STABILITY IN L^1 OF CIRCULAR VORTEX PATCHES

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Abstract. The motion of incompressible and ideal fluids is studied in the plane. The stability in L^1 of circular vortex patches is established among the class of all bounded vortex patches of equal strength without any restriction on the size of the initial perturbation.

For planar incompressible and ideal fluid flow, the theory of Yudovich [9] establishes global well-posedness of the initial value problem with initial vorticity in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Because vorticity is transported in 2d, it remains constant along particle trajectories. If Φ_t is the flow map, then the vorticity is given by $\omega(t, \Phi_t(y)) = \omega(0, y)$, for all $t > 0$ and $y \in \mathbb{R}^2$. When the initial vorticity is a patch of unit strength, represented by the indicator (characteristic) function I_{Ω_0} of a bounded open set $\Omega_0 \subset \mathbb{R}^2$, the resulting vorticity is I_{Ω_t}, with $\Omega_t = \Phi_t(\Omega_0)$. In the special case when Ω_0 is equal to a ball B, the patch is stationary, $\Phi_t(B) = B$, for all $t > 0$. Theorem 3, our main result, gives the stability in $L^1(\mathbb{R}^2)$ of any circular patch within the class of all bounded vortex patches of equal strength. No restriction is placed on the L^1 distance of the perturbation to the ball, and the flow region is not limited to a bounded domain, but rather is the entire space $\mathbb{R}^2$.

Wan and Pulvirenti [8] were the first to study stability of vortex patches in L^1. They considered the case where the flow was contained in a bounded region, and their key estimate, **(J)**, required the initial perturbation to be close to a circular patch in L^1. Our analog of the inequality, given in Lemma 2, removes this assumption. Weaker stability results were given by Saffman [7] and Dritschel [4]. These authors control the measure of the symmetric difference of two patches through a convenient integral, and this idea is incorporated into our argument in Lemma 1.

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Stability in L^1 does not imply that the boundaries of the two patches remain close in any metric. Indeed, numerical simulations give strong evidence of fingering and filamentation, see [1, 3]. Spreading of vorticity may also occur. The best upper bound for the growth rate of the patch diameter is $\mathcal{O}(t\log t)^{1/4}$ given in [5], see also [6]. Nevertheless, in spite of the fact that the patch geometry may be complicated, smoothness of smooth patch boundaries persists for all time, see [2].

For any bounded open set $A \subset \mathbb{R}^2$, denote its mass, momentum, and moment of inertia by

$$|A| = \int_A dx, \quad M(A) = \int_A x\, dx, \quad \text{and} \quad i(A) = \int_A |x|^2 dx,$$

respectively. Our arguments depend heavily upon the fact that these three quantities are conserved in time when $A = \Omega_t$ is a patch moving with the flow.

Lemma 1. *If $A \subset \mathbb{R}^2$ is any bounded open set, then*

$$i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} \geq 0.$$

Equality holds if and only if the set A is a ball.

Proof. For any ball $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$, introduce the quantity

$$Q = Q(A; B_r(x_0)) = \int_{A \triangle B_r(x_0)} \left| |x - x_0|^2 - r^2 \right| \, dx, \tag{1}$$

in which $A \triangle B_r(x_0) = (A \setminus B_r(x_0)) \cup (B_r(x_0) \setminus A)$ denotes the symmetric difference. Note that $Q \geq 0$ and $Q = 0$ if and only if $A = B_r(x_0)$.

The quantity Q can also be written as

$$Q = \int_A (|x - x_0|^2 - r^2)\, dx + \int_{B_r(x_0)} (r^2 - |x - x_0|^2)\, dx,$$

since the portions of these two integrals over the set $A \cap B_r(x_0)$ cancel.

Now, we can expand the first integral in Q and compute the second to obtain

$$Q = i(A) - 2x_0 \cdot M(A) + (|x_0|^2 - r^2)|A| + \frac{\pi}{2} r^4.$$

A rearrangement of terms gives

$$Q = i(A) - \frac{|A|^2}{2\pi} - \frac{|M(A)|^2}{|A|} + \frac{1}{2\pi}\left(\pi r^2 - |A|\right)^2 + |A| \left| x_0 - \frac{M(A)}{|A|} \right|^2. \tag{2}$$

This last expression is minimized by choosing $B_r(x_0)$ with the same mass and center of mass as A:

$$|B_r(x_0)| = \pi r^2 = |A| \quad \text{and} \quad x_0 = \frac{M(A)}{|A|}.$$

With this choice, the Lemma now follows. $\square$

Lemma 2. *If $B = B_r(0)$, then for any bounded open set A,*

$$\|I_A - I_B\|_{L^1}^2 \leq 4\pi\, Q(A;B) \leq 4\pi \sup_{A\triangle B} \left||x|^2 - r^2\right| \, \|I_A - I_B\|_{L^1},$$

in which $Q(A;B)$ is defined by (1).

Proof. Using the identity (2) and then Lemma 1, we have for any bounded open set A',

$$(|A'| - |B|)^2 = (|A'| - \pi r^2)^2 \leq 2\pi\, Q(A';B).$$

Application of this inequality with $A' = A \cup B$ and $A' = A \cap B$ yields the first part of the Lemma as follows

$$\begin{aligned} \|I_A - I_B\|_{L^1}^2 &= (|A \setminus B| + |B \setminus A|)^2 \\ &\leq 2|A \setminus B|^2 + 2|B \setminus A|^2 \\ &= 2(|A \cup B| - |B|)^2 + 2(|A \cap B| - |B|)^2 \\ &\leq 4\pi\, [Q(A \cup B; B) + Q(A \cap B; B)] \\ &= 4\pi\, Q(A;B). \end{aligned}$$

The second half of the Lemma follows directly from the definition of $Q(A;B)$ given in (1). $\square$

Theorem 3. *Let $B = B_r(0)$. Then for any bounded open set $\Omega_0 \subset \mathbb{R}^2$, we have that*

$$\|I_{\Omega_t} - I_B\|_{L^1}^2 \leq 4\pi \sup_{\Omega_0 \triangle B} \left||x|^2 - r^2\right| \, \|I_{\Omega_0} - I_B\|_{L^1},$$

for all $t > 0$.

Proof. The identity (2) shows that the quantity $Q(\Omega_t;B)$ depends only on conserved quantities, and it is therefore also conserved. Thus, we have that $Q(\Omega_t;B) = Q(\Omega_0;B)$, for all $t > 0$, and so the result follows from Lemma 2. $\square$

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