CONVERGENCE OF FINITE DIFFERENCE METHODS FOR POISSON'S EQUATION WITH INTERFACES

XU-DONG LIU AND THOMAS C. SIDERIS

ABSTRACT. In this paper, a weak formulation of the discontinuous variable coefficient Poisson equation with interfacial jumps is introduced. The existence, uniqueness and regularity of solutions of the Poisson equation are obtained. Finite difference methods can be derived from the weak formulation. An abstract framework is given for proving convergence of the finite difference methods for such problems. The finite difference method developed in [9] is proven to be convergent.

1. INTRODUCTION

This paper proves the convergence of the finite difference method introduced in [9] for the Poisson equation with discontinuous coefficients and given interfacial jumps. Based on the Ghost Fluid Method (GFM) of [4], see also [3], [5], the finite difference method is simple, efficient and robust. One of the novelties, and advantages, of the method is the arm-by-arm splitting technique which makes the method in multi-dimension as simple as in one-dimension. Another advantage is that the resulting linear system of the method is the same as the linear system obtained from the simplest standard five point stencil finite difference method for the Poisson equation without discontinuous coefficients or given interfacial jumps. Therefore the resulting linear system is symmetric and positive definite and can be efficiently solved. Inherited from the GFM, this finite difference method captures the sharp solution profile at the interfaces without smearing. This is a necessity in the development of effective schemes for problems involving interfaces. A variety of other approaches to problems with interfaces have been taken, [2], [7], [8], [10], [11], [12]. An important application of the method here is for Hele-Shaw flow, see [6].

The convergence proof starts with the formulation of the problem, in terms of a uniformly elliptic bilinear form. Existence and uniqueness of a weak solution v follow immediately using elementary functional analysis. The solution space is the same as H_0^1 except with a different, but equivalent, inner product induced by the bilinear form. Discretizing this weak formulation in a natural way results a discrete weak problem, which is equivalent

1

to the finite different method in [9]. As expected, the discrete weak formulation inherits the structural conditions (uniform boundedness, extension, consistency) in the discrete sense, hence existence, uniqueness, and uniform boundedness of the family v^h of discrete solution hold.

We then provide an abstract framework for the convergence proof. In addition to the uniform structural conditions for the weak problem and its discretizations, we postulate the existence of a uniformly bounded family of extension operators T^h from the discrete spaces to the solution space, which satisfy a strong approximation property. This natural assumption implies that as the grid size h goes to zero the image of the discrete space fills out the entire solution space. Finally, we impose weak consistency of the discrete problem with the original problem. Under these assumptions, the sequence of extended weak solutions $T^h(v^h)$ converges weakly to the solution v in H_0^1. Guided by the abstract framework, we establish the converge of the finite difference scheme of [9].

There is a similarity between the standard convergence proof of finite element methods and our approach. Both of them use structural conditions. For finite element methods, the structural conditions and weak consistency are inherited directly from the weak problem for the PDE because the discrete bilinear forms are obtained by restriction to finite dimensional subspaces. Cea's Lemma then says that the extended discrete solution is the closest function in the finite dimensional subspace to the true solution. This reduces convergence and error estimation to a problem in approximation theory. For finite difference methods, further approximations are made so that the discrete problem can not be obtained by restriction to a finite dimensional subspace. In our case, the difference between the finite difference scheme and the finite element method is that point values of the coefficients are used rather than cell averages. Because the coefficients in our problem are discontinuous, the solution lies only in H_0^1 and not $H^2 \cap H_0^1$, and therefore, we obtain convergence, but without a rate.

2. Equations and Weak Formulation

Consider a bounded domain, $\Omega \subset \mathbb{R}^n$, with smooth boundary, $\partial\Omega$. Let $\Gamma \subset \Omega$ be a smooth interface of co-dimension $n-1$, represented by the zero level-set of a smooth function $\phi(x)$, which is a signed distance function of the interface locally. We assume that ϕ divides Ω into disjoint subdomains, $\Omega^- = \{\phi < 0\}$ and $\Omega^+ = \{\phi > 0\}$, with $\partial\Omega^- = \Gamma$. Thus, we may write $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$. The unit normal vector of the interface is $n = \nabla\phi/|\nabla\phi|$, for $\phi(x) = 0$, pointing from Ω^- to Ω^+.

We seek solutions of the variable coefficient Poisson equation away from the interface given by

$$(2.1a) \qquad \nabla \cdot (\beta(x)\nabla u(x)) = f(x), \quad x \in \Omega \setminus \Gamma,$$

in which $x = (x_1, \dots, x_n)$ denotes the spatial variables and ∇ is the gradient operator. The coefficient $\beta(x)$ is assumed to be a positive definite,

symmetric $n \times n$ matrix, the components of which are continuously differentiable on the closure of each disjoint subdomain, Ω^- and Ω^+, but they may be discontinuous across the interface Γ. It follows that there are positive constants $m < M$ with $m\,Id \leq \beta(x) \leq M\,Id$, where Id stands for the $n \times n$ identity matrix. We suppose that on the interface, β assumes the limiting values from within Ω^-. The right-hand side $f(x)$ is assumed to lie in $L^2(\Omega)$.

Given functions a and b along the interface Γ, we prescribe the jump conditions

(2.1b) $\qquad \begin{cases} [u]_\Gamma(x) \equiv u^+(x) - u^-(x) = a(x), \\ [(\beta u)_n]_\Gamma(x) \equiv (\beta u)_n^+(x) - (\beta u)_n^-(x) = b(x), \end{cases} \quad x \in \Gamma.$

Note that $(\beta u)_n = n \cdot \beta \nabla u$, and the "$\pm$" subscripts refer to limits taken from within the subdomains $\Omega^\pm$.

Finally, we prescribe boundary conditions

(2.1c) $\qquad\qquad\qquad u(x) = g(x), \quad x \in \partial\Omega,$

for a given function g on the boundary.

We are going to use the usual Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$, but instead of the usual inner product we choose one which is better suited to our problem:

(2.2) $\qquad\qquad\qquad B[u,v] = \int\limits_\Omega \beta \nabla u \cdot \nabla v.$

This induces a norm on $H_0^1(\Omega)$ which is equivalent to the usual one, thanks to the Poincaré inequality and the uniform bounds for the coefficient matrix.

Let R_Γ and $R_{\partial\Omega}$ denote the restriction operators from $H^1(\Omega)$ to $L^2(\Gamma)$ and $L^2(\partial\Omega)$, respectively. Throughout this section, we shall always assume that our boundary data a, b are the restrictions of functions $\widetilde{a}, \widetilde{b} \in H_0^1(\Omega)$, and that g is the restriction of a function $\widetilde{g} \in H^1(\Omega)$:

(2.3) $\qquad\qquad a = R_\Gamma(\widetilde{a}), \quad b = R_\Gamma(\widetilde{b}), \quad \text{and} \quad g = R_{\partial\Omega}(\widetilde{g}).$

To simplify the notation, from now on we will drop the tildes.

First let us consider the version of problem of (2.1a), (2.1b), (2.1c) with homogeneous boundary conditions for u:

(2.4) $\qquad \begin{cases} \nabla \cdot (\beta(x)\nabla u(x)) = f(x), & x \in \Omega \setminus \Gamma \\ [u]_\Gamma(x) = 0, & x \in \Gamma \\ [(\beta u)_n]_\Gamma(x) = b(x), & x \in \Gamma \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$

Definition 2.1. A function $u \in H_0^1(\Omega)$ is a weak solution of (2.4) if it satisfies

(2.5) $\qquad\qquad -B[u,\psi] = \int\limits_\Omega f\psi + \int\limits_{\Omega^-} \nabla \cdot (b\psi n),$

for all $\psi \in H_0^1(\Omega)$.

A classical solution of (2.4), $u|_{\Omega^\pm} \in C^2(\overline{\Omega^\pm})$, is easily seen to satisfy (2.5). The boundary condition $[u]_\Gamma = 0$ is understood with the aid of R_Γ, and it is fulfilled since $u \in H_0^1(\Omega)$.

Theorem 2.1. *Given $f \in L^2(\Omega)$ and $b \in H_0^1(\Omega)$, there exists a unique weak solution of (2.5) in $H_0^1(\Omega)$.*

Proof. The right-hand side of (2.5)

$$F(\psi) = \int\limits_\Omega f\psi + \int\limits_{\Omega^-} \nabla \cdot (b\psi n)$$

is a continuous linear functional on $H_0^1(\Omega)$. By the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that $-B[u, \psi] = F(\psi)$, for all $\psi \in H_0^1(\Omega)$. $\qquad\square$

Next we reduce the general case (2.1a), (2.1b), (2.1c) to the homogeneous case. We will construct a unique solution of the problem in the class

$$H(a, g) = \{u : u - g + a\chi_{\Omega^-} \in H_0^1(\Omega)\},$$

in which χ_{Ω^-} is the characteristic function of Ω^-. If $u \in H(a, g)$ then

$$[u]_\Gamma = a \quad \text{and} \quad R_{\partial\Omega}(u) = g.$$

Note that $H_0^1(\Omega)$ can be identified with $H(0, 0)$, and so the following definition is consistent with the previous one.

Definition 2.2. A function $u \in H(a, g)$ is a weak solution of (2.1a), (2.1b), (2.1c) if $v = u - g + a\chi_{\Omega^-}$ satisfies

(2.6a) $$-B[v, \psi] = F(\psi),$$

for all $\psi \in H_0^1(\Omega)$, where

(2.6b) $$F(\psi) = F_1(\psi) + \ldots + F_4(\psi)$$

$$= \int\limits_\Omega f\psi + \int\limits_\Omega \beta\nabla g \cdot \nabla\psi - \int\limits_{\Omega^-} \beta\nabla a \cdot \nabla\psi + \int\limits_{\Omega^-} \nabla \cdot (b\psi n).$$

A classical solution of (2.1a), (2.1b), (2.1c) is necessarily a weak solution.

Theorem 2.2. *If $f \in L^2(\Omega)$, $g \in H^1(\Omega)$, and a, $b \in H_0^1(\Omega)$, then there exists a unique weak solution of (2.6a), (2.6b), (2.1c) in $H(a, g)$.*

Proof. The right-hand side of (2.6b) is a continuous linear functional on $H_0^1(\Omega)$. By the Riesz representation theorem, there exists a unique $v \in H_0^1(\Omega)$ such that $-B[v, \psi] = F(\psi)$, for all $\psi \in H_0^1(\Omega)$. Define the solution as $u = v + g - a\chi_{\Omega^-}$. $\qquad\square$

Remark 1. We could replace $\int\limits_{\Omega^-} \nabla \cdot (b\psi n)$ by $\int\limits_\Gamma b\psi \, ds$ in (2.6b). Then the requirement for b is reduced to $b \in L^2(\Gamma)$, which is a bit weaker than our assumption (2.3) for b.

3. Abstract Form of Finite Difference Methods

In this section an abstract framework is given for proving convergence of finite difference methods for the elliptic interface problem introduced in the previous section.

For any $h > 0$, let $H_0^{1,h}$ be a finite dimensional vector space with norm $\| \cdot \|_h$. This space should be thought of as a discrete approximation of the Sobolev space $H_0^1(\Omega)$ with grid size measured by the parameter h.

On each finite dimensional space $H_0^{1,h}$, we suppose there exists a bounded **extension operator** $T^h : H_0^{1,h} \to H_0^1(\Omega)$ with the bound

$$(3.1a) \qquad \|T^h(\psi^h)\|_{H_0^1(\Omega)} \leq C_0 \|\psi^h\|_h,$$

for all $\psi^h \in H_0^{1,h}$, with C_0 independent of h.

Furthermore, we assume a **strong approximation property**. That is, given $\psi \in H_0^1(\Omega)$, there exists a sequence $\psi^h \in H_0^{1,h}$ such that

$$(3.1b) \qquad T^h(\psi^h) \to \psi \quad \text{in} \quad H_0^1(\Omega), \quad \text{as} \quad h \to 0.$$

On each vector space $H_0^{1,h}$, we suppose there is a **uniformly bounded family of bilinear forms** $B^h[\cdot,\cdot]$ such that for every u^h, $v^h \in H_0^{1,h}$

$$(3.2a) \qquad C_1 \|u^h\|_h^2 \leq B^h[u^h, u^h] \quad \text{and} \quad |\,B^h[u^h, v^h]\,| \leq C_2 \|u^h\|_h \|v^h\|_h,$$

for positive constants C_1, C_2 independent of h. We also assume there exists a **uniformly bounded family of linear functionals** $F^h(\cdot)$ on $H_0^{1,h}$ such that for every $\psi^h \in H_0^{1,h}$

$$(3.2b) \qquad |\,F^h(\psi^h)\,| \leq C_3 \|\psi^h\|_h,$$

again with a constant C_3 independent of h.

Finally, we impose **weak consistency** with the weak problem (2.6a), (2.6b). For every pair of sequences v^h, $\psi^h \in H_0^{1,h}$ such that $T^h(v^h) \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $T^h(\psi^h) \to \psi$ strongly in $H_0^1(\Omega)$, we have that

$$(3.3a) \qquad B^h[v^h, \psi^h] \to B[v, \psi]$$

and

$$(3.3b) \qquad F^h(\psi^h) \to F(\psi),$$

where B and F are defined by (2.2) and (2.6b).

Under these general assumptions, we have the following:

Lemma 3.1. *For every $h > 0$, there exists a unique solution $v^h \in H_0^{1,h}$ of the discrete problem*

$$(3.4) \qquad -B^h[v^h, \psi^h] = F^h(\psi^h),$$

for every $\psi^h \in H_0^{1,h}$.

The sequence of extensions $T^h(v^h)$ of the family of discrete solutions converges weakly to $v = u - g + a\chi_{\Omega^-}$ in $H_0^1(\Omega)$, where u the solution of the weak problem (2.6a), (2.6b).

Proof. The existence of a unique solution of the discrete problem (3.4) follows by the Lax-Milgram lemma.

By the estimates (3.2a), (3.2b), we obtain a uniform bound for the sequence of discrete solutions v^h

$$\|v^h\|_h \le C,$$

with C independent of h. Thus, using (3.1a) we have the bound

$$\|T^h(v^h)\|_{H_0^1(\Omega)} \le C.$$

By weak compactness in the Hilbert space $H_0^1(\Omega)$, there is a subsequence $T^{h'}(v^{h'})$ converging weakly to some $v \in H_0^1(\Omega)$.

Now let $\psi \in H_0^1(\Omega)$ be given. Using the approximation property (3.1b), choose a sequence $\psi^h \in H_0^{1,h}$ such that

$$T^h(\psi^h) \to \psi \quad \text{in} \quad H_0^1(\Omega).$$

By weak consistency (3.3a), (3.3b) we find that the limit function v satisfies (2.6a), (2.6b). Finally, since this problem has a unique solution, it follows that the full sequence $T^h(v^h)$ converges weakly to v in $H_0^1(\Omega)$. $\qquad\square$

Remark 2. The simplest way in which to obtain a consistent, and therefore convergent, scheme is to define $B^h[v^h, \psi^h] = B[T^h(v^h), T^h(\psi^h)]$ and $F^h(\psi^h) = F(T^h(\psi^h))$. This is essentially the method of finite elements which then hinges on the choice of the extension operator T^h, [1].

However, the scheme under consideration in the next sections is not of this type, insofar as it originates from different approximations for B and F.

4. NUMERICAL METHOD

In this section, we rederive the finite difference scheme from [9] for (2.1a), (2.1b), (2.1c), by discretizing the weak formulation (2.6a), (2.6b).

For the remainder of the paper we assume that the coefficient matrix is of the form $\beta \, Id$, for some scalar function. We also assume that the data functions a, b, g, f, all lie in $C^1(\overline{\Omega})$, with a, b vanishing on $\partial\Omega$. For simplicity, we restrict ourselves to the special case of a rectangular domain $\Omega = (x_W, x_E) \times (y_S, y_N)$ in the plane. Given positive integers I and J, set $\Delta x = (x_E - x_W)/(I + 1)$ and $\Delta y = (y_N - y_S)/(J + 1)$, and define a uniform grid $\Omega^h = \{(x_i, y_j)\}$ where $x_i = x_W + i\Delta x$ and $y_j = y_S + j\Delta y$ for $i = 0, 1, \cdots, I + 1$ and $j = 0, 1, \cdots, J + 1$. The grid size is defined as $h = \min(\Delta x, \Delta y)$. The ratio $\Delta x/\Delta y$ is fixed when the grid size h goes to zero.

The set of grid functions will be denoted by

(4.1a) $H^{1,h} = \{w^h = (w_{i,j}) : 0 \le i \le I + 1, \, 0 \le j \le J + 1\}.$

The discrete solution space is defined as

(4.1b) $H_0^{1,h} = \{\psi^h = (\psi_{i,j}) \in H^{1,h} : \psi_{i,j} = 0 \text{ on the grid boundary}\}.$

To construct the bilinear form on $H_0^{1,h}$, we discretize the coefficient β in two ways as follows,

$$(4.2) \qquad \beta_{i+1/2,j}^1 = \beta(x_{i+1/2}, y_j), \qquad \beta_{i,j+1/2}^2 = \beta(x_i, y_{j+1/2}).$$

For $w^h \in H^{1,h}$ define the usual finite difference operators

$$(\nabla_x w)_{i+1/2,j} = (w_{i+1,j} - w_{i,j})/\Delta x,$$
$$(\nabla_y w)_{i,j+1/2} = (w_{i,j+1} - w_{i,j})/\Delta y.$$

For v^h, $\psi^h \in H_0^{1,h}$, the bilinear form then is given by

$$(4.3) \qquad B^h[v^h, \psi^h] = \sum_{j=1}^J \sum_{i=0}^I \beta_{i+1/2,j}^1 (\nabla_x v)_{i+1/2,j} (\nabla_x \psi)_{i+1/2,j} \Delta x \Delta y$$
$$+ \sum_{i=1}^I \sum_{j=0}^J \beta_{i,j+1/2}^2 (\nabla_y w)_{i,j+1/2} (\nabla_y \psi)_{i,j+1/2} \Delta x \Delta y.$$

As in the continuous case, we use this to define a norm on $H_0^{1,h}$:

$$(4.4) \qquad \|\psi^h\|_h^2 = B^h[\psi^h, \psi^h].$$

Our next task will be to discretize the linear functional F in (2.6b). The data functions naturally give rise to grid functions a^h, b^h, etc., by restriction to the grid Ω^h. The normal vector is discretized by

$$(4.5) \qquad n^h = (n^{(1),h}, n^{(2),h}) = (\phi_x^h, \phi_y^h)/|\nabla \phi^h|$$

where for $i = 1, \cdots, I$, $j = 1, \cdots, J$,

$$(\phi_x)_{i,j} = (\phi_{i+1,j} - \phi_{i-1,j})/(2\Delta x),$$
$$(\phi_y)_{i,j} = (\phi_{i,j+1} - \phi_{i,j-1})/(2\Delta y).$$

Hence n^h is defined on all interior grid points. It will not be used on $\partial\Omega$.

We can immediately define

$$(4.6a) \qquad F_1^h(\psi^h) = \sum_{i=1}^I \sum_{j=1}^J f_{i,j} \psi_{i,j} \Delta x \Delta y,$$

and

$$(4.6b) \qquad F_2^h(\psi^h) = B^h[g^h, \psi^h].$$

The other two pieces require the localization of integrals to the subdomain Ω^-. First, we discretize the characteristic function:

$$(4.6c) \qquad \chi_{i,j} = \begin{cases} 1, & \text{if} \quad \phi_{i,j} \leq 0 \\ 0, & \text{if} \quad \phi_{i,j} > 0. \end{cases}$$

Define

(4.6d) $\chi^1_{i+1/2,j} = \left(\chi_{i,j}(1 - \theta_{i+1/2,j}) + \chi_{i+1,j}\theta_{i+1/2,j} \right)$

$$\theta_{i+1/2,j} = \begin{cases} |\phi_{i+1,j}|/(|\phi_{i,j}| + |\phi_{i+1,j}|), & \text{if } |\phi_{i,j}| + |\phi_{i+1,j}| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The factor $\chi^1_{i+1/2,j}\Delta x$ approximates the length of the portion of the arm from (x_i, y_j) to (x_{i+1}, y_j) within Ω^-. And also, let

(4.6e) $\chi^2_{i,j+1/2} = \left(\chi_{i,j}(1 - \theta_{i,j+1/2}) + \chi_{i,j+1}\theta_{i,j+1/2} \right)$

$$\theta_{i,j+1/2} = \begin{cases} |\phi_{i,j+1}|/(|\phi_{i,j}| + |\phi_{i,j+1}|), & \text{if } |\phi_{i,j}| + |\phi_{i,j+1}| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now we define the remaining two pieces of the linear functional F.

(4.6f) $$F_4^h(\psi^h) = \sum_{j=1}^{J} \sum_{i=0}^{I} \nabla_x(bn^{(1)}\psi)_{i+1/2,j}\, \chi^1_{i+1/2,j}\Delta x\Delta y$$
$$+ \sum_{i=1}^{I} \sum_{j=0}^{J} \nabla_y(bn^{(2)}\psi)_{i,j+1/2}\, \chi^2_{i,j+1/2}\Delta x\Delta y$$

and

(4.6g) $$F_3^h(\psi^h) = -\sum_{j=1}^{J} \sum_{i=0}^{I} \beta^1_{i+1/2,j}(\nabla_x a)_{i+1/2,j}(\nabla_x \psi)_{i+1/2,j}\, \chi^1_{i+1/2,j}\Delta x\Delta y$$
$$- \sum_{i=1}^{I} \sum_{j=0}^{J} \beta^2_{i,j+1/2}(\nabla_y a)_{i,j+1/2}(\nabla_y \psi)_{i,j+1/2}\, \chi^2_{i,j+1/2}\Delta x\Delta y.$$

Using (4.6a),...,(4.6g), define

(4.7) $$F^h = F_1^h + \ldots + F_4^h,$$

With definitions (4.3) and (4.7) the discrete problem is formulated as in (3.4).

Next we show that the discrete weak formulation is the same as the finite difference scheme introduced in [9].

Make the substitution $v^h = u^h - g^h + a^h\chi^h$, using (4.6c), in (3.4) to write

(4.8) $-B^h[u^h, \psi^h] + B^h[g^h, \psi^h] - B^h[a^h\chi^h, \psi^h]$
$$= F_1^h(\psi^h) + \cdots + F_4^h(\psi^h).$$

We note right away that the second term on the left cancels with $F_2^h(\psi^h)$ on the right. We are going to apply summation by parts to remove the difference operators from the test vector ψ^h. The idea is expressed by the

one-dimensional formula

$$(4.9) \qquad -\sum_{i=0}^{I} \alpha_{i+1/2}(\nabla_x \psi)_{i+1/2} = \sum_{i=1}^{I} (\nabla_x \alpha)_i \psi_i,$$

provided that $\psi_0 = \psi_{I+1} = 0$. Here, $(\nabla_x \alpha)_i = (\alpha_{i+1/2} - \alpha_{i-1/2})/\Delta x$.

Using (4.9), the first term in (4.8) can be rewritten as

$$(4.10\text{a}) \quad -B^h[u^h, \psi^h] = \sum_{i=1}^{I} \sum_{j=1}^{J} [\nabla_x(\beta^1 \nabla_x u)_{i,j} + \nabla_y(\beta^2 \nabla_y u)_{i,j}]\psi_{i,j}\Delta x \Delta y.$$

In the same way, we have using definitions (4.6d), (4.6e),

$$(4.10\text{b}) \quad F_4^h(\psi^h) = -\sum_{i=1}^{I} \sum_{j=1}^{J} b_{i,j}[n_{i,j}^{(1)}(\nabla_x \chi^1)_{i,j} + n_{i,j}^{(2)}(\nabla_y \chi^2)_{i,j}]\psi_{i,j}\Delta x \Delta y$$

To treat the remaining term on the left-hand side of (4.8), we use the following product rule for the difference operator

$$\nabla_x(a\chi)_{i+1/2,j} = (\nabla_x a)_{i+1/2,j}\chi^1_{i+1/2,j} + a^1_{i+1/2,j}(\nabla_x \chi)_{i+1/2,j}$$
$$a^1_{i+1/2,j} = a_{i+1,j}(1 - \theta_{i+1/2,j}) + a_{i,j}\theta_{i+1/2,j},$$

in which $\chi^1_{i+1/2,j}$ and $\theta_{i+1/2,j}$ were defined in (4.6d). Similarly, we have using (4.6e)

$$\nabla_y(a\chi)_{i,j+1/2} = \nabla_y a_{i,j+1/2}\chi^2_{i,j+1/2} + a^2_{i,j+1/2}\nabla_x \chi_{i,j+1/2}$$
$$a^2_{i,j+1/2} = a_{i,j+1}(1 - \theta_{i,j+1/2}) + a_{i,j}\theta_{i,j+1/2}.$$

It follows from this and (4.9) that

$$B^h[a^h\chi^h, \psi^h] + F_3^h(\psi^h)$$

$$(4.10\text{c}) \qquad \begin{aligned} &= \sum_{i=0}^{I} \sum_{j=1}^{J} \beta^1_{i+1/2,j} a^1_{i+1/2,j}(\nabla_x \chi)_{i+1/2,j}(\nabla_x \psi)_{i+1/2,j}\Delta x \Delta y \\ &\quad + \sum_{i=1}^{I} \sum_{j=0}^{J} \beta^2_{i,j+1/2} a^2_{i,j+1/2}(\nabla_y \chi)_{i,j+1/2}(\nabla_y \psi)_{i,j+1/2}\Delta x \Delta y \\ &= -\sum_{i=1}^{I} \sum_{j=1}^{J} [\nabla_x(\beta^1 a^1 \nabla_x \chi)_{i,j} + \nabla_y(\beta^2 a^2 \nabla_x \chi)_{i,j}]\psi_{i,j}\Delta x \Delta y \end{aligned}$$

Combining (4.8), (4.10a), (4.10b), (4.10c), we obtain

$$
\sum_{i=1}^{I}\sum_{j=1}^{J}[\nabla_x(\beta^1\nabla_x u)_{i,j} + \nabla_y(\beta^2\nabla_y u)_{i,j}]\psi_{i,j}\Delta x\Delta y
$$

$$
= -\sum_{i=1}^{I}\sum_{j=1}^{J}[\nabla_x(\beta^1 a^1\nabla_x\chi)_{i,j} + \nabla_y(\beta^2 a^2\nabla_x\chi)_{i,j}]\psi_{i,j}\Delta x\Delta y
$$

$$
-\sum_{i=1}^{I}\sum_{j=1}^{J} b_{i,j}[n_{i,j}^{(1)}\nabla_x(\chi^1)_{i,j} + n_{i,j}^{(2)}\nabla_y(\chi^2)_{i,j}]\psi_{i,j}\Delta x\Delta y
$$

$$
+\sum_{i=1}^{I}\sum_{j=1}^{J} f_{i,j}\psi_{i,j}\Delta x\Delta y.
$$

Since this must hold for all test vectors $\psi^h \in H_0^{1,h}$, we have shown that the finite difference scheme

$$
\nabla_x(\beta^1\nabla_x u)_{i,j} + \nabla_y(\beta^2\nabla_y u)_{i,j} = -\nabla_x(\beta^1 a^1\nabla_x\chi)_{i,j} - \nabla_y(\beta^2 a^2\nabla_x\chi)_{i,j}
$$
$$
- b_{i,j}[n_{i,j}^{(1)}(\nabla_x\chi^1)_{i,j} + n_{i,j}^{(2)}(\nabla_y\chi^2)_{i,j}] + f_{i,j}
$$

holds at all interior grid points. Note that this is the scheme that was found in [9], see equation (77) therein.

Remark 3. Here the passage from the weak formulation via summation by parts to the finite difference scheme is analogous to what is often done with PDE's.

5. CONVERGENCE

In this section, we establish the converge of the scheme (3.4).

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a rectangle. Assume that the data functions a, b, g, f all lie in $C^1(\overline{\Omega})$, with a, b vanishing on $\partial\Omega$. Suppose that the coefficients have the form β Id. Then there is a family of linear extensions $T^h : H_0^{1,h} \to H_0^1(\Omega)$ which together with the bilinear forms B^h (4.3), and the linear functionals F^h (4.7) satisfy the structural conditions (3.1a), (3.1b), (3.2a), (3.2b), (3.3a), (3.3b).*

The sequence of extended approximate solutions $T^h(v^h)$ of the discrete weak problem (3.4), (4.3), (4.7) converge weakly to the weak solution of the PDE (2.6a), (2.6b) in $H_0^1(\Omega)$.

Proof. The second statement follows from the first by Lemma 3.1. The next three subsections are devoted to the verification of the structural conditions: uniform boundedness, extension and approximation, and finally consistency. □

5.1. **Uniform boundedness.**

Lemma 5.1. *The family of bilinear forms* (4.3) *satisfy the uniform bounds* (3.2a).

Proof. Because of our choice of norm in (4.4), the lower bound in (3.2a) is immediate. The upper bound in (3.2a) follows easily from the Cauchy-Schwarz inequality.   □

Lemma 5.2. *The family of linear functionals* (4.7) *satisfy the uniform bound* (3.2b).

Proof. We treat the four pieces $F_1^h, \dots, F_4^h$ in turn.

By the Cauchy-Schwarz inequality, we have

$$|F_1^h(\psi^h)| \leq (\sum_{i,j} |f_{i,j}|^2 \Delta x \Delta y)^{1/2} (\sum_{i,j} |\psi_{i,j}|^2 \Delta x \Delta y)^{1/2}.$$

The first factor is just a Riemann sum for the L^2-norm of f, and so it is bounded by $2\|f\|_{L^2(\Omega)}$, for h small enough. The second factor is estimated using the discrete version of the Poincaré inequality

$$(5.1) \qquad (\sum_{i,j} |\psi_{i,j}|^2 \Delta x \Delta y)^{1/2} \leq C(\Omega) (\sum_{i=0}^{I} \sum_{j=1}^{J} |(\nabla_x \psi)_{i+1/2,j}|^2 \Delta x \Delta y)^{1/2},$$

which follows as in the continuous case using summation by parts. This last sum is then bounded by $m^{-1}\|\psi^h\|_h$ because of the uniform lower bound for the coefficients β.

From (4.6f), we can use the discrete product formula to rewrite F_4^h as

$$F_4^h(\psi^h)$$

$$= \sum_{i=0}^{I} \sum_{j=1}^{J} [(bn^{(1)})_{i+1,j} (\nabla_x \psi)_{i+1/2,j} + \nabla_x (bn^{(1)})_{i+1/2,j} \psi_{i,j}] \chi_{i+1/2,j}^1 \Delta x \Delta y$$

$$+ \sum_{i=1}^{I} \sum_{j=0}^{J} [(bn^{(2)})_{i,j+1} (\nabla_x \psi)_{i,j+1/2} + \nabla_x (bn^{(2)})_{i,j+1/2} \psi_{i,j}] \chi_{i,j+1/2}^2 \Delta x \Delta y$$

Since b and $(n^{(1)}, n^{(2)})$ are in C^1 and (χ^1, χ^2) is bounded, using the Cauchy-Schwarz inequality again, we get the bound

$$|F_4^h(\psi^h)| \leq C\|bn\|_{L^2(\Omega)} \|\psi^h\|_h + C\|\nabla \cdot (bn)\|_{L^2(\Omega)} \left(\sum_{i,j} |\psi_{i,j}|^2 \Delta x \Delta y \right)^{1/2}.$$

Applying (5.1) to the last sum above, we obtain the desired bound for F_4^h.

In the same way, we get,

$$|F_2^h(\psi^h)| \leq C\|\beta \nabla g\|_{L^2(\Omega)} \|\psi^h\|_h \quad \text{and} \quad |F_3^h(\psi^h)| \leq C\|\beta \nabla a\|_{L^2(\Omega)} \|\psi^h\|_h.$$

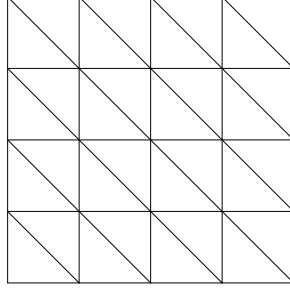   □

FIGURE 1

5.2. **Strong Approximation.** In this section we define a uniformly bounded family of extension operators T^h using the basic approach from the theory of finite elements. Then we verify the strong approximation property.

To this end, for each h consider a triangulation of the domain Ω containing all triangles with vertices

$$\{(x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1})\} \quad \text{or} \quad \{(x_i, y_j), (x_{i-1}, y_j), (x_i, y_{j-1})\}$$

based on the grid Ω^h, as shown in Figure 5.2. For any grid point $(x_i, y_j) \in \Omega^h$, let $\eta_{i,j}^h \in H^1(\Omega)$ be the continuous, piecewise linear function which is equal to 1 at the grid point (x_i, y_j) and equal to 0 at all other grid points. Given $w^h \in H^{1,h}$, the extension operator is defined as

$$T^h(w^h) = \sum_{i,j} w_{i,j} \eta_{i,j}.$$

Then $T^h : H^{1,h} \to H^1(\Omega)$ and $T^h : H_0^{1,h} \to H_0^1(\Omega)$. Although it is not required by the abstract framework, the extensions T^h are linear operators.

Since $T^h(w^h)$ is the unique continuous linear interpolant on each triangle $\{(x_i, y_j), (x_{i\pm1}, y_j), (x_i, y_{j\pm1})\}$, we have explicitly

(5.2)
$$T^h(w^h)(x,y) = w_{i,j} + (\nabla_x w)_{i\pm1/2,j}(x - x_{i\pm1}) + (\nabla_y w)_{i,j\pm1/2}(y - y_{j\pm1}).$$

Given the coefficient function β on Ω, we write using the definition (4.2)

$$\beta^{1,h} = (\beta_{i+1/2,j}^1), \quad \beta^{2,h} = (\beta_{i,j+1/2}^2).$$

In general, given a discrete function $\beta^{1,h}$ defined at the half grid points $(x_{i+1/2}, y_j)$, as above, we define the piecewise constant extension

$$S^{1,h}(\beta^{1,h})(x,y) = \beta_{i+1/2,j}^1$$

on every triangular cell having the horizontal edge from (x_i, y_j) to (x_{i+1}, y_j), see Figure 2. Similarly, given a discrete function $\beta^{2,h}$ defined at the half grid points $(x_i, y_{j+1/2})$, we define
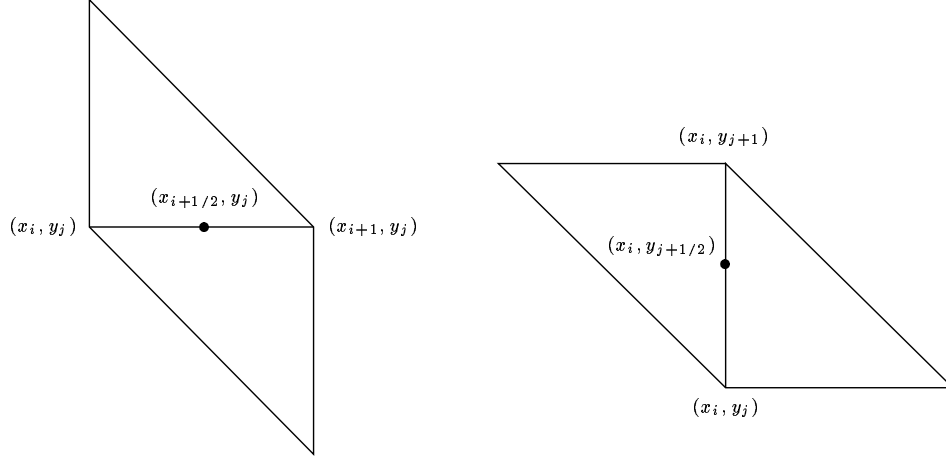
$$S^{2,h}(\beta^{2,h})(x,y) = \beta_{i,j+1/2}^2$$

FIGURE 2

on every triangular cell having the vertical edge from (x_i, y_j) to (x_i, y_{j+1}), see Figure 2.

The key to our estimates will be to replace sums by integrals. The following lemma summarizes the important formulas.

Lemma 5.3. *With the definitions above, we have*

$$(5.3) \qquad B^h[v^h, \psi^h] = \int_\Omega \Big[S^{1,h}(\beta^{1,h}) T^h(v^h)_x T^h(\psi^h)_x$$

$$+ S^{2,h}(\beta^{2,h}) T^h(v^h)_y T^h(\psi^h)_y \Big]$$

$$(5.4) \qquad F_4^h(\psi^h) = \int_\Omega \Big[S^{1,h}(\chi^{1,h}) T^h(b^h n^{(1),h} \psi^h)_x$$

$$+ S^{2,h}(\chi^{2,h}) T^h(b^h n^{(2),h} \psi^h)_y \Big]$$

$$(5.5) \qquad F_3^h(\psi^h) = - \int_\Omega \Big[S^{1,h}(\beta^{1,h}) S^{1,h}(\chi^{1,h}) T^h(a^h)_x T^h(\psi^h)_x$$

$$+ S^{2,h}(\beta^{2,h}) S^{2,h}(\chi^{2,h}) T^h(a^h)_y T^h(\psi^h)_y \Big].$$

Proof. Suppose that $\xi_{i+1/2,j}^1$ is defined for $i = 0, \ldots, I$, $j = 0, \ldots, J+1$, and $\xi_{i+1/2,j}^1 = 0$ for $j = 0$ or $j = J+1$. Each point of the form $(x_{i+1/2}, y_j)$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, lies on the boundary of the two triangular cells sharing the arm from (x_i, y_j) to (x_{i+1}, y_j) , and on these cells we have $\xi_{i+1/2,j}^1 = S^{1,h}(\xi^{1,h})(\text{x,y})$. On the remaining cells, namely those with $j = 0$ or $j = J+1$, we have $\xi_{i+1/2,j}^1 = S^{1,h}(\xi^{1,h}) = 0$. Therefore, since the area of each pair of cells (where $S^{1,h}(\xi^{1,h})$ could be nonzero) is equal to $\Delta x \Delta y$, we

have

$$\sum_{i=0}^{I}\sum_{j=1}^{J}\xi_{i+1/2,j}^{1}\Delta x\Delta y = \int_{\Omega}S^{1,h}(\xi^{1,h}).$$

Similarly, we have

$$\sum_{i=1}^{I}\sum_{j=0}^{J}\xi_{i,j+1/2}^{2}\Delta x\Delta y = \int_{\Omega}S^{2,h}(\xi^{1,h}),$$

provided that $\xi_{i,j+1/2}^{2} = 0$ for $i = 0$ or $i = I + 1$.

Notice that the piecewise continuous extensions $S^{\alpha,h}$ are multiplicative in the sense that, for $\alpha = 1,2$, $S^{\alpha,h}(a^{\alpha,h}b^{\alpha,h}) = S^{\alpha,h}(a^{\alpha,h})S^{\alpha,h}(b^{\alpha,h})$, for arbitrary $a^{\alpha,h}$, $b^{\alpha,h}$ defined at the half grid points. So using the formulas just derived, we have from (4.3) that

$$\begin{aligned}
B^{h}[v^{h},\psi^{h}] &= \int_{\Omega}[S^{1,h}(\beta^{1,h}\nabla_{x}v^{h}\nabla_{x}\psi^{h}) + S^{2,h}(\beta^{2,h}\nabla_{y}v^{h}\nabla_{y}\psi^{h})]\\
&= \int_{\Omega}[S^{1,h}(\beta^{1,h})S^{1,h}(\nabla_{x}v^{h})S^{1,h}(\nabla_{x}\psi^{h})\\
&\qquad\qquad + S^{2,h}(\beta^{2,h})S^{2,h}(\nabla_{y}v^{h})S^{2,h}(\nabla_{y}\psi^{h})].
\end{aligned}$$

By (5.2), we see that $S^{1,h}(\nabla_{x}\psi^{h}) = T^{h}(\psi^{h})_{x}$ and $S^{2,h}(\nabla_{y}\psi^{h}) = T^{h}(\psi^{h})_{y}$, and therefore, we have verified (5.3).

The proofs of the other two formulas are similar and will be omitted. □

We are now ready to study the extensions.

Lemma 5.4. *The family of extensions $T^{h} : H_{0}^{1,h} \to H_{0}^{1}(\Omega)$ satisfy the uniform estimates*

$$\frac{m}{M}\|\psi^{h}\|_{h}^{2} \leq \|T^{h}\psi^{h}\|_{H_{0}^{1}(\Omega)}^{2} \leq \frac{M}{m}\|\psi^{h}\|_{h}^{2}$$

Proof. Let $\psi^{h} \in H_{0}^{1,h}$. By definition, we have

$$\|T^{h}\psi^{h}\|_{H_{0}^{1}(\Omega)}^{2} = B[T^{h}\psi^{h}, T^{h}\psi^{h}] \quad \text{and} \quad \|\psi^{h}\|_{h}^{2} = B^{h}[\psi^{h}, \psi^{h}].$$

Moreover, by (5.3), we have that

$$B^{h}[\psi^{h},\psi^{h}] = \int_{\Omega}[S^{1,h}(\beta^{1,h})T^{h}(\psi^{h})_{x}^{2} + S^{2,h}(\beta^{2,h})T^{h}(\psi^{h})_{y}^{2}].$$

Now by the uniform bounds on β, we have that

$$\beta(x,y) \leq \frac{M}{m}\beta(x',y'),$$

for arbitrary (x,y), $(x',y') \in \Omega$, and so

(5.6) $$\frac{m}{M}\beta \leq S^{1,h}(\beta^{1,h}), S^{2,h}(\beta^{2,h}) \leq \frac{M}{m}\beta.$$

By (5.6), it follows that

$$\frac{m}{M}B^h[\psi^h, \psi^h] \leq B[T^h\psi^h, T^h\psi^h] \leq \frac{M}{m}B^h[\psi^h, \psi^h].$$

$\square$

Lemma 5.5. *The extension operator* $T^h : H_0^{1,h} \to H_0^1(\Omega)$ *has the strong approximation property* (3.1b).

Proof. First let $\psi \in C_0^1(\overline{\Omega})$, and define $\psi^h \in H_0^{1,h}$ to be the grid function with values $\psi_{i,j} = \psi(x_i, y_j)$. Since $\psi \in C^1(\overline{\Omega})$, ψ and its first derivatives are uniformly continuous on $\overline{\Omega}$. It follows from (5.2) the piecewise constant functions $T^h(\psi^h)_x$ and $T^h(\psi^h)_y$ converge uniformly to ψ_x and ψ_y, respectively, therefore also in $L^2(\omega)$:

$$\|\nabla(T^h\psi^h - \psi)\|_{L^2(\Omega)} \to 0,$$

as $h \to 0$. Since β has a uniform lower bound, this implies that

$$\|T^h\psi^h - \psi\|_{H_0^1(\Omega)} \to 0,$$

as $h \to 0$.

Finally, the result for general $\psi \in H_0^1(\Omega)$ follows by density. $\square$

5.3. Weak consistency.

Lemma 5.6. *The bilinear form* (4.3) *satisfies the weak consistency hypothesis* (3.3a).

Proof. Suppose that v^h, $\psi^h \in H_0^{1,h}$ are sequences such that $T^h(v^h) \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $T^h(\psi^h) \to \psi$ strongly in $H_0^1(\Omega)$. Thus, we have that $\nabla T^h(v^h) \rightharpoonup \nabla v$ weakly in $L^2(\Omega)$ and $\nabla T^h(\psi^h) \to \nabla \psi$ strongly in $L^2(\Omega)$.

Recalling the definitions from the previous subsection, we have that the functions $S^{\alpha,h}(\beta^{\alpha,h})$ ($\alpha = 1, 2$) are both uniformly bounded and converge pointwise to β in $\Omega \setminus \Gamma$. Writing

$$S^{1,h}(\beta^{1,h})T^h(\psi^h)_x - \beta\psi_x = [S^{1,h}(\beta^{1,h}) - \beta]\psi_x + S^{1,h}(\beta^{1,h})[T^h(\psi^h)_x - \psi_x],$$

it follows by the dominated convergence theorem that $S^{1,h}(\beta^{1,h})T^h(\psi^h)_x \to \beta\psi_x$ strongly in $L^2(\Omega)$.

Now, using Lemma (5.3) we have that

$$\begin{aligned}
B^h[v^h, \psi^h] &= \int_\Omega [S^{1,h}(\beta^{1,h})T^h(v^h)_x T^h(\psi^h)_x \\
&\qquad\qquad + S^{2,h}(\beta^{2,h})T^h(v^h)_y T^h(\psi^h)_y] \\
&= \langle T^h(v^h)_x, S^{1,h}(\beta^{1,h})T^h(\psi^h)_x \rangle_{L^2(\Omega)} \\
&\qquad\qquad + \langle T^h(v^h)_y, S^{2,h}(\beta^{2,h})T^h(\psi^h)_y \rangle_{L^2(\Omega)} \\
&\to \langle v_x, \beta\psi_x \rangle_{L^2(\Omega)} + \langle v_y, \beta\psi_y \rangle_{L^2(\Omega)} \\
&= B[v, \psi],
\end{aligned}$$

as $h \to 0$. $\square$

Lemma 5.7. *The linear functional F defined in (4.7) satisfies the weak consistency hypothesis (3.3b).*

Proof. Let $\psi \in H_0^1(\Omega)$ and suppose that $\psi^h \in H_0^{1,h}$ is a sequence such that $T^h(\psi^h) \to \psi$ strongly in $H_0^1(\Omega)$. We must show that $F^h(\psi^h) \to F(\psi)$.

To begin, we observe that it is enough to prove this for test functions in $C_0^1(\overline{\Omega})$. For any $\overline{\psi} \in C_0^1(\overline{\Omega})$, we can write

$$|F^h(\psi^h) - F(\psi)| \leq |F^h(\psi^h - \overline{\psi}^h)| + |F^h(\overline{\psi}^h) - F(\overline{\psi})| + |F(\overline{\psi} - \psi)|$$

$$\leq \|F^h\| \|\psi^h - \overline{\psi}^h\|_h + |F^h(\overline{\psi}^h) - F(\overline{\psi})| + \|F\| \, \|\overline{\psi} - \psi\|_{H_0^1(\Omega)}.$$

By Lemma 5.4, we have

$$\|\psi^h - \overline{\psi}^h\|_h \leq \frac{M}{m} \|T^h(\psi^h - \overline{\psi}^h)\|_{H_0^1(\Omega)}$$

$$\leq \frac{M}{m} \left[\|T^h(\psi^h) - \psi\|_{H_0^1(\Omega)} + \|T^h(\overline{\psi}^h) - \overline{\psi}\|_{H_0^1(\Omega)} + \|\overline{\psi} - \psi\|_{H_0^1(\Omega)} \right].$$

By Lemma 5.2, the norms $\|F^h\|$, $\|F\|$ are uniformly bounded. Therefore, we have shown that

$$|F^h(\psi^h) - F(\psi)| \leq |F^h(\overline{\psi}^h) - F(\overline{\psi})| + C \Big[\|T^h(\psi^h) - \psi\|_{H_0^1(\Omega)}$$

$$+ \|T^h(\overline{\psi}^h) - \overline{\psi}\|_{H_0^1(\Omega)} + \|\overline{\psi} - \psi\|_{H_0^1(\Omega)} \Big].$$

We claim that this can be made arbitrarily small for all $h < h_0$. By assumption, we have that $T^h(\psi^h) - \psi \to 0$ in $H_0^1(\Omega)$. By density, $\overline{\psi}$ can be chosen arbitrarily close to ψ in $H_0^1(\Omega)$. By construction, we have that $T^h(\overline{\psi}^h) - \overline{\psi} \to 0$ in $H_0^1(\Omega)$. This covers all but the first term above. The first term can also be made small if the consistency condition is valid for $\overline{\psi} \in C_0^1(\overline{\Omega})$.

We now proceed to verify the consistency of F under the assumption that $\psi \in C_0^1(\overline{\Omega})$ by considering each individual piece.

By (4.6a) and the fact that $\psi \in C_0^1(\overline{\Omega})$, we see that $F_1^h(\psi^h)$ is simply a Riemann sum for $F_1(\psi)$, and thus $F_1^h(\psi^h) \to F_1(\psi)$, as $h \to 0$.

Given $g \in C^1(\overline{\Omega})$, we have that $T^h(g^h) \to g$ in $H_0^1(\Omega)$, as in the proof of Lemma 5.5. Thus, we may apply the result of Lemma 5.6 to conclude that

$$F_2(\psi^h) = B^h[g^h, \psi^h] \to B[g, \psi] = F_4(\psi).$$

According to (5.4), we have

$$F_4^h(\psi^h) \;=\; \int_\Omega [S^{1,h}(\chi^{1,h}) T^h(b^h n^{(1),h} \psi^h)_x + S^{2,h}(\chi^{2,h}) T^h(b^h n^{(2),h} \psi^h)_y].$$

Since $b, n, \psi \in C^1(\overline{\Omega})$, we have as in the proof of Lemma 5.5, that $\nabla T^h(b^h n^h \psi^h) \to \nabla b n \psi$ in L^2. Moreover, $S^{\alpha,h}(\chi^{\alpha,h})$ is uniformly bounded and tends to χ_{Ω^-} pointwise in $\Omega \setminus \Gamma$. Thus, it follows that $F^h(\psi^h) \to F(\psi)$.

The consistency of F_3^h follows from the formula (5.5), in the same way. $\square$

REFERENCES

[1] Ciarlet, P.G. *The finite element method for elliptic problems.* North-Holland, New York (1978).

[2] Johansen, H. and Colella, P., *A Cartesian Grid Embedded Boundary Method for Poisson's Equation on Irregular Domains*, J. Comput. Phys., vol. 147, pp. 60-85 (1998).

[3] Fedkiw, R., Aslam, T., and Xu, Shaojie, *The Ghost Fluid Method for Deflagration and Detonation Discontinuities*, J. Comput. Phys. vol. 154, pp. 393-427 (1999).

[4] Fedkiw, R., Aslam, T., Merriman, B., and Osher, S., *A Non-Oscillatory Eulerian Approach to Interfaces in Multimaterial Flows (The Ghost Fluid Method)*, J. Comput. Phys. vol. 152, pp. 457-492 (1999).

[5] Fedkiw, R., and Liu, X.-D., *The Ghost Fluid Method for Viscous Flows, Progress in Numerical Solutions of Partial Differential Equations*, Arcachon, France, edited by M. Hafez, July 1998.

[6] Hou, T., Li, Z., Osher, S., Zhao, H., *A Hybrid Method for Moving Interface Problems with Application to the Hele-Shaw Flow*, J. Comput. Phys., vol. 134, pp. 236-252 (1997).

[7] LeVeque, R.J. and Li, Z., *The Immersed Interface Method for Elliptic Equations with Discontinuous Coefficients and Singular Sources*, SIAM J. Numer. Anal., vol. 31, pp. 1019-1044 (1994).

[8] Li, Z., *A Fast Iterative Algorithm for Elliptic Interface Problems*, SIAM J. Numer. Anal., vol. 35, no. 1, pp. 230-254, (1998).

[9] X.D. Liu, R. Fedkiw and M. Kang, *A Boundary Condition Capturing Method for Poisson's Equation on Irregular Domains*, J. Comput. Phys. vol. 160, pp. 151-178 (2000).

[10] Mulder, W., Osher, S., and Sethian, J.A., *Computing Interface Motion in Compressible Gas Dynamics*, J. Comput. Phys., vol. 100, pp. 209-228 (1992).

[11] Peskin, C., *Numerical Analysis of Blood Flow in the Heart*, J. Comput. Phys., vol. 25, pp. 220-252 (1977).

[12] Peskin, C. and Printz, B., *Improved Volume Conservation in the Computation of Flows with Immersed Elastic Boundaries*, J. Comput. Phys., vol. 105, pp. 33-46 (1993).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106

E-mail address: xdl@math.ucsb.edu, sideris@math.ucsb.edu