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# GLOBAL AND ALMOST GLOBAL EXISTENCE OF SMALL SOLUTIONS TO A DISSIPATIVE WAVE EQUATION IN 3D WITH NEARLY NULL NONLINEAR TERMS

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Abstract. The existence of global small $\mathcal{O}(\varepsilon)$ solutions to quadratically nonlinear wave equations in three space dimensions under the null condition is shown to be stable under the simultaneous addition of small $\mathcal{O}(\nu)$ viscous dissipation and $\mathcal{O}(\delta)$ non-null quadratic nonlinearities, provided that $\varepsilon\delta/\nu \ll 1$. When this condition is not met, small solutions exist "almost globally", and in certain parameter ranges, the addition of dissipation enhances the lifespan.

## Contents



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1. **Introduction.** We shall be concerned with the nonlinear PDE

$$\partial_t^2\varphi - \Delta\varphi - \nu\partial_t\Delta\varphi = \sum_{\alpha,\beta=0}^{3}\sum_{\ell=1}^{3} C^{\ell}_{\alpha,\beta}\partial_\alpha\varphi\partial_\ell\partial_\beta\varphi, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.1}$$

in which $\partial_0 = \partial_t$, $\partial_\ell = \partial_{x_\ell}$, $\ell = 1,2,3$, and $\mathbb{R}^+ = [0,\infty)$. Greek indices will always range from 0 to 3 and Latin indices from 1 to 3. Hereafter, we shall employ the summation convention. The viscosity parameter will be assumed to satisfy $0 < \nu \le 1$. The equation (1.1) serves as a simplified, dimensionless model for viscoelasticity.

In Theorem B of [8], Ponce proved that the initial value problem for (1.1) has a unique, global, strong solution for initial data that is sufficiently small. His argument is based on the dissipative properties of the linear equation, and although he does not quantify it, the size of the initial data must be small relative to the viscosity parameter. On the other hand, in the hyperbolic case $\nu = 0$, when the nonlinear terms satisfy the Klainerman null condition, there exist global small solutions, see [5], [1]. One would expect this result to be stable under viscous perturbations, and moreover, based on Ponce's result one also expects that the introduction of small viscosity would allow a simultaneous nonlinear perturbation from the null condition. However, proving this requires the by no means routine adaptation of hyperbolic methods to dissipative equations such as (1.1).

The main result of this paper, Theorem 4.1, shows that these suppositions do indeed hold. We define a parameter $\delta$ which measures the deviation of the nonlinearity from being null, and we quantify the limitation on the size of the initial data, roughly $\nu/\delta$, leading to global existence. We also obtain in Theorem 4.3 lower bounds for the lifespan of solutions in cases where our global existence result does not hold. In the hyperbolic case, it is well-known that the lifespan $T(\varepsilon)$ of small solutions of size $\varepsilon$ satisfies an "almost global" lower bound of the form $[T(\varepsilon)]^\varepsilon \ge C > 1$, see [3], [4]. We show that dissipative effects can improve the almost global lifespan of the hyperbolic case if the viscosity $\nu$ is large enough relative to the size of the data.

These results require decay estimates which are uniform with respect to $\nu$, given in Sections 8 and 9. The derivation of the decay estimates extends the weighted $L^2$ approach, introduced in [6] and refined in [10], to the case of partial dissipation, at the expense of introducing space-time weighted norms. To implement the method of [10], it is helpful to reformulate the problem as a first order system, and this is done in the next section. The decay estimates are coupled with energy estimates based on the translational, rotational, and scaling vector fields, derived for two distinct energy levels in Sections 10 and 11. The energy estimates are also nonstandard insofar as occurrences of the scaling vector field are indexed separately because the linear equation is not scaling invariant. A short discussion of local existence is provided in Section 12, for completeness.

There is an extensive literature on existence of solutions to dissipative wave equations in three space dimensions. However, the authors are aware of only one other work, [7], with uniformity in the viscosity parameter. There the underlying nonlinear hyperbolic system is Hamiltonian with a positive definite conserved energy, and the nonlinearity is bounded at infinity. It is possible to use the dissipation to establish local well-posedness in spaces of low regularity and to prove global existence for large data.

2. **PDEs.** It is convenient to rewrite equation (1.1) in first order form. To do so, let $e_0, \dots, e_3$ be the standard basis on $\mathbb{R}^4$ (viewed as column vectors), and define the new unknowns

$$u = u^\alpha e_\alpha = \partial_\alpha \varphi e_\alpha. \tag{2.1a}$$

We shall denote the spatial and space-time gradients of $u$ as $\nabla u$ and $\partial u$, respectively. Thus, we have

$$(\nabla u)_{\alpha k} = \partial_k u^\alpha \quad \text{and} \quad (\partial u)_{\alpha\beta} = \partial_\beta u^\alpha,$$

as well as the relations

$$\partial u = \partial u^\top. \tag{2.1b}$$

We obtain from (1.1), (2.1a), (2.1b), the evolutionary system

$$Lu \equiv \partial_t u - A^j \partial_j u - \nu B \Delta u = N(u, \nabla u), \tag{2.2a}$$

together with the constraints

$$\partial_j u^k = \partial_k u^j. \tag{2.2b}$$

The coefficients are given by

$$A^j = e_0 \otimes e_j + e_j \otimes e_0,\ j = 1, 2, 3; \quad B = e_0 \otimes e_0 \tag{2.2c}$$

and the nonlinearity has the form

$$N(u, \nabla u) = N^0(u, \nabla u) e_0, \quad \text{with} \quad N^0(u, \nabla v) = C^\ell_{\alpha,\beta} u^\alpha \partial_\ell v^\beta. \tag{2.2d}$$

Conversely, given a solution $u$ of (2.2a)-(2.2d), we can recover a solution of $\varphi$ of (1.1) with $\partial \varphi^\top = u$ using

$$\varphi(t, x) = c + \int_0^1 \langle (t, x), u(st, sx) \rangle_{\mathbb{R}^4} ds.$$

The system (2.2a)-(2.2d) can be written more explicitly as

$$\partial_t u^0 - \nabla \cdot \bar{u} - \nu \Delta u^0 = N^0(u, \nabla u) \tag{2.3a}$$

$$\partial_t \bar{u} - (\nabla u^0)^\top = 0 \tag{2.3b}$$

$$\nabla \wedge \bar{u} = 0, \tag{2.3c}$$

in which $\bar{u} = u^j e_j$ is viewed as an element of $\mathbb{R}^3$.

3. **Notation.** We shall employ the vector fields

$$\nabla, \quad \Omega = x \wedge \nabla, \quad S = t\partial_t + r\partial_r, \quad S_0 = r\partial_r. \tag{3.1}$$

We avoid the vector field $\partial_t$ because in order to control $\partial_t^k u(0)$, the second order operator $L$ forces the initial data to have $2k$ derivatives, which clashes with the hyperbolic case $\nu = 0$. The Lorentz rotations $x\partial_t + t\nabla$ are likewise unsuitable. Although the system (2.2a), (2.2b) is not scale invariant, it is possible to effectively use the scaling operator $S$. However, its usage will be indexed separately (see the space $X^{p,q}$ and the energy $\mathcal{E}_{p,q}$ defined below). The rotational operators $\Omega$ are modified in the usual way when used with vector-valued functions, consistent with the rotational invariance of the linear system. Thus, we take

$$\widetilde{\Omega}_i = I\Omega_i + Z_i$$

in which

$$Z_1 = e_2 \otimes e_3 - e_3 \otimes e_2,\ Z_2 = e_3 \otimes e_1 - e_1 \otimes e_3,\ Z_3 = e_1 \otimes e_2 - e_2 \otimes e_1.$$

This definition is dictated by the fact that

$$\partial(\Omega_i\varphi) = \widetilde{\Omega}_i\partial\varphi,$$

for scalar functions $\varphi$. We emphasize that these operators are vectorial, and we use the notation $(\widetilde{\Omega}_i u)^\alpha$ to denote the $\alpha$-th component of the vector $\widetilde{\Omega}_i u \in \mathbb{R}^4$. However, notice that

$$(\widetilde{\Omega}_i u)^0 = \Omega_i u^0,$$

and so all of the vector fields act as scalars on the 0-th component of vectors $u \in \mathbb{R}^4$. We shall frequently rely on the decomposition

$$\nabla = \omega\partial_r - \frac{\omega}{r}\wedge\Omega, \quad \omega = \frac{x}{r}, \quad r = |x|, \quad \partial_r = \omega^j\partial_j. \tag{3.2}$$

It will be convenient to use the abbreviation

$$\Gamma = \{\nabla, \widetilde{\Omega}\},$$

however, the fields $S_0$ and $S$ will always be tracked individually. Thus for example, given integers $0 \le q \le p$, we define the space

$$X^{p,q} = \{u \in H^p(\mathbb{R}^3;\mathbb{R}^4) : \|S_0^k\Gamma^a u\|_{L^2} < \infty, \qquad \text{for all } |a| + k \le p,\ k \le q\}.$$

This is a Hilbert space with the inner product

$$\langle u, v\rangle_{X^{p,q}} = \sum_{\substack{|a|+k\le p\\ k\le q}} \langle S_0^k\Gamma^a u, S_0^k\Gamma^a v\rangle_{L^2}.$$

The index $p$ indicates the total number of allowable derivatives while the index $q$ limits the number of occurrences of $S_0$ which, in practice, could be strictly less than the total $p$. These spaces characterize the initial data that we shall consider, and they may be used to establish local well-posedness for our system for appropriate pairs $(p, q)$, see Section 12.

Given a solution of (2.2a)-(2.2d), our main objective will be to obtain *a priori* estimates for the energy

$$\mathcal{E}_{p,q}[u](t) = \sum_{\substack{|a|+k\le p\\ k\le q}} \left[\frac{1}{2}\|S^k\Gamma^a u(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla S^k(\Gamma^a u)^0(s)\|_{L^2}^2\, ds\right],$$

for two sets of pairs $(p, q)$, informally referred to as high and low. If $u(0) = u_0$, it will be convenient to write

$$\mathcal{E}_{p,q}[u_0] \equiv \mathcal{E}_{p,q}[u](0) = \frac{1}{2}\|u_0\|_{X^{p,q}}^2,$$

(with a slight abuse of notation). Hidano [2] has also used energy norms with a limitation on the scaling operator.

To get *a priori* bounds for the energy, it will be necessary to also obtain dispersive estimates. These will be derived using weighted $L^2$-estimates in two space-time regions which we shall refer to as the interior and exterior regions.

We now define cut-off functions which determine these regions and the quantities which in Section 9 will be shown to have good decay properties. Given a function

$$\psi \in C^\infty(\mathbb{R}), \quad \psi(s) = \begin{cases} 1, & s \le 1/2 \\ 0, & s \ge 1 \end{cases}, \quad \psi' \le 0, \tag{3.3a}$$

we define
$$\zeta(t,x)=\psi\left(\frac{|x|}{\sigma\langle t\rangle}\right)\quad\text{and}\quad \eta(t,x)=1-\psi\left(\frac{2|x|}{\sigma\langle t\rangle}\right), \tag{3.3b}$$
where we have used the common notation $\langle t\rangle=(1+t^2)^{1/2}$. The parameter $\sigma\ll 1$ will be chosen in Lemma 8.1. We have
$$\zeta(t,x)=\begin{cases}1, & |x|\le\sigma\langle t\rangle/2\\ 0, & |x|\ge\sigma\langle t\rangle\end{cases}\quad\text{and}\quad \eta(t,x)=\begin{cases}0, & |x|\le\sigma\langle t\rangle/4\\ 1, & |x|\ge\sigma\langle t\rangle/2\end{cases}.$$
This is not a partition of unity. We can say that
$$1\le\zeta+\eta\quad\text{and}\quad 1-\eta\le\zeta^2. \tag{3.3c}$$
We shall frequently rely on the property that
$$\langle r+t\rangle\Big[|\partial\zeta(t,x)|+|\partial\eta(t,x)|\Big]\lesssim 1. \tag{3.3d}$$

In the interior region, we shall derive estimates in $L^1(\langle t\rangle^\theta dt)$, with $0\le\theta\le 1$, for the quantities
$$\mathcal{Y}^{\text{int}}_{p,q}[u](t)=\sum_{\substack{|a|+k\le p-1\\ k\le q}}\|\zeta\nabla S^k\Gamma^a u(t)\|^2_{L^2}$$
and
$$\mathcal{Z}^{\text{int}}_{p,q}[u](t)=\sum_{\substack{|a|+k\le p-1\\ k\le q}}\|\zeta\Delta S^k(\Gamma^a u)^0(t)\|^2_{L^2},$$
for $q<p$, see Theorem 9.1.

In the exterior region, it is critical to decompose the solution into its orthogonal and tangential components along the light cone, $\mathbb{P}u$ and $\mathbb{Q}u$, respectively. These projections are defined as follows:
$$\mathbb{P}u(t,x)=\frac{1}{2}\,\hat\omega\otimes\hat\omega\; u(t,x)=\frac{1}{2}\left[u^0(t,x)-\omega\cdot\bar u(t,x)\right]\hat\omega \tag{3.4a}$$
$$\mathbb{Q}u(t,x)=(I-\mathbb{P})u(t,x),$$
in which
$$\hat\omega=\begin{bmatrix}1\\-\omega\end{bmatrix}\in\mathbb{R}^4,\quad \omega=\frac{x}{|x|},\quad 0\ne x\in\mathbb{R}^3. \tag{3.4b}$$
Thanks to the fact that $\widetilde\Omega_j\omega=0$ and $\partial_r\omega=0$, we obtain
$$[\widetilde\Omega_j,\mathbb{P}]=[\partial_r,\mathbb{P}]=0\quad\text{and}\quad[\widetilde\Omega_j,\mathbb{Q}]=[\partial_r,\mathbb{Q}]=0. \tag{3.5}$$
The quantities to be bounded in Theorem 9.2 are
$$\mathcal{Y}^{\text{ext}}_{p,q}[u](t)=\sum_{\substack{|a|+k\le p-1\\ k\le q}}\sum_{j=1}^{3}\big[\|\eta\,\langle t-r\rangle\,\mathbb{P}\partial_j S^k\Gamma^a u(t)\|^2_{L^2}$$
$$+\|\eta\,\langle t+r\rangle\,\mathbb{Q}\partial_j S^k\Gamma^a u(t)\|^2_{L^2}\big]$$
and
$$\mathcal{Z}^{\text{ext}}_{p,q}[u](t)=t^2\sum_{\substack{|a|+k\le p-1\\ k\le q}}\|\eta\Delta S^k(\Gamma^a u)^0(t)\|^2_{L^2},$$
again with $q<p$.

Given a quadratic nonlinearity $N$ of the form (2.2d), we associate to it a cubic polynomial

$$P_N(y) = C^{\ell}_{\alpha,\beta} y^\alpha y^\beta y^\ell, \quad y \in \mathbb{R}^4.$$

We say that $N$ is null if

$$P_N(y) = 0, \qquad \text{for all} \quad y \in \mathcal{N} = \{y \in \mathbb{R}^4 : y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0\},$$

where $\mathcal{N}$ is the collection of null vectors in $\mathbb{R}^4$.

As a final notational remark, we shall write $A \lesssim B$ if there exists a generic constant $C$, independent of the initial data and the parameters $\nu$, $\varepsilon^2$, $\delta$ (the latter two defined in Theorem 4.1) such that $A \le CB$. Constants may depend on $\max_{\alpha,\beta,\ell} |C^{\ell}_{\alpha,\beta}|$. The symbol $\mathcal{O}(B)$ denotes any quantity such that $\mathcal{O}(B) \lesssim B$.

## 4. Main Results.

**Theorem 4.1** (Global existence)**.** *Choose $(p, q)$ such that $p \ge 11$, and $p \ge q > p^*$, where $p^* = \left[\frac{p+5}{2}\right]$. Define*

$$\delta = \max\{|\Omega^a P_N(y)| : y \in \mathcal{N},\ \|y\| = 1,\ |a| \le p^*\} \tag{4.1}$$

*and assume that $\delta \le 1$.*

*There are positive constants $C_0$, $C_1 > 1$ with the property that if the initial data $u_0$ satisfies*

$$C_0 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}^{1/2}_{p,q}[u_0]\right) < \varepsilon^2, \tag{4.2a}$$

*for some $\varepsilon^2 \ll 1$, and*

$$C_0^3 \left(\frac{\delta}{\nu}\right)^2 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}^{1/2}_{p,q}[u_0]\right) < 1, \tag{4.2b}$$

*then (2.2a)-(2.2d) has a unique global solution*

$$u \in C(\mathbb{R}^+; X^{p,q})$$

*with $u(0, x) = u_0(x)$,*

$$\sup_{0 \le t < \infty} \mathcal{E}_{p,q}[u](t) \le C_1 \mathcal{E}_{p,q}[u_0] \langle t \rangle^{C_1 \varepsilon} \tag{4.3a}$$

*and*

$$\sup_{0 \le t < \infty} \mathcal{E}_{p^*,p^*}[u](t) < \varepsilon^2. \tag{4.3b}$$

**Remarks.**

- The assumption (4.2a) only requires the norm $\|u_0\|_{X^{p,q}}$ to be finite, but $\|u_0\|_{X^{p^*,p^*}}$ must be correspondingly small.
- The parameter $\delta$ measures the deviation of the nonlinearity from the null condition, see the remark following Lemma 5.3. If $\delta = 0$, then (4.2a), (4.2b) hold for all $0 < \nu \le 1$, and the existence criterion is uniform in $\nu$, consistent with the hyperbolic case $\nu = 0$.
- The restriction $p \ge 11$ is made so that $p^* + 3 \le p$ holds, see Proposition 11.1.
- Successive restrictions on the size of the parameter $\varepsilon$ arise in Theorem 9.1 and in Propositions 10.1 and 11.1. Generic constants are not permitted to depend on $\varepsilon$, so it will be possible to decrease the size of $\varepsilon$ when necessary.

- As soon as the initial data $u_0$ meets the criterion (4.2a) for a single sufficiently small $\varepsilon$, one can take the infimum over all such $\varepsilon$. As a consequence, the bounds (4.3a), (4.3b) hold with $\varepsilon^2$ replaced by

$$C_0 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right).$$

*Outline of Proof.* Given data $u_0 \in X^{p,q}$ satisfying (2.2b), it can be shown using Picard iteration that the IVP for (2.2a)-(2.2d) has a local solution $u \in C([0,T); X^{p,q})$ where $T$ depends only on $\|u_0\|_{X^{p,q}}$, and the fixed constants $\nu$ and $C^{\ell}_{\alpha,\beta}$, see Section 12.

To establish global existence, it is enough to prove that $\|u(t)\|_{X^{p,q}}$ remains finite. This norm can not be directly bounded by $\mathcal{E}_{p,q}^{1/2}[u](t)$ because the $X^{p,q}$ norm is based on $S_0$ while $\mathcal{E}_{p,q}^{1/2}[u](t)$ uses $S$. However, the norm $\|u(t)\|_{X^{p,q}}$ can be controlled by a function which depends only on $\|u_0\|_{X^{p,q}}$, $\|u(t)\|_{X^{p,0}}$, and $T$, see Section 12. By definition,

$$\frac{1}{2}\|u(t)\|^2_{X^{p,0}} \leq \mathcal{E}_{p,0}[u](t) \leq \mathcal{E}_{p,q}[u](t).$$

Thus, it is enough to show that the energy $\mathcal{E}_{p,q}[u](t)$ remains finite.

Given fixed initial data $u_0$ for which (2.2b), (4.2a), and (4.2b) hold, let $\mathcal{T}$ be the set of times $T \in (0,\infty)$ satisfying the properties:

(P1) Equations (2.2a)-(2.2d) have a unique local solution $u \in C([0,T); X^{p,q})$, with $u(0) = u_0$, and

(P2) $\mathcal{E}_{p^*,p^*}[u](t) < \varepsilon^2$, for $0 \leq t < T$.

If $T \in \mathcal{T}$, then $(0,T) \subset \mathcal{T}$, so the set $\mathcal{T}$ is connected. Since $C_0 > 1$, the local existence result and (4.2a) imply that the set $\mathcal{T}$ is nonempty. The set $\mathcal{T}$ is relatively closed in $(0,\infty)$.

If $T \in \mathcal{T}$, then by Proposition 10.1 and (P2)

$$\sup_{0\leq t<T} \mathcal{E}_{p,q}[u](t) \leq C_1 \mathcal{E}_{p,q}[u_0] \langle T\rangle^{C_1\varepsilon} < \infty,$$

so by the local existence theorem, (P1) holds for some $T' > T$. Using the assumptions (4.2a), (4.2b), and (P2) with Proposition 11.1, we get

$$\sup_{0\leq t<T} \mathcal{E}_{p^*,p^*}[u](t) \leq C_0 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) < \varepsilon^2,$$

and so we have by continuity that (P2) holds for $0 \leq t < T''$, with $T < T'' \leq T'$. This shows that $(0,T'') \subset \mathcal{T}$, and so $\mathcal{T}$ is open. The nonempty connected set $\mathcal{T}$ is both open and closed in $(0,\infty)$, and therefore equal to $(0,\infty)$.

□

**Corollary 4.2.** *The solution given in Theorem 4.1 satisfies the estimates*

$$\int_0^\infty \langle t\rangle^\theta \left[\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) + \nu^2 \mathcal{Z}^{\text{int}}_{p^*,p^*-1}[u](t)\right] dt \lesssim \varepsilon^2,$$

*for all* $0 < \theta < 1$, *and*

$$\sup_{0\leq t<\infty} \left[\mathcal{Y}^{\text{ext}}_{p^*,p^*-1}[u](t) + \nu^2 \mathcal{Z}^{\text{ext}}_{p^*,p^*-1}[u](t)\right] \lesssim \varepsilon^2.$$

*Proof.* The first inequality follows from Theorem 9.1 and Proposition 10.1, and the second follows from Theorem 9.2. □

The next result establishes "almost global" existence of small solutions in the case where the second smallness condition (4.2b) does not hold.

**Theorem 4.3** (Almost global existence). *Choose $(p,q)$ with $p \geq 11$ and $p \geq q > p^*$, where $p^* = \left[\frac{p+5}{2}\right]$. Define $\delta \leq 1$ by (4.1).*

*There are positive constants $C_0$, $C_1 > 1$ with the property that if the initial data $u_0$ satisfies (4.2a), for some $\varepsilon^2 \ll 1$, then (2.2a)-(2.2d) has a unique solution*

$$u \in C([0,T_0); X^{p,q})$$

*with $T_0$ defined by*

$$C_1 \langle T_0 \rangle^{C_1 \varepsilon} = \left( \frac{2 \, \max\{\nu, C_1 \varepsilon\}}{C_0 \, \delta \, \mathcal{E}^{1/2}_{p^*,p^*}[u_0]} \right)^2$$

*and*

$$\sup_{0 \leq t < T_0} \mathcal{E}_{p^*,p^*}[u](t) < \varepsilon^2.$$

*Proof.* Suppose that $u_0$ satisfies (4.2a), for $\varepsilon^2 \ll 1$. Consider the set

$$\mathcal{T} = \{T \in (0, T_0) : \text{(P1) and (P2) hold}\}.$$

The set $\mathcal{T}$ is nonempty, connected, and closed relative to $(0, T_0)$.

If $T \in \mathcal{T}$, then Proposition 11.2 and (4.2a) imply that

$$\sup_{0 \leq t < T} \mathcal{E}_{p^*,p^*}[u](t) \leq C_0 \mathcal{E}_{p^*,p^*}[u_0] \left( 1 + \mathcal{E}^{1/2}_{p,q}[u_0] \right) < \varepsilon^2.$$

Thus, $\mathcal{T}$ is open relative to $(0, T_0)$. By connectness, $\mathcal{T} = (0, T_0)$. $\square$

**Remarks.**

- The following table summarizes the results of the Theorems. The basic smallness restriction (4.2a) must always be enforced.

| | | |
|---|---|---|
| $\dfrac{\varepsilon\,\delta}{\nu} < \dfrac{1}{C_0}$ | Global existence | (4.5a) |
| $\dfrac{1}{C_0} < \dfrac{\varepsilon\,\delta}{\nu} < \dfrac{\delta}{C_1}$ | Almost global existence with diffusion enhanced lifespan | (4.5b) |
| $\dfrac{\delta}{C_1} < \dfrac{\varepsilon\,\delta}{\nu}$ | Almost global existence with hyperbolic lifespan | (4.5c) |

- The cases (4.5a), (4.5b) show that diffusive effects are important when $\nu \geq C_1 \varepsilon$. The constants $C_0$, $C_1$, depend on $\max |C^\ell_{\alpha,\beta}|$, and we have not verified that the parameter range in (4.5b) is nontrivial. In any case, it is clear from (4.5a)–(4.5c) that the quantity $\varepsilon\delta/\nu$ controls the transition from global to almost global existence.

For the remainder of the article, we assume that properties (P1) and (P2) hold. In the following sections, we are going to establish a series of *a priori* estimates culminating in Propositions 10.1, 11.1, and 11.2.

5. **Commutation.** Recall the linear operator defined in (2.2a), (2.2c)

$$L = I\partial_t - A^j\partial_j - \nu B\Delta.$$

For any multi-index $a$ and any integer $k \geq 0$, we have

$$LS^k\Gamma^a u = (S+1)^k\Gamma^a Lu - \nu B\Delta[S^k - (S-1)^k]\Gamma^a u, \tag{5.1a}$$

$$\nabla \wedge S^k\Gamma^a\bar{u} = (S+1)^k\Gamma^a\nabla \wedge \bar{u}. \tag{5.1b}$$

The appearance of additional Laplacian terms on the right-hand side of (5.1a) reflects the lack of scaling invariance for the operator $L$.

For the nonlinear form (2.2d), we define the commutators

$$\begin{aligned}
[\partial_i, N](u, \nabla v) &= \partial_i N(u, \nabla v) - N(\partial_i u, \nabla v) - N(u, \nabla\partial_i v)\\
[(S+1), N](u, \nabla v) &= SN(u, \nabla v) - N(Su, \nabla v) - N(u, \nabla Sv)\\
[\Omega_i, N](u, \nabla v) &= \Omega_i N(u, \nabla v) - N(\widetilde{\Omega}_i u, \nabla v) - N(u, \nabla\widetilde{\Omega}_i v).
\end{aligned}$$

**Lemma 5.1.** *The nonlinear commutators satisfy the relations*

$$[\partial, N] = [S, N] = 0$$

*and*

$$[\Omega_i, N](u, \nabla v) = \widetilde{C}^{i,j}_{\alpha,\beta}u^\alpha\partial_j v^\beta, \tag{5.2}$$

*with*

$$\widetilde{C}^{i,j}_{\alpha,\beta} = C^j_{\lambda,\beta}(Z_i)_{\alpha\lambda} + C^j_{\alpha,\lambda}(Z_i)_{\beta\lambda} + C^\lambda_{\alpha,\beta}(Z_i)_{j\lambda}.$$

**Remark.** The rotationally invariant case is characterized by the conditions $[\Omega_i, N] = 0$, $i = 1, 2, 3$.

The higher order commutators $[(S+1)^k\Gamma^a, N]$ are defined inductively, each being a nonlinear form of the type (2.2d). Of course, by Lemma 5.1 a nonlinear commutator could be nonzero only for pure $\widetilde{\Omega}$ derivatives. Using the higher order commutators, we have a Leibnitz-type formula:

**Lemma 5.2.**

$$\begin{aligned}
&(S+1)^k\Gamma^a N(u, \nabla v)\\
&\qquad = \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k}} \frac{a!}{a_1!\, a_2!\, a_3!}\frac{k!}{k_1!\, k_2!}[\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1}u, \nabla S^{k_2}\Gamma^{a_2}v).
\end{aligned}$$

**Remark.** Here again we emphasize that $[\Gamma^{a_3}, N] = 0$, unless the derivative $\Gamma^{a_3}$ involves only $\widetilde{\Omega}$.

**Lemma 5.3.** *For any quadratic nonlinearity of the form* (2.2d)

$$\Omega_i P_N(y) = P_{[\Omega_i, N]}(y), \quad i = 1, 2, 3.$$

*If $N$ is null, then $[\Omega_i, N]$ is also null.*

*Proof.* We have $\widetilde{\Omega}_i y = \Omega_i y + Z_i y = 0$, for all $y \in \mathbb{R}^4$. Thus, from the chain rule and (5.2) we obtain the first statement:

$$\Omega_i P_N(y) = D_y P_N(y)[\Omega_i y] = D_y P_N(y)[-Z_i y] = P_{[\Omega_i, N]}(y).$$

Suppose that $N$ is null. The one-parameter family of rotations $U(s) = \exp(-sZ_i)$ leaves the set of null vectors $\mathcal{N}$ invariant. Thus, for any $y \in \mathcal{N}$, we have

$$0 = \frac{d}{ds} P_N(U(s)y)\bigg|_{s=0} = D_y P_N(y)[-Z_i y] = P_{[\Omega_i, N]}(y).$$

This shows that $P_{[\Omega_i,N]}$ is also null. $\square$

**Remark.** If $N$ is null, then since the operators $\Omega_i$ act tangentially along $\mathcal{N}$, we have $\Omega^a P_N(y) = 0$, for all $a$, and the parameter $\delta$ defined in Theorem 4.1 vanishes.

**Lemma 5.4.** *For any $|a| \le p^*$, we have*

$$\begin{aligned}\langle [\widetilde{\Omega}^a, N](u(x), \nabla v(x)), w(x)\rangle_{\mathbb{R}^4} \\ = \tfrac{1}{4} P_{[\widetilde{\Omega}^a,N]}(\hat{\omega})\ \langle \hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\ \langle \hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4}\ w^0(x) + \mathcal{O}(\mathcal{R}),\end{aligned} \tag{5.3a}$$

*with*

$$\mathcal{R} = \Big[|\mathbb{Q}u(x)|\ |\partial_r v(x)| + |u(x)|\ |\mathbb{Q}\partial_r v(x)| + r^{-1}|u(x)|\ |\Omega v(x)|\ \Big]|w^0(x)|,$$

*and also*

$$\begin{aligned}\Big|\tfrac{1}{4} P_{[\widetilde{\Omega}^a,N]}(\hat{\omega})\ \langle \hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\ \langle \hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4}\ w^0(x)\Big| \\ \lesssim \delta\ |u(x)|\ |\partial_r v(x)|\ |w^0(x)|,\end{aligned} \tag{5.3b}$$

*with $\delta$ defined in (4.1).*

*Proof.* By (2.2d), $N = N^0 e_0$, so using (3.2), we can write

$$\begin{aligned}&\langle [\widetilde{\Omega}^a, N](u(x), \nabla v(x)), w^0(x)\rangle_{\mathbb{R}^4} \\ &\qquad = \langle [\widetilde{\Omega}^a, N](u(x), \omega \otimes \partial_r v(x)), w(x)\rangle_{\mathbb{R}^4} \\ &\qquad\qquad + \mathcal{O}\left(r^{-1}|u(x)||\Omega v(x)||w^0(x)|\right).\end{aligned}$$

With the projections defined in (3.4a), (3.4b), we obtain

$$\begin{aligned}&[\widetilde{\Omega}^a, N](u(x), \omega \otimes \partial_r v(x)) = [\widetilde{\Omega}^a, N](\mathbb{P}u(x), \omega \otimes \mathbb{P}\partial_r v(x)) \\ &\qquad + [\widetilde{\Omega}^a, N](\mathbb{Q}u(x), \omega \otimes \mathbb{P}\partial_r v(x)) + [\widetilde{\Omega}^a, N](u(x), \omega \otimes \mathbb{Q}\partial_r v(x)).\end{aligned}$$

The key term is

$$[\widetilde{\Omega}^a, N](\mathbb{P}u(x), \omega \otimes \mathbb{P}\partial_r v(x)) = \frac{1}{4} P_{[\widetilde{\Omega}^a,N]}(\hat{\omega})\langle \hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\langle \hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4},$$

from which (5.3a) now easily follows.

Notice that Lemma 5.3 gives

$$P_{[\widetilde{\Omega}^a,N]}(\hat{\omega}) = \Omega^a P_N(\hat{\omega}).$$

Now $\hat{\omega}/\sqrt{2}$ belongs to $\{\|y\|_{\mathbb{R}^4} = 1\} \cap \mathcal{N}$, so by homogeneity we have

$$|\Omega^a P_N(\hat{\omega})| \le 2^{3/2}\delta,$$

and (5.3b) follows. $\square$

6. **Sobolev Inequalities.**

**Lemma 6.1.** *Suppose that $u \in X^{2,0}$. Set $r = |x|$. Then*

$$\|u\|_{L^\infty} \lesssim \sum_{|a|\leq 2} \|\nabla^a u\|_{L^2} \tag{6.1a}$$

$$\|r^{-1}u\|_{L^2} \lesssim \|\partial_r u\|_{L^2} \tag{6.1b}$$

$$\|r^{1/2}u\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla \widetilde{\Omega}^a u\|_{L^2} \tag{6.1c}$$

$$\sup_{|x|=r} r|u(x)| \lesssim \left( \sum_{|a|\leq 1} \|\partial_r \widetilde{\Omega}^a u\|_{L^2(|y|\geq r)} \sum_{|a|\leq 2} \|\widetilde{\Omega}^a u\|_{L^2(|y|\geq r)} \right)^{1/2}. \tag{6.1d}$$

*Proof.* The inequality (6.1a) is the standard Sobolev lemma, and (6.1b) is Hardy's inequality. Inequalities (6.1c) and (6.1d) were proven in Lemma 3.3 of [9]. □

**Proposition 6.2.** *Suppose that $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^4$ satisfies*

$$\mathcal{Y}^{\rm int}_{2,0}[u](t) + \mathcal{Y}^{\rm ext}_{2,0}[u](t) + \mathcal{E}_{2,0}[u](t) < \infty.$$

*Then using the weights (3.3b), we have*

$$\|\, \zeta\, u(t)\, \|_{L^\infty} \lesssim \left(\mathcal{Y}^{\rm int}_{2,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{1,0}[u](t) \tag{6.2a}$$

$$\|\, r\zeta\, \nabla u(t)\, \|_{L^\infty} \lesssim \left(\mathcal{Y}^{\rm int}_{3,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{2,0}[u](t) \tag{6.2b}$$

$$\|\, r^{-1}\zeta\, u(t)\, \|_{L^2} \lesssim \left(\mathcal{Y}^{\rm int}_{1,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{0,0}[u](t) \tag{6.2c}$$

$$\|\, \eta\, u(t)\|_{L^\infty} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{2,0}[u](t) \tag{6.2d}$$

$$\|\, \eta\, \mathbb{Q}u(t)\|_{L^\infty} \lesssim \langle t\rangle^{-3/2}\left(\left(\mathcal{Y}^{\rm ext}_{2,0}[u](t)\right)^{1/2} + \mathcal{E}^{1/2}_{1,0}[u](t)\right). \tag{6.2e}$$

*Proof.* Using the cutoff function $\psi$ defined in (3.3a), apply (6.1a) to $\psi u(t)$. This produces

$$\|\psi u(t)\|_{L^\infty} \lesssim \sum_{|a|=1,2} \|\nabla^a u(t)\|_{L^2} + \|u(t)\|_{L^2(|y|\leq 1)}.$$

We apply (6.1c) to the second integral

$$\|u(t)\|_{L^2(|y|\leq 1)} \lesssim \|r^{1/2}u(t)\|_{L^\infty}\|r^{-1/2}\|_{L^2(|y|\leq 1)} \lesssim \sum_{|a|\leq 1} \|\nabla\widetilde{\Omega}^a u(t)\|_{L^2}.$$

Thus, we see that

$$\|\psi u(t)\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla \Gamma^a u(t)\|_{L^2}.$$

On the other hand, we have using (6.1c) again

$$\|(1-\psi)u(t)\|_{L^\infty} \lesssim \|r^{1/2}u(t)\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla\widetilde{\Omega}^a u(t)\|_{L^2}.$$

This shows that

$$\|u(t)\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla \Gamma^a u(t)\|_{L^2}. \tag{6.3}$$

(Clearly, the same bound holds for $\|\langle r\rangle^{1/2}u(t)\|_{L^\infty}$, but we do not need it.)

To prove (6.2a), apply (6.3) to the function $\zeta u(t)$, and use (3.3d).

Applying (6.1d) to $\zeta\nabla u(t)$ yields (6.2b).

The inequality (6.2c) follows by applying (6.1b) to $\zeta u(t)$.

Since

$$\langle t\rangle\|\eta u(t)\|_{L^\infty} \lesssim \|r\eta u(t)\|_{L^\infty},$$

we can get (6.2d), by applying (6.1d) to $\eta u(t)$.

Finally, we prove (6.2e). By (6.1d) applied to $\eta\mathbb{Q}u(t)$, we have

$$\begin{aligned}\langle t\rangle\|\eta\mathbb{Q}u(t)\|_{L^\infty} &\lesssim \|r\eta\mathbb{Q}u\|_{L^\infty}\\ &\lesssim \left(\sum_{|a|\le 1}\|\partial_r\widetilde{\Omega}^a\eta\mathbb{Q}u(t)\|_{L^2}\sum_{|a|\le 2}\|\widetilde{\Omega}^a\eta\mathbb{Q}u(t)\|_{L^2}\right)^{1/2}. \qquad (6.4)\end{aligned}$$

Using (3.3d) and the commutation property (3.5), we see that

$$\sum_{|a|\le 1}\|\partial_r\widetilde{\Omega}^a\eta\mathbb{Q}u(t)\|_{L^2} \lesssim \sum_{|a|\le 1}\|\eta\mathbb{Q}\partial_r\widetilde{\Omega}^a u(t)\|_{L^2} + \langle t\rangle^{-1}\sum_{|a|\le 1}\|\mathbb{Q}\widetilde{\Omega}^a u(t)\|_{L^2}.$$

By linearity, we have $\mathbb{Q}\partial_r = \mathbb{Q}\omega^j\partial_j = \omega^j\mathbb{Q}\partial_j$, so

$$\sum_{|a|\le 1}\|\eta\mathbb{Q}\partial_r\widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \sum_{|a|\le 1}\sum_{j=1}^{3}\|\eta\mathbb{Q}\partial_j\widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \langle t\rangle^{-1}\mathcal{Y}^{\rm ext}_{2,0}[u](t)^{1/2}.$$

Since

$$\sum_{|a|\le 1}\|\mathbb{Q}\widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \sum_{|a|\le 1}\|\widetilde{\Omega}^a u(t)\|_{L^2} \le \mathcal{E}^{1/2}_{1,0}[u](t),$$

we obtain the bound

$$\sum_{|a|\le 1}\|\partial_r\widetilde{\Omega}^a\eta\mathbb{Q}u(t)\|_{L^2} \lesssim \langle t\rangle^{-1}(\mathcal{Y}^{\rm ext}_{2,0}[u](t)^{1/2} + \mathcal{E}^{1/2}_{1,0}[u](t)).$$

Noting that

$$\sum_{|a|\le 2}\|\widetilde{\Omega}^a\eta\mathbb{Q}u(t)\|_{L^2} = \sum_{|a|\le 2}\|\eta\mathbb{Q}\widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \sum_{|a|\le 2}\|\widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \mathcal{E}^{1/2}_{2,0}[u](t),$$

we deduce from (6.4)

$$\langle t\rangle\|\eta\mathbb{Q}u(t)\|_{L^\infty} \lesssim \left(\langle t\rangle^{-1}\left(\mathcal{Y}^{\rm ext}_{2,0}[u](t)^{1/2} + \mathcal{E}^{1/2}_{2,0}[u](t)\right)\ \mathcal{E}^{1/2}_{2,0}[u](t)\right)^{1/2},$$

from which (6.2e) follows by Young's inequality. □

## 7. **Calculus Inequalities.**

**Lemma 7.1.** *Suppose that* $u : [0,T)\times\mathbb{R}^3 \to \mathbb{R}^4$. *If*

$$k_1 + k_2 + |a_1| + |a_2| \le \bar{p} \quad \text{and} \quad k_1 + k_2 \le \bar{q},$$

*then for* $\alpha,\beta = 0,\dots,3$, *we have*[1]

$$\begin{aligned}&\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\ &\lesssim \left(\left(\mathcal{Y}^{\rm int}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+3}{2}],[\frac{\bar{p}}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[u](t),\end{aligned}$$

*provided the right-hand side is finite.*

[1] We remind the reader that $(S^{k_1}\Gamma^{a_1}u(t))^\alpha$ denotes the $\alpha$-th component of the vector $S^{k_1}\Gamma^{a_1}u(t)$.

*In the special case when $k_2 + |a_2| < \bar{p}$, we have*

$$\begin{aligned}
&\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\
&\quad\lesssim \left(\left(\mathcal{Y}^{\mathrm{int}}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+3}{2}],[\frac{\bar{p}}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar{p},\bar{q}}[u](t),
\end{aligned}$$

*provided the right-hand side is finite.*

*Proof.* In the case $k_1 + |a_1| < k_2 + |a_2| + 1$, i.e. $k_1 + |a_1| \le \left[\frac{\bar{p}}{2}\right]$, using the Sobolev inequality (6.2a) we have the following bound:

$$\begin{aligned}
&\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\
&\quad\lesssim \|\zeta S^{k_1}\Gamma^{a_1}u(t)\|_{L^\infty}\|S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^2}\\
&\quad\lesssim \left(\left(\mathcal{Y}^{\mathrm{int}}_{[\frac{\bar{p}+4}{2}],[\frac{\bar{p}}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+2}{2}],[\frac{\bar{p}}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[u](t).
\end{aligned}$$

And in the case $k_2 + |a_2| + 1 \le k_1 + |a_1|$, i.e. $k_2 + |a_2| \le \left[\frac{\bar{p}-1}{2}\right]$, we likewise have

$$\begin{aligned}
&\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\
&\quad\lesssim \|S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^\infty}\|S^{k_1}\Gamma^{a_1}u(t)\|_{L^2}\\
&\quad\lesssim \left(\left(\mathcal{Y}^{\mathrm{int}}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}-1}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+3}{2}],[\frac{\bar{p}-1}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar{p},\bar{q}}[u](t).
\end{aligned}$$

The second statement of the lemma follows similarly from the preceding arguments. □

**Lemma 7.2.** *Suppose that $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^4$. If*

$$k_1 + k_2 + |a_1| + |a_2| \le \bar{p} \quad \text{and} \quad k_1 + k_2 \le \bar{q},$$

*then for $\alpha, \beta = 0, \ldots, 3$, we have*

$$\begin{aligned}
&\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\
&\quad\lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[u](t)\mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[u](t),
\end{aligned}$$

*provided the right-hand side is finite.*

*In the special case when $k_2 + |a_2| < \bar{p}$, we have*

$$\begin{aligned}
&\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\
&\quad\lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[u](t)\mathcal{E}^{1/2}_{\bar{p},\bar{q}}[u](t),
\end{aligned}$$

*provided the right-hand side is finite.*

*Proof.* In the case $k_1 + |a_1| < k_2 + |a_2| + 1$, i.e. $k_1 + |a_1| \le \left[\frac{\bar{p}}{2}\right]$, using the Sobolev inequality (6.2d) we have the following bound:

$$\begin{aligned}
&\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}\\
&\quad\lesssim \|\eta S^{k_1}\Gamma^{a_1}u(t)\|_{L^\infty}\|S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^2}\\
&\quad\lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar{p}+4}{2}],[\frac{\bar{p}}{2}]}[u](t)\mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[u](t).
\end{aligned}$$

And in the case $k_2 + |a_2| + 1 \le k_1 + |a_1|$, i.e. $k_2 + |a_2| \le \left[\frac{\bar{p}-1}{2}\right]$, we similarly have:

$$\begin{aligned}
&\|\eta(S^{k_1}\Gamma^{a_1}u(t))^{\alpha}(S^{k_2}\Gamma^{a_2+1}u(t))^{\beta}\|_{L^2} \\
&\qquad\lesssim \|\eta S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^\infty}\|S^{k_1}\Gamma^{a_1}u(t)\|_{L^2} \\
&\qquad\lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{\bar{p}+5}{2}\right],\left[\frac{\bar{p}-1}{2}\right]}[u](t)\mathcal{E}^{1/2}_{\bar{p},\bar{q}}[u](t).
\end{aligned}$$

The second statement of the lemma follows analogously. □

Two slightly more specialized instances of this basic argument occur in the proof of Proposition 11.1.

8. **Estimates for the Linear Equation.** In this section, we focus on the estimation of solutions of the linear version of the system (2.3a), (2.3b), (2.3c):

$$\partial_t u^0 - \nabla\cdot\bar{u} - \nu\Delta u^0 = G \tag{8.1a}$$

$$\partial_t\bar{u} - (\nabla u^0)^\top = 0 \tag{8.1b}$$

$$\nabla\wedge\bar{u} = 0. \tag{8.1c}$$

**Lemma 8.1.** *Assume that $\sigma$ in (3.3b) is sufficiently small and that $\nu \le 1$. Let $G \in L^2([0,T];L^2(\mathbb{R}^3))$, for some $0 < T < \infty$. If $u = (u^0,\bar{u})$ is a solution of (8.1a), (8.1b), (8.1c) such that*

$$\sup_{0\le t\le T}\mathcal{E}_{1,1}[u](t) < \infty,$$

*then for any $0 \le \theta \le 1$,*

$$\begin{aligned}
&\int_0^T \langle t\rangle^{\theta}\left[\|\zeta\nabla u(t)\|^2_{L^2} + \nu^2\|\zeta\Delta u^0(t)\|^2_{L^2}\right]dt \\
&\qquad\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[u](T) + \int_0^T\langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[u](t)dt \\
&\qquad\qquad + \int_0^T\langle t\rangle^{\theta}\|\zeta G(t)\|^2_{L^2}dt.
\end{aligned}$$

*Proof.* By definition (3.1), the PDEs (8.1a), (8.1b) can be written as

$$\begin{aligned}
t(\nabla\cdot\bar{u} + \nu\Delta u^0) &= Su^0 - r\partial_r u^0 - tG^0 \\
t\nabla u^0 &= S\bar{u} - r\partial_r\bar{u}.
\end{aligned}$$

Squaring and integrating with respect to $\zeta^2 dx$, we obtain

$$\begin{aligned}
&t^2\left[\|\zeta\nabla\cdot\bar{u}\|^2_{L^2} + 2\nu\langle\zeta\nabla\cdot\bar{u},\zeta\Delta u^0\rangle_{L^2} + \nu^2\|\zeta\Delta u^0\|^2_{L^2} + \|\zeta\nabla u^0\|^2_{L^2}\right] \\
&\qquad\lesssim \|\zeta r\partial_r u\|^2_{L^2} + \mathcal{E}_{1,1}[u](t) + t^2\|\zeta G\|^2_{L^2},
\end{aligned} \tag{8.2}$$

where for notational convenience, we have suppressed the $t$ dependence of the integrated terms.

Thanks to (8.1b), the cross term can be rewritten as

$$\begin{aligned}
2\nu\langle\zeta\nabla\cdot\bar{u},\zeta\Delta u^0\rangle_{L^2} &= 2\nu\langle\zeta\nabla\cdot\bar{u},\zeta\partial_t(\nabla\cdot\bar{u})\rangle_{L^2} \\
&= \nu\partial_t\|\zeta\nabla\cdot\bar{u}\|^2_{L^2} - 2\nu\|\partial_t(\zeta^2)\nabla\cdot\bar{u}\|^2_{L^2},
\end{aligned}$$

which can be estimated below by

$$\nu\partial_t\|\zeta\nabla\cdot\bar{u}\|^2_{L^2} - \frac{1}{2}\|\zeta\nabla\cdot\bar{u}\|^2_{L^2} - \frac{C\nu^2}{\langle t\rangle^2}\mathcal{E}_{1,0}[u](t),$$

using (3.3d) and Young's inequality. With this, (8.2) yields

$$t^2\left[\nu\partial_t\|\zeta\nabla\cdot\bar u\|_{L^2}^2+\frac12\|\zeta\nabla\cdot\bar u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\|\zeta\nabla u^0\|_{L^2}^2\right]\\ \lesssim \|\zeta r\partial_r u\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2.$$

Choose $0\le\theta\le1$, multiply the preceding inequality by $\langle t\rangle^{\theta-2}$, integrate in time, and rearrange the result:

$$\begin{aligned}\int_0^T t^2\langle t\rangle^{\theta-2}&\left[\frac12\|\zeta\nabla\cdot\bar u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\|\zeta\nabla u^0\|_{L^2}^2\right]dt\\ &\lesssim\int_0^T\langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r u\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2\right]dt\\ &\qquad-\int_0^T t^2\langle t\rangle^{\theta-2}\nu\partial_t\|\zeta\nabla\cdot\bar u\|_{L^2}^2dt.\end{aligned}\tag{8.3}$$

We now focus on time derivative term on the right. A simple calculation reveals that

$$\frac{d}{dt}\nu t^2\langle t\rangle^{\theta-2}=\nu t\langle t\rangle^{\theta-4}(2+\theta t^2)\le2\nu t\langle t\rangle^{\theta-2}\le\frac14t^2\langle t\rangle^{\theta-2}+4\nu^2\langle t\rangle^{\theta-2},$$

and so, using integration by parts, we get

$$\begin{aligned}-\int_0^T t^2\langle t\rangle^{\theta-2}&\nu\partial_t\|\zeta\nabla\cdot\bar u\|_{L^2}^2dt\\ &\le\int_0^T\left(\frac14t^2\langle t\rangle^{\theta-2}+4\nu^2\langle t\rangle^{\theta-2}\right)\|\zeta\nabla\cdot\bar u\|_{L^2}^2dt\\ &\le\int_0^T\frac14t^2\langle t\rangle^{\theta-2}\|\zeta\nabla\cdot\bar u\|_{L^2}^2dt+C\int_0^T\langle t\rangle^{\theta-2}\mathcal{E}_{1,0}[u](t)dt.\end{aligned}$$

Combining this with the inequality (8.3), we obtain

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\frac14\|\zeta\nabla\cdot\bar u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\|\zeta\nabla u^0\|_{L^2}^2\right]dt\\ \lesssim\int_0^T\langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r u\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2\right]dt.$$

Next, thanks to Lemma 8.2 (to follow), we gain control of the full gradient on the left:

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\|\zeta\nabla u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2\right]dt\\ \lesssim\int_0^T\langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r u\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2\right]dt.$$

Then, inserting $t^2=\langle t\rangle^2-1$ on the left, we may write

$$\int_0^T\langle t\rangle^{\theta}\left[\|\zeta\nabla u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2\right]dt\\ \lesssim\int_0^T\langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2\right]dt.$$

According to definition (3.3b), we have $r \le \sigma\langle t\rangle$ on the support of $\zeta$, so for $\sigma$ sufficiently small, the term

$$\int_0^T \langle t\rangle^{\theta-2}\|\zeta r\partial_r u\|_{L^2}^2 dt \lesssim \int_0^T \sigma^2\langle t\rangle^{\theta}\|\zeta\partial_r u\|_{L^2}^2 dt$$

can be absorbed on the left. This key step yields

$$\int_0^T \langle t\rangle^{\theta}\left[\|\zeta\nabla u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2\right] dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}\left[\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2\right] dt. \quad (8.4)$$

Finally, consider the first term on the right. We have

$$\begin{aligned}\int_0^T \langle t\rangle^{\theta-2}\nu^2\|\zeta\Delta u^0\|_{L^2}^2 dt &= \int_0^T \langle t\rangle^{\theta-2}\nu^2\frac{d}{dt}\int_0^t \|\zeta\Delta u^0\|_{L^2}^2 ds dt\\ &= \langle T\rangle^{\theta-2}\nu^2\int_0^T \|\zeta\Delta u^0\|_{L^2}^2 dt\\ &\quad + (2-\theta)\int_0^T t\langle t\rangle^{\theta-4}\nu^2\int_0^t \|\zeta\Delta u^0\|_{L^2}^2 ds dt\\ &\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[u](T)+\nu\int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,0}[u](t)dt.\end{aligned}$$

Insert this into (8.4). The desired estimate follows immediately since $\nu \le 1$. □

**Remark.** Notice that the time integration used in this lemma arises from the dissipative term in the equation.

In the proof of Lemma 8.1, we used the following simple coercivity estimate:

**Lemma 8.2.** *If $w \in H^1(\mathbb{R}^3,\mathbb{R}^3)$ and $\nabla\wedge w = 0$, then*

$$\frac{1}{2}\|\zeta\nabla w\|_{L^2}^2 - \|\zeta\nabla\cdot w\|_{L^2}^2 \lesssim \langle t\rangle^{-2}\|w\|_{L^2}^2.$$

*Proof.* The constraint $\nabla\wedge w = 0$ implies that

$$|\nabla w|^2 - (\nabla\cdot w)^2 = \partial_i(w^j\partial_j w^i) - \partial_j(w^j\partial_i w^i).$$

Using integration by parts and the property (3.3d), we may write

$$\|\zeta\nabla w\|_{L^2}^2 - \|\zeta\nabla\cdot w\|_{L^2}^2 \lesssim \langle t\rangle^{-1}\|\zeta\nabla w\|_{L^2}\|w\|_{L^2}.$$

The result follows by an application of Young's inequality. □

We now establish a higher order version of Lemma 8.1.

**Proposition 8.3.** *Assume that $\sigma$ in (3.3b) is sufficiently small and that $\nu \le 1$. Fix $0 \le q < p$. Suppose that*

$$S^k G \in L^2([0,T]; X^{p-k-1,0}), \quad k = 0,\dots,q$$

*for some $0 < T < \infty$. If $u$ is a solution of (8.1a), (8.1b), (8.1c) such that*

$$\sup_{0\le t\le T} \mathcal{E}_{p,q+1}[u](t) < \infty,$$

*then for any* $0 \le \theta \le 1$*,*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}_{p,q}^{\rm int}[u](t) + \nu^2 \mathcal{Z}_{p,q}^{\rm int}[u](t)\right] dt \\ \lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p,q}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p,q+1}[u](t)dt \\ + \sum_{\substack{|a|+k\le p-1\\ k\le q}} \int_0^T \langle t\rangle^\theta \|\zeta S^k\Gamma^a G(t)\|_{L^2}^2 dt.$$

*Proof.* We shall prove this by induction on $q$. Fix a multi-index $|a| \le p-1$. By the commutation relations (5.1a), (5.1b), we have that $\Gamma^a u$ solves (8.1a), (8.1b) with $\Gamma^a G$ on the right. Apply Lemma 8.1 to get

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla\Gamma^a u(t)\|_{L^2}^2 + \nu^2\|\zeta\Delta\Gamma^a u^0(t)\|_{L^2}^2\right] dt \\ \lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[\Gamma^a u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[\Gamma^a u](t)dt \\ + \int_0^T \langle t\rangle^\theta \|\zeta\Gamma^a G(t)\|_{L^2}^2 dt.$$

Summing over $|a| \le p-1$ gives the result for $q = 0$.

Now take any $1 \le r < p$, and assume that the result holds when $q = r-1$. Choose $a$ and $k$ such that $k \ne 0$, $|a| + k \le p-1$, and $k \le r$. By (5.1a), (5.1b), we have that $S^k\Gamma^a u$ solves (8.1a), (8.1b) with

$$(S+1)^k\Gamma^a G - \nu B\Delta[S^k - (S-1)^k]\Gamma^a u$$

on the right. Apply Lemma 8.1 again:

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla S^k\Gamma^a u(t)\|_{L^2}^2 + \nu^2\|\zeta\Delta S^k\Gamma^a u^0(t)\|_{L^2}^2\right] dt \\ \lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[S^k\Gamma^a u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[S^k\Gamma^a u](t)dt \\ + \int_0^T \langle t\rangle^\theta \left[\nu^2\|\zeta\Delta[S^k - (S-1)^k](\Gamma^a u)^0\|_{L^2}^2 + \|\zeta S^k\Gamma^a G(t)\|_{L^2}^2\right] dt \\ \lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[S^k\Gamma^a u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[S^k\Gamma^a u](t)dt \\ + \int_0^T \langle t\rangle^\theta \left[\nu^2\mathcal{Z}_{p,r-1}^{\rm int}[u](t) + \|\zeta S^k\Gamma^a G(t)\|_{L^2}^2\right] dt.$$

Notice that this inequality holds for $k=0$, as well, by the result for $q=0$. Perform a summation over $|a|+k\le p-1$, $k\le r$. This yields

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p,r}[u](t)+\nu^2\mathcal{Z}^{\rm int}_{p,r}[u](t)\right]dt$$
$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p,r}[u](T)+\int_0^T\langle t\rangle^{\theta-2}\mathcal{E}_{p,r+1}[u](t)dt$$
$$+\sum_{\substack{|a|+k\le p-1\\ k\le r}}\int_0^T\langle t\rangle^\theta\left[\nu^2\mathcal{Z}^{\rm int}_{p,r-1}[u](t)+\|\zeta S^k\Gamma^a G(t)\|^2_{L^2}\right]dt.$$

The result now follows by the induction hypothesis. ☐

Next, we turn our attention to the exterior region.

**Lemma 8.4.** *Let $G\in C([0,T];L^2(\mathbb{R}^3))$, for some $0<T<\infty$. If $u=(u^0,\bar u)$ is a solution of (8.1a), (8.1b), (8.1c) such that*

$$\sup_{0\le t\le T}\mathcal{E}_{1,1}[u](t)<\infty,$$

*then for all $0\le t\le T$,*

$$\|\eta(r\partial_r u^0+t\nabla\cdot\bar u)\|^2_{L^2}+\|\eta(r\partial_r\bar u+t\nabla u^0)\|^2_{L^2}+(\nu t)^2\|\eta\Delta u^0\|^2_{L^2}$$
$$\lesssim\mathcal{E}_{1,1}[u](t)+t^2\|\eta G\|^2_{L^2}.$$

*Proof.* As in the proof of Lemma 8.1, write

$$r\partial_r u^0+t\nabla\cdot\bar u+t\nu\Delta u^0=Su^0-tG^0$$
$$r\partial_r\bar u+t\nabla u^0=S\bar u,$$

square, and integrate with respect to $\eta^2dx$

$$\|\eta(r\partial_r u^0+t\nabla\cdot\bar u)\|^2_{L^2}+\|\eta(r\partial_r\bar u+t\nabla u^0)\|^2_{L^2}+(\nu t)^2\|\eta\Delta u^0\|^2_{L^2}$$
$$+2\langle\eta(r\partial_r u^0+t\nabla\cdot\bar u),\eta t\nu\Delta u^0\rangle_{L^2}$$
$$\lesssim\|\eta Su\|^2_{L^2}+t^2\|\eta G\|^2_{L^2}. \quad (8.5)$$

We focus on the cross term on the left. Write

$$I=2\langle\eta(r\partial_r u^0+t\nabla\cdot\bar u),\eta t\nu\Delta u^0\rangle_{L^2}=2\nu t\int\eta^2(r\partial_r u^0+t\nabla\cdot\bar u)\Delta u^0dx.$$

The result will follow from (8.5) once we verify that

$$|I|\le\mu(\nu t)^2\|\eta\Delta u^0\|^2_{L^2}+C\mathcal{E}_{1,1}[u](t), \quad (8.6)$$

with say $\mu\le 1/2$.

Using (3.1), (3.2), (8.1b), we have

$$\begin{aligned}\nabla\cdot\bar u&=\omega\cdot\partial_r\bar u-\left(\frac{\omega}{r}\wedge\Omega\right)\cdot\bar u\\&=\omega\cdot\left(\frac1r S\bar u-\frac tr\partial_t\bar u\right)-\left(\frac{\omega}{r}\wedge\Omega\right)\cdot\bar u\\&=-\frac tr\partial_r u^0+\mathcal{O}\left(\frac1r(|\Omega u|+|Su|)\right).\end{aligned}\quad (8.7)$$

So we may write

$$I = 2\nu t \int \eta^2 \left[ \left(1 - \frac{t^2}{r^2}\right) r\partial_r u^0 \Delta u^0 + \mathcal{O}\left(\frac{t}{r}(|\Omega u| + |Su|)\right) \Delta u^0 \right] dx.$$

Insertion of the identity

$$r\partial_r u^0 \Delta u^0 = \nabla \cdot \left[ r\partial_r u^0 \nabla u^0 - \tfrac{1}{2} x |\nabla u^0|^2 \right] + \tfrac{1}{2} |\nabla u^0|^2,$$

followed by integration by parts yields

$$\begin{aligned} I = -2\nu t \int \nabla \left[ \eta^2 \left(1 - \frac{t^2}{r^2}\right)\right] \cdot \left[ r\partial_r u^0 \nabla u^0 - \frac{1}{2} x |\nabla u^0|^2 \right] dx \\ + 2\nu t \int \eta^2 \ \left[ \frac{1}{2} |\nabla u^0|^2 + \mathcal{O}\left(\frac{t}{r}(|\Omega u| + |Su|)\right) \Delta u^0 \right] dx. \end{aligned}$$

By (3.3d) and the fact that $r \gtrsim \langle t \rangle$ on the support of $\eta$, we see that

$$r \left| \nabla \left[ \eta^2 \left(1 - \frac{t^2}{r^2}\right)\right] \right| \lesssim \eta$$

and hence

$$|I| \lesssim \nu t \int \eta |\nabla u^0|^2 dx + \nu t \int \eta (|\Omega u| + |Su|) |\Delta u^0| dx.$$

Using integration by parts and (3.3d), we get

$$\begin{aligned} \nu t \int \eta |\nabla u^0|^2 dx &= -\nu t \int (\eta u^0 \Delta u^0 + u^0 \nabla \eta \cdot \nabla u^0) \, dx \\ &\lesssim \nu t \int \eta |u^0 \Delta u^0| dx + \nu \mathcal{E}_{1,1}[u](t), \end{aligned}$$

so

$$|I| \lesssim \nu t \int \eta (|u^0| + |\Omega u| + |Su|) |\Delta u^0| dx + C\mathcal{E}_{1,1}[u](t).$$

The estimate (8.6) for $I$ now follows by Young's inequality.

□

**Proposition 8.5.** *Fix $0 \le q \le p$. Suppose that*

$$S^k G \in C([0,T], X^{p-k-1,0}), \quad k = 0, \dots, q-1,$$

*for some $0 < T < \infty$. If $u = (u^0, \bar{u})$ is a solution of (8.1a), (8.1b), (8.1c) such that*

$$\sup_{0 \le t \le T} \mathcal{E}_{p,q+1}[u](t) < \infty,$$

*then for all $0 \le t \le T$,*

$$\mathcal{Y}^{\text{ext}}_{p,q}[u](t) + \nu^2 \mathcal{Z}^{\text{ext}}_{p,q}[u](t) \lesssim \mathcal{E}_{p,q+1}[u](t) + \sum_{\substack{|a|+k \le p-1 \\ k \le q}} t^2 \|\eta S^k \Gamma^a G\|^2_{L^2}.$$

*Proof.* First, we note that from (3.4a), and (3.2), we have for each $j$,

$$|\mathbb{P}\partial_j u|^2_{\mathbb{R}^4} = \frac{1}{2}(\partial_j u^0 - \omega \cdot \partial_j \bar{u})^2 \le \frac{1}{2}(\partial_r u^0 - \omega \cdot \partial_r \bar{u})^2 + \mathcal{O}(\frac{1}{r^2}|\Omega u|^2),$$

and by (8.1c),

$$\begin{aligned}|\mathbb{Q}\partial_j u|^2_{\mathbb{R}^4} &= |(I-\mathbb{P})\partial_j u|^2_{\mathbb{R}^4}\\ &= \frac{1}{2}(\partial_j u^0 + \omega\cdot\partial_j \bar{u})^2 + |\omega\wedge\partial_j\bar{u}|^2_{\mathbb{R}^3}\\ &= \frac{1}{2}(\partial_j u^0 + \omega\cdot\partial_j \bar{u})^2 + |\omega\wedge\nabla u^j|_{\mathbb{R}^3}\\ &\le \frac{1}{2}(\partial_r u^0 + \omega\cdot\partial_r \bar{u})^2 + \mathcal{O}(\frac{1}{r^2}|\Omega u|^2).\end{aligned}$$

Therefore, since $r \gtrsim \langle t+r\rangle \ge \langle t-r\rangle$ on $\operatorname{supp}\eta$, we obtain

$$\begin{aligned}\mathcal{Y}^{\text{ext}}_{1,0}[u](t) =& \sum_{j=1}^{3}[\|\eta\langle t-r\rangle\mathbb{P}\partial_j u\|^2_{L^2} + \|\eta\langle t+r\rangle\mathbb{Q}\partial_j u\|^2_{L^2}]\\ \le& \frac{1}{2}\|\eta\langle t-r\rangle(\partial_r u^0 - \omega\cdot\partial_r\bar{u})\|^2_{L^2}\\ &+\frac{1}{2}\|\eta\langle t+r\rangle(\partial_r u^0 + \omega\cdot\partial_r\bar{u})\|^2_{L^2} + C\|\Omega u\|^2_{L^2}\\ \le& \frac{1}{2}\|\eta(t-r)(\partial_r u^0 - \omega\cdot\partial_r\bar{u})\|^2_{L^2}\\ &+\frac{1}{2}\|\eta(t+r)(\partial_r u^0 + \omega\cdot\partial_r\bar{u})\|^2_{L^2} + C[\|\nabla u\|^2_{L^2} + \|\Omega u\|^2_{L^2}].\end{aligned}$$

A bit of algebraic manipulation produces the relation

$$\begin{aligned}&\frac{1}{2}(t-r)^2(\partial_r u^0 - \omega\cdot\partial_r\bar{u})^2 + \frac{1}{2}(t+r)^2(\partial_r u^0 + \omega\cdot\partial_r\bar{u})^2\\ &\qquad\qquad = (r\partial_r u^0 + t\omega\cdot\partial_r\bar{u})^2 + (r\omega\cdot\partial_r\bar{u} + t\partial_r u^0)^2.\end{aligned}$$

Thanks to (3.2), we have

$$\eta|r\partial_r u^0 + t\omega\cdot\partial_r\bar{u}| \le \eta|r\partial_r u^0 + t\nabla\cdot\bar{u}| + \mathcal{O}(|\Omega\bar{u}|)$$

and

$$\eta|r\omega\cdot\partial_r\bar{u} + t\partial_r u^0| \le \eta|r\partial_r\bar{u} + t\nabla u^0|_{\mathbb{R}^3} + \mathcal{O}(|\Omega u^0|).$$

Combining the preceding estimates gives us the bound

$$\mathcal{Y}^{\text{ext}}_{1,0}[u](t) \lesssim \|\eta(r\partial_r u^0 + t\nabla\cdot\bar{u})\|^2_{L^2} + \|\eta(r\partial_r\bar{u} + t\nabla u^0)\|^2_{L^2} + \mathcal{E}_{1,0}[u](t).$$

By Lemma 8.4, we conclude that

$$\mathcal{Y}^{\text{ext}}_{1,0}[u](t) + \nu^2\mathcal{Z}^{\text{ext}}_{1,0}[u](t) \lesssim \mathcal{E}_{1,1}[u](t) + t^2\|\eta G\|^2_{L^2}.$$

Now take any multi-index $a$ with $|a| \le p-1$. By (5.1a), we can apply the preceding inequality to $\Gamma^a u$ to get

$$\mathcal{Y}^{\text{ext}}_{1,0}[\Gamma^a u](t) + \nu^2\mathcal{Z}^{\text{ext}}_{1,0}[\Gamma^a u](t) \lesssim \mathcal{E}_{1,1}[\Gamma^a u](t) + t^2\|\eta\Gamma^a G\|^2_{L^2}.$$

Summation over $|a| \le p-1$ yields

$$\mathcal{Y}^{\text{ext}}_{p,0}[u](t) + \nu^2\mathcal{Z}^{\text{ext}}_{p,0}[u](t) \lesssim \mathcal{E}_{p,1}[u](t) + \sum_{|a|\le p-1} t^2\|\eta\Gamma^a G\|^2_{L^2},$$

which proves the result in the case $q=0$.

The result for $0<q<p$ follows from (5.1b) and induction, as in the proof of Propostion 8.3. □

9. **Decay Estimates.** The next two results establish the dispersive estimates for the nonlinear equation, using a bootstrap argument in connection with Propositions 8.3 and 8.5.

**Theorem 9.1.** *Choose $(p,q)$ so that $p^* = \left[\frac{p+5}{2}\right] < q \le p$. Suppose that $u \in C([0,T);X^{p,q})$ is a solution of (2.2a), (2.2b) with*

$$\sup_{0\le t\le T} \mathcal{E}_{p,q}[u](t) < \infty,$$

*and*

$$\sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t) \le \varepsilon^2 \ll 1. \tag{9.1}$$

*Then*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\mathrm{int}}_{p^*,p^*-1}[u](t) + \nu^2 \mathcal{Z}^{\mathrm{int}}_{p^*,p^*-1}[u](t)\right] dt$$

$$\lesssim \begin{cases} \sup\limits_{0\le t\le T} \langle t\rangle^{-\gamma}\mathcal{E}_{p^*,p^*}[u](t), & 0<\theta+\gamma<1 \\ \log(e+T) \sup\limits_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t), & \theta = 1 \end{cases}, \tag{9.2a}$$

*and*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\mathrm{int}}_{p^*+1,p^*}[u](t) + \nu^2 \mathcal{Z}^{\mathrm{int}}_{p^*+1,p^*}[u](t)\right] dt$$

$$\lesssim \sup_{0\le t\le T} \langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t), \quad 0<\theta+\gamma<1. \tag{9.2b}$$

*Proof.* Fix a pair $(\bar p,\bar q) = (\bar p, \bar p - 1)$ with $2 \le \bar p \le p$. Choose a multi-index $a$ and an integer $k$, with $|a| + k \le \bar p - 1$ and $k \le \bar q = \bar p - 1$. Then, using Lemmas 5.2 and 5.1, we have that $\|\zeta S^k \Omega^a N(u,\nabla u)\|^2_{L^2}$ is bounded by a sum of terms of the form

$$\|\zeta S^{k_1}\Gamma^{a_1} u\, S^{k_2}\Gamma^{a_2+1} u\|^2_{L^2}, \tag{9.3}$$

with $|a_1| + |a_2| \le |a|$ and $k_1 + k_2 \le k$. Thus,

$$k_1 + k_2 + |a_1| + |a_2| \le k + |a| \le \bar p - 1 \quad \text{and} \quad k_1 + k_2 \le \bar p - 1.$$

Lemma 7.1 implies that the terms in (9.3) can be bounded by (a multiple of)

$$\left[\mathcal{Y}^{\mathrm{int}}_{\bar p',\bar q'}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{\bar p'-1,\bar q'}[u](t)\right] \mathcal{E}_{\bar p,\bar q}[u](t),$$

where $\bar p' = [\frac{(\bar p-1)+5}{2}] = [\frac{\bar p}{2}] + 2$ and $\bar q' = [\frac{\bar p - 1}{2}]$.

Therefore, an application of Proposition 8.3 with $G = N(u,\nabla u)$ yields

$$\begin{aligned} &\int_0^T \langle t\rangle^\theta [\mathcal{Y}^{\mathrm{int}}_{\bar p,\bar q}[u](t) + \nu^2 \mathcal{Z}^{\mathrm{int}}_{\bar p,\bar q}[u](t)] dt \\ &\qquad \lesssim \nu \langle T\rangle^{\theta-2}\mathcal{E}_{\bar p,\bar q}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{\bar p,\bar q+1}[u](t) dt \\ &\qquad\quad + \int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\mathrm{int}}_{\bar p',\bar q'}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{\bar p'-1,\bar q'}[u](t)\right] \mathcal{E}_{\bar p,\bar q}[u](t) dt, \end{aligned} \tag{9.4}$$

for any $0 < \theta \le 1$. We are going to apply this for two pairs $(\bar p, \bar q)$.

First, let $(\bar{p},\bar{q}) = (p^*, p^*-1)$. Since $\bar{p} = p^* \geq 5$, we get

$$\bar{p}' = \left[\frac{p^*}{2}\right] + 2 \leq p^*, \quad \bar{q}' = \left[\frac{p^*-1}{2}\right] \leq p^* - 1.$$

In this case, (9.4) yields

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \nu^2 \mathcal{Z}^{\rm int}_{p^*,p^*-1}[u](t)\right] dt$$
$$\lesssim \nu \langle T\rangle^{\theta-2}\mathcal{E}_{p^*,p^*-1}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p^*,p^*}[u](t)dt$$
$$+ \int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right]\mathcal{E}_{p^*,p^*}[u](t)dt.$$

Choose $\gamma \geq 0$ such that $0 < \theta + \gamma \leq 1$. By (9.1), the right-hand side is bounded by

$$\sup_{0\leq t\leq T} \langle t\rangle^{-\gamma}\mathcal{E}_{p^*,p^*}[u](t)\left[\langle T\rangle^{\theta+\gamma-2} + \int_0^T \langle t\rangle^{\theta+\gamma-2}dt\right]$$
$$+ \varepsilon^2 \int_0^T \langle t\rangle^\theta \mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t)dt.$$

For $\varepsilon^2$ sufficiently small, the last term above can be absorbed on the left, and then the inequalities (9.2a) follow immediately.

Next, we use the pair $(\bar{p},\bar{q}) = (p^*+1, p^*)$ in (9.4). Again since $p^* \geq 5$, we have

$$\bar{p}' = \left[\frac{p^*+1}{2}\right] + 2 \leq p^* \quad \text{and} \quad \bar{q}' = \left[\frac{p^*}{2}\right] \leq p^* - 1.$$

We obtain from (9.4)

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p^*+1,p^*}[u](t) + \nu^2 \mathcal{Z}^{\rm int}_{p^*+1,p^*}[u](t)\right] dt$$
$$\lesssim \nu \langle T\rangle^{\theta-2}\mathcal{E}_{p^*+1,p^*}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p^*+1,p^*+1}[u](t)dt$$
$$+ \int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right]\mathcal{E}_{p^*+1,p^*}[u](t)dt.$$

Choose $\gamma > 0$ such that $0 < \theta + \gamma < 1$. Since $p^* + 1 \leq q \leq p$, the right-hand side can be estimated above by

$$\sup_{0\leq t\leq T} \langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t)\Bigg[\nu\langle T\rangle^{\theta+\gamma-2} + \int_0^T \langle t\rangle^{\theta+\gamma-2}dt$$
$$+ \int_0^T \langle t\rangle^{\theta+\gamma} \left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right] dt\Bigg].$$

By (9.2a) and (9.1), the last integral is bounded (by $C\varepsilon^2$), and so the inequality (9.2b) now follows. □

**Theorem 9.2.** *Fix $p \geq 11$. Assume that $p^* = \left[\frac{p+5}{2}\right] < q \leq p$. Suppose that $u \in C([0,T); X^{p,q})$ is a solution of (2.2a), (2.2b) with*

$$\sup_{0\leq t\leq T} \mathcal{E}_{p,q}[u](t) < \infty$$

*and*

$$\sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t) \le 1. \tag{9.5}$$

*Then*

$$\mathcal{Y}^{\text{ext}}_{p,q-1}[u](t) + \nu^2 \mathcal{Z}^{\text{ext}}_{p,q-1}[u](t) \lesssim \mathcal{E}_{p,q}[u](t), \quad 0 \le t \le T.$$

*Proof.* Choose a multi-index $a$ and an integer $k$, such that $|a| + k \le p-1$, $k \le q$. Then, using Lemmas 5.2 and 5.1, we have that the quantity $\|\eta S^k \Omega^a N(u,\nabla u)\|^2_{L^2}$ is bounded by a sum of terms of the form

$$\|\eta S^{k_1}\Gamma^{a_1} u\, S^{k_2}\Gamma^{a_2+1} u\|^2_{L^2}, \tag{9.6}$$

with $|a_1| + |a_2| \le |a|$ and $k_1 + k_2 \le k$. Thus,

$$k_1 + k_2 + |a_1| + |a_2| \le k + |a| \le p - 1 \quad \text{and} \quad k_1 + k_2 \le p - 1.$$

Lemma 7.2 implies that the terms in (9.6) can be bounded by (a multiple of)

$$\langle t\rangle^{-2}\, \mathcal{E}_{p',q'}[u](t)\, \mathcal{E}_{p,q}[u](t),$$

where $p' = [\frac{(p-1)+5}{2}] = [\frac{p}{2}]+2 \le p^*$ and $q' = [\frac{p-1}{2}] < p^*$. Thus, we have $\mathcal{E}_{p',q'}[u](t) \le 1$, by (9.5). The result is now a consequence of Proposition 8.5. $\square$

## 10. **High Energy Estimates.**

**Proposition 10.1.** *Choose $(p,q)$ so that $5 \le p^* = \left[\frac{p+5}{2}\right] \le q \le p$. Suppose that $u \in C([0,T_0); X^{p,q})$ is a solution of (2.2a), (2.2b) with*

$$\sup_{0\le t<T_0} \mathcal{E}_{p^*,p^*}[u](t) \le \varepsilon^2 \ll 1. \tag{10.1}$$

*Then there exists a constant $C_1 > 1$ such that*

$$\begin{aligned} \mathcal{E}_{p,q}[u](t) &\le C_1 \mathcal{E}_{p,q}[u_0]\langle t\rangle^{C_1\varepsilon} \\ \mathcal{E}_{p^*,p^*}[u](t) &\le C_1 \mathcal{E}_{p^*,p^*}[u_0]\langle t\rangle^{C_1\varepsilon}, \end{aligned}$$

*for $0 \le t < T_0$.*

*Proof.* Thanks to the symmetry of the coefficient matrices (2.2c), we obtain the basic energy identity

$$\mathcal{E}_{0,0}[u](T) = \mathcal{E}_{0,0}[u_0] + \int_0^T \langle Lu(t), u(t)\rangle_{L^2} dt, \quad 0 \le T < T_0.$$

For $p \ge q \ge 0$, we can combine this with (5.1a) to get

$$\mathcal{E}_{p,q}[u](T) = \mathcal{E}_{p,q}[u_0] + I + \sum_{\substack{|a|+k\le p\\ k\le q}} \int_0^T \langle (S+1)^k \Gamma^a Lu(t), S^k\Gamma^a u(t)\rangle_{L^2} dt,$$

with

$$I = -\sum_{\substack{|a|+k\le p\\ k\le q}} \int_0^T \langle \nu B\Delta[S^k - (S-1)^k]\Gamma^a u(t), S^k\Gamma^a u(t)\rangle_{L^2} dt.$$

For $q > 0$, using the definition $B = e_0 \otimes e_0$ and integration by parts we get the bound

$$\begin{aligned} I = &\sum_{\substack{|a|+k\le p\\ k\le q}} \int_0^T \langle \nu\nabla[S^k-(S-1)^k](\Gamma^a u)^0(t), \nabla S^k(\Gamma^a u)^0(t)\rangle_{L^2} dt \\ &\lesssim \mathcal{E}^{1/2}_{p,q-1}[u](T)\mathcal{E}^{1/2}_{p,q}[u](T). \end{aligned}$$

It follows from induction on $q$ and Young's inequality that

$$\begin{aligned} \mathcal{E}_{p,q}[u](T) \lesssim\; &\mathcal{E}_{p,q}[u_0] \\ &+ \sum_{\substack{|a|+k\le p\\ k\le q}} \left| \int_0^T \langle (S+1)^k \Gamma^a L u(t), S^k\Gamma^a u(t)\rangle_{L^2} dt\right|. \end{aligned} \tag{10.2}$$

If we combine (10.2) with Lemma 5.2, we find

$$\begin{aligned} \mathcal{E}_{p,q}[u](T) \lesssim\; &\mathcal{E}_{p,q}[u_0] \\ &+ \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p\\ k\le q}} \left| \int_0^T \langle [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1}u, \nabla S^{k_2}\Gamma^{a_2}u), S^k\Gamma^a u\rangle_{L^2} dt\right|. \end{aligned} \tag{10.3}$$

The reader will note that we have deliberately left the absolute value signs on the "outside" of the integrals because we will later exploit cancellations of the coefficients of the nonlinear terms.

Special care must be taken for the terms in the sum with $|a_2| + k_2 = |a| + k = p$. To simplify the notation when analyzing these terms, set $v = S^k\Gamma^a u$. Then using the fact that $\partial_\alpha v^\beta = \partial_\beta v^\alpha$, we may write

$$\begin{aligned} \langle N(u, \nabla S^k\Gamma^a u), &S^k\Gamma^a u\rangle_{\mathbb{R}^4} \\ =&\langle N(u,\nabla v), v\rangle_{\mathbb{R}^4} \\ =&N^0(u,\nabla v)v^0 \\ =&C^\ell_{\alpha,\beta} u^\alpha \partial_\ell v^\beta v^0 \\ =&C^\ell_{\alpha,0} u^\alpha \partial_\ell v^0 v^0 + C^\ell_{\alpha,m} u^\alpha \partial_\ell v^m v^0 \\ =&\frac{1}{2}\Big[ C^\ell_{\alpha,0}\partial_\ell[u^\alpha (v^0)^2] + (C^\ell_{\alpha,m} + C^m_{\alpha,\ell})\partial_\ell(u^\alpha v^m v^0) \\ &\quad - C^\ell_{\alpha,m}\partial_0(u^\alpha v^\ell v^m) - C^\ell_{\alpha,0}\partial_\ell u^\alpha (v^0)^2 \\ &\quad - (C^\ell_{\alpha,m} + C^m_{\alpha,\ell})\partial_\ell u^\alpha v^m v^0 + C^\ell_{\alpha,m}\partial_0 u^\alpha v^\ell v^m \Big] \\ =&\frac{1}{2}\Big[ C^\ell_{\alpha,0}\partial_\ell[u^\alpha (v^0)^2] + (C^\ell_{\alpha,m} + C^m_{\alpha,\ell})\partial_\ell(u^\alpha v^m v^0) \\ &\quad - C^\ell_{\alpha,m}\partial_0(u^\alpha v^\ell v^m)\Big] - \mathcal{O}(|\partial u|\,|v|^2). \end{aligned}$$

Integration over $[0,T]\times\mathbb{R}^3$ yields

$$\begin{aligned}\int_0^T &\langle N(u,\nabla S^k\Gamma^a u), S^k\Gamma^a u\rangle_{L^2}dt\\ &= -\frac{1}{2}\int_{\mathbb{R}^3} C^\ell_{\alpha,m}u^\alpha(T)(S^k\Gamma^a u(T))^\ell(S^k\Gamma^a u(T))^m dx\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3} C^\ell_{\alpha,m}u^\alpha(0)(S^k\Gamma^a u(0))^\ell(S^k\Gamma^a u(0))^m dx\\ &\quad+\mathcal{O}\left(\int_0^T\int_{\mathbb{R}^3}|\partial u||S^k\Gamma^a u|^2dxdt\right).\end{aligned}\tag{10.4}$$

By (6.1a) and the assumption (10.1), we have

$$\|u\|_{L^\infty(\mathbb{R}^3)}\lesssim\|u\|_{H^2(\mathbb{R}^3)}\le\mathcal{E}^{1/2}_{2,0}[u]<\varepsilon\ll 1.\tag{10.5}$$

Using (2.2a) and (10.5), we have

$$|\partial u|\lesssim|\partial_0 u|+|\nabla u|\lesssim(1+|u|)\,|\nabla u|+|\Delta u^0|\lesssim|\nabla u|+|\Delta u|.\tag{10.6}$$

It follows that the right-hand side of (10.4) is bounded by

$$\varepsilon^2\Big(\mathcal{E}_{p,q}[u](T)+\mathcal{E}_{p,q}[u_0]\Big)+\mathcal{O}\left(\int_0^T\int_{\mathbb{R}^3}(|\nabla u|+|\Delta u|)|S^k\Gamma^a u|^2dxdt\right).$$

The remaining terms in (10.3) satisfy

$$\begin{aligned}\sum_{\substack{|a|+k\le p\\ k\le q}}\ \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a_2|+k_2<p}}&\left|\int_0^T\langle[\Gamma^{a_3},N](S^{k_1}\Gamma^{a_1}u,\nabla S^{k_2}\Gamma^{a_2}u),S^k\Gamma^a u\rangle_{L^2}dt\right|\\ &\lesssim\sum_{\substack{|a|+k\le p\\ k\le q}}\ \sum_{\substack{|a_1+a_2|\le|a|\\ k_1+k_2\le k\\ |a_2|+k_2<p}}\int_0^T\int_{\mathbb{R}^3}|S^{k_1}\Gamma^{a_1}u||\nabla S^{k_2}\Gamma^{a_2}u||S^k\Gamma^a u|dxdt.\end{aligned}$$

Altogether, taking $\varepsilon$ sufficiently small we obtain from (10.3)

$$\begin{aligned}\mathcal{E}_{p,q}[u](T)\lesssim\mathcal{E}_{p,q}[u_0]&+\sum_{|a|+k=p}\int_0^T\int_{\mathbb{R}^3}(|\nabla u|+|\Delta u|)|S^k\Gamma^a u|^2dxdt\\ &+\sum_{\substack{|a|+k\le p\\ k\le q}}\ \sum_{\substack{|a_1+a_2|\le|a|\\ k_1+k_2\le k\\ |a_2|+k_2<p}}\int_0^T\int_{\mathbb{R}^3}|S^{k_1}\Gamma^{a_1}u||\nabla S^{k_2}\Gamma^{a_2}u||S^k\Gamma^a u|dxdt.\end{aligned}$$

This immediately leads to

$$\begin{aligned}\mathcal{E}_{p,q}[u](T)\lesssim\mathcal{E}_{p,q}[u_0]&+\sum_{|a|+k=p}\int_0^T(\|\nabla u(t)\|_{L^\infty}+\|\Delta u(t)\|_{L^\infty})\mathcal{E}_{p,q}[u](t)dt\\ &+\sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}}\int_0^T\|\,|S^{k_1}\Gamma^{a_1}u(t)|\,|S^{k_2}\Gamma^{a_2+1}u(t)|\,\|_{L^2}\mathcal{E}^{1/2}_{p,q}[u](t)dt.\end{aligned}$$

Using (3.3c), (6.2a), (6.2d), and the fact that $5 \le \left[\frac{p+5}{2}\right]$, we obtain

$$\begin{aligned}\|\nabla u(t)\|_{L^\infty} + &\|\Delta u(t)\|_{L^\infty} \\ &\lesssim \|\zeta \nabla u(t)\|_{L^\infty} + \|\zeta \Delta u(t)\|_{L^\infty} \\ &\quad + \|\eta \nabla u(t)\|_{L^\infty} + \|\eta \Delta u(t)\|_{L^\infty} \\ &\lesssim \left(\mathcal{Y}^{\mathrm{int}}_{4,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{4,0}[u](t) \\ &\lesssim \left(\mathcal{Y}^{\mathrm{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t).\end{aligned}$$

By (3.3c) and Lemmas 7.1 and 7.2, we get the same bound

$$\begin{aligned}&\sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \|\, |S^{k_1}\Gamma^{a_1} u(t)|\, |S^{k_2}\Gamma^{a_2+1} u(t)|\, \|_{L^2} \\ &\quad\lesssim \sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \|\zeta\, |S^{k_1}\Gamma^{a_1} u(t)|\, |S^{k_2}\Gamma^{a_2+1} u(t)|\, \|_{L^2} \\ &\qquad + \sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \|\eta\, |S^{k_1}\Gamma^{a_1} u(t)|\, |S^{k_2}\Gamma^{a_2+1} u(t)|\, \|_{L^2} \\ &\quad\lesssim \left[\left(\mathcal{Y}^{\mathrm{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t)\right] \mathcal{E}^{1/2}_{p,q}[u](t).\end{aligned}$$

Inserting this into the previous energy inequality yields

$$\begin{aligned}\mathcal{E}_{p,q}[u](T) \lesssim\, & \mathcal{E}_{p,q}[u_0] \\ & + \int_0^T \left[\left(\mathcal{Y}^{\mathrm{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t)\right] \mathcal{E}_{p,q}[u](t)dt.\end{aligned}$$

An application of Gronwall's inequality produces

$$\begin{aligned}&\mathcal{E}_{p,q}[u](T) \\ &\quad\lesssim \mathcal{E}_{p,q}[u_0] \exp \int_0^T \left[\left(\mathcal{Y}^{\mathrm{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t)\right] dt. \qquad (10.7)\end{aligned}$$

We point out that (10.7) holds for any pair $(p, q)$ as long as $p \ge q \ge \left[\frac{p+5}{2}\right] \ge 5$, which requires only $p \ge 5$.

Recalling the definition $p^* = \left[\frac{p+5}{2}\right]$, we obtain using Theorem 9.1

$$\begin{aligned}\int_0^T &\left(\mathcal{Y}^{\mathrm{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} dt \\ &\le \left(\int_0^T \langle t\rangle \mathcal{Y}^{\mathrm{int}}_{p^*,p^*-1}[u](t)dt\right)^{1/2} \left(\int_0^T \langle t\rangle^{-1} dt\right)^{1/2} \\ &\lesssim \left(\sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t) \log(e+T)\right)^{1/2} (\log(e+T))^{1/2} \\ &\lesssim \sup_{0\le t\le T} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \log(e+T).\end{aligned}$$

Likewise, we have the bound

$$\int_0^T \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{p+5}{2}],[\frac{p+5}{2}]}[u](t) \lesssim \sup_{0\le t\le T} \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\log(e+T).$$

Now thanks to the assumption (10.1), the inequality (10.7) implies that

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0]\exp\left[C\varepsilon\log(e+T)\right] \le \mathcal{E}_{p,q}[u_0]\langle T\rangle^{C_1\varepsilon}.$$

Returning to (10.7), we can repeat this argument with the pair $(p,q)=(p^*,p^*)$ because $p^*\ge 5$ implies that $p^*\ge\left[\frac{p*+5}{2}\right]$. Therefore, we also obtain the bound

$$\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0]\langle T\rangle^{C_1\varepsilon},$$

after a possible increase in the size of the constant $C_1$. Note however, that the choice of $C_1$ is independent of $\varepsilon$. The size of $C_1$ may always be increased, while the size of $\varepsilon$ may always be decreased. The statement of Proposition 10.1 now follows. □

**Corollary 10.2.** *Under the hypotheses of Proposition 10.1, we have*

$$\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0] + \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \left|\int_0^T \langle[\Gamma^{a_3},N](S^{k_1}\Gamma^{a_1}u,\nabla S^{k_2}\Gamma^{a_2}u),S^k\Gamma^a u\rangle_{L^2}dt\right|.$$

*Proof.* This is simply (10.3) from the proof of Proposition 10.1 in the case where $(p,q)=(p^*,p^*)$. □

## 11. Low Energy Estimates.

**Proposition 11.1.** *Choose $(p,q)$ such that $p\ge 11$, and $p\ge q>p^*$, where $p^*=\left[\frac{p+5}{2}\right]$. Let $\delta\le 1$ be defined as in (4.1). Suppose that $u\in C([0,T_0);X^{p,q})$ is a solution of (2.2a), (2.2b) with*

$$\sup_{0\le t<T_0}\mathcal{E}_{p^*,p^*}[u](t)\le\varepsilon^2\ll 1. \tag{11.1}$$

*There exists a constant $C_0>1$ such that if*

$$C_0^3\left(\frac{\delta}{\nu}\right)^2\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}^{1/2}_{p,q}[u_0]\right)<1, \tag{11.2}$$

*then*

$$\sup_{0\le t<T_0}\mathcal{E}_{p^*,p^*}[u](t)\le C_0\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}^{1/2}_{p,q}[u_0]\right). \tag{11.3}$$

*Proof.* We continue from the inequality of Corollary 10.2. Using the cut-off functions defined in (3.3b) and recalling (3.3c), we can write:

$$\mathcal{E}_{p^*,p^*}[u](T) \;\lesssim\; \mathcal{E}_{p^*,p^*}[u_0]+I_1+I_2, \tag{11.4}$$

with

$$I_1=\sum_{\substack{|a_1+a_2|\le|a|\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}}\int_0^T\int_{\mathbb{R}^3}\zeta^2|S^{k_1}\Gamma^{a_1}u|\,|\nabla S^{k_2}\Gamma^{a_2}u|\,|S^k\Gamma^a u|\,dxdt$$

and

$$I_2 = \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \left| \int_0^T \int_{\mathbb{R}^3} \eta \langle [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1}u, \nabla S^{k_2}\Gamma^{a_2}u), S^k\Gamma^a u\rangle_{\mathbb{R}^4} dxdt \right|,$$

for $0 \le T < T_0$.

*Interior Low Energy.* The first integral $I_1$ on the right of (11.4) is bounded by

$$|I_1| \lesssim \sum_{\substack{k_1+k_2+|a_1|+|a_2|\le p^*\\ k_1+k_2\le p^*}} \int_0^T \| \zeta^2 |S^{k_1}\Gamma^{a_1}u|\, |\nabla S^{k_2}\Gamma^{a_2}u|\, \|_{L^2}\, \mathcal{E}^{1/2}_{p^*,p^*}[u](t)dt.$$

To estimate this, we follow the same strategy as in Lemma 7.1.

In the case $k_1+|a_1| < k_2+|a_2|+1$, i.e. $k_1+|a_1| \le \left[\frac{p^*}{2}\right]$, we have using (6.2a)

$$\begin{aligned}
\|\zeta^2 |S^{k_1}\Gamma^{a_1}u||\nabla S^{k_2}\Gamma^{a_2}u|\, \|_{L^2} &\lesssim \|\zeta S^{k_1}\Gamma^{a_1}u\|_{L^\infty} \|\zeta \nabla S^{k_2}\Gamma^{a_2}u\|_{L^2}\\
&\lesssim \left[ \left( \mathcal{Y}^{\text{int}}_{\left[\frac{p^*+4}{2}\right],\left[\frac{p^*}{2}\right]}[u](t) \right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p^*+2}{2}\right],\left[\frac{p^*}{2}\right]}[u](t) \right]\\
&\qquad\qquad \times \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2}.
\end{aligned}$$

Next, we consider the terms in $I_1$ with $k_2+|a_2|+1 \le k_1+|a_1|$, i.e. $k_2+|a_2| \le \left[\frac{p^*-1}{2}\right]$. By Hardy's inequality (6.2c) and the Sobolev inequality (6.2b), we can write:

$$\begin{aligned}
&\|\zeta^2\, |S^{k_1}\Gamma^{a_1}u|\, |\nabla S^{k_2}\Gamma^{a_2}u|\, \|_{L^2}\\
&\quad \lesssim \|r^{-1}\zeta S^{k_1}\Gamma^{a_1}u\|_{L^2} \|r\zeta \nabla S^{k_2}\Gamma^{a_2}u\|_{L^\infty}\\
&\quad \lesssim \left[ \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \right]\\
&\qquad \times \left[ \left( \mathcal{Y}^{\text{int}}_{\left[\frac{p^*+5}{2}\right],\left[\frac{p^*-1}{2}\right]}[u](t) \right)^{1/2} + \langle t\rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p^*+3}{2}\right],\left[\frac{p^*-1}{2}\right]}[u](t) \right].
\end{aligned}$$

Thanks to the assumption $p \geq 11$, we have $\left[\frac{p^*+5}{2}\right] \leq p^*$, so altogether for the interior low energy we have:

$$\begin{aligned}
I_1 \lesssim & \int_0^T \left[\left(\mathcal{Y}^{\text{int}}_{\left[\frac{p^*+5}{2}\right],\left[\frac{p^*}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p^*+3}{2}\right],\left[\frac{p^*}{2}\right]}[u](t)\right] \\
& \qquad \times \left[\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\right]\mathcal{E}^{1/2}_{p^*,p^*}[u](t)dt \\
\lesssim & \int_0^T \left[\left(\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\right] \\
& \qquad \times \left[\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\right]\mathcal{E}^{1/2}_{p^*,p^*}[u](t)dt \\
\lesssim & \int_0^T \left(\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)\right)^{1/2}\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)dt \\
& + \int_0^T \langle t\rangle^{-1}\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2}\mathcal{E}_{p^*,p^*}[u](t)dt \\
& + \int_0^T \langle t\rangle^{-2}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt.
\end{aligned}$$

By Theorem 9.1 and Proposition 10.1, we can estimate these three integrals as follows. Note that $\varepsilon$ in (11.1) must again be taken small enough, $2C_1\varepsilon < 1$ is sufficient, in addition to our earlier restrictions.

First integral:

$$\begin{aligned}
& \int_0^T \left(\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)\right)^{1/2}\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)dt \\
& \lesssim \left(\sup_{0\leq t\leq T}\langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t)\right)^{1/2} \\
& \qquad \times \int_0^T \langle t\rangle^{C_1\varepsilon/2}\left(\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)\right)^{1/2}\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2}dt \\
& \lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u_0]\left(\int_0^T \langle t\rangle^{C_1\varepsilon/2}\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)dt\right)^{1/2} \\
& \qquad \times \left(\int_0^T \langle t\rangle^{C_1\varepsilon/2}\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)dt\right)^{1/2} \\
& \lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u_0]\left(\sup_{0\leq t\leq T}\langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t)\right)^{1/2} \\
& \qquad \times \left(\sup_{0\leq t\leq T}\langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p,q}[u](t)\right)^{1/2} \\
& \lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0].
\end{aligned}$$

Second integral:

$$\int_0^T \langle t\rangle^{-1}\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2}\mathcal{E}_{p^*,p^*}[u](t)dt$$

$$
\begin{aligned}
&\lesssim \sup_{0\le t\le T}\langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t)\int_0^T\langle t\rangle^{-1+C_1\varepsilon}\left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2}dt\\
&\lesssim \mathcal{E}_{p^*,p^*}[u_0]\left(\int_0^T\langle t\rangle^{-2+C_1\varepsilon}dt\right)^{1/2}\\
&\qquad\qquad\times\left(\int_0^T\langle t\rangle^{C_1\varepsilon}\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)dt\right)^{1/2}\\
&\lesssim \mathcal{E}_{p^*,p^*}[u_0]\left(\sup_{0\le t\le T}\langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p,q}[u](t)\right)^{1/2}\\
&\lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0].
\end{aligned}
$$

Third integral:

$$
\begin{aligned}
&\int_0^T\langle t\rangle^{-2}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt\\
&\qquad\lesssim\left(\sup_{0\le t\le T}\langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t)\right)^{3/2}\int_0^T\langle t\rangle^{-2+\frac{3}{2}C_1\varepsilon}dt\\
&\qquad\lesssim \mathcal{E}^{3/2}_{p^*,p^*}[u_0]\\
&\qquad\lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0].
\end{aligned}
$$

Combining these estimates, we have

$$
I_1\lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0]. \tag{11.5}
$$

*Exterior Low Energy.* Thanks to Lemma 5.4, (5.3a), the second integral $I_2$ on the right of (11.4) is estimated by

$$
I_2 \ \lesssim\ I_2'+I_2'', \tag{11.6a}
$$

with

$$
\begin{aligned}
I_2' = \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}}\Bigg|&\int_0^T\int_{\mathbb{R}^3}\tfrac14\ \eta\ P_{[\widetilde{\Omega}^a,N]}(\hat\omega)\ \langle\hat\omega,S^{k_1}\Gamma^{a_1}u\rangle_{\mathbb{R}^4}\\
&\times\ \langle\hat\omega,\partial_r S^{k_2}\Gamma^{a_2}u\rangle_{\mathbb{R}^4}\ (S^k\Gamma^a u)^0\ dxdt\Bigg|
\end{aligned} \tag{11.6b}
$$

and

$$I_2'' = \sum_{\substack{k_1+k_2+|a_1|+|a_2|\le p^* \\ k_1+k_2\le p^*}} \int_0^T \Big[ \| \,\eta|\mathbb{Q}S^{k_1}\Gamma^{a_1}u|\,|\partial_r S^{k_2}\Gamma^{a_2}u|\,\|_{L^2} + \|\,\eta|S^{k_1}\Gamma^{a_1}u|\,|\mathbb{Q}\partial_r S^{k_2}\Gamma^{a_2}u|\,\|_{L^2} + \|\,r^{-1}\eta|S^{k_1}\Gamma^{a_1}u|\,|S^{k_2}\widetilde{\Omega}\Gamma^{a_2}u|\,\|_{L^2} \Big] \times \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\,dt. \tag{11.6c}$$

Before estimating the main term $I_2'$ above, we dispatch the easiest terms $I_2''$. We claim that within the the range of indices of the sum,

$$k_1+k_2+|a_1|+|a_2|\le p^*,\quad k_1+k_2\le p^*,$$

the quantities

$$\begin{aligned} Q_1 &= \|\,\eta|\mathbb{Q}S^{k_1}\Gamma^{a_1}u|\,|\partial_r S^{k_2}\Gamma^{a_2}u|\,\|_{L^2}\\ Q_2 &= \|\,\eta|S^{k_1}\Gamma^{a_1}u|\,|\mathbb{Q}\partial_r S^{k_2}\Gamma^{a_2}u|\,\|_{L^2}\\ Q_3 &= \|\,r^{-1}\eta|S^{k_1}\Gamma^{a_1}u|\,|S^{k_2}\widetilde{\Omega}\Gamma^{a_2}u|\,\|_{L^2}, \end{aligned}$$

which appear in (11.6c), satisfy the estimate

$$Q_1+Q_2+Q_3 \lesssim \langle t\rangle^{-3/2}\,\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\,\mathcal{E}^{1/2}_{p,q}[u](t). \tag{11.7}$$

In the following, we shall make use of the fact that $p\ge 11$ implies $p^*+3\le p$ and $\left[\frac{p^*+5}{2}\right]\le p^*$. Recall also that $p^*<q$.

Using (6.2e) and Theorem 9.2, we have

$$\begin{aligned} Q_1 &= \|\,\eta|\mathbb{Q}S^{k_1}\Gamma^{a_1}u|\,|\partial_r S^{k_2}\Gamma^{a_2}u|\,\|_{L^2}\\ &\le \|\eta\mathbb{Q}S^{k_1}\Gamma^{a_1}u\|_{L^\infty}\|\partial_r S^{k_2}\Gamma^{a_2}u\|_{L^2}\\ &\lesssim \langle t\rangle^{-3/2}\left(\left(\mathcal{Y}^{\text{ext}}_{k_1+|a_1|+2,k_1}[u](t)\right)^{1/2}+\mathcal{E}^{1/2}_{k_1+|a_1|+1,k_1}[u](t)\right)\\ &\qquad\times\mathcal{E}^{1/2}_{k_2+|a_2|+1,k_2}[u](t)\\ &\lesssim \langle t\rangle^{-3/2}\,\mathcal{E}^{1/2}_{k_1+|a_1|+2,k_1+1}[u](t)\,\mathcal{E}^{1/2}_{k_2+|a_2|+1,k_2}[u](t). \end{aligned}$$

Since $k_1+k_2+|a_1|+|a_2|\le p^*$, either

$$k_1+|a_1|+2\le\left[\frac{p^*+3}{2}\right]\le p^*\quad\text{and}\quad k_2+|a_2|+1\le p^*+1\le p$$

or

$$k_1+|a_1|+2\le p^*+2\le p\quad\text{and}\quad k_2+|a_2|+1\le\left[\frac{p^*+3}{2}\right]\le p^*.$$

Therefore (11.7) holds for $Q_1$.

The details are quite similar for the term $Q_2$:

$$Q_2 = \|\,\eta|S^{k_1}\Gamma^{a_1}u|\,|\mathbb{Q}\partial_r S^{k_2}\Gamma^{a_2}u|\,\|_{L^2}$$

$$
\begin{aligned}
&\leq \|S^{k_1}\Gamma^{a_1}u\|_{L^2}\|\eta\mathbb{Q}\partial_r S^{k_2}\Gamma^{a_2}u\|_{L^\infty}\\
&\lesssim \mathcal{E}^{1/2}_{k_1+|a_1|,k_1}[u](t)\\
&\qquad \times \langle t\rangle^{-3/2}\left(\left(\mathcal{Y}^{\rm ext}_{k_2+|a_2|+3,k_2}[u](t)\right)^{1/2}+\mathcal{E}^{1/2}_{k_2+|a_2|+2,k_2}[u](t)\right)\\
&\lesssim \langle t\rangle^{-3/2}\,\mathcal{E}^{1/2}_{k_1+|a_1|,k_1}[u](t)\,\mathcal{E}^{1/2}_{k_2+|a_2|+3,k_2+1}[u](t)\\
&\lesssim \langle t\rangle^{-3/2}\,\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\,\mathcal{E}^{1/2}_{p^*+3,p^*+1}[u](t)\\
&\lesssim \langle t\rangle^{-3/2}\,\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\,\mathcal{E}^{1/2}_{p,q}[u](t),
\end{aligned}
$$

since $p^*+3\leq p$ and $p^*<q$. Thus, the claimed estimate (11.7) is valid for $Q_2$, as well.

By Lemma 7.2, we have

$$
\begin{aligned}
Q_3 &= \|\, r^{-1}\eta|S^{k_1}\Gamma^{a_1}u|\;|S^{k_2}\widetilde{\Omega}\Gamma^{a_2}u|\,\|_{L^2}\\
&\lesssim \langle t\rangle^{-1}\|\,\eta|S^{k_1}\Gamma^{a_1}u|\;|S^{k_2}\widetilde{\Omega}\Gamma^{a_2}u|\,\|_{L^2}\\
&\lesssim \langle t\rangle^{-2}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p^*+1,p^*}[u](t)\\
&\lesssim \langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p,q}[u](t),
\end{aligned}
$$

which verifies the claim (11.7) for $Q_3$.

Thus, from (11.6c), (11.7), we have that

$$
I_2''\lesssim\int_0^T\langle t\rangle^{-3/2}\,\mathcal{E}_{p^*,p^*}[u](t)\,\mathcal{E}^{1/2}_{p,q}[u](t)\,dt. \tag{11.8}
$$

We now turn to $I_2'$ given in (11.6b). By (5.3b), it is estimated by

$$
\begin{aligned}
I_2' &\lesssim \sum_{\substack{a_1+a_2+a_3=a\\k_1+k_2=k\\|a|+k\leq p^*\\k\leq p^*}}\int_0^T\delta\,\|\,\eta\,|S^{k_1}\Gamma^{a_1}u|\;|\partial_r S^{k_2}\Gamma^{a_2}u|\;|S^k(\Gamma^a u)^0|\,\|_{L^1}dt\\
&\equiv\int_0^T Q_0\,dt.
\end{aligned} \tag{11.9}
$$

We are going to estimate these terms for the range of indices in the sum.

First, from (2.3b) we note the simple relation (similar to (8.7))

$$
\partial_r u=\partial_r u^0e_0+\partial_r\bar u=\partial_r u^0e_0+\frac1r S\bar u-\frac tr\partial_t\bar u=\partial_r u^0e_0+\frac1r S\bar u-\frac tr(\nabla u^0)^\top,
$$

and thus,

$$
\eta|\partial_r u|\lesssim\eta(|\nabla u^0|+\langle t\rangle^{-1}|Su|).
$$

So for the terms under consideration, we may write

$$
\begin{aligned}
Q_0 &\lesssim \delta\|\,|S^{k_1}\Gamma^{a_1}u|\;|\nabla S^{k_2}(\Gamma^{a_2}u)^0|\;|S^k(\Gamma^a u)^0|\,\|_{L^1}\\
&\qquad+\delta\langle t\rangle^{-1}\|\,\eta\,|S^{k_1}\Gamma^{a_1}u|\;|S^{k_2+1}\Gamma^{a_2}u|\,\|_{L^2}\|S^k\Gamma^a u\|_{L^2}.
\end{aligned} \tag{11.10a}
$$

Using a slight variant of Lemma 7.2 with $(\bar{p},\bar{q})=(p^*,p^*)$, we see that the second term in (11.10a) satisfies

$$\begin{aligned}
&\delta\langle t\rangle^{-1}\|\ \eta\ |S^{k_1}\Gamma^{a_1}u|\ |S^{k_2+1}\Gamma^{a_2}u|\ \|_{L^2}\|S^k\Gamma^a u\|_{L^2}\\
&\quad\lesssim \delta\langle t\rangle^{-2}\ \mathcal{E}^{1/2}_{[\frac{p^*+5}{2}],[\frac{p^*-1}{2}]}[u](t)\ \mathcal{E}^{1/2}_{p^*+1,p^*+1}[u](t)\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\\
&\quad\lesssim \delta\langle t\rangle^{-2}\ \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p^*+1,p^*+1}[u](t)\\
&\quad\lesssim \delta\langle t\rangle^{-2}\ \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p,q}[u](t).
\end{aligned}\tag{11.10b}$$

The first and dominant term in (11.10a) measures the deviation from the null condition, and it will be estimated with the aid the diffusion term in the energy. We use (6.1b) to get

$$\begin{aligned}
&\|\ |S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}(\Gamma^{a_2}u)^0|\ |S^k(\Gamma^a u)^0|\ \|_{L^1}\\
&\quad\lesssim \|\ r\ |S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}(\Gamma^{a_2}u)^0|\ \|_{L^2}\|r^{-1}S^k(\Gamma^a u)^0\|_{L^2}\\
&\quad\lesssim \|\ r\ |S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}(\Gamma^{a_2}u)^0|\ \|_{L^2}\|\nabla S^k(\Gamma^a u)^0\|_{L^2}.
\end{aligned}\tag{11.10c}$$

Now, we employ the usual strategy. For the terms under consideration, we have

$$\text{Case (a):}\quad k_1+|a_1|\le\left[\frac{p^*}{2}\right]\quad\text{and}\quad k_2+|a_2|\le p^*$$

or

$$\text{Case (b):}\quad k_2+|a_2|\le\left[\frac{p^*}{2}\right]\quad\text{and}\quad k_1+|a_1|\le p^*.$$

So by (6.1d), we obtain

$$\begin{aligned}
&\|\ r\ |S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}(\Gamma^{a_2}u)^0|\ \|_{L^2}\\
&\quad\lesssim\begin{cases}\|rS^{k_1}\Gamma^{a_1}u\|_{L^\infty}\|\nabla S^{k_2}(\Gamma^{a_2}u)^0\|_{L^2}, & \text{Case (a)}\\ \|S^{k_1}\Gamma^{a_1}u\|_{L^2}\|r\nabla S^{k_2}(\Gamma^{a_2}u)^0\|_{L^\infty}, & \text{Case (b)}\end{cases}\\
&\quad\lesssim\begin{cases}\|S^{k_1}\Gamma^{a_1+2}u\|_{L^2}\|\nabla S^{k_2}(\Gamma^{a_2}u)^0\|_{L^2}, & \text{Case (a)}\\ \|S^{k_1}\Gamma^{a_1}u\|_{L^2}\|\nabla S^{k_2}(\Gamma^{a_2+2}u)^0\|_{L^2}, & \text{Case (b)}\end{cases}\\
&\quad\lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\|\nabla S^{k_2}(\Gamma^b u)^0\|_{L^2},
\end{aligned}\tag{11.10d}$$

with $k_2+|b|\le p^*$.

Combining (11.10a), (11.10b), (11.10c), (11.10d), we end up with

$$\begin{aligned}
Q_0\lesssim{}& \delta\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\|\nabla S^{k_2}(\Gamma^b u)^0\|_{L^2}\|\nabla S^k(\Gamma^a u)^0\|_{L^2}\\
&+\delta\langle t\rangle^{-2}\ \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p,q}[u](t),
\end{aligned}$$

where $k_2+|b|\le p^*$. Thus, from (11.9), we have shown that

$$\begin{aligned}
I_2'\lesssim{}&\sum_{k+|a|\le p^*}\int_0^T\delta\ \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\ \|\nabla S^k(\Gamma^a u)^0\|^2_{L^2}\ dt\\
&+\int_0^T\delta\langle t\rangle^{-2}\ \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p,q}[u](t)\ dt.
\end{aligned}\tag{11.11}$$

Inserting the estimates (11.11) and (11.8) into (11.6a), we find that

$$I_2 \lesssim \int_0^T \langle t\rangle^{-3/2}\, \mathcal{E}_{p^*,p^*}[u](t)\, \mathcal{E}_{p,q}^{1/2}[u](t)\, dt + \sum_{k+|a|\le p^*} \int_0^T \delta\, \mathcal{E}_{p^*,p^*}^{1/2}[u](t)\, \|\nabla S^k(\Gamma^a u)^0\|_{L^2}^2\, dt.$$

By Proposition 10.1, we get

$$\begin{aligned} I_2 &\lesssim \sup_{0\le t\le T}\left(\langle t\rangle^{-\frac{3}{2}C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}_{p,q}^{1/2}[u](t)\right)\int_0^T \langle t\rangle^{-\frac{3}{2}(1-C_1\varepsilon)}dt \\ &\quad + \delta \sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}^{1/2}[u](t)\sum_{k+|a|\le p^*}\int_0^T \|\nabla S^k(\Gamma^a u)^0\|_{L^2}^2\, dt \\ &\lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0] + \frac{\delta}{\nu}\sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}^{3/2}[u](t), \end{aligned} \tag{11.12}$$

provided $C_1\varepsilon < 1/3$.

We deduce from (11.4), (11.5), (11.12) that

$$\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}_{p,q}^{1/2}[u_0]\right) + \frac{\delta}{\nu}\sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}^{3/2}[u](t),$$

for every $0\le T<T_0$. Thus, there exists a constant $C_0>4$ such that

$$\mathcal{E}_{p^*,p^*}[u](T) \le \frac{C_0}{4}\left[\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}_{p,q}^{1/2}[u_0]\right) + \frac{\delta}{\nu}\sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}^{3/2}[u](t)\right],$$

for every $0\le T<T_0$. In other words, the function

$$S(T) = \sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}[u](t)$$

satisfies

$$S(T) \le A_0 + B_0 S(T)^{3/2}, \quad 0\le T<T_0, \tag{11.13}$$

with

$$A_0 = \frac{C_0}{4}\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}_{p,q}^{1/2}[u_0]\right) \quad \text{and} \quad B_0 = \frac{C_0\delta}{4\nu}.$$

We now conclude the proof with a standard argument, using (11.13) to show that (11.2) implies (11.3).

Suppose that $S(T)<4A_0$. Then by (11.13), we have

$$S(T) \le A_0 + (4A_0)^{1/2}B_0\, S(T).$$

If $(4A_0)^{1/2}B_0 < 1/2$, i.e. (11.2) holds, then $S(T)<2A_0$. Thus, since $S(0)<4A_0$,[2] we obtain by continuity that $S(T)<2A_0<4A_0$, for all $0\le T<T_0$, i.e. (11.3) holds. □

**Remark.** The reader will note that (11.12) is the only point in this paper where $\nu>0$ is relied upon. In particular, we never use the estimates for $\nu^2\mathcal{Z}^{\mathrm{int}}$ and $\nu^2\mathcal{Z}^{\mathrm{ext}}$ given in Corollary 4.2.

We now consider the situation when the condition (11.2) does not hold.

---

[2]We may assume that $A_0>0$, for otherwise the solution is identically zero.

**Proposition 11.2.** *Choose $(p,q)$ with $p \geq 11$ and $p \geq q > p^*$, where $p^* = \left[\frac{p+5}{2}\right]$. Let $\delta \leq 1$ be defined by (4.1). Suppose that $u \in C([0,T_0);X^{p,q})$ is a solution of (2.2a), (2.2b) with*

$$\sup_{0\leq t<T_0} \mathcal{E}_{p^*,p^*}[u](t) \leq \varepsilon^2 \ll 1.$$

*There exist constants $C_0, C_1 > 1$ such that if*

$$C_1\langle T_0\rangle^{C_1\varepsilon} \leq \left(\frac{2\ \max\{\nu, C_1\varepsilon\}}{C_0\ \delta\ \mathcal{E}^{1/2}_{p^*,p^*}[u_0]}\right)^2, \tag{11.14}$$

*then*

$$\sup_{0\leq t<T_0} \mathcal{E}_{p^*,p^*}[u](t) \leq C_0\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}^{1/2}_{p,q}[u_0]\right).$$

**Remark.** The constant $C_0$ may be assumed to be the same in Propositions 11.1 and 11.2. The constant $C_1$ is the one given by Proposition 10.1.

*Proof.* We continue with the same notation as used in the proof of Proposition 11.1. All of the estimates derived there up to and including (11.13) are valid under the current hypotheses, insofar as the assumption (11.2) is used only in the final paragraph of the proof.

Using (11.13) with Proposition 10.1, we have

$$\begin{aligned} S(T) &\leq A_0 + B_0(C_1\mathcal{E}_{p^*,p^*}[u_0]\langle T_0\rangle^{C_1\varepsilon})^{1/2}\ S(T) \\ &\leq A_0 + B_1(C_1\langle T_0\rangle^{C_1\varepsilon})^{1/2}\ S(T), \end{aligned} \tag{11.15}$$

with

$$B_1 = B_0\mathcal{E}^{1/2}_{p^*,p^*}[u_0] = \frac{C_0\delta}{4\nu}\mathcal{E}^{1/2}_{p^*,p^*}[u_0].$$

Alternatively, we may avoid using the dissipation when estimating $I_2'$ in (11.6b). Consider the terms in the sum for $I_2'$ with $k_2+|a_2| \neq p^*$. By Lemma 7.2, these can be estimated by

$$\delta\int_0^T \langle t\rangle^{-1}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt. \tag{11.16}$$

The remaining terms have the form

$$\begin{aligned} &\sum_{\substack{k+|a|=p^*\\ k\leq p^*}}\int_0^T\int_{\mathbb{R}^3}\tfrac{1}{4}\eta P_N(\hat\omega)\langle\hat\omega,u\rangle_{\mathbb{R}^4}\langle\hat\omega,\partial_r S^k\Gamma^a u\rangle_{\mathbb{R}^4}(S^k\Gamma^a u)^0 dxdt \\ &\qquad = \sum_{\substack{k+|a|=p^*\\ k\leq p^*}}\int_0^T\int_{\mathbb{R}^3}\tfrac{1}{4}\eta P_N(\hat\omega)\hat\omega^\gamma\hat\omega^\mu\hat\omega^j u^\gamma\partial_j(S^k\Gamma^a u)^\mu(S^k\Gamma^a u)^0 dxdt. \end{aligned}$$

By (5.1a), (5.1b), $v = S^k\Gamma^a u$ satisfies $\partial_j v^\mu = \partial_\mu v^j$, so we can write:

$$\begin{aligned} \partial_j(S^k\Gamma^a u)^\mu(S^k\Gamma^a u)^0 =& \tfrac{1}{2}\partial_j\left[(S^k\Gamma^a u)^\mu(S^k\Gamma^a u)^0\right] \\ &+\tfrac{1}{2}\partial_\mu\left[(S^k\Gamma^a u)^j(S^k\Gamma^a u)^0\right] \\ &-\tfrac{1}{2}\partial_0\left[(S^k\Gamma^a u)^\mu(S^k\Gamma^a u)^j\right]. \end{aligned}$$

Using integration by parts, these terms are estimated by

$$\int_{\mathbb{R}^3} |u(T)|\, |S^k\Gamma^a u(T)|^2\, dx \;+\; \int_{\mathbb{R}^3} |u(0)|\, |S^k\Gamma^a u(0)|^2\, dx \\ +\; \int_0^T \int_{\mathbb{R}^3} \max_{\gamma,\mu,j} |\partial(\eta P_N(\hat\omega)\hat\omega^\gamma\hat\omega^\mu\hat\omega^j)|\, |u|\, |S^k\Gamma^a u|^2\, dxdt \\ +\; \int_0^T \int_{\mathbb{R}^3} \eta\, |P_N(\hat\omega)|\, |\partial u|\, |S^k\Gamma^a u|^2\, dxdt.$$

By (11.1), (3.3d), and (4.1), this can be bounded by

$$\varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \int_0^T \langle t\rangle^{-3/2} \|r^{1/2}u\|_{L^\infty}\mathcal{E}_{p^*,p^*}[u](t)dt \\ + \delta\int_0^T \|\eta\partial u\|_{L^\infty}\mathcal{E}_{p^*,p^*}[u](t)dt.$$

From (6.1c), (10.6), and (6.2d), this in turn is estimated by

$$\varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \int_0^T \langle t\rangle^{-3/2} \mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt \\ + \delta\int_0^T \langle t\rangle^{-1}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt.$$

And so, in view of (11.16), the preceding expression serves as a bound for $I_2'$. By Propositions 7.2 and 10.1, we have

$$\begin{aligned} I_2' &\lesssim \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] \\ &\quad + (C_1\mathcal{E}_{p^*,p^*}[u_0])^{3/2}\int_0^T \langle t\rangle^{-3/2(1-C_1\varepsilon)}dt \\ &\quad + \delta\sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}[u](t)(C_1\mathcal{E}_{p^*,p^*}[u_0])^{1/2}\int_0^T \langle t\rangle^{-1+C_1\varepsilon/2}dt \\ &\lesssim \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] \\ &\quad + \frac{\delta}{C_1\varepsilon} S(T)(C_1\mathcal{E}_{p^*,p^*}[u_0]\,\langle T\rangle^{C_1\varepsilon})^{1/2}. \end{aligned} \tag{11.17}$$

Combined with (11.4) and (11.6a), (11.17) leads to the bound

$$S(T) \le A_0 + B_2\left(C_1\langle T_0\rangle^{C_1\varepsilon}\right)^{1/2} S(T), \tag{11.18}$$

with $A_0$ as above and

$$B_2 = \frac{C_0\delta}{4C_1\varepsilon}\mathcal{E}^{1/2}_{p^*,p^*}[u_0].$$

Putting (11.15), (11.18) together, we have derived

$$S(T) \le A_0 + \min\{B_1, B_2\}\left(C_1\langle T_0\rangle^{C_1\varepsilon}\right)^{1/2} S(T).$$

If (11.14) holds, then

$$\min\{B_1,B_2\}(C_1\langle T_0\rangle^{C_1\varepsilon})^{1/2} \le 1/2,$$

and we obtain the desired conclusion

$$S(T) \le 2A_0 \le 4A_0 = C_0\mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}^{1/2}_{p,q}[u_0]\right),$$

for $0 \leq T < T_0$. $\square$

12. **Remarks on Local Existence.**

**Lemma 12.1.** *The operator*

$$P(\nabla) = A^j \partial_j u + \nu B \Delta u$$

*from* (2.2a), (2.2c) *generates a $C^0$ semigroup $U(t)$ on $X^{p,q}$.*

*Proof.* The explicit formula for the Fourier transform of $U(t)$ is

$$\begin{aligned}
\widehat{U}(t,\xi) &= \exp tP(i\xi) \\
&= \exp t \begin{bmatrix} -\nu|\xi|^2 & i\xi^\top \\ i\xi & 0 \end{bmatrix} \\
&= \begin{bmatrix} b'(t,\xi) & i\xi^\top b(t,\xi) \\ \\ i\xi b(t,\xi) & -\xi \otimes \xi \int_0^t b(s,\xi) ds + I \end{bmatrix},
\end{aligned}$$

where $b(t,\xi)$ solves the ODE

$$\begin{aligned}
&D_t^2 b(t,\xi) + \nu|\xi|^2 D_t b(t,\xi) + |\xi|^2 b(t,\xi) = 0, \\
&b(t,\xi) = 0, \quad D_t b(0,\xi) = 1.
\end{aligned}$$

The $C^\infty$ function $b(t,\xi)$ is given by

$$b(t,\xi) = \frac{e^{\lambda_1(|\xi|^2)t} - e^{\lambda_2(|\xi|^2)t}}{\lambda_1(|\xi|^2) - \lambda_2(|\xi|^2)}$$

with

$$\begin{aligned}
\lambda_1(|\xi|^2) &= \frac{1}{2}\left(-\nu|\xi|^2 + \sqrt{\nu^2|\xi|^4 - 4|\xi|^2}\right) \sim -\frac{1}{\nu} \\
\lambda_2(|\xi|^2) &= \frac{1}{2}\left(-\nu|\xi|^2 - \sqrt{\nu^2|\xi|^4 - 4|\xi|^2}\right) \sim -\nu|\xi|^2,
\end{aligned}$$

as $|\xi| \to \infty$. $\square$

**Lemma 12.2.** *If $u_0 \in X^{p,q}$ and $f = (f^0, \dots, f^3) = (f_0, \bar{f}) \in L^2([0,T], X^{p,q})$, then the IVP*

$$\partial_t u - P(\nabla)u = [f^0 + \nabla \cdot \bar{f}]e_0, \quad u(0) = u_0$$

*has a unique solution $u \in C^0([0,T], X^{p,q})$ given by*

$$u(t) = U(t)u_0 + \int_0^t U(t-s)[f^0(s) + \nabla \cdot \bar{f}(s)]e_0 ds.$$

*This solution satisfies the estimate*

$$\begin{aligned}
&\|u(t)\|_{X^{p,q}}^2 + \nu \int_0^t \|\nabla u^0(s)\|_{X^{p,q}}^2 ds \\
&\qquad \lesssim \langle t \rangle^q \left[\|u_0\|_{X^{p,q}}^2 + \int_0^t (\|f^0(s)\|_{X^{p,q}}^2 + \nu^{-1}\|\bar{f}(s)\|_{X^{p,q}}^2) ds\right]. \quad (12.1)
\end{aligned}$$

*Proof.* This can be shown using the Fourier transform, energy estimates, and induction on $q$. $\square$

**Theorem 12.3.** *If $u_0 \in X^{p,q}$, $p \geq 4$, satisfies* (2.2b), *then there exists a $T > 0$ depending only on $\|u(0)\|_{X^{p,q}}$, $\nu$, and $\max_{\alpha,\beta,\ell} |C^\ell_{\alpha,\beta}|$ such that the IVP for* (2.2a), (2.2b) *has a unique solution $u \in C([0,T], X^{p,q})$.*

*Proof.* For $|a| + k \leq p$, $k \leq q$, and $u \in X^{p,q}$, we may write (as in Proposition 10.1)

$$S_0^k \Gamma^a N(u, \nabla u) = f^0 + \nabla \cdot \bar{f},$$

with $f = (f^0, \bar{f}) \in X^{p,q}$. Thanks to the energy estimate (12.1), the map

$$F(u)(t) = U(t)u_0 + \int_0^t U(t-s)N(u(s), \nabla u(s))ds$$

is a contraction on $C([0,T], B_1(u(0)))$, provided $T$ is sufficiently small, where $B_1(u_0)$ denotes the closed ball of radius one with center $u_0$ in $X^{p,q}$. □

**Proposition 12.4.** *There is a continuous function $\Phi : \mathbb{R}^3 \to \mathbb{R}^+$ such that the local solution of Theorem 12.3 satisfies*

$$\|u(t)\|_{X^{p,q}} \leq \Phi(\|u_0\|_{X^{p,q}}, t, \sup_{0 \leq s \leq t} \|u(s)\|_{X^{p,0}}), \quad 0 \leq t \leq T.$$

*Proof.* This is proven by induction on $q$. For $|a| + k \leq q$, the equation satisfied by $v = S_0^k \Gamma^a u$ is linear in $v$. □

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