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ALMOST GLOBAL EXISTENCE FOR 2-D INCOMPRESSIBLE ISOTROPIC ELASTODYNAMICS

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ABSTRACT. We consider the Cauchy problem for 2-D incompressible isotropic elastodynamics. Standard energy methods yield local solutions on a time interval $[0, T/\epsilon]$ for initial data of the form ϵU_0, where T depends only on some Sobolev norm of U_0. We show that for such data there exists a unique solution on a time interval $[0, \exp T/\epsilon]$, provided that ϵ is sufficiently small. This is achieved by careful consideration of the structure of the nonlinearity. The incompressible elasticity equation is inherently linearly degenerate in the isotropic case; in other words, the equation satisfies a null condition. This is essential for time decay estimates. The pressure, which arises as a Lagrange multiplier to enforce the incompressibility constraint, is estimated in a novel way as a nonlocal nonlinear term with null structure. The proof employs the generalized energy method of Klainerman, enhanced by weighted L^2 estimates and the ghost weight introduced by Alinhac.

1. INTRODUCTION

The long-time behavior of elastic waves for isotropic incompressible materials is studied in 2-D. The equations of incompressible elastodynamics display a linear degeneracy in the isotropic case; i.e., the equation inherently satisfies a null condition. By taking advantage of this structure, we prove that the 2-D incompressible isotropic nonlinear elastic system is almost globally well-posed for small initial data. More precisely, we prove that for initial data of the form ϵU_0, there exists a unique solution for a time interval $[0, \exp(T(U_0)/\epsilon)]$, where $T(U_0)$ depends only on a certain weighted Sobolev norm of the U_0.

To place our result in context, we review a few highlights from the existence theory of nonlinear wave equations and elastodynamics. The initial value problem for small solutions of 3-D quasilinear wave equations with quadratic nonlinearities is almost globally well-posed [11], and in general this is sharp [8]. If, in addition, the nonlinearity satisfies the null condition, global existence was shown independently in [4] using conformal compactification and in [14] using the generalized energy method. The generalized energy method of Klainerman can be adapted to prove

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almost global existence for certain nonrelativistic systems of 3-D nonlinear wave equations, using scaling invariance to get weighted L^2-estimates, as was first done in [15]. This approach was subsequently developed to obtain global existence under the null condition in [32]; see also [34] for a different method. A unified treatment for obtaining weighted L^2 estimates for certain hyperbolic systems appeared in [30].

The existence question is more delicate in 2-D because, even with the null condition, quadratic nonlinearities have critical decay. A series of articles considered the case of cubically nonlinear equations satisfying the null condition; see for example [6, 12, 21]. Alinhac [2] was the first to establish global existence for null bilinear forms. His argument combines vector fields with what he called the ghost energy method, but crucially it also relies on finite propagation speed through a certain Hardy-type inequality.

The long-time existence theory for isotropic elastodynamics largely follows the paradigm of nonlinear wave equations. Almost global existence of small displacement solutions for 3-D compressible elastodynamics was first shown in [10], and counterexamples to global existence appeared in [9, 33]. The almost global existence proof was simplified considerably in [15] by enhancing the vector field approach with a new weighted L^2 estimate based on scaling invariance to compensate for the absence of Lorentz invariance. In [27], it was first noticed that there was a null structure compatible with the system of isotropic elastodynamics which can be used to establish global existence of small displacement solutions in 3-D. More comprehensive versions of this result appeared in [1, 28]. Compressible isotropic elastodynamics (in free space) is characterized by two wave families: fast pressure waves and slower shear waves. The aforementioned null condition limits the self-interaction of pressure waves. The equations possess an inherent null structure for shear waves. Thus, in the incompressible case where pressure waves are not present, it is plausible to expect global existence of small displacement solutions in 3-D without an additional null condition assumption, although the absence of finite propagation speed presents an obstacle. Nevertheless, this intuition was confirmed in [29, 31].

Our method for proving almost global existence for incompressible isotropic elastodynamics in 2-D is based on the ideas of [31], with three new ingredients: the treatment of the pressure term, the reliance on the structure of the nonlinear terms to obtain weighted L^2 estimates, and the use of the ghost weight in combination with the energy estimates. The equations are written as a first-order system with constraints whose unknowns are the material space-time gradient of the deformation, expressed in spatial coordinates (Section 2). The advantage of this choice is that the Lagrange multiplier which enforces the incompressibility constraint appears as a pressure gradient.

The generalized energy method forms the backbone of the argument. This enables control in $L^2(\mathbb{R}^2)$ of derivatives of the solution with respect to the vector fields which arise as infinitesimal generators corresponding to the fundamental invariance of the equations under translation, rotation, and scaling. This invariance gives rise to the basic commutation identities for the vector fields (Section 4). The rotational vector field yields $|x|^{-1/2}$ spatial decay by means of Sobolev-type inequalities (Section 3), which is weaker than the fully Lorentz invariant case. Using scaling invariance, the spatial decay is improved to $t^{-1/2}$ time decay in $L^\infty(\mathbb{R}^2)$ by means of a series of weighted estimates for the gradient of the solution which follow

from an algebraic manipulation of the equations and the constraints. This algebraic procedure (which is an implementation of the method of [30]) allows control of the solution gradient in $L^2(\mathbb{R}^2)$ with the weight $(t-|x|)^{-1}$, and it also gives t^{-1} decay in $L^2(\mathbb{R}^2)$ of certain special quantities (Sections 6, 8). For small energy solutions, the weighted estimates are closed by a bootstrapping argument (Section 7) which relies on the structure of the nonlinear terms (unlike 3-D). The pressure gradient is estimated as a solution of a nonlinear Poisson equation (Section 5). The nonlinearities of this elliptic equation have a null structure. Thus, the pressure term is essentially treated in a novel way as a nonlocal null form. This absence of finite propagation speed prevents us from achieving global existence because the Hardy-type inequality for functions of compact support used in [2] is not available. Energy estimates are performed using the ghost weight of [2] (Section 9) which provides a convenient solution to the technical problem that the weighted estimates hold only for the solution gradient.

Finally, the general isotropic case is easily treated by our method since it can be regarded as a higher order nonlinear correction to the Hookean case (Section 10).

Before ending the introduction, let us mention some related works on viscoelasticity, where there is viscosity in the momentum equation. In 2-D, the global well-posedness with near equilibrium states is due to [22] (see also [20]), and in 3-D to [3] and [19], independently. The initial boundary value problem is considered in [23]; the compressible case can be found in [7, 26]. For more results near equilibrium, readers are referred to [5, 24, 25, 35, 36]. In [16] a class of large solutions in two space dimensions is established via the strain-rotation decomposition (which is based on earlier results in [17] and [18]). In all of these works, the initial data is restricted by the viscosity parameter. The work [13] was the first to establish global existence for 3-D incompressible viscoelastic materials uniformly in the viscosity parameter.

2. Preliminaries and main results

Classically, the motion of an elastic body is described as a second-order evolution equation in Lagrangian coordinates. In the incompressible case, the equations are more conveniently written as a first-order system with constraints in Eulerian coordinates. We start with a time-dependent family of orientation-preserving diffeomorphisms $x(t,\cdot)$, $0 \le t < T$. Material points X in the reference configuration are deformed to the spatial positions $x(t,X)$ at time t. Derivatives with respect to the spatial coordinates will be written as $(\partial_t, \nabla) = \partial$. Let $X(t,x)$ be the corresponding reference map: $X(t,\cdot)$ is the inverse of $x(t,\cdot)$.

Lemma 2.1. *Given a family of deformations $x \in C^2([0,T)\times\mathbb{R}^2;\mathbb{R}^2)$, define the velocity and deformation gradient as follows:*

$$v(t,x) = \frac{dx(t,X)}{dt}\Big|_{X=X(t,x)}, \quad F(t,x) = \frac{\partial x(t,X)}{\partial X}\Big|_{X=X(t,x)}.$$

Then for $(t,x) \in [0,T)\times\mathbb{R}^2$, we have

$$\partial_t F + v\cdot\nabla F = \nabla v F, \tag{2.1}$$

$$F_{mj}\partial_m F_{ik} = F_{lk}\partial_l F_{ij}; \tag{2.2}$$

and if $x(t,X)$ is incompressible, that is, $\det F(t,x) \equiv 1$, then

$$\nabla\cdot v = \partial_i v_i = 0 \quad \text{and} \quad (\nabla\cdot F^{\top})_j = \partial_i F_{ij} = 0. \tag{2.3}$$

Proof. The equations (2.1), (2.2) express the commutativity of material derivates $D_t D_{X_j} = D_{X_j} D_t$, $D_{X_j} D_{X_k} = D_{X_k} D_{X_j}$ in spatial coordinates. The second statement follows from Nanson's formula. Details are given in [19]. □

Here and in what follows, we use the summation convention over repeated indices. The identities (2.2) and (2.3) will be used frequently in the sequel.

The next lemma shows that we can recover the family of deformations from the first order system (2.1), given consistent initial data for F.

Lemma 2.2. *Let $F_0 \in C^1(\mathbb{R}^2; \mathbb{M}^{2\times 2})$. Suppose that F_0 satisfies* (2.2), $\det F_0(x) > 0$, *and $\|F_0 - I\|_{C^1} \ll 1$. Assume that*

$$v \in C^1([0,T)\times\mathbb{R}^2;\mathbb{R}^2), \quad F \in C^1([0,T)\times\mathbb{R}^2;\mathbb{R}^2\otimes\mathbb{R}^2)$$

satisfy (2.1) *with $F(0,x) = F_0(x)$. Then there exists a family of deformations $x \in C^2([0,T)\times\mathbb{R}^2;\mathbb{R}^2)$ such that*

$$F(t,x(t,X)) = \frac{\partial x(t,X)}{\partial X}.$$

If, in addition, $\det F_0(x) = 1$ *and* $\nabla\cdot v = 0$, *then $x(t,\cdot)$ is an incompressible family.*

Proof. Since $F_0(x)$ is invertible, we may define $H(x) = F_0(x)^{-1}$. We claim that H is the gradient of a deformation. By assumption, F_0 satisfies (2.2). Multiply this by $H_{ai}H_{jb}H_{kc}$, sum over repeated indices, and use the fact that $H\partial_m F_0 + \partial_m H F_0 = 0$, for any derivative. This leads to the equations $\partial_k H_{ij} = \partial_j H_{ik}$. So if

$$\varphi(x) = \int_0^1 H(sx)x\, ds,$$

then $\varphi \in C^2(\mathbb{R}^2;\mathbb{R}^2)$ and $\partial_j\varphi_i = H_{ij}$. Since $\|F_0 - I\|_{C^1} \ll 1$, we have that $\|\partial\varphi - I\|_{C^1} = \|H - I\|_{C^1} \ll 1$. It follows that φ is a deformation.

Let $x(t,X)$ be the flow of the vector field $v(t,x)$ with initial data $x(0,X) = \varphi^{-1}(X)$. Then $x(t,X)$ is a family of deformations and by (2.1) we see that

$$\frac{d}{dt}[Dx(t,X) - F(t,x(t,X))] = \nabla v(t,x(t,X))[Dx(t,X) - F(t,x(t,X))]$$

and

$$\begin{aligned}[Dx(t,X) - F(t,x(t,X))]\Big|_{t=0} &= D\varphi^{-1}(X) - F_0(\varphi^{-1}(X)) \\ &= D\varphi^{-1}(X) - (\partial\varphi)^{-1}(\varphi^{-1}(X)) = 0.\end{aligned}$$

The first statement follows by uniqueness.

Setting $J(t,X) = \det F(t,x(t,X))$, we have that

$$D_t J(t,X) = \nabla\cdot v(t,x(t,X))J(t,X).$$

So if $\nabla\cdot v = 0$ and $J(0,X) = 1$, we get that $J(t,X) = 1$; i.e. $x(t,X)$ is an incompressible family. □

To best illustrate our methods and ideas, we shall first consider the equations of motion for incompressible Hookean elasticity, which corresponds to the Hookean strain energy function $W(F) = \frac{1}{2}|F|^2$:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \nabla\cdot(FF^\top), \\ \partial_t F + v\cdot\nabla F = \nabla v F, \\ \nabla\cdot v = 0, \quad \nabla\cdot F^\top = 0. \end{cases} \tag{2.4}$$

As will be seen in Section 10, where the case of general energy function is discussed, there is no essential loss of generality in considering this simplest case. Since the 3-D case has been treated in [29,31], below we will focus on 2-D.

Denote the rotation operator by

$$\Omega = x_2\partial_1 - x_1\partial_2 = x^\perp \cdot \nabla$$

and the scaling operator by

$$S = t\partial_t + x_1\partial_1 + x_2\partial_2 = t\partial_t + r\partial_r, \quad S_0 = r\partial_r.$$

We shall frequently use the decomposition

$$\nabla = (x/r)\partial_r + (x^\perp/r^2)\Omega. \tag{2.5}$$

As in previous works, we define

$$\begin{cases} \widetilde{\Omega} f = \Omega f, \quad \forall \text{ scalar } f, \\ \widetilde{\Omega} v = \Omega v + (e_2 \otimes e_1 - e_1 \otimes e_2)v, \quad \forall\, v \in \mathbb{R}^2, \\ \widetilde{\Omega} G = \Omega G + [(e_2 \otimes e_1 - e_1 \otimes e_2), G], \quad \forall\, G \in \mathbb{R}^2 \otimes \mathbb{R}^2, \end{cases}$$

where $[A, B] = AB - BA$ denotes the standard Lie bracket product.

Let Γ be any of the following differential operators:

$$\Gamma \in \{\partial_t, \partial_1, \partial_2, \widetilde{\Omega}, S\}. \tag{2.6}$$

Define the generalized energy by

$$E_k(t) = \sum_{|\alpha|\le k} \big(\|\Gamma^\alpha v(t,\cdot)\|_{L^2}^2 + \|\Gamma^\alpha G(t,\cdot)\|_{L^2}^2\big). \tag{2.7}$$

We also define the weighted energy norm

$$X_k(t) = \sum_{|\alpha|\le k-1} \big(\|\langle t-r\rangle \nabla\Gamma^\alpha v\|_{L^2} + \|\langle t-r\rangle \nabla\Gamma^\alpha G\|_{L^2}\big), \tag{2.8}$$

in which we denote $\langle\sigma\rangle = \sqrt{1+\sigma^2}$.

To describe the space of the initial data, we introduce

$$\Lambda = \{\nabla, S_0, \Omega\}$$

and

$$H^k_\Lambda = \{f : \sum_{|\alpha|\le k} \|\Lambda^\alpha f\|_{L^2} < \infty\}.$$

Then we define

$$H^k_\Lambda(T) = \{(v,G) : [0,T) \to \mathbb{R}^2 \times (\mathbb{R}^2 \otimes \mathbb{R}^2) : (v,G) \in \bigcap_{j=0}^{k} C^j([0,T); H^{k-j}_\Lambda\}$$

with the norm

$$\sup_{0\le t<T} E_k(t)^{1/2}.$$

The main result of this paper is

Theorem 2.3. *Let $(v_0, G_0) \in H^k_\Lambda$, with $k \geq 5$. Suppose that $(v_0, F_0) = (v_0, I + G_0)$ satisfy the constraints (2.2), (2.3) and $\|(v_0, G_0)\|_{H^k_\Lambda} < \epsilon$.*

There exist two positive constants C_0 and ϵ_0 which depend only on k such that if $\epsilon \leq \epsilon_0$, then the system of incompressible Hookean elastodynamics (2.4) with initial data $(v_0, F_0) = (v_0, I + G_0)$ has a unique solution $(v, F) = (v, I + G)$, with $(v, G) \in H^k_\Lambda(T)$, $T \geq \exp(C_0/\epsilon)$ and $E_k(t) \leq (10\epsilon)^2$, for $0 \leq t < T$.

3. L^∞ ESTIMATES

In this section we derive several weighted $L^\infty - L^2$ Sobolev imbedding inequalities. These will be useful in proving decay of solutions in L^∞.

Lemma 3.1. *For all radial functions $f \in H^1(\mathbb{R}^2)$, $\lambda = 1, 2$, there hold*

$$r^\lambda |f(r)|^2 \lesssim \|r^{\lambda-1}\partial_r f\|^2_{L^2} + \|f\|^2_{L^2}, \tag{3.1}$$

$$r\langle t-r\rangle^\lambda |f(r)|^2 \lesssim \|\langle t-r\rangle \partial_r f\|^2_{L^2} + \|\langle t-r\rangle^{\lambda-1} f\|^2_{L^2}, \tag{3.2}$$

provided that the right-hand side is finite.

Proof. It suffices to show the lemma for radial $f \in C^1_0(\mathbb{R}^2)$ and then use a completion argument.

The first inequality is shown as follows:

$$\begin{aligned} r^\lambda |f(r)|^2 &= -r^\lambda \int_r^\infty \partial_\rho [|f(\rho)|^2] d\rho \\ &= -r^\lambda \int_r^\infty 2 f(\rho) \partial_r f(\rho) d\rho \\ &\lesssim \int_r^\infty \rho^{\lambda-1} |f(\rho)| |\partial_r f(\rho)| \rho d\rho \\ &\lesssim \|r^{\lambda-1}\partial_r f\|_{L^2} \|f\|_{L^2}. \end{aligned}$$

The calculation for the other inequality is similar:

$$\begin{aligned} r\langle t-r\rangle^\lambda |f(r)|^2 &= -r \int_r^\infty \partial_\rho [\langle t-\rho\rangle^\lambda |f(\rho)|^2] d\rho \\ &\lesssim r \int_r^\infty [\langle t-\rho\rangle^\lambda |f(\rho)| |\partial_r f(\rho)| + \langle t-\rho\rangle^{\lambda-1} |f(\rho)|^2] d\rho \\ &\lesssim \int_r^\infty [\langle t-\rho\rangle^2 |\partial_r f(\rho)|^2 + \langle t-\rho\rangle^{2(\lambda-1)} |f(\rho)|^2] \rho d\rho \\ &= \|\langle t-r\rangle \partial_r f\|^2_{L^2} + \|\langle t-r\rangle^{\lambda-1} f\|^2_{L^2}. \end{aligned}$$

□

The next lemma is just the Sobolev imbedding theorem $H^1 \hookrightarrow L^\infty$ on the circle $\mathbb{S}^1$.

Lemma 3.2. *For each $x = r\omega \in \mathbb{R}^2$ and $f(r\omega) \in H^1(\mathbb{S}^1)$, there holds*

$$|f(r\omega)|^2 \lesssim \sum_{a=0,1} \int_{\mathbb{S}^1} |\Omega^a f(r\omega)|^2 d\sigma.$$

Proof. Write $\omega = (\cos\theta, \sin\theta)$. We have

$$\begin{aligned}|f(r\omega)|^2 &= |f(r\omega)|^2\cos^2\theta + |f(r\omega)|^2\sin^2\theta\\ &= \int_{\frac{\pi}{2}}^{\theta}\frac{d}{d\phi}\big[|f(r\cos\phi, r\sin\phi)|^2\cos^2\phi\big]d\phi\\ &\quad + \int_0^{\theta}\frac{d}{d\phi}\big[|f(r\cos\phi, r\sin\phi)|^2\sin^2\phi\big]d\phi\\ &\lesssim \int_{\mathbb{S}^1}[|\Omega f(r\omega)||f(r\omega)| + |f(r\omega)|^2]d\sigma,\end{aligned}$$

from which the result follows by the Cauchy-Schwarz inequality. □

These two results combine to yield

Lemma 3.3. *For all $f \in H^2(\mathbb{R}^2)$, $\lambda = 1, 2$, there hold*

$$r^{\lambda}|f(x)|^2 \lesssim \sum_{a=0,1}\|r^{\lambda-1}\partial_r\Omega^a f\|_{L^2}^2 + \|\Omega^a f\|_{L^2}^2, \tag{3.3}$$

$$r\langle t-r\rangle^{\lambda}|f(x)|^2 \lesssim \sum_{a=0,1}[\|\langle t-r\rangle\partial_r\Omega^a f\|_{L^2}^2 + \|\langle t-r\rangle^{\lambda-1}\Omega^a f\|_{L^2}^2], \tag{3.4}$$

provided the right-hand side is finite.

Proof. First apply Lemma 3.2 to the left-hand side, then apply Lemma 3.1 to the integrand. □

Remark. The estimates of Lemma 3.3 hold in the vector- and matrix-valued cases using $\widetilde{\Omega}$ in place of Ω. This can be seen by applying Lemma 3.3 component-wise and then using the fact that, for example, $|\Omega v| \le |\widetilde{\Omega} v| + |v|$, for vectors v.

Remark. The case $\lambda = 2$ in (3.4) will be applied only to derivatives ∇f.

Lemma 3.4. *For all $f \in H^2(\mathbb{R}^2)$, there holds*

$$\langle t\rangle\|f\|_{L^\infty(r\le\langle t/2\rangle)} \lesssim \sum_{|\alpha|\le 2}\|\langle t-r\rangle\partial^\alpha f\|_{L^2},$$

provided the right-hand side is finite.

Proof. Let $\varphi \in C_0^\infty$ satisfy $\varphi(s) = 1$ for $s \le 1$, $\varphi(s) = 0$ for $s \ge 3/2$. Note that $\langle t\rangle \lesssim \langle t-r\rangle$ on supp $\varphi(r/\langle t/2\rangle)$. Thus, for $|x| \le \langle t/2\rangle$, we have by Sobolev imbedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ that

$$\begin{aligned}\langle t\rangle|f(x)| &= \langle t\rangle\varphi(r/\langle t/2\rangle)|f(x)|\\ &\lesssim \langle t\rangle\|\varphi(r/\langle t/2\rangle)f\|_{H^2}\\ &\lesssim \langle t\rangle\sum_{|\alpha|\le 2}\|\partial^\alpha f\|_{L^2(r\le\frac{3}{2}\langle t/2\rangle)}\\ &\lesssim \sum_{|\alpha|\le 2}\|\langle t-r\rangle\partial^\alpha f\|_{L^2}.\end{aligned}$$

□

Remark. This result will only be applied to derivatives ∇f.

4. Commutation

Write $F = I + G$. For any multi-index α, we have the following commutation properties when applying Γ^α to the equation (2.4) (see, for instance, [13, 31] for details). This will be essential in all of the subsequent estimations:

$$\begin{cases} \partial_t \Gamma^\alpha v - \nabla \cdot \Gamma^\alpha G = -\nabla \Gamma^\alpha p + f_\alpha, \\ \partial_t \Gamma^\alpha G - \nabla \Gamma^\alpha v = g_\alpha, \\ \nabla \cdot \Gamma^\alpha v = 0, \quad \nabla \cdot \Gamma^\alpha G^\top = 0, \end{cases} \tag{4.1}$$

where

$$\begin{cases} f_\alpha = \sum\limits_{\beta+\gamma=\alpha} [-\Gamma^\beta v \cdot \nabla \Gamma^\gamma v + \nabla \cdot (\Gamma^\gamma G \Gamma^\beta G^\top)], \\ g_\alpha = \sum\limits_{\beta+\gamma=\alpha} [-\Gamma^\beta v \cdot \nabla \Gamma^\gamma G + \nabla \Gamma^\gamma v \Gamma^\beta G]. \end{cases} \tag{4.2}$$

From (2.2), we also have

$$\nabla^\perp \cdot \Gamma^\alpha G = h_\alpha, \tag{4.3}$$

where

$$(h_\alpha)_i = \sum_{\beta+\gamma=\alpha} \big[\Gamma^\beta G_{m1} \partial_m \Gamma^\gamma G_{i2} - \Gamma^\beta G_{m2} \partial_m \Gamma^\gamma G_{i1}\big]. \tag{4.4}$$

5. Bound for the pressure gradient

The following lemma shows that the pressure gradient may be treated as a nonlinear term. The first estimate appeared in [31]. The second is a novel refinement which saves one derivative over the first bound and which allows us to exploit the null structure. This is essential in Section 9 when we estimate the ghost weighted energy.

Lemma 5.1. *Let $(v, F) = (v, I + G)$, $(v, G) \in H^k_\Lambda(T)$, solve the equation (2.4) and the constraint (2.2). Then we have*

$$\|\nabla \Gamma^\alpha p\|_{L^2} \lesssim \|f_\alpha\|_{L^2}, \tag{5.1}$$

$$\|\nabla \Gamma^\alpha p\|_{L^2} \lesssim \sum_i \sum_{\substack{\beta+\gamma=\alpha \\ |\beta| \le |\gamma|}} \|\partial_j \Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j \Gamma^\beta G_{ik} \Gamma^\gamma G_{jk}\|_{L^2}, \tag{5.2}$$

for all $|\alpha| \le k - 1$.

Proof. Taking the divergence of the first equation of (4.1) and then using the constraints given in the third equation of (4.1), we find

$$\Delta \Gamma^\alpha p = \nabla \cdot f_\alpha + \nabla \cdot (\nabla \cdot \Gamma^\alpha G) - \partial_t \nabla \cdot \Gamma^\alpha v = \nabla \cdot f_\alpha.$$

By (4.2) and the constraint equations in (4.1), we have

$$\begin{aligned}\nabla \cdot f_\alpha &= -\sum_{\beta+\gamma=\alpha} \partial_i\partial_j(\Gamma^\beta v_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik}\Gamma^\gamma G_{jk}) \\ &= -\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\le|\gamma|}} \partial_i\partial_j(\Gamma^\beta v_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik}\Gamma^\gamma G_{jk}) \\ &\quad -\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|>|\gamma|}} \partial_i\partial_j(\Gamma^\beta v_i \Gamma^\gamma v_j - \Gamma^\beta G_{ik}\Gamma^\gamma G_{jk}) \\ &= -\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\le|\gamma|}} \partial_i(\partial_j\Gamma^\beta v_i \Gamma^\gamma v_j - \partial_j\Gamma^\beta G_{ik}\Gamma^\gamma G_{jk}) \\ &\quad -\sum_{\substack{\beta+\gamma=\alpha \\ |\beta|>|\gamma|}} \partial_j(\Gamma^\beta v_i \partial_i\Gamma^\gamma v_j - \Gamma^\beta G_{ik}\partial_i\Gamma^\gamma G_{jk}).\end{aligned}$$

The result now follows since

$$\nabla\Gamma^\alpha p = \Delta^{-1}\nabla(\nabla \cdot f_\alpha)$$

and since $\Delta^{-1}\nabla\otimes\nabla$ is bounded in L^2. □

6. Estimates for special quantities, I

Lemma 6.1. *Suppose that* $(v,F)=(v,I+G)$, $(v,G)\in H^k_\Lambda(T)$, *solves* (2.4), (2.2). *Define*

$$L_k = \sum_{|\alpha|\le k} [|\Gamma^\alpha v| + |\Gamma^\alpha G|]$$

and

$$N_k = \sum_{|\alpha|\le k-1} [t|f_\alpha| + t|g_\alpha| + (t+r)|h_\alpha| + t|\nabla\Gamma^\alpha p|]. \tag{6.1}$$

Then for all $|\alpha|\le k-1$,

$$r|\partial_r\Gamma^\alpha v\cdot\omega| \lesssim L_k \tag{6.2}$$

$$r|\partial_r\Gamma^\alpha G^\top\omega| \lesssim L_k \tag{6.3}$$

$$r|\partial_r\Gamma^\alpha G\omega - \nabla\cdot\Gamma^\alpha G| \lesssim L_k \tag{6.4}$$

$$r|\partial_r\Gamma^\alpha G\omega^\perp| \lesssim L_k + N_k \tag{6.5}$$

$$(t\pm r)|\nabla\Gamma^\alpha v \pm \nabla\cdot\Gamma^\alpha G\otimes\omega| \lesssim L_k + N_k. \tag{6.6}$$

Proof. By the decomposition (2.5) we have

$$\nabla\Gamma^\alpha v = \partial_r\Gamma^\alpha v\otimes\omega + \frac{1}{r}\Omega\Gamma^\alpha v\otimes\omega^\perp. \tag{6.7}$$

Taking the trace of this identity yields

$$\nabla\cdot\Gamma^\alpha v = \partial_r\Gamma^\alpha v\cdot\omega + \frac{1}{r}\Omega\Gamma^\alpha v\cdot\omega^\perp,$$

and so, by the divergence-free velocity constraint of (4.1), we obtain (6.2). It also follows from (6.7) that

$$r|\nabla\Gamma^\alpha v - \partial_r\Gamma^\alpha v\otimes\omega| \lesssim L_k, \tag{6.8}$$

which will be used shortly in proving (6.6).

Again by (2.5), we may write for any matrix-valued function H:

$$\begin{aligned}\partial_r H &= \partial_r H\, I\\ &= \partial_r H[\omega\otimes\omega+\omega^\perp\otimes\omega^\perp]\\ &= \partial_r H\omega\otimes\omega + \partial_r H\omega^\perp\otimes\omega^\perp\\ &= [\nabla\cdot H - \frac{1}{r}\Omega H\omega^\perp]\otimes\omega + [\nabla^\perp\cdot H + \frac{1}{r}\Omega H\omega]\otimes\omega^\perp.\end{aligned} \tag{6.9}$$

Multiplying both sides of (6.9) times the vector ω, we obtain

$$\partial_r H\omega = \nabla\cdot H - \frac{1}{r}\Omega H\omega^\perp.$$

Apply this to $H=\Gamma^\alpha G^\top$, and use the other divergence-free constraint from (4.1). We thereby obtain

$$r|\partial_r\Gamma^\alpha G^\top\omega| = |\Omega\Gamma^\alpha G^\top\omega^\perp| \lesssim L_k, \tag{6.10}$$

which is (6.3).

Next, apply (6.9) to $H=\Gamma^\alpha G$ and use the constraint (4.3):

$$r\nabla\cdot\Gamma^\alpha G\otimes\omega - r\partial_r\Gamma^\alpha G = \Omega\Gamma^\alpha G\omega^\perp\otimes\omega - [rh_\alpha+\Omega\Gamma^\alpha G\omega]\otimes\omega^\perp.$$

From this it follows that

$$r|\nabla\cdot\Gamma^\alpha G - \partial_r\Gamma^\alpha G\omega| = r|[\nabla\cdot\Gamma^\alpha G\otimes\omega - \partial_r\Gamma^\alpha G]\omega| = |\Omega\Gamma^\alpha G\omega^\perp| \lesssim L_k,$$

proving (6.4), and also

$$r|\nabla\cdot\Gamma^\alpha G\otimes\omega - \partial_r\Gamma^\alpha G| \lesssim L_k + N_k. \tag{6.11}$$

As an immediate consequence of this last inequality, we get

$$r|\partial_r\Gamma^\alpha G\omega^\perp| = r|[\nabla\cdot\Gamma^\alpha G\otimes\omega - \partial_r\Gamma^\alpha G]\omega^\perp| \lesssim L_k + N_k,$$

which proves (6.5). We are now ready to prove (6.6).

Using the PDEs (4.1) and the definition $S=t\partial_t+r\partial_r$, we can write

$$\begin{aligned}&t\nabla\Gamma^\alpha v + r\partial_r\Gamma^\alpha G = S\Gamma^\alpha G - tg_\alpha,\\ &t\nabla\cdot\Gamma^\alpha G + r\partial_r\Gamma^\alpha v = S\Gamma^\alpha v - tf_\alpha + t\nabla\Gamma^\alpha p.\end{aligned}$$

This is rearranged as follows:

$$\begin{aligned}&t\nabla\Gamma^\alpha v + r\nabla\cdot\Gamma^\alpha G\otimes\omega = r[\nabla\cdot\Gamma^\alpha G\otimes\omega - \partial_r\Gamma^\alpha G] + S\Gamma^\alpha G - tg_\alpha,\\ &t\nabla\cdot\Gamma^\alpha G\otimes\omega + r\nabla\Gamma^\alpha v = r[\nabla\Gamma^\alpha v - \partial_r\Gamma^\alpha v\otimes\omega] + [S\Gamma^\alpha v - tf_\alpha + t\nabla\Gamma^\alpha p]\otimes\omega.\end{aligned}$$

Notice that by (6.8) and (6.11), the right-hand sides of these identities are bounded by L_k+N_k. Therefore, the bounds (6.6) result from the combination of these two identities. □

Lemma 6.2. *Let $(v,G)\in H^k_\Lambda(T)$. Then for $|\alpha|\le k-2$, we have*

$$\|r\Gamma^\alpha v\cdot\omega\|_{L^\infty} + \|r\Gamma^\alpha G^\top\omega\|_{L^\infty} \lesssim E^{1/2}_{|\alpha|+2}.$$

Proof. We shall make use of the fact that

$$\Omega(v(x)\cdot\omega) = (\widetilde{\Omega}v(x))\cdot\omega, \quad \widetilde{\Omega}(G(x)^\top\omega) = (\widetilde{\Omega}G(x))^\top\omega, \quad \omega = x/r, \tag{6.12}$$

by the scalar, vectorial, and matricial definitions of $\widetilde{\Omega}$ in Section 2.

By (3.3) with $\lambda = 2$ and (6.12), we have

$$\begin{aligned}
&\|r\Gamma^\alpha v\cdot\omega\|_{L^\infty} + \|r\Gamma^\alpha G^\top\omega\|_{L^\infty}\\
&\qquad\lesssim \sum_{a=0,1}[\|r\partial_r\Omega^a(\Gamma^\alpha v\cdot\omega)\|_{L^2} + \|r\partial_r\widetilde{\Omega}^a(\Gamma^\alpha G^\top\omega)\|_{L^2}\\
&\qquad\qquad + \|\Omega^a(\Gamma^\alpha v\cdot\omega)\|_{L^2} + \|\widetilde{\Omega}^a(\Gamma^\alpha G^\top\omega)\|_{L^2}]\\
&\qquad= \sum_{a=0,1}[\|r(\partial_r\widetilde{\Omega}^a\Gamma^\alpha v)\cdot\omega\|_{L^2} + \|r\partial_r(\widetilde{\Omega}^a\Gamma^\alpha G^\top)\omega\|_{L^2}\\
&\qquad\qquad + \|(\widetilde{\Omega}^a\Gamma^\alpha v)\cdot\omega\|_{L^2} + \|(\widetilde{\Omega}^a\Gamma^\alpha G^\top)\omega\|_{L^2}].
\end{aligned}$$

The result now follows by (6.2), (6.3). □

7. Weighted L^2 estimate

In this section, we show that the weighted norm is controlled by the energy for small solutions.

Lemma 7.1. *Suppose that* $(v,F) = (v, I+G)$, $(v,G)\in H^k_\Lambda(T)$, $k\geq 4$, *solves* (2.4), (2.2). *Then*

$$\|N_k(t)\|_{L^2} \lesssim E_k(t) + E_k(t)^{1/2}X_k(t).$$

Proof. By (6.1) and Lemma 5.1, we have that

$$\begin{aligned}
\|N_k(t)\|_{L^2} &\leq \sum_{|\alpha|\,\leq\, k-1}[t\|\nabla\Gamma^\alpha p(t)\|_{L^2} + t\|f_\alpha(t)\|_{L^2} + t\|g_\alpha(t)\|_{L^2} + \|(t+r)h_\alpha(t)\|_{L^2}]\\
&\lesssim \sum_{|\alpha|\,\leq\, k-1}[t\|f_\alpha(t)\|_{L^2} + t\|g_\alpha(t)\|_{L^2} + \|(t+r)h_\alpha(t)\|_{L^2}].
\end{aligned}$$

In estimating these terms, we shall consider two regions: $r\leq\langle t/2\rangle$ and $r\geq\langle t/2\rangle$.

Estimates of nonlinearities for $r\leq\langle t/2\rangle$. Examining definitions (4.2), (4.4), we find that

$$\begin{aligned}
&\sum_{|\alpha|\,\leq\, k-1}[t\|f_\alpha\|_{L^2(r\leq\langle t/2\rangle)} + t\|g_\alpha\|_{L^2(r\leq\langle t/2\rangle)} + \|(t+r)h_\alpha\|_{L^2(r\leq\langle t/2\rangle)}]\\
&\qquad\lesssim \sum_{\substack{\beta+\gamma=\alpha\\ |\alpha|\,\leq\, k-1}}\langle t\rangle\|(|\Gamma^\beta v| + |\Gamma^\beta G|)(|\nabla\Gamma^\gamma v| + |\nabla\Gamma^\gamma G|)\|_{L^2(r\leq\langle t/2\rangle)}.
\end{aligned}$$

To simplify the notation a bit, we shall write

$$|(\Gamma^k v, \Gamma^k G)| = \sum_{|\alpha|\leq k}[|\Gamma^\alpha v| + |\Gamma^\alpha G|]$$

(and similar). We make use of the fact that since $\beta+\gamma=\alpha$, $|\alpha|\le k-1$, and $k\ge 4$, either $|\beta|+2\le k$ or $|\gamma|+3\le k$. Thus, we have by Lemma 3.4:

$$\begin{aligned}
&\langle t\rangle\|(|\Gamma^\beta v|+|\Gamma^\beta G|)(|\nabla\Gamma^\gamma v|+|\nabla\Gamma^\gamma G|)\|_{L^2(r\le\langle t/2\rangle)}\\
&\quad\lesssim\langle t\rangle\|(\Gamma^{k-2}v,\Gamma^{k-2}G)\|_{L^\infty}\|(\nabla\Gamma^{k-1}v,\nabla\Gamma^{k-1}G)\|_{L^2(r\le\langle t/2\rangle)}\\
&\qquad+\langle t\rangle\|(\Gamma^{k}v,\Gamma^{k}G)\|_{L^2}\|(\nabla\Gamma^{k-3}v,\nabla\Gamma^{k-3}G)\|_{L^\infty(r\le\langle t/2\rangle)}\\
&\quad\lesssim\|(\Gamma^{k-2}v,\Gamma^{k-2}G)\|_{L^\infty}\|\langle t-r\rangle(\nabla\Gamma^{k-1}v,\nabla\Gamma^{k-1}G)\|_{L^2(r\le\langle t/2\rangle)}\\
&\qquad+\|(\Gamma^{k}v,\Gamma^{k}G)\|_{L^2}\sum_{|\lambda|\le 2}\|\langle t-r\rangle\partial^\lambda(\nabla\Gamma^{k-3}v,\nabla\Gamma^{k-3}G)\|_{L^2}\\
&\quad\lesssim E_k(t)^{1/2}X_k(t).
\end{aligned}$$

Estimates of nonlinearities for $r\ge\langle t/2\rangle$. Using (2.5), we replace all derivatives which occur in (4.2), (4.4) by their radial and angular parts. We find that

$$\begin{aligned}
|f_\alpha|+|g_\alpha|+|h_\alpha|\lesssim&\sum_{\substack{\beta+\gamma=\alpha\\|\alpha|\le k-1}}(|\Gamma^\beta v\cdot\omega|+|\Gamma^\beta G^\top\omega|)(|\partial_r\Gamma^\gamma v|+|\partial_r\Gamma^\gamma G|)\\
&+\sum_{\substack{\beta+\gamma=\alpha\\|\alpha|\le k-1}}(|\Gamma^\beta v|+|\Gamma^\beta G|)\frac{1}{r}(|\Omega\Gamma^\gamma v|+|\Omega\Gamma^\gamma G|).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\sum_{|\alpha|\le k-1}[t\|f_\alpha\|_{L^2(r\ge\langle t/2\rangle)}+t\|g_\alpha\|_{L^2(r\ge\langle t/2\rangle)}+\|(t+r)h_\alpha\|_{L^2(r\ge\langle t/2\rangle)}]\\
&\lesssim\sum_{\substack{\beta+\gamma=\alpha\\|\alpha|\le k-1}}\|r(|\Gamma^\beta v\cdot\omega|+|\Gamma^\beta G^\top\omega|)(|\partial_r\Gamma^\gamma v|+|\partial_r\Gamma^\gamma G|)\|_{L^2(r\ge\langle t/2\rangle)}\\
&\qquad+\sum_{\substack{\beta+\gamma=\alpha\\|\alpha|\le k-1}}\|(|\Gamma^\beta v|+|\Gamma^\beta G|)(|\Omega\Gamma^\gamma v|+|\Omega\Gamma^\gamma G|)\|_{L^2(r\ge\langle t/2\rangle)}.
\end{aligned}\tag{7.1}$$

We claim that all terms on the right-hand side of (7.1) can be bounded by E_k.

Consider the term in the first sum which has $\beta=\alpha$, $\gamma=0$. It can be estimated as follows:

$$\begin{aligned}
&\|r(|\Gamma^\alpha v\cdot\omega|+|\Gamma^\alpha G^\top\omega|)(|\partial_r v|+|\partial_r G|)\|_{L^2(r\ge\langle t/2\rangle)}\\
&\lesssim\left(\sup_{r\ge\langle t/2\rangle}r^2\int_{\mathbb{S}^1}(|\Gamma^\alpha v(r\omega)\cdot\omega|^2+|\Gamma^\alpha G^\top(r\omega)\omega|^2)d\sigma\right)^{1/2}\\
&\qquad\times\left(\int_{\langle t/2\rangle}^\infty\sup_{\mathbb{S}^1}(|\partial_r v(r\omega)|^2+|\partial_r G(r\omega)|^2)rdr\right)^{1/2}.
\end{aligned}\tag{7.2}$$

Using (3.1) with $\lambda=2$ from Lemma 3.1, the first of these integrals is bounded by

$$\|r\partial_r\Gamma^\alpha v\cdot\omega\|_{L^2}+\|r\partial_r\Gamma^\alpha G^\top\omega\|_{L^2}+\|\Gamma^\alpha v\|_{L^2}+\|\Gamma^\alpha G\|_{L^2}.$$

Noticing the form of the first two terms, we can use (6.2), (6.3) to bound this by $E_{|\alpha|+1}^{1/2}\le E_k^{1/2}$.

The second integral in (7.2) is bounded as follows using Lemma 3.2:

$$\sum_{a=0,1} (\|\Omega^a \partial_r v\|_{L^2} + \|\Omega^a \partial_r G\|_{L^2}) \lesssim E_2^{1/2}.$$

This proves that the term in (7.2) is bounded by E_k, as claimed.

For the remaining terms in the first sum in (7.1), we have $\beta \neq \alpha$. We estimate these as follows:

$$\begin{aligned}(7.3)\quad &\|r(|\Gamma^\beta v \cdot \omega| + |\Gamma^\beta G^\top \omega|)(|\partial_r \Gamma^\gamma v| + |\partial_r \Gamma^\gamma G|)\|_{L^2(r \geq \langle t/2 \rangle)} \\ &\qquad \lesssim (\|r\Gamma^\beta v \cdot \omega\|_{L^\infty} + \|r\Gamma^\beta G^\top \omega\|_{L^\infty})(\|\partial_r \Gamma^\gamma v\|_{L^2} + \|\partial_r \Gamma^\gamma G\|_{L^2}).\end{aligned}$$

The terms in L^∞ are estimated by $E_{|\beta|+2}^{1/2}$, using Lemma 6.2, and thus, since $|\beta| \leq |\alpha| - 1 \leq k - 2$, the expression (7.3) is bounded by

$$E_{|\beta|+2}^{1/2} E_{|\gamma|+1}^{1/2} \lesssim E_k.$$

Altogether, the first sum of terms in (7.1) is bounded by E_k.

The second group of terms on the right of (7.1) is also bounded by E_k, using the same strategy as in the case $r \leq \langle t/2 \rangle$. □

Lemma 7.2. *Suppose that* $(v, G) \in H^k_\Lambda(T)$ *and* $(v, F) = (v, I + G)$ *solves* (2.4), (2.2). *Then for* $|\alpha| \leq k - 1$, *there holds*

$$\|(t - r)\nabla \Gamma^\alpha G\|_{L^2} \lesssim E_k(t)^{1/2} + \|(t - r)\nabla \cdot \Gamma^\alpha G\|_{L^2} + \|(t + r)h_\alpha\|_{L^2}.$$

Proof. For any $\mathbb{R}^2 \otimes \mathbb{R}^2$-valued function H, we have that

$$|\nabla H|^2 - [|\nabla \cdot H|^2 + |\nabla^\perp \cdot H|^2] = -2[\partial_1(H_{i1}\partial_2 H_{i2}) - \partial_2(H_{i1}\partial_1 H_{i2})].$$

Thus, using integration by parts and Young's inequality, we get

$$\begin{aligned}\|(t - r)\nabla H\|_{L^2}^2 &- \left[\|(t - r)\nabla \cdot H\|_{L^2}^2 + \|(t - r)\nabla^\perp \cdot H\|_{L^2}^2\right] \\ &= -\int 2(t - r)^2[\partial_1(H_{i1}\partial_2 H_{i2}) - \partial_2(H_{i1}\partial_1 H_{i2})]dx \\ &= -\int 4(t - r)[\omega_1(H_{i1}\partial_2 H_{i2}) - \omega_2(H_{i1}\partial_1 H_{i2})]dx \\ &\leq \frac{1}{2}\|(t - r)\nabla H\|_{L^2}^2 + C\|H\|_{L^2}^2,\end{aligned}$$

and so we obtain

$$\|(t - r)\nabla H\|_{L^2}^2 \lesssim \|(t - r)\nabla \cdot H\|_{L^2}^2 + \|(t - r)\nabla^\perp \cdot H\|_{L^2}^2 + \|H\|_{L^2}^2.$$

The lemma follows by applying this inequality to $H = \Gamma^\alpha G$ and then using (4.3). □

Theorem 7.3. *Suppose that* $(v, F) = (v, I + G)$, $(v, G) \in H^k_\Lambda(T)$, $k \geq 4$, *solves* (2.4), (2.2). *If* $E_k(t) \ll 1$, *then* $X_k(t) \lesssim E_k(t)^{1/2}$.

Proof. Starting with definition (2.8) and using the fact that $\langle t-r\rangle \le 1+|t-r|$, we obtain from Lemma 7.2:

$$\begin{aligned} X_k^2 &\lesssim \sum_{|\alpha|\le k-1} [\|\langle t-r\rangle \nabla\Gamma^\alpha v\|_{L^2}^2 + \|\langle t-r\rangle \nabla\Gamma^\alpha G\|_{L^2}^2] \\ &\lesssim E_k + \sum_{|\alpha|\le k-1} [\|(t-r)\nabla\Gamma^\alpha v\|_{L^2}^2 + \|(t-r)\nabla\Gamma^\alpha G\|_{L^2}^2] \\ &\lesssim E_k + \sum_{|\alpha|\le k-1} [\|(t-r)\nabla\Gamma^\alpha v\|_{L^2}^2 + \|(t-r)\nabla\cdot\Gamma^\alpha G\|_{L^2}^2 + \|N_k\|_{L^2}^2]. \end{aligned}$$

Since

$$\nabla\Gamma^\alpha v = \frac{1}{2}[\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega] + \frac{1}{2}[\nabla\Gamma^\alpha v - \nabla\cdot\Gamma^\alpha G\otimes\omega]$$

and

$$\nabla\cdot\Gamma^\alpha G = \frac{1}{2}[\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega]\omega - \frac{1}{2}[\nabla\Gamma^\alpha v - \nabla\cdot\Gamma^\alpha G\otimes\omega]\omega,$$

we see that

$$\begin{aligned} &|t-r|\,[|\nabla\Gamma^\alpha v| + |\nabla\cdot\Gamma^\alpha G|] \\ &\qquad\lesssim |t+r|\,|\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega| + |t-r|\,|\nabla\Gamma^\alpha v - \nabla\cdot\Gamma^\alpha G\otimes\omega|. \end{aligned}$$

It follows from (6.6) that

$$\|(t-r)\nabla\Gamma^\alpha v\|_{L^2}^2 + \|(t-r)\nabla\cdot\Gamma^\alpha G\|_{L^2}^2 \lesssim E_k + \|N_k\|^2,$$

and thus we obtain

$$X_k \lesssim E_k^{1/2} + \|N_k\|_{L^2}.$$

Then applying Lemma 7.1, we get

$$X_k \lesssim E_k^{1/2} + E_k + E_k^{1/2} X_k(t).$$

Under the assumption that $E_k \ll 1$, we obtain

$$X_k \lesssim E_k^{1/2} + E_k \lesssim E_k^{1/2}.$$

□

8. Estimates for special quantities, II

With the results of the previous section in hand, we can now complete the estimation of $\Gamma^\alpha v + \Gamma^\alpha G\omega$ and $\Gamma^\alpha G\omega^\perp$.

Lemma 8.1. *Let $k \ge 4$, $E_k \ll 1$, $\omega = x/|x|$. Then we have*

$$\|r(\partial_r\Gamma^\alpha v + \partial_r\Gamma^\alpha G\omega)\|_{L^2} + \|r\partial_r\Gamma^\alpha G\omega^\perp\|_{L^2} \lesssim E_k^{1/2}, \qquad |\alpha| \le k-1, \tag{8.1}$$

$$\|r(\Gamma^\alpha v + \Gamma^\alpha G\omega)\|_{L^\infty} + \|r\Gamma^\alpha G\omega^\perp\|_{L^\infty} \lesssim E_k^{1/2}, \qquad |\alpha| \le k-2. \tag{8.2}$$

Proof. First, note that

$$\partial_r\Gamma^\alpha v + \partial_r\Gamma^\alpha G\omega = (\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega)\omega + (\partial_r\Gamma^\alpha G\omega - \nabla\cdot\Gamma^\alpha G).$$

Therefore, by (6.4) and (6.6), we have

$$\begin{aligned} &r|\partial_r\Gamma^\alpha v + \partial_r\Gamma^\alpha G\omega| \\ &\qquad\le (r+t)|\nabla\Gamma^\alpha v + \nabla\cdot\Gamma^\alpha G\otimes\omega| + r|\partial_r\Gamma^\alpha G\omega - \nabla\cdot\Gamma^\alpha G| \\ &\qquad\lesssim L_k + N_k. \end{aligned}$$

Combining this with (6.5), we get the estimate

$$\|r(\partial_r\Gamma^\alpha v+\partial_r\Gamma^\alpha G\omega)\|_{L^2}+\|r\partial_r\Gamma^\alpha G\omega^\perp\|_{L^2}\le E_k^{1/2}+\|N_k\|_{L^2}.$$

Estimate (8.1) now follows by Lemma 7.1 and Theorem 7.3.

To obtain the other estimate, we observe that (similar to (6.12))

$$\widetilde{\Omega}(\Gamma^\alpha G\omega)=(\widetilde{\Omega}\Gamma^\alpha G)\omega,\qquad \widetilde{\Omega}(\Gamma^\alpha G\omega^\perp)=(\widetilde{\Omega}\Gamma^\alpha G)\omega^\perp. \tag{8.3}$$

By (3.3) with $\lambda=2$, (8.3), (8.1), we have

$$\begin{aligned}\|r(\Gamma^\alpha v+\Gamma^\alpha G\omega)\|_{L^\infty}&+\|r\Gamma^\alpha G\omega^\perp\|_{L^\infty}\\ &\lesssim\sum_{a=0,1}[\|r(\partial_r\widetilde{\Omega}^a\Gamma^\alpha v+\partial_r\widetilde{\Omega}^a\Gamma^\alpha G\omega)\|_{L^2}+\|\widetilde{\Omega}^a\Gamma^\alpha v+\widetilde{\Omega}^a\Gamma^\alpha G\omega\|_{L^2}\\ &\qquad+\|r\partial_r\widetilde{\Omega}^a\Gamma^\alpha G\omega^\perp\|_{L^2}+\|\widetilde{\Omega}^a\Gamma^\alpha G\omega^\perp\|_{L^2}]\lesssim E_k^{1/2}.\end{aligned}$$

□

9. Energy estimate with a ghost weight

We are ready to estimate the generalized energy and prove Theorem 2.3. Recall the assumptions that $(v_0,G_0)\in H^k_\Lambda$, with $k\ge 5$, that $(v_0,F_0)=(v_0,I+G_0)$ satisfy the constraints (2.2), (2.3), and that $\|(v_0,G_0)\|_{H^k_\Lambda}<\epsilon\le\epsilon_0$, where ϵ_0 is a sufficiently small positive constant to be chosen below. The local well-posedness of the elasticity system is standard (by the energy method), which gives that the system of incompressible Hookean elastodynamics (2.4) with initial data $(v_0,F_0)=(v_0,I+G_0)$ has a unique solution $(v,F)=(v,I+G)$, with $(v,G)\in H^k_\Lambda(T)$, at least for a short time $T>0$. Moreover, by continuity, we may assume that

$$\|(v,G)\|_{H^k_\Lambda(T)}=\sup_{0\le t<T}E_k(t)^{1/2}\le 10\epsilon. \tag{9.1}$$

We are going to show that if (9.1) holds on an interval $[0,T)$, with $T\le\exp(C_0/\epsilon)$ for a certain positive constant C_0, then

$$\sup_{0\le t<T}E_k(t)^{1/2}\le\sqrt{10}\epsilon.$$

Thus, the existence interval on which (9.1) holds includes the interval $[0,\exp(C_0/\epsilon)]$.

Assume that we have a solution for which (9.1) holds on an interval $[0,T)$ and $\epsilon\le\epsilon_0\ll 1$. Choose ϵ_0 small enough so that all lemmas established in earlier sections hold. We are going to prove the following a priori estimate:

$$\widetilde{E}_k'(t)\le C_1(1+t)^{-1}\widetilde{E}_k(t)^{3/2},\quad e^{-\frac{\pi}{2}}E_k(t)\le\widetilde{E}_k(t)\le E_k(t),\quad 0\le t<T.$$

This estimate gives the desired conclusion:

$$\begin{aligned}E(t)\le e^{\frac{\pi}{2}}\widetilde{E}_k(t)&\le e^{\frac{\pi}{2}}\big(\widetilde{E}_k(0)+C_1(10\epsilon)^3\ln(1+t)\big)\\ &\le 5\epsilon^2\big(1+C_1 10^3\epsilon\ln(1+T)\big)<10\epsilon^2,\end{aligned}$$

provided $T<\exp(C_0/\epsilon)$, with $C_0=1/(2\cdot 10^3\cdot C_1)$, and $\epsilon\le\epsilon_0\ll 1$.

We now proceed to establish the required a priori estimate.

Choose $q = q(t-r)$, with $q(\sigma) = \int_0^\sigma \langle z\rangle^{-2} dz$ so that $q'(\sigma) = \langle\sigma\rangle^{-2}$ and $|q(\sigma)| \le \frac{\pi}{2}$. Let $|\alpha| \le k$. Taking the inner product of the first and second equation in (4.1) with $e^{-q}\Gamma^\alpha v$ and $e^{-q}\Gamma^\alpha G$ respectively and then adding them up, we find

$$\int \Big(e^{-q}\partial_t(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2) - 2e^{-q}(\Gamma^\alpha v_i \partial_j \Gamma^\alpha G_{ij} + \Gamma^\alpha G_{ij}\partial_j\Gamma^\alpha v_i)\Big)dx$$
$$= -2\int e^{-q}\Gamma^\alpha v \cdot \nabla\Gamma^\alpha p dx + 2\int e^{-q}(f_\alpha \cdot \Gamma^\alpha v + (g_\alpha)_{ij}\Gamma^\alpha G_{ij})dx.$$

Integration by parts gives that

$$\begin{aligned}(9.2)\qquad &\frac{d}{dt}\int e^{-q}(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)dx\\ &= -\int e^{-q}\big[\partial_t q(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2) - 2\partial_j q\Gamma^\alpha v_i\Gamma^\alpha G_{ij}\big]dx\\ &\quad - 2\int e^{-q}\Gamma^\alpha v\cdot\nabla\Gamma^\alpha p dx\\ &\quad + 2\int e^{-q}(f_\alpha\cdot\Gamma^\alpha v + (g_\alpha)_{ij}\Gamma^\alpha G_{ij})dx\\ &= -\int \frac{e^{-q}}{\langle t-r\rangle^2}\big(|\Gamma^\alpha v + \Gamma^\alpha G\omega|^2 + |\Gamma^\alpha G\omega^\perp|^2\big)dx\\ &\quad - 2\int e^{-q}\Gamma^\alpha v\cdot\nabla\Gamma^\alpha p dx\\ &\quad + 2\int e^{-q}\big[f_\alpha\cdot\Gamma^\alpha v + (g_\alpha)_{ij}\Gamma^\alpha G_{ij}\big]dx.\end{aligned}$$

We emphasize that here we do not use integration by parts in the term involving pressure. We also point out that we cannot use the approach of Lemma 7.1 to estimate the nonlinear terms because we now have that $|\alpha| \le k$ rather than $|\alpha| \le k-1$ as we had earlier.

Let us first treat the last term in (9.2). Recall that f_α and g_α are given by (4.2). Since v and $G^\top$ are divergence-free, we get

$$\begin{aligned}(9.3)\qquad &\int e^{-q}\big[f_\alpha\cdot\Gamma^\alpha v + (g_\alpha)_{ij}\Gamma^\alpha G_{ij}\big]dx\\ &= \sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\int e^{-q}\Gamma^\alpha v_i\big[\partial_j\Gamma^\gamma G_{ik}\Gamma^\beta G_{jk} - \Gamma^\beta v_j\partial_j\Gamma^\gamma v_i\big]dx\\ &\quad + \sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\int e^{-q}\Gamma^\alpha G_{ik}\big[\partial_j\Gamma^\gamma v_i\Gamma^\beta G_{jk} - \Gamma^\beta v_j\partial_j\Gamma^\gamma G_{ik}\big]dx\\ &\quad + \frac{1}{2}\int e^{-q}\partial_j\big[2\Gamma^\alpha v_i\Gamma^\alpha G_{ik}G_{jk} - v_j(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)\big]dx.\end{aligned}$$

To estimate the last term in (9.3), we first compute that

$$\int e^{-q}\partial_j\big[2\Gamma^\alpha v_i\Gamma^\alpha G_{ik}G_{jk} - v_j(|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)\big]dx$$
$$= -\int\langle t-r\rangle^{-2}e^{-q}\big[2\Gamma^\alpha v_i\Gamma^\alpha G_{ik}G_{jk}\omega_j - (|\Gamma^\alpha v|^2 + |\Gamma^\alpha G|^2)v_j\omega_j\big]dx.$$

This is estimated by

$$(\|\langle t-r\rangle^{-2}v\cdot\omega\|_{L^\infty}+\|\langle t-r\rangle^{-2}G^\top\omega\|_{L^\infty})E_k(t).$$

Now by Lemma 6.2, we have

$$\begin{aligned}\|\langle t-r\rangle^{-2}v\cdot\omega\|_{L^\infty}&\le\|\langle t-r\rangle^{-2}v\cdot\omega\|_{L^\infty(r\le\langle t/2\rangle)}+\|\langle t-r\rangle^{-2}v\cdot\omega\|_{L^\infty(r\ge\langle t/2\rangle)}\\&\lesssim\langle t\rangle^{-2}\|v\|_{L^\infty(r\le\langle t/2\rangle)}+\|v\cdot\omega\|_{L^\infty(r\ge\langle t/2\rangle)}\\&\lesssim\langle t\rangle^{-2}\|v\|_{L^\infty(r\le\langle t/2\rangle)}+\langle t\rangle^{-1}\|rv\cdot\omega\|_{L^\infty(r\ge\langle t/2\rangle)}\\&\lesssim\langle t\rangle^{-1}E_k^{1/2}.\end{aligned}$$

A similar estimate holds for the term with $G^\top\omega$. We have shown that the last term in (9.3) is bounded by $\langle t\rangle^{-1}E_k^{3/2}$.

Next, we are going to estimate the first and second terms on the right-hand side of (9.3). Since $k\ge5$, it is always the case that $|\gamma|\le k-1$ and either $|\beta|\le k-2$ or $|\gamma|\le k-3$. For $r<\langle t/2\rangle$, we use Lemma 3.4 and Theorem 7.3 to get

$$\begin{aligned}(9.4)\qquad&\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\Big|\int_{r<\langle t/2\rangle}e^{-q}\Gamma^\alpha v_i\big[\partial_j\Gamma^\gamma G_{ik}\Gamma^\beta G_{jk}-\Gamma^\beta v_j\partial_j\Gamma^\gamma v_i\big]dx\\&\qquad+\int_{r<\langle t/2\rangle}e^{-q}\Gamma^\alpha G_{ik}\big[\partial_j\Gamma^\gamma v_i\Gamma^\beta G_{jk}-\Gamma^\beta v_j\partial_j\Gamma^\gamma G_{ik}\big]dx\Big|\\&\qquad\lesssim\langle t\rangle^{-1}E_k(t)^{1/2}\Big(\|\langle t-r\rangle(\nabla\Gamma^{k-1}v,\nabla\Gamma^{k-1}G)\|_{L^2(r<\langle t/2\rangle)}\\&\qquad\qquad\cdot\|(\Gamma^{k-2}v,\Gamma^{k-2}G)\|_{L^\infty}\\&\qquad\qquad+\|\langle t\rangle(\nabla\Gamma^{k-3}v,\nabla\Gamma^{k-3}G)\|_{L^\infty(r<\langle t/2\rangle)}\|\Gamma^k(v,G)\|_{L^2}\Big)\\&\qquad\lesssim\langle t\rangle^{-1}E_k(t)^{1/2}\Big(X_k(t)^{1/2}E_k(t)^{1/2}\Big)\lesssim\langle t\rangle^{-1}E_k(t)^{3/2}.\end{aligned}$$

For $r>\langle t/2\rangle$, we write using (2.5):

$$\begin{aligned}(9.5)\qquad&\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\Big|\int_{r>\langle t/2\rangle}e^{-q}\Gamma^\alpha v_i\big[\partial_j\Gamma^\gamma G_{ik}\Gamma^\beta G_{jk}-\Gamma^\beta v_j\partial_j\Gamma^\gamma v_i\big]dx\\&\qquad+\int_{r>\langle t/2\rangle}e^{-q}\Gamma^\alpha G_{ik}\big[\partial_j\Gamma^\gamma v_i\Gamma^\beta G_{jk}-\Gamma^\beta v_j\partial_j\Gamma^\gamma G_{ik}\big]dx\Big|\\&\qquad\le\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\Big|\int_{r>\langle t/2\rangle}e^{-q}\Gamma^\alpha v_i\big[\omega_j\partial_r\Gamma^\gamma G_{ik}\Gamma^\beta G_{jk}-\Gamma^\beta v_j\omega_j\partial_r\Gamma^\gamma v_i\big]dx\\&\qquad\qquad+\int_{r>\langle t/2\rangle}e^{-q}\Gamma^\alpha G_{ik}\big[\omega_j\partial_r\Gamma^\gamma v_i\Gamma^\beta G_{jk}-\Gamma^\beta v_j\omega_j\partial_r\Gamma^\gamma G_{ik}\big]dx\Big|+\int_{r>\langle t/2\rangle}R_\alpha dx\\&\qquad=\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\Big|\int_{r>\langle t/2\rangle}e^{-q}\big\langle(\Gamma^\alpha v,\Gamma^\alpha G),B[(\Gamma^\beta v,\Gamma^\beta G),(\partial_r\Gamma^\gamma v,\partial_r\Gamma^\gamma G)]\big\rangle dx\Big|\\&\qquad\qquad+\int_{r>\langle t/2\rangle}R_\alpha dx,\end{aligned}$$

in which

$$B[(v_1,G_1),(v_2,G_2)] = \big(G_2G_1^\top\omega - (v_1\cdot\omega)v_2, v_2\otimes G_1^\top\omega - (v_1\cdot\omega)G_2\big)$$

and

$$|R_\alpha| \lesssim \frac{1}{r}\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}} |(\Gamma^\alpha v,\Gamma^\alpha G)|\,|(\Gamma^\beta v,\Gamma^\beta G)|\,|(\widetilde{\Omega}\Gamma^\gamma v,\widetilde{\Omega}\Gamma^\gamma G)|.$$

To analyze the structure of these terms when $r > \langle t/2\rangle$, we decompose vector/matrix pairs (v,G) as

$$\begin{aligned}(v,G) &= \sum_{k=-1}^{1}\Pi_k(v,G),\\ \Pi_1(v,G) &= \tfrac12((v+G\omega),(v+G\omega)\otimes\omega),\\ \Pi_{-1}(v,G) &= \tfrac12((v-G\omega),-(v-G\omega)\otimes\omega),\\ \Pi_0(v,G) &= (0,G\omega^\perp\otimes\omega^\perp).\end{aligned}$$

We have the following cancellations:

$$\begin{aligned}B[\Pi_k(v_1,G_1),\Pi_k(v_2,G_2)] &= 0, &&k=\pm1,\\ B[\Pi_0(v_1,G_1),\Pi_k(v_2,G_2)] &= 0, &&k=\pm1,\\ B[\Pi_k(v_1,G_1),\Pi_0(v_2,G_2)] &= 0, &&k=\pm1.\end{aligned}$$

In particular, there is no self-interaction of the "bad" quantity Π_{-1}. Thus, we have

$$\begin{aligned}B[(v_1,G_1),(v_2,G_2)] &= B[\Pi_1(v_1,G_1),\Pi_{-1}(v_2,G_2)]\\ &\quad + B[\Pi_{-1}(v_1,G_1),\Pi_1(v_2,G_2)] + B[\Pi_0(v_1,G_1),\Pi_0(v_2,G_2)].\end{aligned}$$

The conclusion of this discussion is that the terms in (9.5) are bounded by

$$\begin{aligned}&\sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}}\int_{r>\langle t/2\rangle}|(\Gamma^\alpha v,\Gamma^\alpha G)|\,\Big\{|\Gamma^\beta v+\Gamma^\beta G\omega|\,|(\partial_r\Gamma^\gamma v,\partial_r\Gamma^\gamma G)|\\ &\qquad + |(\Gamma^\beta v,\Gamma^\beta G)|\,|\partial_r\Gamma^\gamma v+\partial_r\Gamma^\gamma G\omega|\\ &\qquad + |(\Gamma^\beta v,\Gamma^\beta G)|\,|\partial_r\Gamma^\gamma G\omega^\perp|\\ &\qquad + \frac{1}{r}|(\Gamma^\beta v,\Gamma^\beta G)|\,|(\widetilde{\Omega}\Gamma^\gamma v,\widetilde{\Omega}\Gamma^\gamma G)|\Big\}dx\\ \le& C\|(\Gamma^k v,\Gamma^k G)\|_{L^2}\\ &\times\Big\{\langle t\rangle^{-1}\|r(\Gamma^{k-2}v+\Gamma^{k-2}G\omega)\|_{L^\infty(r>\langle t/2\rangle)}\|(\partial_r\Gamma^{k-1}v,\partial_r\Gamma^{k-1}G)\|_{L^2}\\ &+\langle t\rangle^{-1/2}\Big\|\frac{\Gamma^k v+\Gamma^k G\omega}{\langle t-r\rangle}\Big\|_{L^2}\\ &\qquad\cdot\|r^{1/2}\langle t-r\rangle(\partial_r\Gamma^{k-3}v,\partial_r\Gamma^{k-3}G)\|_{L^\infty(r>\langle t/2\rangle)}\\ &+\langle t\rangle^{-1}\|(\Gamma^{k-2}v,\Gamma^{k-2}G)\|_{L^\infty}\\ &\qquad\cdot\|r|\partial_r\Gamma^{k-1}v+\partial_r\Gamma^{k-1}G\omega|+r|\partial_r\Gamma^{k-1}G\omega^\perp|\|_{L^2(r>\langle t/2\rangle)}\\ &+\langle t\rangle^{-1}\|(\Gamma^k v,\Gamma^k G)\|_{L^2}\\ &\qquad\cdot\|r|\partial_r\Gamma^{k-3}v+\partial_r\Gamma^{k-3}G\omega|+r|\partial_r\Gamma^{k-3}G\omega^\perp|\|_{L^\infty(r>\langle t/2\rangle)}+\langle t\rangle^{-1}E_k\Big\}.\end{aligned}$$

By (8.2), we have that

$$\|r(\Gamma^{k-2}v+\Gamma^{k-2}G\omega)\|_{L^\infty(r>\langle t/2\rangle)}\lesssim E_k^{1/2}.$$

By (3.4), we obtain

$$\langle t\rangle^{-1/2}\left\|\frac{\Gamma^k v+\Gamma^k G\omega}{\langle t-r\rangle}\right\|_{L^2}\|r^{1/2}\langle t-r\rangle(\partial_r\Gamma^{k-3}v,\partial_r\Gamma^{k-3}G)\|_{L^\infty(r>\langle t/2\rangle)}$$
$$\leq\langle t\rangle^{-1/2}\left\|\frac{\Gamma^k v+\Gamma^k G\omega}{\langle t-r\rangle}\right\|_{L^2}E_k^{1/2}.$$

By the conventional Sobolev imbedding, (8.1), and (8.2) we have

$$\|(\Gamma^{k-2}v,\Gamma^{k-2}G)\|_{L^\infty}\|r|\partial_r\Gamma^{k-1}v+\partial_r\Gamma^{k-1}G\omega|+r|\partial_r\Gamma^{k-1}G\omega^\perp|\|_{L^2}\lesssim E_k$$

and

$$\|(\Gamma^k v,\Gamma^k G)\|_{L^2}\|r|\partial_r\Gamma^{k-3}v+\partial_r\Gamma^{k-3}G\omega|+r|\partial_r\Gamma^{k-3}G\omega^\perp|\|_{L^\infty}\lesssim E_k.$$

It follows, therefore, that the integral (9.3) is bounded by

$$\mu\left\|\frac{\Gamma^k v+\Gamma^k G\omega}{\langle t-r\rangle}\right\|_{L^2}^2+C_\mu\langle t\rangle^{-1}E_k^{3/2},$$

for an arbitrarily small $\mu>0$.

It remains to treat the pressure term in (9.2). However, thanks to (5.2), this term is handled exactly as the preceding ones.

Finally, we gather our estimates for (9.2) to get

$$\widetilde{E}_k'(t)+\left\|\frac{\Gamma^k v+\Gamma^k G\omega}{\langle t-r\rangle}\right\|_{L^2}^2\leq\mu\left\|\frac{\Gamma^k v+\Gamma^k G\omega}{\langle t-r\rangle}\right\|_{L^2}^2+C_\mu\langle t\rangle^{-1}E_k^{3/2},$$

with

$$\widetilde{E}_k(t)=\sum_{|\alpha|\leq k}\int e^{-q}(|\Gamma^\alpha v|^2+|\Gamma^\alpha G|^2)dx.$$

Notice that $e^{-\frac{\pi}{2}}E_k(t)\leq\widetilde{E}_k(t)\leq E_k(t)$. We obtain, for μ small, the a priori bound

$$\widetilde{E}_k'(t)\leq C_1\langle t\rangle^{-1}\widetilde{E}_k(t)^{3/2},\quad 0\leq t<T.$$

As discussed at the beginning of the section, this implies that $E_k(t)$ remains bounded by $10\epsilon^2$ on a time interval of order $T\sim\exp(C_0/\epsilon)$.

10. General isotropic elastodynamics

For general isotropic elastodynamics, the energy functional has the form $W=W(F)$ with

$$W(F)=W(QF)=W(FQ)\tag{10.1}$$

for all rotation matrices: $Q=Q^\top$, $\det Q=1$. The first relation is due to frame indifference, while the second one expresses the isotropy of materials. This implies that W depends on F through the principal invariants of $FF^\top$, namely tr $FF^\top$ and $\det FF^\top$ in 2-D. Setting $\tau=\frac{1}{2}$ tr $FF^\top$ and $\delta=\det F=(\det FF^\top)^{1/2}$, we may assume that $W(F)=\bar{W}(\tau,\delta)$, for some smooth function $\bar{W}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$.

Since

$$\frac{\partial\tau}{\partial F}=F\quad\text{and}\quad\frac{\partial\delta}{\partial F}=\delta F^{-T},$$

the Piola-Kirchhoff stress has the form

$$S(F) \equiv \frac{\partial W(F)}{\partial F} = \bar{W}_\tau(\tau,\delta)\, F + \bar{W}_\delta(\tau,\delta)\delta F^{-T}.$$

We assume that the reference configuration is stress free, $S(I) = 0$, so that

$$\bar{W}_\tau(1,1) + \bar{W}_\delta(1,1) = 0. \tag{10.2}$$

The Cauchy stress tensor is

$$T(F) \equiv \delta^{-1} S(F) F^\top = \delta^{-1}\bar{W}_\tau(\tau,\delta) FF^\top + \bar{W}_\delta(\tau,\delta) I,$$

and the term $\nabla \cdot FF^\top$ in (2.4) is replaced by $\nabla \cdot T(F)$. Let us now proceed to examine this term.

Write

$$\begin{aligned} T(F) =& \bar{W}_\tau(1,1) FF^\top \\ &+ [\delta^{-1}\bar{W}_\tau(\tau,\delta) - \bar{W}_\tau(1,1)][FF^\top - I] \\ &+ [\delta^{-1}\bar{W}_\tau(\tau,\delta) - \bar{W}_\tau(1,1) + \bar{W}_\delta(\tau,\delta)] I \\ \equiv& \sum_{a=1}^{3} T_a(F). \end{aligned}$$

Assume that

$$\bar{W}_\tau(1,1) > 0. \tag{10.3}$$

Then $T_1(F)$ gives rise to a Hookean term. Notice that assumption (10.3) rules out the hydrodynamical case $\bar{W}_\tau = 0$. The principal invariants can be expanded about the identity as follows:

$$\tau = \frac{1}{2}\mathrm{tr}\ FF^\top = \frac{1}{2}\mathrm{tr}\ (I+G)(I+G^\top) = 1 + \mathrm{tr}\ G + \frac{1}{2}\mathrm{tr}\ GG^\top \tag{10.4}$$

and

$$\delta = \det F = \det(I+G) = 1 + \mathrm{tr}\ G + \det G. \tag{10.5}$$

In the case of incompressible motion, we have $\delta = 1$, so from (10.5), we get

$$\mathrm{tr}\ G + \det G = 0,$$

and hence from (10.4)

$$\tau - 1 = \frac{1}{2}\ \mathrm{tr}\ GG^\top - \det G = \mathcal{O}(|G|^2).$$

Thus, we see that for $|G| \ll 1$,

$$T_2(F) = \mathcal{O}[(\tau - 1)|G|] = \mathcal{O}(|G|^3),$$

which produces nonlinearities which are of cubic order or higher. Finally, $T_3(F)$ leads to a gradient term which can be included in the pressure. The conclusion is that the general incompressible isotropic case differs from the Hookean case by a nonlinear perturbation which is cubic in the displacement gradient G. Such terms present no further obstacles in the proof of almost global existence in 2-D; in particular, they have the requisite symmetry properties for energy calculations (see [31]).

Theorem 10.1. *Let $(v_0, G_0) \in H^k_\Lambda$, with $k \geq 5$. Suppose that $(v_0, F_0) = (v_0, I+G_0)$ satisfy the constraints* (2.2), (2.3) *and $\|(v_0, G_0)\|_{H^k_\Lambda} < \epsilon$.*

Assume that the smooth strain energy function $W(F)$ is isotropic, frame indifferent, and satisfies (10.2), (10.3).

There exist two positive constants C_0 and ϵ_0 which depend only on k such that if $\epsilon \leq \epsilon_0$, then the system of incompressible isotropic elastodynamics

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \nabla \cdot T(F), \\ \partial_t F + v \cdot \nabla F = \nabla v F, \\ \nabla \cdot v = 0, \quad \nabla \cdot F^\top = 0, \end{cases} \tag{10.6}$$

with initial data $(v_0, F_0) = (v_0, I + G_0)$, has a unique solution $(v, F) = (v, I + G)$, with $(v, G) \in H^k_\Lambda(T)$, $T \geq \exp(C_0/\epsilon)$, and $E_k(t) \leq 2\epsilon^2$, for $0 \leq t < T$.

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