorphically on the spectrum of $\mathcal{L}_0$. Then we obtain that for $Z < 0$ there are exactly two negative eigenvalues of $\mathcal{L}_{1,Z}$ and exactly one for $Z > 0$. We refer the reader to Subsection 6.1 for the precise details on these statements.

Our main stability result is the following:

Theorem 1.1. *We consider the family of periodic-peak $\varphi_{\omega,Z}$ in (1.26). Then, for ω large one has:*

1. *For $Z > 0$, the dnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable in $H^1_{per}([-\pi, \pi])$ by the flow detemined by the NLS-δ equation (1.15).*
2. *For $Z < 0$, the dnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is unstable in $H^1_{per}([-\pi, \pi])$ by the flow detemined by the NLS-δ equation (1.15).*
3. *For $Z < 0$, the dnoidal-peak standing wave $e^{i\omega t}\varphi_{\omega,Z}$ is stable in $H^1_{per,even}([-\pi, \pi])$ by the flow detemined by the NLS-δ equation (1.15).*

The restriction about ω being large in Theorem 1.1 is due to technical reasons determined by the implicit function theorem (see Section 4) and in proving the strictly increasing property of the mapping $\omega \to \|\varphi_{\omega,Z}\|^2$ (see Theorem 5.5 in subsection 5.4).

This paper is organized as follows. Section 3 is devoted to establish a spectral theory for the operator $-\partial_x^2 - Z\delta$ for $Z \neq 0$. Our analysis is based in the theory of von Neumann for self-adjoint extensions. We also establish the periodic well-posedness theory for (1.10), $p = 2$, in $H^1_{per}([-\pi, \pi])$. Section 4 describes the construction, via the implicit function theorem, of a smooth curve of periodic-peak for equation (1.16). Finally, in Section 5, the stability and instability theory of the dnoidal-peak is established.

2. Notation

For any complex number $z \in \mathbb{C}$, we denote by $\Re z$ and $\Im z$ the real part and imaginary part of z, respectively. For $s \in \mathbb{R}$, the Sobolev space $H^s_{\rm per}([-L,L])$ consists of all periodic distributions f such that $\|f\|^2_{H^s} = 2L \sum\limits_{k=-\infty}^{\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty$ (see [34]). We will use the notation $H^s_{\rm per}$ or $H^s_{\rm per}([0,2L])$ for $H^s_{\rm per}([-L,L])$ in several places and $H^0_{\rm per} = L^2_{\rm per}$. We remark that $L^2_{\rm per}$ and $H^1_{\rm per}$ are regarded as real Hilbert space with inner products

$$\langle f,g\rangle_{L^2} = \Re \int_{-L}^{L} f(x)\overline{g(x)}dx, \quad \langle f,g\rangle_{H^1} = \langle f,g\rangle_{L^2} + \langle \partial_x f, \partial_x g\rangle_{L^2}. \tag{2.1}$$

We denote $\|f\|_{L^2} = \|f\|$ and $\langle f,g\rangle_{L^2} = \langle f,g\rangle$. For Ω being an open set of $\mathbb{R}$, $H^n(\Omega)$, $n \in \mathbb{N}$, represents the classical local Sobolev space. $[H^s_{per}]'$, the topological dual of H^s_{per}, is isometrically isomorphic to H^{-s}_{per} for all $s \in \mathbb{R}$. The duality is implemented concretely by the pairing

$$(f,g) = 2L \sum_{k=-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad for \quad f \in H^{-s}_{per}, \ \ g \in H^s_{per}.$$

Thus, if $f \in L^2_{per}$ and $g \in H^s_{per}$, with $s \geq 0$, it follows that $(f,g) = \langle f,g\rangle$. The convolution for $f, g \in L^2_{per}$ is defined by $f \star g(x) = \frac{1}{2L}\int_{-L}^{L} f(x-y)g(y)dy$. The normal elliptic integral of first type (see [13]) is defined by

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi,k)$$

where $y = \sin\varphi$ and $k \in (0,1)$. k is called the modulus and φ the argument. When $y = 1$, we denote $F(\pi/2,k)$ by $K = K(k)$. The three basic Jacobian elliptic functions are denoted by $sn(u;k)$, $cn(u;k)$ and $dn(u;k)$ (called, snoidal, cnoidal and dnoidal, respectively), and are defined via the previous elliptic integral. More precisely, let $u(y;k) := u = F(\varphi,k)$ then $y = sin\varphi := sn(u;k) = sn(u)$ and $cn(u;k) := \sqrt{1-y^2} = \sqrt{1-sn^2(u;k)}$, $dn(u;k) := \sqrt{1-k^2y^2} = \sqrt{1-k^2sn^2(u;k)}$. In particular, we have that $1 \geqq dn(u;k) \geqq k' \equiv \sqrt{1-k^2}$ and the following asymptotic formulas: $sn(x;1) = tanh(x)$, $cn(x;1) = sech(x)$ and $dn(x;1) = sech(x)$.

3. The one-center periodic δ-interaction in one dimension and the global well-posedness in H^1_{per}

In this section for convenience of the reader we establish initially a precise formulation for the *periodic* point interaction determined by the formal linear differential operator

$$-\frac{d^2}{dx^2} + \gamma\delta, \tag{3.1}$$

defined on functions on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. γ represents the coupling constant or strength attached to the point source located at $x = 0$. After that we show a global well-posedness theory in $H^1_{per}([-\pi,\pi])$ for the NLS-δ equation (1.15).

We note that there are many approach for studying the operator in (3.1), for instance, by the use of quadratic forms or by the self-adjoint extensions of symmetric operators. We also note that the quantum mechanics model in (3.1) has been studied into a more general framework when it is associated with the Kronig-Penney model in solid state physics (see Chapter III.2 in Albeverio *et al.* [3]) or when it is associated to singular rank one perturbations (Albeverio *et al.* [4]).

Our main purpose in the following is to obtain many basic structures of model in (3.1) which will be necessary in our stability theory for the periodic-peak solutions (1.23)-(1.23). For instance, to determinate the domain and the real-analytic property (on the parameter Z) of the the self-adjoint operators $\mathcal{L}_{1,Z}, \mathcal{L}_{2,Z}$ in (1.27)-(1.28). So, we will show an explicit formula for the resolvents of (3.1) in terms of the interactions strengths, γ, and a specific description of the spectrum. So, for $A = -\frac{d^2}{dx^2}$ being considered with domain $D(A) = H^2_{\text{per}}([-\pi,\pi])$ and the symmetric restriction $A^0 \equiv A|_{D(A^0)}$ with dense domain $D(A^0) = \{\psi \in D(A) : (\delta,\psi) \equiv \psi(0) = 0\}$, we obtain that the *deficiency subspaces* of A^0,

$$\mathcal{D}_+ = \text{Ker}(A^{0^*} - i), \quad \text{and} \quad \mathcal{D}_- = \text{Ker}(A^{0^*} + i), \tag{3.2}$$

has dimension (*deficiency indices*) equal to 1. It is no difficult to see that these subspaces are generated, respectively, by $g_{+i} \equiv (A-i)^{-1}\delta$ and $g_{-i} \equiv (A+i)^{-1}\delta$, called *deficiency elements* (see Lemma 1.2.3 in [4]).

Next we present explicitly all the self-adjoint extensions of the symmetric operator A^0, which will be parametrized by the strength γ. From the von Neumann's theory of self-adjoint extensions for symmetric operators (see [44]) we obtain that all the closed symmetric extensions of A^0 are self-adjoint and coincides with the restriction of the operator A^{0^*}. Moreover, for $\theta \in [0,2\pi)$ the self-adjoint extension $A^0(\theta)$ of A^0 is defined as follows;

$$\begin{cases} D(A^0(\theta)) = \{\psi + \lambda g_i + \lambda e^{i\theta} g_{-i} : \psi \in D(A^0), \lambda \in \mathbb{C}\}, \\ A^0(\theta)(\psi + \lambda g_i + \lambda e^{i\theta} g_{-i}) = A^{0^*}(\psi + \lambda g_i + \lambda e^{i\theta} g_{-i}) = A^0\psi + i\lambda g_i - i\lambda e^{i\theta} g_{-i}. \end{cases} \tag{3.3}$$

Now we give the profile of the deficiency elements $g_{\pm i}$ with $\|g_{\pm i}\| = 1$. Since $g_i = \overline{g_{-i}}$, we shall determine a formula for $g_{-i} \in L^2_{\text{per}}([0,2\pi])$. So, since g_{-i} represents the fundamental solution associated to $A+i$ we have for $\mathcal{K}_i \in L^2_{\text{per}}([-\pi,\pi])$ such that

$$\widehat{\mathcal{K}_i}(k) = \frac{1}{k^2+i}, \tag{3.4}$$

that $g_{-i} = \frac{1}{2\pi}\mathcal{K}_i(x)$ (in the distributional sense, $g_{-i} = (A+i)^{-1}\delta = \delta \star \mathcal{K}_i = \frac{1}{2\pi}\mathcal{K}_i$). Then, if we denote $\beta = \frac{1+i}{\sqrt{2}}$ ($\beta^2 = i$) we obtain via the variational parameters method that

$$\mathcal{K}_i(x) = \frac{2\pi}{2\beta\sinh(\beta\pi)} \cosh\Big(2\pi\beta\Big(\frac{x}{2\pi} - \Big[\frac{x}{2\pi}\Big] - \frac{1}{2}\Big)\Big), \qquad x \in \mathbb{R}. \tag{3.5}$$

Here $[\cdot]$ stands for the integer part. Lastly, we obtain the following expression for the deficiency element g_{-i}. For $\sigma = 1/(2\beta\sinh(\beta\pi))$ and $x \in [-\pi,\pi]$

$$g_{-i}(x) = \sigma\Big[\cosh\Big(\frac{|x|-\pi}{\sqrt{2}}\Big)\cos\Big(\frac{|x|-\pi}{\sqrt{2}}\Big) + i\sinh\Big(\frac{|x|-\pi}{\sqrt{2}}\Big)\sin\Big(\frac{|x|-\pi}{\sqrt{2}}\Big)\Big]. \tag{3.6}$$

See Figure 1 and Figure 2 below for the profile of the real and imaginary parts of g_{-i}, $\Re(g_{-i})$ and $\Im(g_{-i})$, respectively (We note that $\Re(g_{-i})$ has the peaks in $\pm 2n\pi$, $n \in \mathbb{Z}$, and $\Im(g_{-i})$ is a smooth periodic function). Now, for $\|g_i\|^2 = \|g_{-i}\|^2 = a_0$ with

$$a_0 = \frac{\sqrt{2}}{4}\frac{\sinh(\sqrt{2}\pi) + \sin(\sqrt{2}\pi)}{\cosh(\sqrt{2}\pi) - \cos(\sqrt{2}\pi)}$$

we obtain the normalized deficiency elements $\widetilde{g}_{\pm i} = \frac{g_{\pm i}}{\|g_{\pm i}\|}$. But for convenience of notation we will continue to use $g_{\pm i}$. Thus, from the von Neumann formulas (3.3) we obtain from (3.6) that for $\zeta \in D(A^0(\theta))$, in the form $\zeta = \psi + \lambda g_i + \lambda e^{i\theta} g_{-i}$, we have the basic expression

$$\zeta'(0+) - \zeta'(0-) = -\lambda(1 + e^{i\theta}). \tag{3.7}$$

Next we find γ such that $\gamma\zeta(0) = -\lambda(1 + e^{i\theta})$. Indeed, after some calculations we find the formula

$$\gamma(\theta) = \frac{-2\cos(\theta/2)}{\Re[\coth(\beta\pi)e^{i(\frac{\theta}{2}-\frac{\pi}{4})}]}, \tag{3.8}$$

Therefore, if θ varies in $[0, 2\pi)$, $\gamma = \gamma(\theta)$ varies in $\mathbb{R} \cup \{+\infty\}$. For the unique $\theta_0 \in [0, 2\pi)$ such that $\Re[\coth(\beta\pi)e^{i(\frac{\theta_0}{2}-\frac{\pi}{4})}] = 0$ we have $\lim_{\theta\uparrow\theta_0}\gamma(\theta) = +\infty$.

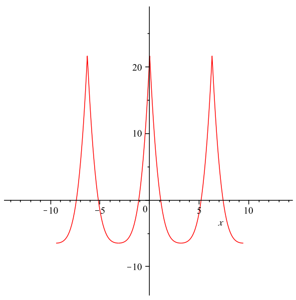

FIGURE 1. Graphic of the function $\Re(g_{-i})$ given by (3.6)

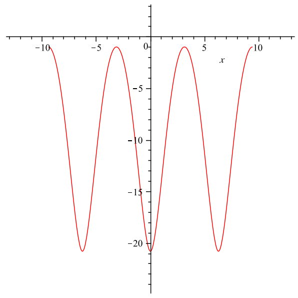

FIGURE 2. Graphic of the function $\Im(g_{-i})$ given by (3.6)

So, from now on we parametrize all self-adjoint extensions of A^0 with the help of γ. Thus we get,

Theorem 3.1. *All self-adjoint extensions of A^0 are given for $-\infty < \gamma \leqq +\infty$ by the following formulas: for $\gamma \in (-\infty, \infty)$*

$$\begin{cases} -\Delta_\gamma = -\dfrac{d^2}{dx^2} \\ D(-\Delta_\gamma) = \{\zeta \in H^1_{per}([-\pi,\pi]) \cap H^2((0,2\pi)) : \zeta'(0+) - \zeta'(0-) = \gamma\zeta(0)\}. \end{cases} \tag{3.9}$$

The special case $\gamma = 0$ just leads to the operator $-\Delta$ in $L^2_{per}([-\pi,\pi])$,

$$-\Delta = -\frac{d^2}{dx^2}, \qquad D(-\Delta) = H^2_{per}([-\pi,\pi]). \tag{3.10}$$

For $\gamma = +\infty$ we have $-\Delta_{+\infty} = -\frac{d^2}{dx^2}$, with a Dirichlet-periodic boundary condition at zero,

$$D(-\Delta_{+\infty}) = \{\zeta \in H^1_{per}([-\pi,\pi]) \cap H^2((0,2\pi)) : \zeta(0) = 0\}. \tag{3.11}$$

Proof. By the arguments sketched above we obtain easily that $A^0(\theta) \subset -\Delta_\gamma$, with $\gamma = \gamma(\theta)$ given in (3.8). But $-\Delta_\gamma$ is symmetric in the corresponding domain $D(-\Delta_\gamma)$ for all $-\infty < \gamma \leqq +\infty$, which implies the relation $A^0(\theta) \subset -\Delta_\gamma \subset (-\Delta_\gamma)^* \subset A^0(\theta)$. It completes the proof of the Theorem. □

Remarks:

(1) From (3.9) we obtain that for $\zeta \in D(-\Delta_\gamma)$, $\zeta \in H^2((-\pi,\pi) - \{0\})$ and $\zeta \in H^2((2n\pi, 2(n+1)\pi))$, $n \in \mathbb{Z}$.

(2) (3.9) is the precise formulation of the *formal linear differential operator* $-\frac{d^2}{dx^2} + \gamma\delta$, namely, for $\zeta \in D(-\Delta_\gamma)$, $(-\frac{d^2}{dx^2} + \gamma\delta)\zeta(x) = -\zeta''(x)$ for every $x \neq 2n\pi$, $n \in \mathbb{Z}$.

Next for describing the resolvent of the self-adjoint operators $-\Delta_\gamma$, we will use the general Krein's resolvent formula (see Theorem 1.2.1 in [4]). Since the proofs in the periodic case are similar to those obtained on the line via the formula (3.12), we refer the reader to Chapters I.3 and III.2 in [3].

By defining the integral kernel $J_k \in L^2_{per}([-\pi,\pi])$ by

$$J_k(\xi) = \frac{2\pi}{2ik\sinh(ik\pi)}\cosh\Big(ik\Big(|\xi|-\pi\Big)\Big), \quad \text{for } \xi \in [-\pi,\pi], \tag{3.12}$$

with $k \neq n$, $n \in \mathbb{Z}$, we have,

Theorem 3.2. *For $-\infty < \gamma \leqq +\infty$, the resolvent of $-\Delta_\gamma$ in $L^2_{per}([-\pi,\pi])$ is given for $k \neq n$, $n \in \mathbb{Z}$, and $k^2 \in \rho(-\Delta_\gamma)$ by*

$$(-\Delta_\gamma - k^2)^{-1} = (-\Delta - k^2)^{-1} - \frac{1}{4\pi^2}\frac{2i\gamma k}{\gamma\coth(ik\pi)+2ik}\langle\cdot,\overline{J_k}\rangle J_k, \tag{3.13}$$

Therefore, $-\Delta_\gamma$ has a compact resolvent for $-\infty < \gamma \leqq +\infty$.

Next, we establish the basic spectral properties of $-\Delta_\gamma$ which will be relevant for our well-posedness results below. We note that a more general result into the framework of the Kronig-Penney model in solid state physics ([36]) can be established (see Chapter III.2, Theorem 2.3.1 in [3]).

Theorem 3.3. *Let $-\infty < \gamma \leqq +\infty$. Then the spectrum of $-\Delta_\gamma$ is discrete $\{\theta_{j,\gamma}\}_{j\geqq 1}$ and such that $\theta_{1,\gamma} < \theta_{2,\gamma} \leqq \theta_{3,\gamma} \leqq \cdots$. In particular, we have the following:*

1) *If $-\infty < \gamma < 0$, $-\Delta_\gamma$ has precisely one negative, simple eigenvalue, i.e.,*

$$\sigma_p(-\Delta_\gamma)\cap(-\infty,0) = \{-\mu_\gamma^2\} \tag{3.14}$$

where μ_γ is positive and satisfies $\gamma = -2\mu_\gamma tanh(\mu_\gamma\pi)$. The function

$$\psi_\gamma(\xi) = \frac{2\pi}{2\|J_{i\mu_\gamma}\|\mu_\gamma\sinh(\mu_\gamma\pi)}\cosh\Big(\mu_\gamma\Big(|\xi|-\pi\Big)\Big) \tag{3.15}$$

for $\xi \in [-\pi,\pi]$, is the strictly positive (normalized) eigenfunction associated to the eigenvalue $-\mu_\gamma^2$. The nonnegative eigenvalues (are non-degenerate) are ordered in the increasing form $0 < \kappa_1^2 < 1 < \kappa_2^2 < 2^2 < \cdots < \kappa_j^2 < j^2 < \cdots$, where for $j \geqq 1$, κ_j is the only solution of the equation $cot(\kappa\pi) = \frac{2\kappa}{\gamma}$ in the interval $(j-\frac{1}{2},j)$. The eigenfunction associated with κ_j^2 is $J_{\kappa_j} \in D(-\Delta_\gamma)$. The sequence $\{j^2\}_{j\geqq 1}$ is the classical set of eigenvalues associated to the operator $-\Delta$ with associated eigenfunctions $\{\sin(jx) : j \geqq 1\} \subset D(A^0) \subset D(-\Delta_\gamma)$.
2) *If $\gamma > 0$, $-\Delta_\gamma$ has nonnegative eigenvalues and the positive eigenvalues (are nondegenerate) are ordered in the increasing form $0 < k_1^2 < 1 < k_2^2 < 2^2 < \cdots < k_j^2 < j^2 < \cdots$, where for $j \geqq 0$, the eigenvalue k_{j+1} is the only solution of the equation $cot(k\pi) = \frac{2k}{\gamma}$ in the interval $(j, j+\frac{1}{2})$. The eigenfunction associated with k_{j+1}^2 is $J_{k_{j+1}} \in D(-\Delta_\gamma)$. The sequence $\{j^2\}_{j\geqq 1}$ is the same as in the item 1) above.*
3) *Zero is not eigenvalue of $-\Delta_\gamma$ for all $\gamma \neq 0$.*
4) *For $\gamma = +\infty$, $\sigma(-\Delta_{+\infty}) = \{j^2\}_{j\geqq 1}$ and with associated eigenfunctions $\{\sin(jx) : j \geqq 1\} \subset -\Delta_{+\infty}$. The eigenvalues are nondegenerate.*

Next we establish some remarks that deserve to be commented.

Remarks:

(1) It is well known from the formula in (3.12) that the resolvent for $-\Delta = -\frac{d^2}{dx^2}$ in $L^2_{per}([-\pi,\pi])$ is given by $(-\Delta - k^2)^{-1}f = J_k \star f$, with $k \neq n$, $n \in \mathbb{Z}$.
(2) $J_k \notin D(-\Delta_\gamma)$ for k such that $\gamma\coth(ik\pi) \neq -2ik$. Indeed, $J_k'(0+) - J_k'(0-) = -2\pi \neq \gamma J_k(0)$.
(3) $J_k \in H^1_{\rm per}([-\pi,\pi]) \cap H^2((2\pi n, 2(n+1)\pi))$, and satisfies $(-\Delta - k^2)J_k(x) = 0$ for $x \in (-\pi,\pi) - \{0\}$ with $J_k'(\pm\pi) = 0$.

(4) Formula (3.13) shows immediately the analytic property of the resolvent-mapping giving by $\gamma \to (-\Delta_\gamma - k^2)^{-1}$. So, it property can be used for showing that the mapping of self-adjoint operators $Z \to \mathcal{L}_{1,Z}$ and $Z \to \mathcal{L}_{2,Z}$ in (1.27)-(1.28), are real-analytic in the sense of Kato (see section 6 below).

(5) The domain $D(-\Delta_\gamma)$, $-\infty < \gamma \leqq +\infty$, consists of all elements ζ of the type

$$\zeta(x) = \psi(x) - \frac{2\gamma\beta}{\gamma \coth(\beta\pi) + 2\beta}\,\psi(0) g_{-i}(x), \qquad x \in \mathbb{R} - 2\pi\mathbb{Z} \tag{3.16}$$

for $\psi \in H^2_{per}([-\pi, \pi])$ and $\beta^2 = i$. The decomposition (3.16) is unique with $(-\Delta_\gamma + i)\zeta = (-\Delta + i)\psi$. So, we obtain that if $\zeta \in D(-\Delta_\gamma)$ and $\zeta(0) = 0$ then $\zeta \in H^2_{per}([-\pi, \pi])$.

The following proposition is concerned with the well-posedness of equation (1.10) in $H^1_{per}([-\pi, \pi])$.

Proposition 3.1. *For any $u_0 \in H^1_{per}([-\pi, \pi])$, there exists $T > 0$ and a unique solution u of* (1.10) *such that $u \in C([-T, T]; H^1_{per}([-\pi, \pi])) \cap C^1([-T, T]; H^{-1}_{per}([-\pi, \pi]))$ and $u(0) = u_0$. For each $T_0 \in (0, T)$ the mapping*

$$u_0 \in H^1_{per}([-\pi, \pi]) \to u \in C([-T_0, T_0]; H^1_{per}([-\pi, \pi]))$$

is continuous. Moreover, since u satisfies the conservation of the energy and the charge defined in (1.29), *namely, $E(u(t)) = E(u_0)$, $\quad Q(u(t)) = Q(u_0)$, for all $t \in [0, T)$, we can choose $T = +\infty$.*

If an initial data u_0 is even the solution $u(t)$ is also even.

Proof. We apply Theorem 3.7.1 of [16] to our problem. Indeed, from Theorem 3.3 we have $-\Delta_{-Z} \geqq -\beta_0$, where $\beta_0 = \mu^2_{-Z}$, if $Z > 0$ and $\beta_0 = 0$ if $Z < 0$. So, for the self-adjoint operator $\mathcal{A} \equiv \Delta_{-Z} - \beta_0$ on $X = L^2_{per}([-\pi, \pi])$ with domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(-\Delta_{-Z})$ we have $\mathcal{A} \leqq 0$. Now, from the min-max principle we obtain that for $Z > 0$

$$\lambda = \inf \left\{ \|v_x\|^2 - Z|v(0)|^2 : \|v\| = 1, v \in H^1_{per}([-\pi, \pi]) \right\}$$

is given by $\lambda = -\beta_0$. Therefore we may take for every $Z \in \mathbb{R}$ the space $X_{\mathcal{A}} = H^1_{per}([-\pi, \pi])$ with norm $\|u\|^2_{X_{\mathcal{A}}} = \|u_x\|^2 + (\beta_0 + 1)\|u\|^2 - Z|u(0)|^2$, which is equivalent to $H^1_{per}([0, 2\pi])$ norm. So, it is very easy to see that the uniqueness of solutions and the conditions (3.7.1), (3.7.3)-(3.7.6) in [16] hold choosing $r = \rho = 2$. Finally, the condition (3.7.2) in [16] with $p = 2$ is satisfied because of $\mathcal{A}$ is a self-adjoint operator on $L^2_{per}([-\pi, \pi])$. $\square$

4. Periodic travelling-wave for the NLS-δ model (1.15)

In this section we construct positive periodic solutions for the elliptic equation (1.16) such that the conditions in (1.17) are satisfied. Indeed, our approach will show that is possible to have periodic peak solutions with an arbitrary minimal period $2L$ and belonging to the domain of the operator $-\frac{d^2}{dx^2} - Z\delta$, $Z \neq 0$. Our analysis is based in the theory of elliptic integral, the theory of Jacobi elliptic functions and the implicit function theorem.

4.1. **The quadrature method.** We start by writing (1.17)-(3) in quadratic form. Indeed, for $\varphi = \varphi_{\omega,Z}$ and $x \neq \pm 2nL$ we obtain

$$[\varphi'(x)]^2 = \frac{1}{2}[-\varphi^4(x) + 2\omega\varphi^2(x) + 4B_\varphi] \equiv \frac{1}{2}F(\varphi(x)), \tag{4.1}$$

where $F(t) = -t^4 + 2\omega t^2 + 4B_\varphi$ and B_φ is a integration constant. We factor $F(\cdot)$ as

$$F(\varphi) = (\eta_1^2 - \varphi^2)(\varphi^2 - \eta_2^2) = 2[\varphi']^2, \tag{4.2}$$

where η_1, η_2 are the positive zeros of the polynomial F. We assume without loss of generality that $\eta_1 > \eta_2 > 0$. So, $\eta_2 \leqq \varphi(\xi) \leqq \eta_1$ and

$$2\omega = \eta_1^2 + \eta_2^2, \qquad 4B_\varphi = -\eta_1^2\eta_2^2. \tag{4.3}$$

We note from (1.9) with $k^2 = \frac{\eta_1^2-\eta_2^2}{\eta_1^2}$, that the dnoidal-profile $\phi_{\omega,0}(\xi+b)$ satisfies equation (4.2) for every $\xi \in \mathbb{R}$ and any shift-value b.

Next, since φ is continuous one has $[\varphi'(0+)]^2 = \frac{1}{2}F(\varphi(0))$ and $[\varphi'(0-)]^2 = \frac{1}{2}F(\varphi(0))$. Then $|\varphi'(0+)| = |\varphi'(0-)|$, which as we will show below implies that $\varphi'(0+) = -\varphi'(0-)$, and so from (1.17)-(4)

$$\varphi'(0+) = -\frac{Z}{2}\varphi(0). \tag{4.4}$$

The case $\varphi'(0+) = \varphi'(0-)$ can not occur. Indeed, from (1.17)-(4) it follows $\varphi(0) = 0$ and so $\varphi'(0)$ exists. Therefore from (4.2) $[\varphi'(0)]^2 = -\eta_1^2\eta_2^2/2$ which is a contradiction.

Next, we obtain restrictions on the value of $\varphi(0)$. From (4.1) and (4.4) we need to have

$$\frac{Z^2}{4}\varphi^2(0) = \frac{1}{2}F(\varphi(0)) > 0, \tag{4.5}$$

and so $\eta_1 > \varphi(0) > \eta_2$. Next, since $\max_{t\in\mathbb{R}} F(t) = \omega^2 + 4B_\varphi$ (which is attained for $t > 0$ in $t_0 = \sqrt{\omega}$), we obtain the condition

$$\frac{Z^2}{4}\varphi^2(0) \leqq \frac{\omega^2 + 4B_\varphi}{2} = \frac{(\omega - \eta_2^2)^2}{2}, \tag{4.6}$$

and from (4.5)

$$\varphi^2(0) = \frac{-(2\omega - \frac{Z^2}{2}) \pm \sqrt{(2\omega - \frac{Z^2}{2})^2 + 16B_\varphi}}{-2}. \tag{4.7}$$

Since $\varphi(0) \in \mathbb{R}$ we need to have $(2\omega - \frac{Z^2}{2})^2 + 16B_\varphi > 0$. We start by considering the case of sign " $-$ " in the square root in (4.7), then:

(1) For $2\omega - \frac{Z^2}{2} > 0$, it follows from (4.3) that $(2\omega - \frac{Z^2}{2})^2 > 4\eta_1^2\eta_2^2$ and so

$$\eta_1 - \eta_2 > |Z|/\sqrt{2}. \tag{4.8}$$

(2) From (4.7) we have as $Z \to 0$ the asymptotic behavior $\varphi^2(0) \to \eta_1^2$.

(3) For $2\omega - \frac{Z^2}{2} < 0$ we obtain from (4.7) that $16B_\varphi > 0$, which is not possible from (4.3).

Now, we consider the case of sign " $+$ " in the square root in (4.7), then:

(1) For $2\omega - \frac{Z^2}{2} < 0$ we have $\varphi^2(0) < 0$, which is a contradiction.

(2) For $2\omega - \frac{Z^2}{2} > 0$ we still have relation (4.8), but as $Z \to 0$ we obtain $\varphi^2(0) \to \eta_2^2$.

We are interested only in the sign " $-$ " in (4.7) for our stability theory.

4.2. **Profile of positive periodic peaks for $Z > 0$.** Next we will find a even periodic profile solution, $\phi_{\omega,Z}$ for (1.16) such that the peaks will be happen in points of the form $\pm 2ns$, $n \in \mathbb{Z}$, $2s$ a specific minimal period, $\eta_1 > \phi_{\omega,Z}(0) \geqq \phi_{\omega,Z}(\xi) \geqq \eta_2$ for all ξ, and

$$\lim_{Z\to 0^+} \phi_{\omega,Z}(\xi) = \phi_{\omega,0}(\xi), \qquad \text{for } \xi \text{ fixed in } (0, 2L), \tag{4.9}$$

where $\phi_{\omega,0}$ is the dnoidal traveling wave defined in (1.9) with a minimal period $2L$ (see Theorem 4.1 below). In subsection 4.5 we will show that s can be chosen equal to L.

We can see from (4.2) and (4.7) (with the sign "-") that for

$$a = \mathrm{dn}^{-1}\Big(\frac{\phi(0)}{\eta_1}; k\Big), \tag{4.10}$$

with k defined by

$$k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \tag{4.11}$$

the following even peak-function

$$\phi_{\omega,Z}(\xi) = \phi(\xi;\eta_1,\eta_2,Z) = \eta_1 \mathrm{dn}\Big(\frac{\eta_1}{\sqrt{2}}|\xi| + a; k\Big), \tag{4.12}$$

for $\xi \in [-s,s]$, with s defined by $s \equiv \frac{\sqrt{2}}{\eta_1}(K-a)$, satisfies the equation

$$-\phi''_{\omega,Z}(\xi) + \omega\phi_{\omega,Z}(\xi) = \phi^3_{\omega,Z}(\xi), \qquad \text{for } \xi \in (-s,0)\cup(0,s).$$

Moreover, since $\mathrm{dn}(\frac{\eta_1}{\sqrt{2}}s + a; k) = \mathrm{dn}(K;k) = \sqrt{1-k^2} = \eta_2/\eta_1$ we obtain the equality $\phi_{\omega,Z}(\pm s) = \eta_2$. Furthermore, from (4.5) we obtain the jump condition, $\phi'_{\omega,Z}(0+) - \phi'_{\omega,Z}(0-) = 2\phi'_{\omega,Z}(0+) = -Z\phi_{\omega,Z}(0)$.

We note tha the shift-value a in (4.10) depends of the values of Z and ω. Moreover, since $1 > \phi(0)/\eta_1 > k' \equiv \sqrt{1-k^2}$ and $1 \geqq dn(x;k) \geqq k'$ for all $x \in \mathbb{R}$, $k \in (0,1)$, with $k' = \mathrm{dn}(K;k)$, it follows that a is well-defined and $a \in [0,K]$. Lastly, by using that dn has a minimal period $2K$ the relation

$$\phi_{\omega,Z}(2s) = \eta_1\mathrm{dn}(2K-a) = \eta_1\mathrm{dn}(-a) = \eta_1\mathrm{dn}(a) = \phi(0) \tag{4.13}$$

implies that the profile $\phi_{\omega,Z}$ in (4.12) can be extend to all the line as a continuous periodic function satisfying the conditions in (1.17) with a minimal period $2s$ and with peak points in $\pm 2ns$, $n \in \mathbb{Z}$ (see Figure 3 below with $s = 2$).

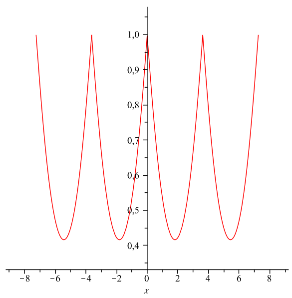

FIGURE 3. Profile of the periodic dnoidal-peak ϕ in (4.12).

Next, **we recall** the following theorem in Angulo [5] which justify the point convergence in (4.9). It result will be useful more later. For $\eta \in (0,\sqrt{\omega})$, we define for $M(\eta,\omega) \equiv 1/\sqrt{2\omega - \eta^2}$,

$$F(\eta,\omega) = 2\sqrt{2}M(\eta,\omega)K(k(\eta,\omega)). \tag{4.14}$$

Theorem 4.1. *Let $L > 0$ fixed. Consider $\omega_0 > \frac{\pi^2}{2L^2}$ and $\eta_0 = \eta(\omega_0) \in (0,\sqrt{\omega_0})$ such that $F(\eta_0,\omega_0) = 2L$. Then there are intervals $J_0(\omega_0)$ around ω_0 and $N_0(\eta_0)$ around η_0, and a unique smooth function $\Lambda_0 : J_0(\omega_0) \to N_0(\eta_0)$ such that $\Lambda_0(\omega_0) = \eta_0$ and for $\eta \equiv \Lambda_0(\omega)$ one has $F(\eta,\omega) = 2L$. Moreover, $N_0(\eta_0) \times J_0(\omega_0) \subseteq \{(\eta,\omega) : \omega > \frac{\pi^2}{2L^2},\ \eta \in (0,\sqrt{\omega})\}$. Furthermore, $J_0(\omega_0) = (\frac{\pi^2}{2L^2}, +\infty)$ and for $\eta_1 = \eta_1(\omega) = \sqrt{2\omega - \eta^2}$, the dnoidal wave solution $\phi_{\omega,0}$ defined in (1.9) has fundamental period $2L$ and satisfies the equation*

$$-\phi''_{\omega,0}(x) + \omega\phi_{\omega,0}(x) - \phi^3_{\omega,0}(x) = 0 \ \text{ for all } \ x \in \mathbb{R}.$$

Also, $\omega \in J_0(\omega_0) \to \phi_{\omega,0} \in H^n_{per}([0,2L])$ is a smooth function for all $n \in \mathbb{N}$.

From Theorem 4.1 we have the following properties of $\phi_{\omega,Z}$ in (4.12) with $\omega > Z^2/4$: From (4.7) (with sign "−"), $\phi_{\omega,Z}(0) \to \eta_1$ as $Z \to 0^+$. Then from (4.10) and from the value of s follow for $Z \to 0^+$ that $a \to dn^{-1}(1;k) = 0$ and $2s \to 2\sqrt{2}K/\eta_1 = 2L$. So, at least formally, we have that

$$\lim_{Z\to 0^+} \phi_{\omega,Z}(\xi) = \phi_{\omega,0}(\xi).$$

The last equality is in the sense that for $\omega > \frac{\pi^2}{2L^2}$ fixed and $\xi \in (0, 2L)$ (see Theorem 4.1), there is a $\delta > 0$ such that for $Z \in (0, \delta)$ and $Z^2/4 < \omega$ we have that the periodic-peaks $\phi_{\omega,Z}$, with minimal period $2s$ are defined in ξ. We note that this type of convergence is not convenient for our purposes, because the period of $\phi_{\omega,Z}$ is changing .

4.3. **Positive periodic peaks for $Z < 0$.** We shall find a even periodic-peak, $\zeta_{\omega,Z}$, with peaks in $\pm 2nr$, $2r$ a specific minimal period, $n \in \mathbb{Z}$, $\eta_1 > \zeta(0) > \eta_2$, $\eta_1 \geqq \zeta(\xi) \geqq \eta_2$ for all ξ, and

$$\lim_{Z\to 0^-} \zeta_{\omega,Z}(\xi) = \phi_{\omega,0}(\xi), \qquad \text{for } \xi \text{ fixed in } (0, 2L), \tag{4.15}$$

where $\phi_{\omega,0}$ is the dnoidal traveling wave defined in (1.9) with a minimal period $2L$. Indeed, by choosing a and k as in (4.10) and (4.11) we define the following even peak-function

$$\zeta_{\omega,Z}(\xi) = \eta_1 dn\Big(\frac{\eta_1}{\sqrt{2}}|\xi| - a; k\Big). \tag{4.16}$$

for $\xi \in [-r, r]$, with r defined by $r \equiv \frac{\sqrt{2}}{\eta_1}(K + a)$. So, we have that $-\zeta''_{\omega,Z}(\xi) + \omega\zeta_{\omega,Z}(\xi) = \zeta^3_{\omega,Z}(\xi)$, for $\xi \in (-r, 0) \cup (0, r)$. Furthermore, from the equality in (4.5) we obtain the condition required in (1.17). We also have that $\zeta_{\omega,Z}(r) = \eta_1 dn(K; k) = \eta_2$ and for $p_0 = \sqrt{2}a/\eta_1$, $\zeta_{\omega,Z}(p_0) = \eta_1$. So, from (4.2) follows $\zeta'_{\omega,Z}(\pm r) = \zeta'_{\omega,Z}(\pm p_0) = 0$. Moreover, p_0 is the only point in $(0, r)$ where the derivative of $\zeta_{\omega,Z}$ is zero. In fact, since $\zeta'_{\omega,Z}(\xi) = 0$ if and only if $sn(\frac{\eta_1}{\sqrt{2}}\xi - a)cn(\frac{\eta_1}{\sqrt{2}}\xi - a) = 0$ and we have that $\frac{\eta_1}{\sqrt{2}}\xi - a \in (-K, K)$ then follows that $\frac{\eta_1}{\sqrt{2}}\xi = a$.

Now, by using that $\zeta_{\omega,Z}(0) = \eta_1 dn(a) = \phi_{\omega,Z}(0)$ the relation $\zeta_{\omega,Z}(2r) = \eta_1 dn(a) = \zeta_{\omega,Z}(0)$ implies that $\zeta_{\omega,Z}$ can be extend to all the line as a continuous periodic function satisfying the conditions in (1.17) with a minimal period $2r$ and with peak points in $\pm 2nr$, $n \in \mathbb{Z}$ (see Figure 4 below with $r = 2$). In the next subsection 4.4 we will show that it is possible to choose $r = L$ for any L.

Lastly, from Theorem 4.1 and the convergences $\phi_{\omega,Z}(0) \to \eta_1$, $a \to 0$, and $2r \to 2L^+$ as $Z \to 0^-$, we have, at least formally, that equality in (4.15) is true.

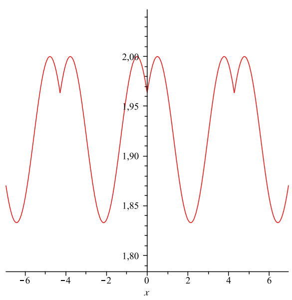

FIGURE 4. Profile of the periodic dnoidal-peak ζ in (4.16).

Remark: For the "convergence" of the periodic-peak $\phi_{\omega,Z}$ and $\zeta_{\omega,Z}$ to the solitary wave peak (1.14), with $p = 2$ we consider for a determined parameter (η_2 is our case) the minimal period $2s$ or $2r$ sufficiently large. Indeed, from (4.11) and (4.3) we obtain for all Z that $k^2(\eta_2) \to 1$, and $\eta_1^2 = 2\omega - \eta_2^2 \to 2\omega$ as

$\eta_2 \to 0$. We also have in this limit from (4.7) (with "-") that $\phi^2_{\omega,Z}(0) = \zeta^2_{\omega,Z}(0) \to 2\omega - \frac{Z^2}{2}$. Now, the identities $sn^{-1}(y;k) = dn^{-1}(\sqrt{1-k^2y^2};k)$ and $sn^{-1}(y;1) = \tanh^{-1}(y)$ (see pg. 31 in [13]) imply that

$$a = dn^{-1}\Big(\frac{\phi_{\omega,Z}(0)}{\eta_1};k\Big) = sn^{-1}\Big(\frac{1}{k}\sqrt{1-\frac{\phi^2_{\omega,Z}(0)}{\eta_1^2}};k\Big) \to \tanh^{-1}(Z/2\sqrt{\omega}), \tag{4.17}$$

as $\eta_2 \to 0$. Lastly, since $\mathrm{dn}(y;1) = \mathrm{sech}(y)$ we obtain the convergence (uniformly on compact-set) $\phi_{\omega,Z}(\xi) \to \phi_{\omega,Z,2}(\xi)$, as $\eta_2 \to 0$. We note that $s \to +\infty$ since $K(k(\eta_2)) \to +\infty$ as $\eta_2 \to 0$.

4.4. **Dnoidal-peak solutions to the NLS-δ with an arbitrary minimal period.** In subsections 4.2 and 4.3 we found dnoidal-peak profiles (4.12) and (4.16) with a minimal period $2s$ and $2r$. Next we shall see that the equality $s = L$ and $r = L$ can be obtained by any *a priori* L. In the analysis below we consider the case $Z > 0$, but a similar result can be established for $Z < 0$.

From the analysis in subsection 4.1, we start by defining the general notations to be used in the next subsections. For $4\omega > Z^2$ it follows from (4.11), (4.3), (4.8) and $\eta_1 > \eta_2 > 0$ that for all Z,

$$0 < \eta_2 < \theta(\omega, Z) < \sqrt{\omega} < \lambda(\omega, Z) < \eta_1 < \sqrt{2\omega} \tag{4.18}$$

where for $g(\omega, Z) = \sqrt{(8\omega - Z^2)/8}$ we have

$$\theta(\omega, Z) = -\frac{\sqrt{2}}{4}|Z| + g(\omega, Z) \quad \text{and} \quad \lambda(\omega, Z) = \frac{\sqrt{2}}{4}|Z| + g(\omega, Z). \tag{4.19}$$

For $\eta \in (0, \theta(\omega, Z))$ we define the functions:

$$k^2(\eta, \omega) = \frac{2\omega - 2\eta^2}{2\omega - \eta^2} \in (0, 1), \tag{4.20}$$

$k'^2(\eta, \omega) = 1 - k^2(\eta, \omega)$, and for $M(\eta, \omega) = 1/\sqrt{2\omega - \eta^2}$, the period function

$$T_-(\eta, \omega, Z) = 2\sqrt{2}M(\eta, \omega)[K(k(\eta, \omega)) - a(\eta, \omega, Z)] \tag{4.21}$$

where

$$a(\eta, \omega, Z) = \mathrm{dn}^{-1}(M(\eta, \omega)\Phi(\eta, \omega, Z); k(\eta, \omega)), \tag{4.22}$$

with $\Phi(\eta, \omega, Z)$ defined by (see (4.7))

$$\Phi^2(\eta, \omega, Z) = \frac{(2\omega - \frac{Z^2}{2}) + \sqrt{(2\omega - \frac{Z^2}{2})^2 - 4\eta^2(2\omega - \eta^2)}}{2}. \tag{4.23}$$

We note that the functions M, a and Φ defined above are independent of the sign of Z. We will denote them by $M(\eta)$, $a(\eta)$, $\Phi(\eta)$ or $M(\omega)$, $a(\eta, \omega)$, $\Phi(\eta, \omega)$ depending of the context. Moreover, the mapping $Z \to a(\cdot, \cdot, Z)$ is analytic.

Remark: For $\eta \in (0, \theta(\omega, Z))$ we obtain the *a priori* condition (4.6), namely, $\frac{Z^2}{4}\Phi^2 \leqq (\omega - \eta^2)^2/2$.

In the following lemma we establish several properties of the periodic function T_- which are main in the existence of periodic peak with an arbitrary minimal period L and in the existence of a smooth curve of positive periodic peak depending of the phase-velocity ω.

Lemma 4.1. *For $Z \neq 0$ and $\omega > Z^2/4$ fixed, the mappings for $\eta \in (0, \theta(\omega, Z))$*

$$\eta \to a(\eta), \quad \eta \to \Phi(\eta), \quad \text{and} \quad \eta \to M(\eta)\Phi(\eta) \tag{4.24}$$

are well defined. Moreover, they are strictly increasing, strictly decreasing and strictly decreasing functions respectively. Also, one has that

$$\lim_{\eta \to 0} T_-(\eta) = +\infty, \tag{4.25}$$

and

$$\lim_{\eta\to\theta} T_-(\eta) = 2\sqrt{2}[\lambda(\omega,Z)]^{-1}[K(k_0)-a_0] \equiv T_0(\omega,Z), \tag{4.26}$$

where $a_0 \equiv a_0(\omega,Z) \in (0, K(k_0))$ *and* $k_0 = k_0(\omega,Z)$ *are defined by*

$$dn(a_0;k_0) = \frac{1}{2}[\lambda(\omega,Z)]^{-1}(4\omega - Z^2)^{1/2}, \qquad k_0^2 = \frac{1}{2}|Z|[\lambda(\omega,Z)]^{-2}(8\omega - Z^2)^{1/2}. \tag{4.27}$$

Lastly, the mapping $\eta \in (0, \theta(\omega,Z)) \to T_-(\eta)$ *is a strictly decreasing function and so* $T_-(\eta) \in (T_0(\omega,Z), +\infty)$*. Moreover, for* $\eta \in (0,\theta(\omega,Z))$ *it follows that for* $\eta_1 \equiv \sqrt{2\omega-\eta^2}$ *we have* $\eta_1 - \eta > |Z|/\sqrt{2}$ *(see (4.8)) and* $\omega \to a(\omega)$ *is a strictly decreasing function with*

$$\lim_{\omega\to+\infty} a(\omega) = 0. \tag{4.28}$$

Proof. Initially from (4.18) the relation $0 > \frac{Z^2}{2} - \sqrt{2}|Z|\sqrt{(8\omega - Z^2)/8} > 2\eta^2 - 2\omega + \frac{Z^2}{2}$, implies that $1 > M(\eta,\omega)\Phi(\eta) > k'(\eta,\omega)$, and so a is well defined. Now, from (4.20) and (4.23) we have $\lim_{\eta\to 0} a(\eta) = \alpha < \infty$, with α satisfying $\text{sech}(\alpha) = \sqrt{1 - \frac{Z^2}{4\omega}}$ and (4.17)). So, since $K(k) \to +\infty$ as $k \to 1$ we obtain that the period function $T_-(\eta)$ satisfies (4.25).

Now, for $\eta \to \theta$ one has $k^2(\eta) \to k_0^2$ defined in (4.27). Since the mappping $\eta \to k^2(\eta)$ is strictly decreasing it follows that $k(\eta) \in (k_0, 1)$, for all $\eta \in (0, \theta(\omega,Z))$. We note that the condition $\omega > Z^2/4$ implies that the right hand side of (4.27) is bigger than $k_0' \equiv \sqrt{1-k_0^2}$ and so a_0 is well-defined. The above considerations yield the limit in (4.26).

The decreasing property of the last two functions in (4.24) follows immediately. Next we see that $\eta \in (0,\theta(\omega,Z)) \to a(\eta)$ is a strictly increasing function. We denote by $\psi(\eta)$ the strictly decreasing function $M(\eta)\Phi(\eta)$, then by (4.22) and the formula 710.53 in [13] we obtain

$$0 > \frac{d}{d\eta}\psi(\eta) = \frac{d}{d\eta}dn(a;k) = -k^2 sn(a)cn(a)\,\frac{da}{d\eta} + \frac{k sn(a)cn(a)}{k'^2}\Big[E(a) - k'^2 a - dn(a)\,\frac{sn(a)}{cn(a)}\Big]\frac{dk}{d\eta}, \tag{4.29}$$

with $E(u) = E(u;k) = \int_0^u dn^2(y;k)dy$. Since $a \in [0,K]$ and $\frac{dk}{d\eta} < 0$ from (4.29) we only need to see that the expression between the square brackets is negative for obtaining that $\frac{da}{d\eta} > 0$. Indeed, for $F(u) = E(u) - k'^2 u - dn(u)\,\frac{sn(u)}{cn(u)}$ we have $F(0) = 0$ and $F(K) = -\infty$. Moreover, from [13] (pg. 20) we obtain for $u \in (0,K)$, $F'(u) = dn^2(u) - k'^2 + k^2 sn^2(u) - \frac{dn^2(u)}{cn^2(u)} = -\frac{k'^2}{cn^2(u)} < 0$. Therefore, $F(a) < 0$ for $a \in (0,K)$.

Now, since $M^2(\omega)\Phi^2(\omega) \to 1$ and $k^2(\omega) \to 1$ as $\omega \to +\infty$, we obtain $\lim_{\omega\to+\infty} a(\omega) = dn^{-1}(1;1) = sech^{-1}(1) = 0$. Next we see that $\omega \to a(\omega)$ is a strictly decreasing function. If we denote by $f(\omega) = M(\omega)\Phi(\omega)$ we obtain similarly to (4.29) that

$$\frac{d}{d\omega}f(\omega) = \frac{d}{d\omega}dn(a;k) = \frac{\partial}{\partial a}dn(a;k)\frac{da}{d\omega} + \frac{\partial}{\partial k}dn(a;k)\frac{dk}{d\omega}. \tag{4.30}$$

So, since $\frac{\partial}{\partial k}dn(a;k) < 0$, $\frac{dk}{d\omega} > 0$ and $\frac{\partial}{\partial a}dn(a;k) < 0$ it is sufficient to show that $\frac{d}{d\omega}f(\omega) > 0$. Indeed, since $f(\omega) > 0$ will be see that $\frac{d}{d\omega}[f(\omega)]^2 > 0$. So, from (4.23) we obtain that

$$\frac{d}{d\omega}[f(\omega)]^2 > 0 \quad \text{if and only if} \quad \frac{\Phi^2(\omega)}{\eta_1^2} < \frac{\Phi^2(\omega) - \eta^2}{2\Phi^2(\omega) + \frac{Z^2}{2} - 2\omega}.$$

By our construction in subsection 4.1 we have that $\Phi^2(\omega) < \eta_1^2$, so only remains to show that the positive function $g(\omega) = 2\Phi^2(\omega) + \frac{Z^2}{2} - 2\omega$ satisfies $g(\omega) < \Phi^2(\omega) - \eta^2$. Indeed, after some algebra we obtain that

$$g(\omega) < \Phi^2(\omega) - \eta^2 \Leftrightarrow (2\omega - 2\eta^2) > \sqrt{\Big(2\omega - \frac{Z^2}{2}\Big)^2 - 4\eta^2(2\omega - \eta^2)},$$

which is true if and only if $\omega > \frac{Z^2}{8}$. Since $\omega > \frac{Z^2}{4}$, we finish the proof.

The fact that the mapping $\eta \in (0, \theta(\omega, Z)) \to T_-(\eta)$ is a strictly decreasing function follows from the analysis in Theorem 4.2 below. □

Next we show that there is a $2L$-periodic peak solution for equation (1.16) with the profile (4.12). It existence will be crucial in the next subsection for applying the implicit function theorem.

We start our analysis by studying the behavior of the mapping $\omega \in (\frac{Z^2}{4}, +\infty) \to T_0(\omega, Z)$ given in (4.26), with Z fixed. From (4.27) one has for $\omega \to +\infty$ that $k_0^2 \to 0$, then $K(k_0) \to \frac{\pi}{2}$ and so from the definition of $\lambda(\omega, Z)$ and a_0 we obtain immediately that

$$\lim_{\omega\to+\infty} T_0(\omega, Z) = 0. \tag{4.31}$$

Hence for $L > 0$ fixed there exists $\omega > \frac{Z^2}{4}$ such that $2L > T_0(\omega, Z)$. Consequently, from Lemma 4.1 there is a unique $\eta = \eta(\omega) \in (0, \theta(\omega, Z))$ such that

$$2s = T_-(\eta) = 2L. \tag{4.32}$$

Lastly, from the above analysis, if we define $\eta_1 \equiv \sqrt{2\omega - \eta^2}$ for η satisfying (4.32), k^2 and a via the relations in (4.20) and (4.22), respectively, we obtain the that peak-function $\phi_{\omega,Z}$ in (4.12) can be extend to all the line as a even periodic function with a minimal period $2s = 2L$ and in the interval $[0, 2L]$ it is symmetric with regard to the line $\xi = L$. Hence, we have obtained a periodic dnoidal-peak solution for equation (1.16) which satisfies all the properties in (1.17) and it belongs to the domain of $-\frac{d^2}{dx^2} - Z\delta$.

Remarks:

(1) for Z fixed, $\omega \to T_0(\omega, Z)$ is a strictly decreasing function.

(2) For ω fixed and $\omega > \frac{Z^2}{4}$, $T_0(\omega, Z) \to \sqrt{2}\pi/\sqrt{\omega}$, as $Z \to 0^+$, and $Z \to T_0(\omega, Z)$ is a strictly increasing function. Then from the relation $2L > T_0(\omega, Z)$ we obtain that ω must satisfy $\omega > \frac{\pi^2}{2L^2}$ (see Theorem 4.1 for the case $Z = 0$ in (1.16)).

4.5. **Smooth curve of periodic peaks to the NLS-δ with $Z \neq 0$.** In this section we construct a smooth curve of positive periodic peak solutions of (1.16), $\omega \to \varphi_{\omega,Z}$, with Z fixed. These solutions $\varphi = \varphi_{\omega,Z}$ have *a priori* fundamental period $2L$, satisfy the conditions in (1.17) **(with $\pi = L$)**, and $\varphi_{\omega,Z} \in D(-\frac{d^2}{dx} - Z\delta)$. Moreover, for $\omega > \frac{Z^2}{4}$ and ω fixed and large one has that

$$\lim_{Z\to 0} \varphi_{\omega,Z}(x) = \phi_{\omega,0}(x) \qquad \text{for } \ x \in [-L, L], \tag{4.33}$$

where $\phi_{\omega,0}$ is defined in (1.9). Our analysis will show also that the mapping $Z \to \varphi_{\omega,Z}$ is analytic. This will be essential in our stability theory. In addition, we shall need to show that the map $\omega \to \eta(\omega) \in (0, \theta(\omega, Z))$ is smooth.

First we consider the case $Z \neq 0$ and small.

4.5.1. *Smooth curve of periodic peaks to the NLS-δ with $Z > 0$.* We shall show that for $Z > 0$ fixed, there exists a smooth curve $\omega \to \phi_{\omega,Z} \in H^1_{per}([-L, L])$ satisfying the conditions in (1.17). Moreover, the convergence in (4.33) can be justified at least for $Z \to 0^+$. The proof will be a consequence of the implicit function theorem, Lemma 4.1 and Theorem 4.1. We recall that $\omega > Z^2/4$.

Theorem 4.2. *Let $L > 0$ fixed, δ small, $\delta < \frac{\pi^2}{2L^2}$, and $Z \in (-\delta, \delta)$. Let $\omega_0 > \frac{\pi^2}{2L^2}$ and η_0 be the unique $\eta_0 \in (0, \sqrt{\omega_0})$ such that $F(\eta_0, \omega_0) = 2L$. Then,*

(1) *there are an rectangle $R = J(\omega_0) \times (-\delta_0, \delta_0)$ around $(\omega_0, 0)$, an interval $N_1(\eta_0)$ around η_0, and a unique smooth function $\Lambda_1 : R \to N_1(\eta_0)$ such that $\Lambda_1(\omega_0, 0) = \eta_0$ and*
$$\frac{2\sqrt{2}}{\eta_1}[K(k) - a(\omega, Z))] = 2L, \tag{4.34}$$
where $\eta_1^2 = \eta_{1,Z}^2 \equiv 2\omega - \eta_{2,Z}^2$ for $(\omega, Z) \in R$ and $\eta_{2,Z} = \Lambda_1(\omega, Z)$.

(2) *$J(\omega_0) = (\frac{\pi^2}{2L^2}, +\infty)$ and $k \in (k_0, 1)$, k_0 defined in (4.27).*

(3) *$N_1(\eta_0)\times \subset \mathbb{G} = \{(\eta, \omega, Z) : \omega > \frac{\pi^2}{2L^2}, 2L > T_0(\omega, Z), \eta \in (0, \theta(\omega, Z))\}$.*

(4) *For $Z = 0$ we have $a(\omega, 0) = 0$ and so from Theorem 4.1 it follows that $\Lambda_1(\omega, 0) = \Lambda_0(\omega)$. Therefore,*
$$\lim_{Z\to 0^+} \eta_{2,Z}(\omega) = \eta(\omega).$$

(5) *For $Z \in (0, \delta_0)$ we denote $\eta_{2,Z}$ by $\eta_{2,+}$. Then the dnoidal-peak solution $\phi_{\omega,Z}$ in (4.12) with η_1 being $\eta_{1,+} = (2\omega^2 - \eta_{2,+}^2)^{1/2}$, has minimal period $2L$ and satisfies for $\omega > \frac{\pi^2}{2L^2}$,*
$$\lim_{Z\to 0^+} \phi_{\omega,Z}(x) = \phi_{\omega,0}(x), \quad \text{for } x \in [-L, L].$$

(6) *$Z \to \phi_{\omega,Z} \in H^1_{per}([-L, L])$ is real-analytic.*

Proof. The proof is a consequence of the implicit function theorem applied to the periodic-mapping
$$T_-(\eta, \omega, Z) = 2\sqrt{2}M(\eta, \omega)[K(k(\eta, \omega)) - a(\eta, \omega, Z)] \equiv F(\eta, \omega) - 2\sqrt{2}M(\eta, \omega)a(\eta, \omega, Z)$$
with domain $\mathbb{G}$. From (4.31) follows $\mathbb{G} \neq \emptyset$. Moreover, if $(\eta_0, \omega_0, Z) \in \mathbb{G}$ then for all $\omega > \omega_0$ we obtain $(\eta_0, \omega, Z) \in \mathbb{G}$ ($T_-(\eta_0, \omega_0, 0) = 2L$). Next, we claim that $\partial_\eta T_-(\eta_0, \omega_0, 0) < 0$. Indeed, from Theorem 2.1 in Angulo [5] we have $\partial_\eta F(\eta, \omega) < 0$ since $\partial_\eta k(\eta, \omega)$ is a strictly decreasing function of η, since $\partial_\eta a(\eta, \omega, Z) > 0$ (see Lemma 4.1) we prove the claim. Theorem 4.1 implies item (2) above.

Finally, since the functions a in (4.22) and Λ_1 are analytic, the mapping $(\omega, Z) \to \phi_{\omega,Z}$ is analytic for $(\omega, Z) \in \{(\omega, Z) : \omega > Z^2/4\}$, Z small. This finishes the proof of the Theorem. □

Corollary 4.1. *Consider the mapping $\Lambda_1 : R \to N(\eta_0)$ obtained in Theorem 4.2. Then for $Z > 0$ fixed and ω large, the mapping $\omega \to \eta_{2,+}(\omega) = \Lambda_1(\omega, Z)$ is a strictly decreasing function. Moreover, for $k(\omega) = k(\eta_{2,+}(\omega), \omega)$ and $a(\omega) = a(\eta_{2,+}(\omega), \omega)$ defined in (4.20) and (4.22) respectively, one has $\frac{d}{d\omega}k(\omega) > 0$ and $\frac{d}{d\omega}a(\omega) < 0$.*

Proof. Let $Z > 0$ fixed. Since $T_-(\Lambda_1(\omega, Z), \omega, Z) = 2L$ one has that $\frac{\partial T_-}{\partial \eta}\eta'_{2,+}(\omega) = -\frac{\partial T_-}{\partial \omega}$. Using the relation $\partial_\omega T_-(\eta, \omega, Z) < 0$ (see Appendix) we obtain $\eta'_{2,+}(\omega) < 0$. Next, for $a(\omega) \equiv a(\Lambda_1(\omega, Z), \omega, Z)$ we obtain $\frac{d}{d\omega}a(\omega) = \frac{\partial a}{\partial \Lambda_1}\frac{d\Lambda_1}{d\omega} + \frac{\partial a}{\partial \omega} < 0$, since $\frac{\partial a}{\partial \Lambda_1} > 0$ and $\frac{\partial a}{\partial \omega} < 0$ (see Lemma 4.1). Finally, from the formula in (4.20) it follows immediately that $k(\omega)$ is a strictly increasing function. This completes the proof of the Corollary. □

In the next section, we will need to use that the mapping $Z \to \phi_{\omega,Z}$ is analytic for $Z > 0$ (we recall that this property is local type). So, by using an argument similar to that provided in the proof of Theorem 4.2 and the analysis in subsection 4.4 we obtain :

Theorem 4.3. *Let $L > 0$ fixed and $Z_0 > 0$. Consider $\omega_0 > \frac{Z_0^2}{4}$ such that $2L > T_0(\omega_0, Z_0)$ and $\omega_0 > \frac{\pi^2}{2L^2}$. Let $\eta_{2,0} = \eta_{2,0}(\omega_0, Z_0) \in (0, \theta(\omega_0, Z_0))$ the unique value such that $T_-(\eta_{2,0}, \omega_0, Z_0) = 2L$. Then,*

(1) *there are an rectangle $S(\omega_0, Z_0) = H(\omega_0) \times I(Z_0)$ around (ω_0, Z_0), an interval $N_2(\eta_{2,0})$ around $\eta_{2,0}$, and a unique smooth function $\Lambda_2 : S(\omega_0, Z_0) \to N_1(\eta_{2,0})$ such that $\Lambda_2(\omega_0, Z_0) = \eta_{2,0}$ and $T_-(\eta_{2,+}, \omega, Z) = 2L$ for $\eta_{2,+} = \Lambda_2(\omega, Z)$.*

(2) *$H(\omega_0)$ can be choosen as $(\mu(Z,L),+\infty)$, where $\mu(Z,L)>\frac{Z_0^2}{4}$ and $\mu(Z,L)>\frac{\pi^2}{2L^2}$. For $Z=0$ we have $\mu(0,L)=\frac{\pi^2}{2L^2}$.*

(3) *the dnoidal-peak solution in (4.12), $\phi_{\omega,Z}(\xi)\equiv\phi(\xi;\eta_{1,+})$, determined by $\eta_{1,+}\equiv(2\omega-\eta_{2,+}^2)^{1/2}$ satisfies the properties in (1.17). Moreover, the mapping $Z\to\phi_{\omega,Z}\in H^1_{per}([-L,L])$ is real-analytic.*

Corollary 4.2. *For $Z>0$ fixed and ω large, the mapping $\omega\to\eta_{2,+}(\omega)=\Lambda_2(\omega,Z)$ is a strictly decreasing function. Moreover, for $k(\omega)=k(\eta_{2,+}(\omega),\omega)$ and $a(\omega)=a(\eta_{2,+}(\omega),\omega)$ defined in (4.20) and (4.22) respectively, one has that $\frac{d}{d\omega}k(\omega)>0$ and $\frac{d}{d\omega}a(\omega)<0$*

Corollary 4.3. *For $Z\geqq 0$ fixed, consider the mapping $a:(\mu(Z,L),+\infty)\to\mathbb{R}$ defined in Corollary 4.1 and Corollary 4.2. Then $a(\omega)\to 0$ as $\omega\to+\infty$.*

Proof. From Corollary 4.2 it follows that for $\omega>\omega_1$, $0\leqq\omega^{-1}\eta_{2,+}^2(\omega)\leqq\omega^{-1}\eta_{2,+}^2(\omega_1)$. Thus, $k^2(\omega)\to 1^+$ and $M(\eta_{2,+},\omega)\Phi(\omega)\to 1^+$ as $\omega\to+\infty$. Therefore, (4.22) yields the identity $\lim_{\omega\to+\infty}a(\omega)=sech^{-1}(1)=0$. □

4.5.2. *Smooth curve of periodic peaks to the NLS-δ with $Z<0$.* The following Theorem shows that for $Z<0$, fixed, there exists a smooth curve $\omega\to\zeta_{\omega,Z}\in H^1_{per}([-L,L])$, with ω large, satisfying the conditions in (1.17) and that the convergence in (4.33) for $Z\to 0^-$ is possible. The proof is similar to that of Theorem 4.2 and Theorem 4.3, so we shall only describe the main points in the argument. We start by defining

$$T_+(\eta_2,\omega)=2\sqrt{2}M(\eta_2,\omega)[K(k(\eta_2,\omega))+a(\eta_2,\omega)] \tag{4.35}$$

and $T_1(\omega,Z)=2\sqrt{2}[\lambda(\omega,Z)]^{-1}[K(k_0)+a_0]$, where $T_1(\omega,Z)=\lim_{\eta_2\to\theta}T_+(\eta_2,\omega)$. Since $\lim_{\omega\to+\infty}a_0(\omega)=0$ it follows that $\lim_{\omega\to+\infty}T_1(\omega,Z)=0$. Therefore, since the mapping $\omega\to T_1(\omega,Z)$ is a strictly decreasing function we obtain a unique $\omega_1>\frac{Z^2}{4}$ such that $2L>T_1(\omega_1,Z)$ and for every $\omega>\omega_1$, $2L>T_1(\omega,Z)$. Now, for ω chosen in this form one finds a unique $\eta_{2,1}=\eta_{2,1}(\omega)\in(0,\theta(\omega,Z))$ such $T_+(\eta_{2,1},\omega)=2L$, because of $\lim_{\eta_2\to 0}T_+(\eta_2,\omega)=+\infty$ and $\partial_\eta T_+<0$ for ω large (see Appendix). Moreover, since $T_1(\omega,Z)=T_0(\omega,Z)+\frac{4\sqrt{2}}{\lambda(\omega,Z)}a_0\to\frac{\sqrt{2}}{\sqrt{\omega}}\pi$ as $Z\to 0^-$, we obtain *a priori* the condition $\omega>\frac{\pi^2}{2L^2}$.

We have the following theorem of existence.

Theorem 4.4. *Let $L>0$ fixed and $Z_0<0$. Consider ω_1 large such that $2L>T_1(\omega_1,Z_0)$. In particular, $\omega_1>\frac{Z_0^2}{4}$ and $\omega_1>\frac{\pi^2}{2L^2}$. Let $\eta_{2,1}=\eta_{2,1}(\omega_1,Z_0)\in(0,\theta(\omega_1,Z_0))$ the unique value such that $T_+(\eta_{2,1})=2L$. Then,*

(1) *there are an rectangle $W(\omega_1,Z_0)=Q(\omega_1)\times V(Z_0)$ around (ω_1,Z_0), an interval $N_2(\eta_{2,1})$ around $\eta_{2,1}$, and a unique smooth function $\Lambda_3:W(\omega_1,Z_0)\to N_2(\eta_{2,1})$ such that $\Lambda_3(\omega_1,Z_0)=\eta_{2,1}$ and $T_+(\eta_{2,-},\omega,Z)=2L$ for $\eta_{2,-}=\Lambda_3(\omega,Z)$,*

(2) *$Q(\omega_1)$ can be choosen as $(\nu(Z,L),+\infty)$, where $\nu(Z,L)>\frac{Z_0^2}{4}$ and $\nu(Z,L)>\frac{\pi^2}{2L^2}$. For $Z=0$ we have $\nu(0,L)=\frac{\pi^2}{2L^2}$,*

(3) *for $Z=0$ we have $a(\omega,0)=0$ and so from Theorem 4.1 we have $\Lambda_3(\omega,0)=\Lambda_0(\omega)$. Therefore, $\lim_{Z\to 0^-}\eta_{2,-}(\omega)=\eta(\omega)$,*

(4) *the dnoidal-peak solution in (4.16), $\zeta_{\omega,Z}(\xi)\equiv\zeta(\xi;\eta_{1,-})$, determined by $\eta_{1,-}\equiv(2\omega-\eta_{2,-}^2)^{1/2}$ satisfies the properties in (1.17). Moreover, the mapping $Z\to\zeta_{\omega,Z}\in H^1_{per}([-L,L])$ is real-analytic,*

(5) *$\lim_{Z\to 0^-}\zeta_{\omega,Z}(\xi)=\phi_{\omega,0}(\xi)$, for $\xi\in[-L,L]$.*

Corollary 4.4. *For $Z<0$ fixed and ω large, the mapping $\omega\to\eta_{2,-}(\omega)=\Lambda_3(\omega,Z)$ is a strictly decreasing function. Moreover, for $k=k(\Lambda_3(\omega,Z),\omega)$ and $a=a(\Lambda_3(\omega,Z),\omega)$ defined in (4.20) and (4.22) respectively, one has that $\frac{d}{d\omega}k(\omega)>0$ and $\frac{d}{d\omega}a(\omega)<0$*

Proof. For T_+ defined in (4.35), it follows that $\partial_\eta T_+ < 0$ and $\partial_\omega T_+ < 0$ for ω large (see Appendix). Then for Λ_3 satisfying $T_+(\Lambda_3(\omega, Z), \omega) = 2L$ we obtain that $\Lambda_3'(\omega) < 0$ and $a'(\omega) < 0$. $\square$

Corollary 4.5. *For $Z \leqq 0$ fixed, consider the mapping $a : (\nu(Z, L), +\infty) \to \mathbb{R}$ determined by Theorem 4.4. Then $a(\omega) \to 0$ as $\omega \to +\infty$.*

5. Stability of Dnoidal-Peak for the NLS-δ model (1.15)

In this section we study the stability of the orbit

$$\Omega_{\varphi_{\omega,Z}} = \{e^{i\theta}\varphi_{\omega,Z} : \theta \in [0, 2\pi)\}, \tag{5.1}$$

generated by the smooth curve of dnoidal-peak $\omega \to \varphi_{\omega,Z}$, where $\varphi_{\omega,Z}$ is defined as in (1.26). Moreover,

$$\lim_{Z\to 0} \varphi_{\omega,Z}(\xi) = \phi_{\omega,0}(\xi), \quad \text{for } \xi \in [-\pi, \pi], \tag{5.2}$$

where $\phi_{\omega,0}$ being the dnoidal-wave solution to the cubic Schrödinger equation determined by Theorem 4.1.

We start obtaining the spectral information associated to the operators in (1.27) and (1.28) necessary to establish our stability theorem. We denote $L^2_{per}([-\pi, \pi])$ and $H^1_{per}([-\pi, \pi])$ simply by H^1_{per} and L^2_{per}, respectively.

5.1. **The basic linear operators $\mathcal{L}_{1,Z}$ and $\mathcal{L}_{2,Z}$.** For $u \in H^1_{per}$ we write $u = u_1 + iu_2$. Let $H_{\omega,Z}$ be defined by

$$H_{\omega,Z}u = \mathcal{L}_{1,Z}u_1 + i\,\mathcal{L}_{2,Z}u_2 \tag{5.3}$$

where the linear operators $\mathcal{L}_{i,Z}$, $i = 1, 2$, are defined as in (1.28) and (1.27), respectively. Next, for $Z \in \mathbb{R}$ and the subspace $\mathcal{D}$ defined as

$$\mathcal{D} = \{\zeta \in H^1_{per} \cap H^2((2n\pi, 2(n+1)\pi)) : \zeta'(0+) - \zeta'(0-) = -Z\zeta(0)\}, \quad n \in \mathbb{Z}, \tag{5.4}$$

we have that $\mathcal{L}_{i,Z}$ are self-adjoint operators on L^2_{per} with domain $D(\mathcal{L}_{i,Z}) = \mathcal{D}$ (see the Stability Self-Adjoint Theorem in Kato [35]).

We note that the linear operators $\mathcal{L}_{i,Z}$ are related with the the second variation of $G_{\omega,Z} = E + \omega Q$ at $\varphi_{\omega,Z}$. More exactly, let $u = \zeta + i\psi$ with $\zeta, \psi \in \mathcal{D}$ and $v = v_1 + iv_2 \in H^1_{per}$ then

$$\langle G''_{\omega,Z}(\varphi_{\omega,Z})u, v\rangle = \langle H_{\omega,Z}u, v\rangle = \langle \mathcal{L}_{1,Z}\zeta + i\mathcal{L}_{2,Z}\psi, v\rangle = \langle \mathcal{L}_{1,Z}\zeta, v_1\rangle + \langle \mathcal{L}_{2,Z}\psi, v_2\rangle. \tag{5.5}$$

Indeed, we define $\mathcal{Q}(\zeta, v_1) = \omega \int \zeta v_1 dx - 3\int \varphi^2_{\omega,Z}\zeta v_1 dx$. Thus,

$$\langle \mathcal{L}_{1,Z}\zeta, v_1\rangle = [\zeta'(0+) - \zeta'(0-)]v_1(0) + \langle \zeta', v_1'\rangle + \mathcal{Q}(\zeta, v_1) = -Z\zeta(0)v_1(0) + \langle \zeta', v_1'\rangle + \mathcal{Q}(\zeta, v_1). \tag{5.6}$$

Similarly, we obtain $\langle \mathcal{L}_{2,Z}\psi, v_2\rangle = -Z\psi(0)v_2(0) + \langle \psi', v_2'\rangle + \omega\langle \psi, v_2\rangle - \langle \varphi^2_{\omega,Z}\psi, v_2\rangle$. A simple calculation shows that $\langle G''_{\omega,Z}(\varphi_{\omega,Z})(\zeta, \psi), (v_1, v_2)\rangle = \langle \mathcal{L}_{1,Z}\zeta, v_1\rangle + \langle \mathcal{L}_{2,Z}\psi, v_2\rangle$.

5.2. **Some spectral structure of $\mathcal{L}_{1,Z}$ and $\mathcal{L}_{2,Z}$.** This subsection is concerned with some specific spectral structure of the linear operators $\mathcal{L}_{i,Z}$. By convenience we will denote $\mathcal{L}_{i,Z}$ only by $\mathcal{L}_i$.

Proposition 5.1. *Let $Z \in \mathbb{R}$ and $\omega > Z^2/4$. Then,*

1. *$\mathcal{L}_2$ is a nonnegative operator with a discrete spectrum, $\sigma(\mathcal{L}_2) = \{\lambda_n : n \geqq 0\}$, ordered in the increasing form $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \cdots$. The eigenvalue zero is simple with eigenfunction $\varphi_{\omega,Z}$.*
2. *$\mathcal{L}_1$ is a operator with a discrete spectrum, $\sigma(\mathcal{L}_1) = \{\alpha_n : n \geqq 0\}$, ordered in the increasing form $\alpha_0 < \alpha_1 \leqq \alpha_2 \leqq \alpha_3 \leqq \alpha_4 \cdots$*

Proof. From Section 3, Theorem 3.3, it follows that the operators $\mathcal{L}_i$ have a compact resolvent and so its spectrum is discrete. Since $\varphi_{\omega,Z} \in \mathcal{D}$ and satisfies equation (1.16) we obtain that $\mathcal{L}_2\varphi_{\omega,Z} = 0$ for all Z. Moreover, $\varphi_{\omega,Z}$ being positive it corresponds to the first eigenvalue of $\mathcal{L}_2$ which is simple. □

Next we have the following kernel-structure of $\mathcal{L}_1$.

Theorem 5.1. *Let $Z \in \mathbb{R} - \{0\}$ and ω large. Then $\mathcal{L}_1$ has a trivial kernel.*

Proof. Let $v \in \mathcal{D}$ such that $\mathcal{L}_1 v = 0$ and $Z > 0$ (so $\varphi_{\omega,Z} = \phi_{\omega,Z}$). Therefore v satisfies the following problem

$$\begin{cases} v \in H^2(0, 2\pi) \\ \mathcal{L}_1 v(x) = 0 \quad \text{for } \ x \in (0, 2\pi). \end{cases} \tag{5.7}$$

From item (3) in (1.17) it follows that $c\phi'_{\omega,Z}(x)$ for $x \in (0, 2\pi)$ and any $c \in \mathbb{R}$ satisfies (5.7). We note that $\phi'_{\omega,Z}$ is a 2π-periodic odd function with jump-discontinuity in $\pm 2n\pi$, $n \in \mathbb{Z}$.

Next, we consider the transformation $\Lambda(x) = v(\beta x)$, for $\beta = \frac{\sqrt{2}}{\eta_1}$, $\ x \in (0, 2(K - a))$, with a defined in (4.22). Then from Theorems 4.2 and 4.3 we have $\beta x \in (0, 2\pi)$ and so (5.7) implies that

$$\Lambda''(x) + [\sigma - 6k^2 sn^2(x + a; k)]\Lambda(x) = 0 \quad \text{for } \ x \in (0, 2(K - a)), \tag{5.8}$$

where $\sigma = (6\eta_{1,+}^2 - 2\omega)/\eta_{1,+}^2 = 4 + k^2$. Now, for $\Upsilon(x) = \Lambda(x - a)$ with $x \in (a, 2K - a)$ we have that Υ satisfies the following Lamé's equation

$$\Upsilon''(x) + [\sigma - 6k^2 sn^2(x; k)]\Upsilon(x) = 0, \quad \text{for } \ x \in (a, 2K - a). \tag{5.9}$$

Now, from Theorem 5.2 below the periodic eigenvalue problem in $L^2_{per}([0, 2K])$

$$\begin{cases} \Phi''(x) + [\lambda - 6k^2 sn^2(x; k)]\Phi(x) = 0, \quad x \in (0, 2K) \\ \Phi(0) = \Phi(2K(k)), \ \ \Phi'(0) = \Phi'(2K(k)), \ \ k \in (0, 1) \end{cases} \tag{5.10}$$

has the first three eigenvalues $\lambda_0, \lambda_1, \lambda_2$ simple and the rest of the eigenvalues are distributed in the form $\lambda_3 \leqq \lambda_4 < \lambda_5 \leqq \lambda_6 < \cdots$ and satisfying $\lambda_3 = \lambda_4, \lambda_5 = \lambda_6, \ldots$, i.e., they are double eigenvalues and so for these values of λ all solutions of (5.10) have period $2K(k)$. In particular, $\lambda_1 = 4 + k^2$ and $\Phi_1(x) = sn(x; k)cn(x; k) = C_0 \frac{d}{dx} dn(x; k)$, for $x \in [0, 2K(k)]$, $k \in (0, 1)$. Now, from Floquet theory (see pg. 7 in [21]) the other solution for the Lamé's equation in (5.10) with $\lambda = \lambda_1$ is of the form $\Psi(x) = x\Phi_1(x) + p_2(x)$, where $p_2(x)$ is even with period $2K(k)$. In fact, the variational parameter method shows that for $E(x) = E(x; k) = \int_0^x dn^2(y)dy$ (the normal elliptic integral of the second kind),

$$\Psi_1(x) = 2x\Phi_1(x) - \frac{2 - k^2}{1 - k^2} E(x)\Phi_1(x) + \frac{1}{1 - k^2} dn(x)[sn^2(x) - (1 - k^2)cn^2(x)] \tag{5.11}$$

satisfies (5.10) with $\lambda = \lambda_1$. Next, by considering the Zeta function de Jacobi, $Z(x) = Z(x; k)$, defined by

$$Z(x) = \int_0^x \Big[dn^2(y; k) - \frac{E(k)}{K(k)}\Big] dy$$

which is an odd periodic function de x with period $2K(k)$, we can rewrite Ψ_1 as

$$\begin{aligned} \Psi_1(x) &= x\Big[2 - \frac{2 - k^2}{1 - k^2}\frac{E(k)}{K(k}\Big]\Phi_1(x) + \Big[\frac{1}{1 - k^2} dn(x)[sn^2(x) - (1 - k^2)cn^2(x)] - \frac{2 - k^2}{1 - k^2} Z(x)\Phi_1(x)\Big] \\ &\equiv x\beta\Phi_1(x) + p_1(x). \end{aligned} \tag{5.12}$$

Hence we obtain the representation $\Psi = \frac{1}{\beta}\Psi_1 = x\Phi_1 + p_2$, with $p_2 \equiv \frac{1}{\beta} p_1$ and p_2 being an even periodic function with periodic $2K(k)$. Since $\{\Phi_1, \Psi\}$ is a linearly independent (LI) set of solutions for the Lamé's equation in (5.9) on $\mathbb{R}$, then it is a base of solutions for (5.9) on the interval $(a, 2K-a)$. Hence the following functions on $(0, 2\pi)$, for $\eta_1 = \eta_{1,+}$,

$$\begin{cases} \Lambda_1(x) = D\phi'_{\omega,Z}(x), \quad \text{and} \\ \Lambda_2(x) = \Big(\frac{\eta_1}{\sqrt{2}}x + a\Big)\Lambda_1(x) + p_2\Big(\frac{\eta_1}{\sqrt{2}}x + a\Big) \end{cases} \tag{5.13}$$

with $D = -\frac{\sqrt{2}}{\eta_1^2 k^2}$, are a LI set of solutions for (5.7) on $(0, 2\pi)$. Therefore, there are $\alpha, \gamma \in \mathbb{R}$ such that

$$v(x) = \alpha\Lambda_1(x) + \gamma\Lambda_2(x), \quad x \in (0, 2\pi). \tag{5.14}$$

Next we show that $\alpha = -\gamma K$. The continuity of v implies that

$$\begin{cases} v(0) = v(0+) = \alpha\Lambda_1(0+) + \gamma[a\Lambda_1(0+) + p_2(a)], \quad \text{and} \\ v(2\pi) = v(2\pi-) = \alpha\Lambda_1(0-) + \gamma[(2K-a)\Lambda_1(0-) + p_2(2K-a)], \end{cases} \tag{5.15}$$

where we are used that $2(K-a) = \frac{2\pi}{\sqrt{2}}\eta_1$. Then, since $v(0) = v(2\pi)$, p_2 is symmetric with regard to $x = K$ and $\Lambda_1(0-) = -\Lambda_1(0+) \neq 0$ follow from (5.15) that $\alpha = -\gamma K$. Therefore, we obtain for v satisfying (5.7) and $v(0) = v(2\pi)$ that

$$v(x) = \gamma[-K\Lambda_1(x) + \Lambda_2(x)], \quad \text{for all } \ x \in (0, 2\pi). \tag{5.16}$$

Hence, the subspace $\mathcal{K} = \{v \in H^2(0, 2\pi) : v \ \text{ satisfies } \ (5.7) \ \text{ and } \ v(0) = v(2\pi)\}$ is one-dimensional.

Next we show that for $v \in \mathcal{D}$ satisfying $\mathcal{L}_1 v = 0$ we have that v is an even function. We start by establishing a similar problem to (5.7) on $(-2\pi, 0)$. Hence the following functions on $(-2\pi, 0)$, for $\eta_1 = \eta_{1,+}$,

$$\begin{cases} \widetilde{\Lambda_1}(x) = D\phi'_{\omega,Z}(x), \quad \text{and} \\ \widetilde{\Lambda_2}(x) = \Big(\frac{\eta_1}{\sqrt{2}}x - a\Big)\widetilde{\Lambda_1}(x) + p_2\Big(\frac{\eta_1}{\sqrt{2}}x - a\Big) \end{cases} \tag{5.17}$$

are a LI set of solutions on $(-2\pi, 0)$. Therefore, there are $r, q \in \mathbb{R}$ such that

$$v(x) = r\widetilde{\Lambda_1}(x) + q\widetilde{\Lambda_2}(x), \quad x \in (-2\pi, 0). \tag{5.18}$$

Now, from (5.14) and (5.18) we obtain that $v(\pi) = \gamma p_2(K)$ and $v(-\pi) = qp_2(K)$, then since v is 2π-periodic and $p_2(K) \neq 0$ (Λ_1 and Λ_2 can not have zeros in a same point) we obtain $\gamma = q$. Moreover, from the continuity of v in $x = 0$ we obtain that

$$\begin{cases} v(0+) = \alpha\Lambda_1(0+) + \gamma[a\Lambda_1(0+) + p_2(a)], \quad \text{and} \\ v(0-) = r\Lambda_1(0-) + \gamma[-a\Lambda_1(0-) + p_2(-a)]. \end{cases} \tag{5.19}$$

Therefore, $r = -\alpha$. So, we have for $x \in (0, 2\pi)$ that

$$\begin{cases} v(-x) = r\widetilde{\Lambda_1}(-x) + \gamma\Big[\Big(-\frac{\eta_1}{\sqrt{2}}x - a\Big)\widetilde{\Lambda_1}(-x) + p_2\Big(-\frac{\eta_1}{\sqrt{2}}x - a\Big)\Big] \\ \qquad\quad = -r\Lambda_1(x) + \gamma\Big[\Big(\frac{\eta_1}{\sqrt{2}}x + a\Big)\Lambda_1(x) + p_2\Big(\frac{\eta_1}{\sqrt{2}}x + a\Big)\Big] = v(x). \end{cases} \tag{5.20}$$

Now, since v is just even we obtain that $v'(0+) = -v'(0-)$ and so from the condition $v'(0+) - v'(0-) = -Zv(0)$ we obtain

$$v'(0+) = -\frac{Z}{2}v(0). \tag{5.21}$$

Next we show that equality (5.21) implies that γ in (5.16) is equal to zero. Indeed, for $\phi(x) = \phi_{\omega,Z}(x)$ the relations

$$sn^2 - (1-k^2)cn^2 = \frac{2-k^2}{k^2}(1 - dn^2) - (1-k^2), \quad \Lambda_1(0+) = \frac{Z}{2}\frac{\sqrt{2}}{k^2\eta_1^2}\phi(0), \quad \text{and} \quad dn(a) = \frac{\phi(0)}{\eta_1},$$

imply from (5.16) that

$$v(0)=\gamma\Big[D(a-K)\phi'(0+)+\frac{1}{\beta}p_1(a)\Big] \tag{5.22}$$

where

$$p_1(a)=\frac{1}{1-k^2}\frac{\phi(0)}{\eta_1}\Big[\frac{2-k^2}{k^2}\Big(1-\frac{\phi^2(0)}{\eta_1^2}\Big)-(1-k^2)\Big]-\frac{Z}{2}\frac{\sqrt{2}}{k^2\eta_1^2}\frac{2-k^2}{1-k^2}Z(a)\phi(0). \tag{5.23}$$

Now we calculate $v'(0+)$. Since $\phi''(0+)=\lim_{x\to 0^+}\phi''(x)=(\omega-\phi^2(0))\phi(0)$ and $\phi'(0+)=-\frac{Z}{2}\phi(0)$ follow from (5.16) and (5.13),

$$v'(0+)=\gamma\Big[D(a-K)(\omega-\phi^2(0))\phi(0)-DZ\frac{\eta_1}{2\sqrt{2}}\phi(0)+\frac{\eta_1}{\sqrt{2}}p_2'(a)\Big]. \tag{5.24}$$

Now, from the relations

$$p_1'(x)=sn(x)cn(x)\Big[k^2+\Big(\frac{E(k)}{K(k)}-1\Big)\frac{2-k^2}{1-k^2}\Big]+\frac{4-2k^2}{1-k^2}sn(x)cn(x)dn^2(x)-\frac{2-k^2}{1-k^2}Z(x)[1-2sn^2(x)]dn(x) \tag{5.25}$$

and $k^2sn^2+dn^2=1$, we obtain

$$\begin{aligned}\frac{\eta_1}{\sqrt{2}}p_1'(a)&=\frac{Z}{2}\frac{1}{k^2\eta_1}\Big[k^2+\Big(\frac{E(k)}{K(k)}-1\Big)\frac{2-k^2}{1-k^2}\Big]\phi(0)+\frac{Z}{2}\frac{1}{k^2\eta_1}\frac{4-2k^2}{1-k^2}\frac{\phi^3(0)}{\eta_1^2}\\&\quad-\frac{2-k^2}{1-k^2}Z(a)\Big[\frac{k^2-2}{k^2}+\frac{2}{k^2}\frac{\phi^2(0)}{\eta_1^2}\Big]\frac{\phi(0)}{\sqrt{2}}.\end{aligned} \tag{5.26}$$

Next we suppose $\gamma\neq 0$. From (5.21), (5.22)-(5.23)-(5.24)-(5.26) we obtain from the equalities $\omega=\frac{(2-k^2)\eta_1^2}{2}$, $dn^2(a)=\frac{\phi^2(0)}{\eta_1^2}$, and some cancelations the following equality

$$\begin{aligned}\Big[\beta(K-a)+\frac{2-k^2}{1-k^2}Z(a)\Big]\Big[\frac{Z^2}{4}+\phi^2(0)-\omega\Big]&=\frac{\eta_1}{2\sqrt{2}}Z\Big[\beta+\frac{2-k^2}{1-k^2}\Big(dn^2(a)+\frac{E}{K}\Big)\Big]\\&=\frac{\eta_1}{2\sqrt{2}}Z\Big[2+\frac{2-k^2}{1-k^2}dn^2(a)\Big],\end{aligned} \tag{5.27}$$

where in the last equality we use the value of β. Next, by using that the Jacobian Zeta function $Z(a)$ can be rewrite in the form

$$Z(a)=E(a)-a\frac{E(k)}{K(k)}$$

and once again the value of β we obtain from (5.27),

$$\Big[2(K(k)-a)+\frac{2-k^2}{1-k^2}(E(a;k)-E(k))\Big]\Big[\frac{Z^2}{4}+\phi^2(0)-\omega\Big]=\frac{\eta_1}{2\sqrt{2}}Z\Big[2+\frac{2-k^2}{1-k^2}dn^2(a;k)\Big]. \tag{5.28}$$

Now we see that equality in (5.28) give us a contradiction. Indeed, from (4.7) (with the sign "$-$") we have that $\frac{Z^2}{4}+\phi^2(0)-\omega>0$. Next we show that there is a $\delta>0$ (independent of k) such that for every $\zeta\in[0,\delta]$ we have

$$g(\zeta,k)=\frac{2-k^2}{1-k^2}dn^2(\zeta;k)>2\qquad\text{for all }\ k\in(0,1). \tag{5.29}$$

Initially we see that the equation $g(\zeta,k)=2$ has solution if and only if $\zeta\in(\pi/4,+\infty)$. Indeed, since $dn^{-1}(y;k)=sn^{-1}(\sqrt{\frac{1-y^2}{k^2}};k)$ ([13]-pg. 31) we have that ζ needs to satisfy

$$\zeta=sn^{-1}\Big(\frac{1}{\sqrt{2-k^2}};k\Big). \tag{5.30}$$

Now since $sn^{-1}(1/\sqrt{2};0) = sin^{-1}(1/\sqrt{2}) = \frac{\pi}{4}$, $sn^{-1}(1;1) = tanh^{-1}(1) = +\infty$, and the right-hand side of (5.30) is a strictly increasing mapping of $k \in (0,1)$ we obtain our affirmation. Thus since $g(0,k) > 2$ for every $k \in (0,1)$ the intermediate value theorem shows the inequality in (5.29).

Next we show that the following function

$$F(\zeta,k) = 2(K(k)-\zeta) + \frac{2-k^2}{1-k^2}(E(\zeta;k)-E(k)) \tag{5.31}$$

for $(\zeta,k) \in I = [0,\delta]\times[0,1]$, with $\delta < \pi/4$, assumes its maximum value on the boundary of the rectangle I and it which is zero. Therefore,

$$F(\zeta,k) < 0 \qquad \text{for all } (\zeta,k) \in (0,\delta)\times(0,1). \tag{5.32}$$

We note initially from the relations $E(\zeta;1) = \int_0^\zeta sech^2(x)dx < 1$, $E(1) = 1$ and $\lim_{k\to 1^+}(1-k^2)K(k) = 0$, that for every $\zeta \in [0,\delta]$ we have $F(\zeta,1) = \lim_{k\to 1^+} F(\zeta,k) = -\infty$. Moreover, from (5.29) and the relations ([13]-pg. 282)

$$\frac{\partial F}{\partial \zeta} = -2 + \frac{2-k^2}{1-k^2}dn^2(\zeta;k) > 0, \quad \frac{\partial F}{\partial k} = \frac{k(E(k)-K(k))}{1-k^2} - \frac{2k}{(1-k^2)^2}(E(k)-E(\zeta;k)) < 0 \tag{5.33}$$

we obtain that F do not have critical point in the interior of I. Now, since $K(0) = E(0) = \frac{\pi}{2}$ and $E(\zeta;0) = \zeta$ it follows that for every $\zeta \in [0,\delta]$, $F(\zeta,0) = 0$ and so from (5.33) we have that $F(\delta,k) < 0$ and $F(0,k) < 0$ for all $k \in (0,1)$. Thus we obtain inequality in (5.32).

Now, from the theory in Section 4 we know that a and k are smooth functions of ω with $a(\omega) \to 0$ as $\omega \to +\infty$, therefore if follows from (5.32) that the right-hand side of (5.28) is negative for ω large, which a contradiction.

Therefore $\gamma = 0$ and so for $Z > 0$ one has that $Ker(\mathcal{L}_1) = \{0\}$. The case $Z < 0$ follows similarly. This finishes the proof. □

The next result will be used more later, but it is also interesting by itself.

Proposition 5.2. *Let $Z \in \mathbb{R} - \{0\}$. If λ is an simple eigenvalue for $\mathcal{L}_1$ then the eigenfunction associated is either even or odd.*

Proof. let $v \in D(\mathcal{L}_1) - \{0\}$ such that $\mathcal{L}_1 v = \lambda v$. Then, since $\varphi_{\omega,Z}$ is even, we also have for $\zeta(x) \equiv v(-x)$ the relation $\mathcal{L}_1\zeta(x) = \lambda\zeta(x)$. Then there exists $\beta \in \mathbb{R}$ such that $v(x) = \beta v(-x)$ for $x \in \mathbb{R}$. If $v(0) \neq 0$ then $\beta = 1$ and thus v is even. If $v(0) = 0$ from (5.4) we have that $v \in H^2_{per}$ and so $v'(x)$ exists for $x \in \mathbb{R}$. Then we get that $v'(0) = -\beta v'(0)$ and from the Cauchy uniqueness principle $v'(0) \neq 0$ (in other way, $v \equiv 0$). Therefore $\beta = -1$ and so v is a odd function. □

5.3. **Counting the negative eigenvalues for $\mathcal{L}_{1,Z}$.** In this subsection we use the theory of perturbation for linear operators to determinate the number of negative eigenvalues of $\mathcal{L}_{1,Z}$ for $Z \neq 0$. We will use the theory of analytic perturbation for linear operators (see [35] and [44]) and some arguments found in [37]. Our study will be divided into four steps:

(I) From our analysis in Section 4 it follows that by fixing ω large one has the convergence in (5.2) being in H^1_{per}.

(II) The linear operators $\mathcal{L}_i$ in (1.27)- (1.28) are the self-adjoint operators on L^2_{per} associated with the following bilinear forms defined for $v, w \in H^1_{per}$,

$$\begin{aligned} \mathcal{Q}^1_{\omega,Z}(v,w) &= \langle v_x, w_x\rangle + \omega\langle v,w\rangle - Zv(0)w(0) - \langle 3\varphi^2_{\omega,Z}v, w\rangle \\ \mathcal{Q}^2_{\omega,Z}(v,w) &= \langle v_x, w_x\rangle + \omega\langle v,w\rangle - Zv(0)w(0) - \langle \varphi^2_{\omega,Z}v, w\rangle. \end{aligned} \tag{5.34}$$

Indeed, since the proof for $\mathcal{L}_1$ is similar to the one of $\mathcal{L}_2$, we only deal with $\mathcal{L}_1$. Since the form $\mathcal{Q}^1_{\omega,Z}$ has domain $H^1_{per}\times H^1_{per}$ and it is symmetric, bounded from below and closed, from the theory of representation of forms by operators (The First Representation Theorem in [35], VI. Section 2.1), one has that there is a self-adjoint operator $\widetilde{\mathcal{L}_1}: D(\widetilde{\mathcal{L}_1}) \subset L^2_{per} \to L^2_{per}$ such that

$$D(\widetilde{\mathcal{L}_1}) = \{v \in H^1_{per} : \exists w \in L^2_{per} \ s.t. \ \forall z \in H^1_{per},\ \mathcal{Q}^1_{\omega,Z}(v,z) = \langle w, z\rangle\}, \tag{5.35}$$

and for $v \in D(\widetilde{\mathcal{L}_1})$ we define $\widetilde{\mathcal{L}_1}v \equiv w$, where w is the (unique) function of L^2_{per} which satisfies $\mathcal{Q}^1_{\omega,Z}(v,z) = \langle w, z\rangle$ for all $z \in H^1_{per}$. A similar operator $\widetilde{\mathcal{L}_2}$ and domain $D(\widetilde{\mathcal{L}_2})$ associated to $\mathcal{Q}^2_{\omega,Z}$ is obtained.

Next, we describe explicitly the self-adjoint operators $\widetilde{\mathcal{L}_1}$ and $\widetilde{\mathcal{L}_2}$.

Proposition 5.3. *The domain for both $\widetilde{\mathcal{L}_1}$ and $\widetilde{\mathcal{L}_2}$ in L^2_{per} is*

$$D_Z = \{\zeta \in H^1_{per} \cap H^2((0,2\pi)) :\ \zeta'(0+) - \zeta'(0-) = -Z\zeta(0)\}, \tag{5.36}$$

and for $v \in D_Z$ one has that

$$\widetilde{\mathcal{L}_1}v = -\frac{d^2}{dx^2}v + \omega v - 3\varphi^2_{\omega,Z}v, \quad \widetilde{\mathcal{L}_2}v = -\frac{d^2}{dx^2}v + \omega v - \varphi^2_{\omega,Z}v. \tag{5.37}$$

Proof. Since the proof of $\widetilde{\mathcal{L}_2}$ is similar to the one of $\widetilde{\mathcal{L}_1}$, we only deal with $\widetilde{\mathcal{L}_1}$. We consider $\mathcal{Q}^1_{\omega,Z} = \mathcal{Q}^1_Z + \mathcal{Q}^1_\omega$ with $\mathcal{Q}^1_Z : H^1_{\rm per} \times H^1_{\rm per} \to \mathbb{R}$ and $\mathcal{Q}^1_\omega : L^2_{\rm per} \times L^2_{\rm per} \to \mathbb{R}$ defined by

$$\mathcal{Q}^1_Z(v,z) = \langle v_x, z_x\rangle - Zv(0)z(0), \quad \mathcal{Q}^1_\omega(v,z) = \omega\langle v, z\rangle - 3\langle\varphi^2_{\omega,Z}v, z\rangle. \tag{5.38}$$

We denote by $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) the self-adjoint operator on L^2_{per} (see Kato [35], VI. Section 2.1) associated with $\mathcal{Q}^1_Z$ (resp. $\mathcal{Q}^1_\omega$). Thus, $D(\mathcal{T}_1) = D(\widetilde{\mathcal{L}_1})$ ($D(\mathcal{T}_2) = L^2_{per}$). We claim that $\mathcal{T}_1$ is a self-adjoint extension of the operator A^0 defined in Section 3. Let $v \in H^2_{\rm per}$ such that $v(0) = 0$, and define $w \equiv -v_{xx} \in L^2_{\rm per}$. Then for every $z \in H^1_{\rm per}$ we have $\mathcal{Q}^1_Z(v,z) = (w,z)$. Thus, $v \in D(\mathcal{T}_1)$ and $\mathcal{T}_1 v = w = -\frac{d^2}{dx^2}v$. Hence, $A^0 \subset \mathcal{T}_1$. So, using Theorem 3.1 there exists $\beta \in \mathbb{R}$ such that $D(\mathcal{T}_1) = D(-\Delta_\beta)$ which yields the claim. Next we shall show that $\beta = -Z$. Take $v \in D(\mathcal{T}_1)$ with $v(0) \neq 0$. Following the ideas in (5.6) we obtain $\langle \mathcal{T}_1 v, v\rangle = [v'(0+) - v'(0-)]v(0) + \|v_x\|^2 = \|v_x\|^2 + \beta[v(0)]^2$, which should be equal to $\mathcal{Q}^1_Z(v,v) = \|v_x\|^2 - Z[v(0)]^2$. Therefore $\beta = -Z$, and the Proposition is proved. □

(III) By Proposition 5.3 we can drop the tilde over $\widetilde{\mathcal{L}_1}$ and $\widetilde{\mathcal{L}_2}$ and work with the operators $\mathcal{L}_{1,Z}$ and $\mathcal{L}_{2,Z}$. The following Lemma verifies the analyticity of the families of operators $\mathcal{L}_{i,Z}$.

Lemma 5.1. *As a function of Z, $(\mathcal{L}_{1,Z})$ and $(\mathcal{L}_{2,Z})$ are two real-analytic families of self-adjoint operators of type (B) in the sense of Kato.*

Proof. From Proposition 5.3, Theorem VII-4.2 in [35], it suffices to prove that the families of bilinear forms $(\mathcal{Q}^1_{\omega,Z})$ and $(\mathcal{Q}^2_{\omega,Z})$ defined in (5.34) are real-analytic family of type (b). Indeed, it is immediate that they are bounded from below and closed. Moreover, Theorems 4.3-4.4 and the decomposition of $\mathcal{Q}^1_{\omega,Z}$ into $\mathcal{Q}^1_Z$ and $\mathcal{Q}^1_\omega$, implies that $Z \to \langle \mathcal{Q}^1_{\omega,Z}v, v\rangle$ is analytic. The proof of the analyticity of the family $(\mathcal{Q}^2_{\omega,Z})$ is similar to the one of $(\mathcal{Q}^1_{\omega,Z})$. □

Remark: The explicit resolvent formula for $-\Delta_{-Z}$ in (3.13) can be used to give another proof of the fact that the families $(\mathcal{L}_{i,Z})$ are real-analytic in the sense of Kato.

The following result is a consequence of the classical Floquet theory (see [40], [33] and [5]) and it gives a precise description of the spectrum of the self-adjoint operator in (1.31) which we want to perturb.

Theorem 5.2. *The operator $\mathcal{L}_0$ has exactly one negative simple isolated first eigenvalue τ_0. The second eigenvalue is zero, and it is simple with associated eigenfunction $\frac{d}{dx}\phi_{\omega,0}$. The rest of the spectrum is positive and discrete.*

Remark: The Theorem 5.2 can also be shown by using a Fourier approach developed by Angulo&Natali in [8].

Proposition 5.4. *There exist $Z_0 > 0$ and two analytic functions $\Pi : (-Z_0, Z_0) \to \mathbb{R}$ and $\Omega : (-Z_0, Z_0) \to L^2_{per}$ such that*

(i) $\Pi(0) = 0$ *and* $\Omega(0) = \frac{d}{dx}\phi_{\omega,0}$.
(ii) *For all $Z \in (-Z_0, Z_0)$, $\Pi(Z)$ is the simple isolated second eigenvalue of $\mathcal{L}_{1,Z}$ and $\Omega(Z)$ is an associated eigenvector for $\Pi(Z)$.*
(iii) *Z_0 can be chosen small enough such that, except the two first eigenvalues, the spectrum of $\mathcal{L}_{1,Z}$ is positive.*

Proof. From Theorem 5.2 we separate the spectrum $\sigma(\mathcal{L}_0)$ of $\mathcal{L}_0$ in (1.31) into two parts $\sigma_0 = \{\tau_0, 0\}$ and σ_1 by a closed curve Γ (for example a circle) such that σ_0 belongs to the inner domain of Γ and σ_1 to the outer domain of Γ (note that $\sigma_1 \subset (b, +\infty)$ for $b > 0$). From Lemma 5.1 follows that that $\Gamma \subset \rho(\mathcal{L}_{1,Z})$ for sufficiently small $|Z|$ and $\sigma(\mathcal{L}_{1,Z})$ is likewise separated by Γ into two parts so that the part of $\sigma(\mathcal{L}_{1,Z})$ inside Γ consists of a finite system of eigenvalues with total multiplicity (algebraic) two. Therefore we obtain from the Kato-Rellich Theorem (see Theorem XII.8 in [44]) the existence of two analytic functions Π, Ω defined in a neighborhood of zero such that we obtain the items (i), (ii) and (iii). This completes the proof of the Proposition. □

Next we shall study how the perturbed second eigenvalue $\Pi(Z)$ changes depending on the sign of Z. For Z small we have the following picture.

Theorem 5.3. *There exists $0 < Z_1 < Z_0$ such that $\Pi(Z) < 0$ for any $Z \in (-Z_1, 0)$ and $\Pi(Z) > 0$ for any $Z \in (0, Z_1)$. Therefore, for Z negative and small $\mathcal{L}_{1,Z}$ has exactly two negative eigenvalues and for Z positive and small $\mathcal{L}_{1,Z}$ has exactly one negative eigenvalue.*

Proof. From Taylor's theorem we the following expansions

$$\Pi(Z) = \beta Z + O(Z^2), \quad \text{and} \quad \Omega(Z) = \phi'_{\omega,0} + Z\psi_0 + O(Z^2) \tag{5.39}$$

where $\phi'_{\omega,0} = \frac{d}{dx}\phi_{\omega,0}$, $\beta \in \mathbb{R}$ ($\beta = \Pi'(0)$) and $\psi_0 \in L^2_{per}$ ($\psi_0 = \Omega'(0)$). The desired result will follow if we show that $\beta > 0$. From Theorems 4.2, 4.3 and 4.4 there exists $\chi_0 \in H^1_{per}$ such that for Z close to zero

$$\varphi_{\omega,Z} = \phi_{\omega,0} + Z\chi_0 + O(Z^2). \tag{5.40}$$

Now, using (5.40) to substitute into (1.16) and differentiating with respect to Z, we obtain

$$\langle \mathcal{L}_0\chi_0, \psi\rangle = \phi_{\omega,0}(0)\psi(0) + O(Z), \tag{5.41}$$

for any $\psi \in H^1_{per}$.

We develop β with respect to Z. We compute $\langle \mathcal{L}_{1,Z}\Omega(Z), \phi'_{\omega,0}\rangle$ in two different ways.

(1) Since $\mathcal{L}_{1,Z}\Omega(Z) = \Pi(Z)\Omega(Z)$ it follows from (5.39) that

$$\langle \mathcal{L}_{1,Z}\Omega(Z), \phi'_{\omega,0}\rangle = \beta Z\|\phi'_{\omega,0}\|^2 + O(Z^2). \tag{5.42}$$

(2) Since $\mathcal{L}_{1,Z}$ is self-adjoint, $\phi'_{\omega,0} \in \mathcal{D}(\mathcal{L}_{1,Z})$ and

$$\mathcal{L}_{1,Z}\phi'_{\omega,0} = -6Z\phi_{\omega,0}\phi'_{\omega,0}\chi_0 + O(Z^2), \tag{5.43}$$

we obtain from (5.39) and (5.43) that

$$\langle \mathcal{L}_{1,Z}\Omega(Z), \phi'_{\omega,0}\rangle = -6Z\langle \phi'_{\omega,0}, \chi_0\phi_{\omega,0}\phi'_{\omega,0}\rangle + O(Z^2). \tag{5.44}$$

It is easy to see that $\mathcal{L}_0(\omega\phi_{\omega,0} - \phi^3_{\omega,0}) = 6\phi_{\omega,0}(\phi'_{\omega,0})^2$, which combined with (5.44) and (5.41) gives us the last equality

$$\langle \mathcal{L}_{1,Z}\Omega(Z), \phi'_{\omega,0}\rangle = -Z[\omega\phi^2_{\omega,0}(0) - \phi^4_{\omega,0}(0)] + O(Z^2). \tag{5.45}$$

Finally, a combination of (5.42) and (5.45) leads to

$$\beta = -\frac{\omega\phi^2_{\omega,0}(0) - \phi^4_{\omega,0}(0)}{\|\phi'_{\omega,0}\|^2} + O(Z). \tag{5.46}$$

Now, from Theorem 4.1 we have $\phi_{\omega,0}(0) \in (0, \sqrt{\omega})$ and so $\beta > 0$ for Z small. Hence, the first equality in (5.39) completes the proof. □

(IV) Now we are in position for counting the number of negative eigenvalues of $\mathcal{L}_{i,Z}$ for all Z, using a classical continuation argument based on the Riesz-projection. We denote the number of negatives eigenvalues of $\mathcal{L}_{i,Z}$ by $n(\mathcal{L}_{i,Z})$.

Theorem 5.4. *Let ω be large. Then*
(1) *for $Z > 0$, $n(\mathcal{L}_{1,Z}) = 1$,*
(2) *for $Z < 0$, $n(\mathcal{L}_{1,Z}) = 2$.*

Proof. **We recall that for ω large and $Z \neq 0$,** $Ker(\mathcal{L}_{1,Z}) = \{0\}$. Let $Z < 0$ and define Z_∞ by

$$Z_\infty = \inf\{z < 0 : \mathcal{L}_{1,Z} \text{ has exactly two negative eigenvalues for all } Z \in (z, 0)\}.$$

From Theorem 5.3 one has that Z_∞ is well defined and $Z_\infty \in [-\infty, 0)$. We claim that $Z_\infty = -\infty$. Suppose that $Z_\infty > -\infty$. Let $N = n(\mathcal{L}_{1,Z_\infty})$ and Γ a closed curve (for example a circle or a rectangle) such that $0 \in \Gamma \subset \rho(\mathcal{L}_{1,Z_\infty})$ and such that all the negatives eigenvalues of $\mathcal{L}_{1,Z_\infty}$ belong to the inner domain of Γ. From Lemma 5.1 it follows that there is a $\delta > 0$ such that for $Z \in [Z_\infty - \delta, Z_\infty + \delta]$ we have $\Gamma \subset \rho(\mathcal{L}_{1,Z})$ and for $\xi \in \Gamma$, $Z \to (\mathcal{L}_{1,Z} - \xi)^{-1}$ is analytic. Therefore the existence of an analytic family of Riesz-projections, $Z \to P(Z)$, given by $P(Z) = -\frac{1}{2\pi i}\int_\Gamma(\mathcal{L}_{1,Z} - \xi)^{-1}d\xi$, implies that $\dim(\text{Rank } P(Z)) = \dim(\text{Rank } P(Z_\infty)) = N$, for all $Z \in [Z_\infty - \delta, Z_\infty + \delta]$. Now by definition of Z_∞, there exists $z_0 \in (Z_\infty, Z_\infty + \delta)$ and $\mathcal{L}_{1,Z}$ has exactly two negative eigenvalues for all $Z \in (z_0, 0)$. Therefore $\mathcal{L}_{1,Z_\infty+\delta}$ has two negative eigenvalues and $N = 2$, hence $\mathcal{L}_{1,Z}$ has two negative eigenvalues for $Z \in (Z_\infty - \delta, 0)$ contradicting the definition of Z_∞. Therefore, we have established the claim $Z_\infty = -\infty$. A similar analysis is applied to the case $Z > 0$. This finishes the proof of the Theorem. □

Remark: We can choose Γ independently of the parameter $Z < 0$ in the beginning of the proof of Theorem 5.4 in the following manner : since for all Z, $\varphi_{\omega,Z} \leqq \eta_{1,+} \leqq \sqrt{2\omega}$, for $\|f\| = 1$ and $f \in \mathcal{D}$, $\langle \mathcal{L}_{1,Z}f, f\rangle \geqq -3\int \varphi^2_{\omega,Z}f^2dx \geqq -6\omega$. Therefore, $\inf\sigma(\mathcal{L}_{1,Z}) \geqq -6\omega$ for all $Z < 0$. So, Γ can be chosen as the rectangle $\Gamma = \partial R$ for R being $R = \{z \in \mathbb{C} : z = z_1 + iz_2, (z_1, z_2) \in [-6\omega - 1, 0] \times [-a, a]$, for some $a > 0\}$.

Proposition 5.5. *The function $\Omega(Z)$ defined in Proposition 5.4 and associated to the second negative eigenvalue of $\mathcal{L}_{1,Z}$ can be extended to $(-\infty, \infty)$. Moreover, $\Omega(Z) \in H^1_{per}$ is an odd function for $Z \in (-\infty, \infty)$.*

Proof. From Lemma 5.1 and Theorem XII.7 in [44] the set $\Gamma_0 = \{(Z, \lambda)|Z \in \mathbb{R}, \lambda \in \rho(\mathcal{L}_{1,Z})\}$ is open and $(Z, \lambda) \in \Gamma_0 \to (\mathcal{L}_{1,Z} - \lambda)^{-1}$ is a analytic function in both variables. So, the argument in Proposition 5.4 implies that the functions $\Omega(Z)$ and $\Pi(Z)$ are analytic for every $Z \in \mathbb{R}$. Next we consider $Z < 0$ (the case $Z > 0$ is similar). From Proposition 5.2 and Proposition 5.4 the eigenvectors $\Omega(Z)$ are even or odd and $\Omega(0) = \frac{d}{dx}\phi_{\omega,0}$ is odd. Then, from the equality $\lim_{Z\to 0}\langle \Omega(Z), \Omega(0)\rangle = \|\Omega(0)\|^2 \neq 0$, one has that $\langle \Omega(Z), \Omega(0)\rangle \neq 0$ for Z close to 0. Thus $\Omega(Z)$ is odd. Let z_∞ be $z_\infty = \{z < 0 : \Omega(Z)$ is odd for any $Z \in (z, 0]\}$. Suposse now that $z_\infty > -\infty$. If $\Omega(z_\infty)$ is odd, then by continuity there exists $\delta > 0$ such that $\Omega(z_\infty - \delta)$ is odd which is a contradiction. Thus Proposition 5.2 implies that $\Omega(z_\infty)$ is even. Now, since $\Omega(z_\infty)$ is the limit of odd functions we obtain that $\Omega(z_\infty)$ is odd. Therefore $\Omega(z_\infty) \equiv 0$, which is a contradiction because $\Omega(z_\infty)$ is an eigenvector. This concludes the proof of the Proposition. □

5.4. **Convexity condition.** Here, we shall prove the increasing property of the mapping $\omega \to \|\varphi_{\omega,Z}\|^2$, for all Z, which suffices for our stability/instability results for the orbit defined in (1.18).

Theorem 5.5. *Let $Z \in \mathbb{R} - \{0\}$ and ω large. Then the dnoidal-peak smooth curve $\omega \to \varphi_{\omega,Z}$ given in* (1.26) *satisfies $\frac{d}{d\omega}\|\varphi_{\omega,Z}\|^2 > 0$.*

Proof. For $Z > 0$ we have $\varphi_{\omega,Z} = \phi_{\omega,Z}$. Then via a change of variable and from Theorem 4.3 we have for $a = a(\omega)$, $\eta_1 = \eta_{1,+}$, $k = k(\omega)$ and $K - a = \frac{\eta_1}{\sqrt{2}}\pi$ the equality

$$\|\phi_{\omega,Z}\|^2 = \eta_1^2 \int_{-\pi}^{\pi} dn^2\Big(\frac{\eta_1}{\sqrt{2}}|\xi| + a; k\Big)d\xi = 2\sqrt{2}\eta_1[E(k) - E(a)] = 2\sqrt{2}\eta_1[E(k) - E(\varphi_a, k)]. \tag{5.47}$$

Here $E(\varphi_a, k) = \int_0^{\varphi_a} \sqrt{1 - k^2\sin^2\theta}\ d\theta = \int_0^a dn^2(u; k)\ du = E(a)$, with $\sin\varphi_a = sn(a)$ and $E(k) = E(\pi/2, k)$. Next, we consider the identity

$$\frac{d}{d\omega}\|\phi_\omega\|^2 = 2\sqrt{2}\frac{d\eta_1}{d\omega}[E(k) - E(\varphi_a, k)] + 2\sqrt{2}\eta_1\Big[\Big(E'(k) - \frac{\partial E}{\partial k}\Big)\frac{dk}{d\omega} - \frac{\partial E}{\partial\varphi_a}\frac{d\varphi_a}{d\omega}\Big]. \tag{5.48}$$

We shall calculate the differentiation terms in (5.48).

(1) From the definition of $E(\cdot,\cdot)$ one has that $\frac{\partial E}{\partial\varphi_a}(\varphi_a, k) = \sqrt{1 - k^2 sn^2(a)} = dn(a)$.

(2) From [13] we obtain $\partial_k E(\varphi_a, k) = (E(a) - a)/k$.

(3) Next, since $sn(u + K) = \frac{cn(u)}{dn(u)} \equiv cd(u)$ one has that $\varphi_a(\omega) = \sin^{-1}[cd(\eta_1\pi/\sqrt{2})]$. So,

$$\frac{d}{d\omega}\varphi_a = \frac{dn}{k' sn}\frac{d}{d\omega}cd\Big(\frac{\eta_1}{\sqrt{2}}\pi; k). \tag{5.49}$$

Now, from using [13] again one finds that

$$\begin{aligned}\frac{d}{d\omega}cd\Big(\frac{\eta_1}{\sqrt{2}}\pi; k) &= \frac{\pi}{\sqrt{2}}\frac{\partial}{\partial u}cd\Big(\frac{\eta_1}{\sqrt{2}}\pi; k\Big)\frac{d\eta_1}{d\omega} + \frac{\partial}{\partial k}cd\Big(\frac{\eta_1}{\sqrt{2}}\pi; k\Big)\frac{dk}{d\omega}\\ &= -\frac{k'^2\pi}{\sqrt{2}}\frac{d\eta_1}{d\omega}\frac{sn}{dn^2} + \frac{sn}{kdn^2}\Big[E\Big(\frac{\eta_1}{\sqrt{2}}\pi\Big) - k'^2\frac{\eta_1}{\sqrt{2}}\pi\Big]\frac{dk}{d\omega}.\end{aligned}$$

So, from (5.49) and from the equality $dn(u + K) = k'/(dnu)$

$$\frac{d}{d\omega}\varphi_a = dna\Big[-\frac{\pi}{\sqrt{2}}\frac{d\eta_1}{d\omega} + \frac{1}{kk'^2}\Big[E\Big(\frac{\eta_1}{\sqrt{2}}\pi\Big) - k'^2\frac{\eta_1}{\sqrt{2}}\pi\Big]\frac{dk}{d\omega}\Big] \tag{5.50}$$

(4) Combining the identities

$$\frac{d}{dk}K(k) = \frac{E(k) - k'^2K(k)}{kk'^2}, \qquad \frac{\pi}{\sqrt{2}}\frac{d\eta_1}{d\omega} = \frac{d}{dk}K(k)\frac{dk}{d\omega} - a'(\omega),$$

and $E\Big(\frac{\eta_1}{\sqrt{2}}\pi\Big) - E(k) + k'^2 a = \int_{K-a}^{K}[k'^2 - dn^2(u)]du = -k^2\int_{K-a}^{K} cn^2(u)du$ it follows that

$$\frac{d}{d\omega}\varphi_a = dn(a)\Big[a'(\omega) - \frac{k}{k'^2}\int_{K-a}^{K} cn^2(u)du\frac{dk}{d\omega}\Big] \equiv dn(a)A(\omega). \tag{5.51}$$

We observe that $A(\omega) < 0$ and so $\frac{d}{d\omega}\varphi_a < 0$.

Then, gathering the information in (5.48) and from (1)-(4) above we obtain that

$$\begin{aligned}\frac{d}{d\omega}\|\phi_\omega\|^2 &= \frac{4}{\pi}\Big[K'(k)[E(k)-E(a)]+E'(k)[K(k)-a]\Big]\frac{dk}{d\omega}\\ &-\frac{4}{\pi}a'(\omega)[E(k)-E(a)]+\frac{4}{\pi}[K(k)-a]\frac{a-E(a)}{k}\frac{dk}{d\omega}-2\sqrt{2}\eta_1 dn^2(a)A(\omega).\end{aligned}\tag{5.52}$$

Now, since $a-E(a)=\int_0^a[1-dn^2(u)]du=k^2\int_0^a sn^2(u)du>0$, $E(k)-E(a)>0$, $a'(\omega)<0$, $A(\omega)<0$ and $\frac{dk}{d\omega}>0$ we obtain that the expression on the second line in (5.52) is positive. Therefore from (5.52) one concludes that

$$\begin{aligned}\frac{\pi}{4}\frac{d}{d\omega}\|\phi_{\omega,Z}\|^2 &> \frac{d}{d\omega}[K(k)E(k)]-E(a)\frac{d}{d\omega}K(k)-a\frac{d}{d\omega}E(k)\\ &> \frac{d}{d\omega}[K(k)E(k)]-a\frac{d}{d\omega}[K(k)+E(k)] > \frac{d}{d\omega}[K(k)E(k)-\frac{1}{2}(K(k)+E(k))]\end{aligned}\tag{5.53}$$

where ω is chosen large enough such that $a(\omega)\leqq\frac{1}{2}$. We note that here we have used that the mapping $k\to K(k)+E(k)$ is increasing and so $\frac{d}{d\omega}[K(k)+E(k)]=\frac{d}{dk}[K(k)+E(k)]\frac{dk}{d\omega}>0$. Since $\frac{d}{dk}\Big[K(k)E(k)-\frac{1}{2}(K(k)+E(k))\Big]>0$, it follows from (5.53) that $\frac{d}{d\omega}\|\phi_{\omega,Z}\|^2>0$ for ω large.

Next, we consider the case $Z<0$. For $\varphi_{\omega,Z}=\zeta_{\omega,Z}$ and $\beta=\sqrt{2}/\eta_1$ one has that

$$\|\zeta_{\omega,Z}\|^2=\eta_1^2\int_{-\pi}^{\pi}dn^2\Big(\frac{\eta_1}{\sqrt{2}}|\xi|-a\Big)d\xi=\frac{4}{\beta}\int_{-a}^{\frac{\pi}{\beta}-a}dn^2(y)dy=\frac{4}{\beta}\int_{-a}^{K}dn^2(y)dy\equiv G(\beta),\tag{5.54}$$

using that $K+a=\frac{\eta_1}{\sqrt{2}}\pi$. So, $\frac{d}{d\omega}\|\zeta_{\omega,Z}\|^2=G'(\beta)\frac{d\beta}{d\omega}=-\frac{\sqrt{2}}{\eta_1^2}\frac{d\eta_1}{d\omega}G'(\beta)$, where

$$G'(\beta)=4\beta^{-2}\Big[-\int_{-a}^{K}dn^2(y)dy+\beta\frac{d}{d\beta}\int_{-a}^{K}dn^2(y)dy\Big]\equiv 4\beta^{-2}H(\beta).\tag{5.55}$$

The idea now is to show that $H(\beta)<0$. Indeed, from Section 4 we have $\omega\to\eta_2(\omega)$ is a positive decreasing function, then for $\omega\to+\infty$ follows $\eta_2^2/2\omega\to 0$. So, Theorem 4.4 implies that $k^2\to 1$ and $\eta_1^2/2\omega\to 1$ for $\omega\to+\infty$. Thus, $\beta\to 0$ as $\omega\to+\infty$. Hence, $a(\beta)=a(\eta_1^{-1}(\sqrt{2}/\beta))\to 0$ as $\beta\to 0$ (see Corollaries 4.3 and 4.5). Since $dn(x;1)=sech(x)$ and $K(1)=+\infty$ we obtain $H(0)=-\int_0^\infty sech^2(y)dy<0$. Therefore $H(\beta)<0$ for β close to zero. This completes the proof of the Theorem. □

5.5. **Stability results.** From the last subsections our stability results associated to the orbit in (5.1) generated by the dnoidal-peak solution profile $\varphi_{\omega,Z}$ in (1.26) can be now established. As it was pointed the abstract theory of Grillakis, Shatah and Strauss [28] shall be use, and so we briefly discuss the criterion for obtaining stability or instability in our case. Consider the linear operator $H_{\omega,Z}$ defined in (5.3) and denote by $n(H_{\omega,Z})$ the number of negative eigenvalues of $H_{\omega,Z}$. Define

$$p_Z(\omega_0)=\begin{cases}1, & \text{if }\ \partial_\omega\|\varphi_{\omega,Z}\|^2>0,\ \ at\ \ \omega=\omega_0,\\ 0, & \text{if }\ \partial_\omega\|\varphi_{\omega,Z}\|^2<0,\ \ at\ \ \omega=\omega_0.\end{cases}\tag{5.56}$$

Then, having established the Assumption 1, Assumption 2 and Assumption 3 of [28], namely, the existence of global solutions (Proposition 3.1), the existence of a smooth curve of standing-wave, $\omega\to\varphi_{\omega,Z}$ (Theorem 4.3 - Theorem 4.4), and $Ker(\mathcal{L}_{1,Z})=\{0\}$, $Ker(\mathcal{L}_{2,Z})=[\varphi_{\omega,Z}]$, the next Theorem follows from the Instability Theorem and Stability Theorem in [28].

Theorem 5.6. *Let ω_0 be large.*

(1) *If $n(H_{\omega_0,Z})=p_Z(\omega_0)$, then the dnoidal-peak standing wave $e^{i\omega_0 t}\varphi_{\omega_0,Z}$ is stable in $H^1_{per}([-\pi,\pi])$ by the flow determined by the NLS-δ equation (1.15).*

(2) *If $n(H_{\omega_0,Z}) - p_Z(\omega_0)$ is odd, then the dnoidal-peak standing wave $e^{i\omega_0 t}\varphi_{\omega_0,Z}$ is unstable in $H^1_{per}([-\pi,\pi])$ by the flow determined by the NLS-δ equation (1.15).*

Now we can prove our main result Theorem 1.1

Proof. From Theorem 5.5 follows that $p_Z(\omega) = 1$ for all $Z \in \mathbb{R} - \{0\}$ and ω large. Next, from Proposition 5.1 we have that $\mathcal{L}_{2,Z}$ has zero as a simple eigenvalue and from Theorem 5.1 we have $\mathcal{L}_{1,Z}$ has a trivial kernel. Thus, from Theorem 5.6, Theorem 5.4 we obtain the item (1) and item (2).

Proposition 5.5 assures that the second eigenvalue of $\mathcal{L}_{1,Z}$ considered in the whole space $L^2_{per}([-\pi,\pi])$ is associated with an odd eigenfunction, and thus dissapears when the problem is restricted to subspace of even periodic functions. Moreover, since $\varphi_{\omega,Z}$ is an even function and trivially satisfies that $\langle \mathcal{L}_{1,Z}\varphi_{\omega,Z}, \varphi_{\omega,Z}\rangle < 0$, for $Z < 0$, we obtain that the first negative eigenvalue of $\mathcal{L}_{1,Z}$ is still present when the problem is restricted to the subspace of even periodic function of $H^1_{per}([-\pi,\pi])$, namely, $H^1_{per,even}([-\pi,\pi])$. So we obtain in this case that $n(H_{\omega,Z}|_{H^1_{per,even}([-\pi,\pi])}) = 1$. Therefore item (3) of the Theorem follows from item (1) of Theorem 5.6 and Proposition 3.1. This finishes the proof of the Theorem. □

6. Appendix

We shall establish two main properties of the period functions T_- and T_+ defined in (4.21) and (4.35), respectively. Namely, $\partial_\omega T_- < 0$ and $\partial_\eta T_+ < 0$.

1) $\partial_\omega T_- < 0$ for ω large: We denote $g = \frac{1}{2\sqrt{2}}\sqrt{2\omega - \eta^2}$, then $gT_- = K - a$. So,

$$g\partial_\omega T_- = -\frac{1}{2\omega - \eta^2}(K - a) + \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)}\frac{dk}{d\omega} - \frac{d}{d\omega}a(\omega). \tag{6.1}$$

Next, for $f(\omega) = M(\omega)\Phi(\omega)$ and $\varphi = sin^{-1}\Big(\sqrt{\frac{1-f^2}{k^2}}\Big)$, we obtain from the relation $a = dn^{-1}(f;k) = F(\varphi;k)$ that (see pgs. 282 and 284 in [13])

$$\frac{d}{d\omega}a(\omega) = \frac{1}{dn(a)}\frac{\partial\varphi}{\partial\omega} + \Big[\frac{E(a) - (1-k^2)a}{k(1-k^2)} - \frac{ksn(a)cn(a)}{(1-k^2)dn(a)}\Big]\frac{dk}{d\omega}. \tag{6.2}$$

Now, since $\frac{dk}{d\omega} = \frac{1-k^2}{(2\omega-\eta^2)k}$ follows from (6.2) that $g\partial_\omega T_- < 0$ if and only if

$$-(K - a) + E(k) - E(a) - (2\omega - \eta^2)\frac{k^2}{dn(a)}\frac{\partial\varphi}{\partial\omega} + \frac{k^2 sn(a)cn(a)}{dn(a)} < 0. \tag{6.3}$$

Next,

$$\frac{\partial\varphi}{\partial\omega} = \frac{1}{2}\frac{1}{cn(a)sn(a)}\Big[-\frac{1}{k^2}\frac{d}{d\omega}(f^2) - \frac{2sn^2(a)}{k}\frac{dk}{d\omega}\Big] \tag{6.4}$$

implies that (6.3) is equivalent to

$$-(K - a) + E(k) - E(a) + \frac{2\omega - \eta^2}{2dn(a)cn(a)sn(a)}\frac{d}{d\omega}(f^2) + \frac{sn(a)dn(a)}{cn(a)} < 0. \tag{6.5}$$

So, since $a(\omega) \to 0$ and $k \to 1$ as $\omega \to +\infty$ we have from (6.5) that it is sufficient to show that $\lim_{\omega\to+\infty}\frac{2\omega-\eta^2}{sn(a)}\frac{d}{d\omega}(f^2) = 0$, because of $K(1) = +\infty$, $E(1) = 1$, $E(0) = 0$, $dn(0) = cn(0) = 1$ and $sn(0) = 0$. Indeed, from the definition of f and (4.23) follow that

$$\lim_{\omega\to+\infty}(2\omega - \eta^2)^2\frac{d}{d\omega}(f^2) = Z^2. \tag{6.6}$$

So, since $f^2(\omega) \to 1$ as $\omega \to +\infty$ and $sn(a) = \frac{1}{k}\sqrt{1-f^2}$ we obtain that the l' Hospital's rule implies $\lim_{\omega\to+\infty} \frac{2\omega-\eta^2}{sn(a)} \frac{d}{d\omega}(f^2) = Z^2 \lim_{\omega\to+\infty} \frac{1}{(2\omega-\eta^2)\sqrt{1-f^2}} = Z^2 \cdot \frac{4}{Z^2} \cdot \lim_{\omega\to+\infty} \sqrt{1-f^2} = 0$.

2) $\partial_\eta T_+ < 0$ for ω large: From $gT_+ = K + a$ we obtain

$$g\partial_\eta T_+ = \frac{\eta}{2\omega-\eta^2}(K+a) + \frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\frac{dk}{d\eta} + \frac{d}{d\eta}a. \tag{6.7}$$

Next, by using a similar formula to (6.2) for $\frac{d}{d\eta}a$ and $\frac{dk}{d\eta} = \frac{-2\omega(1-k^2)}{(2\omega-\eta^2)\eta k}$ follow from (6.7) that $g\partial_\eta T_+ < 0$ if and only if

$$\frac{\eta^2}{\omega}K(k) - E(k) + \frac{\eta^2}{\omega}a - E(a) + \frac{1}{2}\frac{\eta k^2(2\omega-\eta^2)}{\omega dn(a)}\frac{\partial\varphi}{\partial\eta} + \frac{k^2 sn(a)cn(a)}{dn(a)} < 0. \tag{6.8}$$

Now, since $\frac{\eta^2}{\omega} = \frac{2(1-k^2)}{2-k^2}$ and $k \to 1$, $\frac{\eta^2}{\omega}K(k) - E(k) \to -1$ and $a(\omega) \to 0$ as $\omega \to +\infty$, we have that for obtaining (6.8) it is sufficient to show that

$$\lim_{\omega\to+\infty} \frac{2\omega-\eta^2}{\omega}\frac{\partial\varphi}{\partial\eta} = 0. \tag{6.9}$$

Indeed, since $sn(0) = 0$, $\frac{dk}{d\eta} \to 0$, $(2\omega-\eta^2)\frac{d}{d\eta}(f^2) \to -4\eta$ as $\omega \to +\infty$,

$$\frac{\partial\varphi}{\partial\eta} = \frac{1}{2cn(a)sn(a)}\Big[-\frac{1}{k^2}\frac{d}{d\eta}(f^2) - \frac{2sn^2(a)}{k}\frac{dk}{d\eta}\Big], \tag{6.10}$$

$\lim_{\omega\to+\infty} \frac{1}{\omega a} = \lim_{\omega\to+\infty} \frac{2}{\omega^2 \frac{d}{d\omega}(f^2)}\sqrt{1-f^2} = 0$, by l'Hospital's rule and (6.6), we obtain (6.9).

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