

RIGIDITY OF QUASI-EINSTEIN METRICS

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ABSTRACT. We call a metric quasi-Einstein if the m -Bakry-Emery Ricci tensor is a constant multiple of the metric tensor. This is a generalization of Einstein metrics, which contains gradient Ricci solitons and is also closely related to the construction of the warped product Einstein metrics. We study properties of quasi-Einstein metrics and prove several rigidity results. We also give a splitting theorem for some Kähler quasi-Einstein metrics.

1. INTRODUCTION

Einstein metrics and their generalizations are important both in mathematics and physics. A particular example is from the study of smooth metric measure spaces. Recall a smooth metric measure space is a triple $(M^n, g, e^{-f} dvol_g)$, where M is a complete n -dimensional Riemannian manifold with metric g , f is a smooth real valued function on M , and $dvol_g$ is the Riemannian volume density on M . A natural extension of the Ricci tensor to smooth metric measure spaces is the m -Bakry-Emery Ricci tensor

$$(1.1) \quad \text{Ric}_f^m = \text{Ric} + \text{Hess}f - \frac{1}{m} df \otimes df \quad \text{for } 0 < m \leq \infty.$$

When f is constant, this is the usual Ricci tensor. We call a triple (M, g, f) (a Riemannian manifold (M, g) with a function f on M) (m) -quasi-Einstein if it satisfies the equation

$$(1.2) \quad \text{Ric}_f^m = \text{Ric} + \text{Hess}f - \frac{1}{m} df \otimes df = \lambda g$$

for some $\lambda \in \mathbb{R}$. This equation is especially interesting in that when $m = \infty$ it is exactly the gradient Ricci soliton equation; when m is a positive integer, it corresponds to warped product Einstein metrics (see Section 2 for detail); when f is constant, it gives the Einstein equation. We call a quasi-Einstein metric trivial when f is constant (the rigid case).

Many geometric and topological properties of manifolds with Ricci curvature bounded below can be extended to manifolds with m -Bakry-Emery Ricci tensor bounded from below when m is finite or m is infinite and f is bounded, see the survey article [18] and the references there for details.

Quasi-Einstein metrics for finite m and for $m = \infty$ share some common properties. It is well-known now that compact solitons with $\lambda \leq 0$ are trivial [8]. The same result is proven in [10] for quasi-Einstein metrics on compact manifolds with finite m . Compact shrinking Ricci solitons have positive scalar curvature [8, 5]. Here we show

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Proposition 1.1. *A quasi-Einstein metric with $1 \leq m < \infty$ and $\lambda > 0$ has positive scalar curvature.*

In dimension 2 and 3 compact Ricci solitons are trivial [6, 8]. More generally compact shrinking Ricci solitons with zero Weyl tensor are trivial [5, 15, 13]. We prove a similar result in dimension 2 for m finite.

Theorem 1.2. *All 2-dimensional quasi-Einstein metrics on compact manifolds are trivial.*

In fact, from the correspondence with warped product metrics, 2-dimensional quasi-Einstein metrics (with finite m) can be classified, see [1, Theorem 9.119]. (The proof was not published though.)

In Section 3 we also extend several properties for Ricci solitons ($m = \infty$) to quasi-Einstein metrics (general m).

On the other hand we show Kähler quasi-Einstein metrics behave very differently when m is finite and m is infinite.

Theorem 1.3. *Let (M^n, g) be an n -dimensional complete simply-connected Riemannian manifold with a Kähler quasi-Einstein metric for finite m . Then $M = M_1 \times M_2$ is a Riemannian product, and f can be considered as a function of M_2 , where M_1 is an $(n-2)$ -dimensional Einstein manifold with Einstein constant λ , and M_2 is a 2-dimensional quasi-Einstein manifold.*

Combine this with Theorem 1.2 we immediately get

Corollary 1.4. *There are no nontrivial Kähler quasi-Einstein metrics with finite m on compact manifolds.*

Note that all known examples of (nontrivial) compact shrinking soliton are Kähler, see the survey article [3].

Ricci solitons play a very important role in the theory of Ricci flow and are extensively studied recently. Warped product Einstein metrics have considerable interest in physics and many Einstein metrics are constructed in this form, especially on noncompact manifolds [1]. It was asked in [1] whether one could find Einstein metrics with nonconstant warping function on compact manifolds. From Corollary 1.4 only non-Kähler ones are possible. Indeed in [12] warped product Einstein metrics are constructed on a class of S^2 bundles over Kähler-Einstein bases warped with S^m for $m \geq 2$, giving compact nontrivial quasi-Einstein metrics for positive integers $n \geq 4$, $m \geq 2$. When $m = 1$, there are no nontrivial quasi-Einstein metrics on compact manifolds, see Remark 3.5 and Proposition 2.1. When $n = 3$, $m \geq 2$ it remains open.

In Section 4 we also give a characterization of quasi-Einstein metrics with finite m which are Einstein at the same time.

2. WARPED PRODUCT EINSTEIN METRICS

In this section we show that when m is a positive integer the quasi-Einstein metrics (1.2) correspond to some warped product Einstein metrics, mainly due to the work of [10].

Recall that given two Riemannian manifolds (M^n, g_M) , (F^m, g_F) and a positive smooth function u on M , the warped product metric on $M \times F$ is defined by

$$(2.3) \quad g = g_M + u^2 g_F.$$

We denote it as $M \times_u F$. Warped product is very useful in constructing various metrics.

When $0 < m < \infty$, consider $u = e^{-\frac{f}{m}}$. Then we have

$$\begin{aligned}\nabla u &= -\frac{1}{m} e^{-\frac{f}{m}} \nabla f, \\ \frac{m}{u} \text{Hess } u &= -\text{Hess } f + \frac{1}{m} df \otimes df.\end{aligned}$$

Therefore (1.2) can be rewritten as

$$(2.4) \quad \text{Ric} - \frac{m}{u} \text{Hess } u = \lambda g.$$

Hence we can use equation (2.4) to study (1.2) when m is finite and vice verse. Taking trace of (2.4) we have

$$(2.5) \quad \Delta u = \frac{u}{m} (R - \lambda n).$$

Since $u > 0$ this immediately gives the following result which is similar to the $m = \infty$ (soliton) case.

Proposition 2.1. *A compact quasi-Einstein metric with constant scalar curvature is trivial.*

In [10] it is shown that a Riemannian manifold (M, g) satisfies (2.4) if and only if the warped product metric $M \times_u F^m$ is Einstein, where F^m is an m -dimensional Einstein manifold with Einstein constant μ satisfies

$$\mu = u\Delta u + (m-1)|\nabla u|^2 + \lambda u^2.$$

(In [10] it is only stated for compact Riemannian manifold, while compactness is redundant. Also the Laplacian there and here have different sign.) Therefore we have the following nice characterization of the quasi-Einstein metrics as the base metrics of warped product Einstein metrics.

Theorem 2.2. *(M, g) satisfies the quasi-Einstein equation (1.2) if and only if the warped product metric $M \times_{e^{-\frac{f}{m}}} F^m$ is Einstein, where F^m is an m -dimensional Einstein manifold with Einstein constant μ satisfying*

$$(2.6) \quad \mu e^{\frac{2}{m}f} = \lambda - \frac{1}{m} (\Delta f - |\nabla f|^2).$$

3. FORMULAS AND RIGIDITY FOR QUASI-EINSTEIN METRICS

In this section we generalize the calculations in [14] for Ricci solitons to the metrics satisfying the quasi-Einstein equation (1.2).

Recall the following general formulas, see e.g. [14, Lemma 2.1] for a proof.

Lemma 3.1. *For a function f in a Riemannian manifold*

$$(3.7) \quad 2(\text{div Hess } f)(\nabla f) = \frac{1}{2} \Delta |\nabla f|^2 - |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla \Delta f \rangle,$$

$$(3.8) \quad \text{div } \nabla \nabla f = \text{Ric} \nabla f + \nabla \Delta f.$$

The trace form of (1.2)

$$(3.9) \quad R + \Delta f - \frac{1}{m} |\nabla f|^2 = \lambda n,$$

where R is the scalar curvature, will be used later.

Using these formulas and the contracted second Bianchi identity

$$(3.10) \quad \nabla R = 2 \operatorname{div} \operatorname{Ric},$$

we can show the following formulas for quasi-Einstein metrics, which generalize some of the formulas in Section 2 of [14].

Lemma 3.2. *If $\operatorname{Ric}_f^m = \lambda g$, then*

$$(3.11) \quad \frac{1}{2} \Delta |\nabla f|^2 = |\operatorname{Hess} f|^2 - \operatorname{Ric}(\nabla f, \nabla f) + \frac{2}{m} |\nabla f|^2 \Delta f,$$

$$(3.12) \quad \frac{1}{2} \nabla R = \frac{m-1}{m} \operatorname{Ric}(\nabla f) + \frac{1}{m} (R - (n-1)\lambda) \nabla f,$$

$$(3.13) \quad \begin{aligned} \frac{1}{2} \Delta R - \frac{m+2}{2m} \nabla_{\nabla f} R &= \frac{m-1}{m} \operatorname{tr}(\operatorname{Ric} \circ (\lambda I - \operatorname{Ric})) - \frac{1}{m} (R - n\lambda)(R - (n-1)\lambda) \\ &= -\frac{m-1}{m} \left| \operatorname{Ric} - \frac{1}{n} Rg \right|^2 - \frac{m+n-1}{mn} (R - n\lambda) \left(R - \frac{n(n-1)}{m+n-1} \lambda \right). \end{aligned}$$

Proof. From (3.7) we have

$$(3.14) \quad \frac{1}{2} \Delta |\nabla f|^2 = 2(\operatorname{div} \operatorname{Hess} f)(\nabla f) + |\operatorname{Hess} f|^2 - \operatorname{Ric}(\nabla f, \nabla f) - \langle \nabla f, \nabla \Delta f \rangle.$$

By taking the divergence of (1.2), we have

$$(3.15) \quad \operatorname{div} \operatorname{Ric} + \operatorname{div} \operatorname{Hess} f - \frac{1}{m} \Delta f \operatorname{div} f - \frac{1}{m} (\nabla_{\nabla f} \nabla f)^* = 0,$$

where X^* is the dual 1-form of the vector field X . Using (3.10) we get

$$(3.16) \quad 2 \operatorname{div} \operatorname{Hess} f(\nabla f) = -\langle \nabla R, \nabla f \rangle + \frac{2}{m} \Delta f |\nabla f|^2 + \frac{2}{m} \operatorname{Hess} f(\nabla f, \nabla f).$$

Now taking the covariant derivative of (3.9) yields

$$(3.17) \quad \nabla R + \nabla \Delta f - \frac{1}{m} \nabla |\nabla f|^2 = 0.$$

Plug this into (3.16) and then plug (3.16) into (3.14) we get

$$\begin{aligned} \frac{1}{2} \Delta |\nabla f|^2 &= \langle \nabla \Delta f - \frac{1}{m} \nabla |\nabla f|^2, \nabla f \rangle + \frac{2}{m} \Delta f |\nabla f|^2 + \frac{2}{m} \operatorname{Hess} f(\nabla f, \nabla f) \\ &\quad + |\operatorname{Hess} f|^2 - \operatorname{Ric}(\nabla f, \nabla f) - \langle \nabla f, \nabla \Delta f \rangle \\ &= |\operatorname{Hess} f|^2 - \operatorname{Ric}(\nabla f, \nabla f) + \frac{2}{m} |\nabla f|^2 \Delta f, \end{aligned}$$

which is (3.11).

For (3.12), using (3.10), (3.15), (3.8), and (3.17), we get

$$\begin{aligned}
\nabla R &= 2 \operatorname{div} \operatorname{Ric} \\
&= -2 \operatorname{div} \operatorname{Hess} f + \frac{2}{m} \Delta f \nabla f + \frac{2}{m} \nabla_{\nabla f} \nabla f \\
&= -2 \operatorname{Ric}(\nabla f) - 2 \nabla \Delta f + \frac{2}{m} \Delta f \nabla f + \frac{2}{m} \nabla_{\nabla f} \nabla f \\
&= -2 \operatorname{Ric}(\nabla f) + 2 \nabla R - \frac{2}{m} \nabla |\nabla f|^2 + \frac{2}{m} \Delta f \nabla f + \frac{2}{m} \nabla_{\nabla f} \nabla f.
\end{aligned}$$

Solving for ∇R and noting that $\nabla |\nabla f|^2 = 2 \nabla_{\nabla f} \nabla f$, we get

$$\nabla R = 2 \operatorname{Ric}(\nabla f) - \frac{2}{m} \Delta f \nabla f + \frac{2}{m} \nabla_{\nabla f} \nabla f.$$

From (1.2), we have that

$$\nabla_{\nabla f} \nabla f = \left(\lambda + \frac{1}{m} |\nabla f|^2 \right) \nabla f - \operatorname{Ric}(\nabla f),$$

so by using this substitution and (3.9) for Δf , we arrive at (3.12),

$$\frac{1}{2} \nabla R = \frac{m-1}{m} \operatorname{Ric}(\nabla f) + \frac{1}{m} (R - (n-1)\lambda) \nabla f.$$

Taking the divergent of the above equation we have

$$(3.18) \quad \frac{1}{2} \Delta R = \frac{m-1}{m} \operatorname{div} (\operatorname{Ric}(\nabla f)) + \frac{1}{m} \operatorname{div} ((R - (n-1)\lambda) \nabla f).$$

Now

$$\begin{aligned}
\operatorname{div} (\operatorname{Ric}(\nabla f)) &= \langle \operatorname{div} \operatorname{Ric}, \nabla f \rangle + \operatorname{tr} (\operatorname{Ric} \circ \operatorname{Hess} f) \\
&= \langle \frac{1}{2} \nabla R, \nabla f \rangle + \operatorname{tr} \left(\operatorname{Ric} \circ \left(\frac{1}{m} df \otimes df + \lambda g - \operatorname{Ric} \right) \right) \\
&= \langle \frac{1}{2} \nabla R, \nabla f \rangle + \frac{1}{m} \operatorname{Ric}(\nabla f, \nabla f) + \operatorname{tr} (\operatorname{Ric} \circ (\lambda g - \operatorname{Ric})) \\
(3.19) \quad &= \langle \frac{1}{2} \nabla R, \nabla f \rangle + \frac{1}{m-1} \left(\langle \frac{1}{2} \nabla R, \nabla f \rangle - \frac{1}{m} (R - (n-1)\lambda) |\nabla f|^2 \right) \\
&\quad + \operatorname{tr} (\operatorname{Ric} \circ (\lambda g - \operatorname{Ric})),
\end{aligned}$$

where the last equation comes from (3.12). Also

$$(3.20) \quad \operatorname{div} ((R - (n-1)\lambda) \nabla f) = (R - (n-1)\lambda) \Delta f + \langle \nabla R, \nabla f \rangle.$$

Plugging (3.19) and (3.20) into (3.18) and using (3.9) we arrive at

$$\frac{1}{2} \Delta R - \frac{m+2}{2m} \nabla_{\nabla f} R = \frac{m-1}{m} \operatorname{tr} (\operatorname{Ric} \circ (\lambda I - \operatorname{Ric})) - \frac{1}{m} (R - n\lambda)(R - (n-1)\lambda).$$

Let λ_i be the eigenvalues of the Ricci tensor, we get

$$\begin{aligned}
\operatorname{tr} (\operatorname{Ric} \circ (\lambda I - \operatorname{Ric})) &= \sum \lambda_i (\lambda - \lambda_i) \\
&= - \left| \operatorname{Ric} - \frac{1}{n} Rg \right|^2 + R \left(\lambda - \frac{1}{n} R \right),
\end{aligned}$$

which yields (3.13). \square

As in [14], these formulas give important information about quasi-Einstein metrics. Combining the first equation (3.11) in Lemma 3.2 with the maximal principle we have

Proposition 3.3. *If a compact Riemannian manifold satisfying (1.2) and*

$$\text{Ric}(\nabla f, \nabla f) \leq \frac{2}{m} |\nabla f|^2 \Delta f$$

then the function f is constant so it is Einstein.

Equation (3.12) gives

Proposition 3.4. *When $m \neq 1$, a quasi-Einstein metric has constant scalar curvature if and only if*

$$\text{Ric}(\nabla f) = -\frac{1}{m-1} (R - (n-1)\lambda) \nabla f.$$

Remark 3.5. *When $m = 1$, the constant μ in (2.6) is zero, combining with (3.9), we get $R = (n-1)\lambda$. The scalar curvature is always constant.*

Equation (3.13) gives the following results.

Proposition 3.6. *If a Riemannian manifold M satisfies (1.2) with $m \geq 1$ and*
a) $\lambda > 0$ and M is compact then the scalar curvature is bounded below by

$$(3.21) \quad R \geq \frac{n(n-1)}{m+n-1} \lambda.$$

Equality holds if and only if $m = 1$.

- b) $\lambda = 0$, the scalar curvature is constant and $m > 1$, then M is Ricci flat.*
- c) $\lambda < 0$ and the scalar curvature is constant, then*

$$n\lambda \leq R \leq \frac{n(n-1)}{m+n-1} \lambda$$

and when $m > 1$, R equals either of the extreme values iff M is Einstein.

Remark 3.7. *When m is finite, a manifold with quasi-Einstein metric and $\lambda > 0$ is automatically compact [16].*

Remark 3.8. *Let $m = \infty$, we recover the well know result [8] that compact shrinking Ricci soliton has positive scalar curvature, and some results in [14] about gradient Ricci solitons with constant scalar curvature.*

Proof. a) Since M is compact, applying the equation (3.13) to a minimal point of R , we have

$$-\frac{m+n-1}{mn} (R_{\min} - n\lambda) (R_{\min} - \frac{n(n-1)}{m+n-1} \lambda) \geq \frac{m-1}{m} \left| \text{Ric} - \frac{1}{n} Rg \right|^2 \geq 0.$$

So

$$\frac{n(n-1)}{m+n-1} \lambda \leq R_{\min} \leq n\lambda$$

which gives (3.21).

- b) c) Since R is constant, from (3.13)

$$-\frac{m+n-1}{mn} (R - n\lambda) (R - \frac{n(n-1)}{m+n-1} \lambda) = \frac{m-1}{m} \left| \text{Ric} - \frac{1}{n} Rg \right|^2 \geq 0.$$

So if $\lambda = 0$, $m > 1$, then $\text{Ric} = \frac{1}{n}Rg$ and $R = 0$, thus it is Ricci flat. If $\lambda < 0$, $R \in [n\lambda, \frac{n(n-1)}{m+n-1}\lambda]$. \square

4. TWO DIMENSIONAL QUASI-EINSTEIN METRICS

First we recall a characterization of warped product metrics found in [4] (see also [15]).

Theorem 4.1 (Cheeger-Colding). *A Riemannian manifold (M^n, g) is a warped product $(a, b) \times_u N^{n-1}$ if and only if there is a nontrivial function h such that $\text{Hess } h = kg$ for some function $k : M \rightarrow \mathbb{R}$. ($u = h'$ up to a multiplicative constant)*

From this we can give a characterization of quasi-Einstein metrics which are Einstein.

Proposition 4.2. *A complete finite m quasi-Einstein metric (M^n, g, u) is Einstein if and only if u is constant or M is diffeomorphic to \mathbb{R}^n with the warped product structure $\mathbb{R} \times_{a^{-1}e^{ar}} N^{n-1}$, where N^{n-1} is Ricci flat, a is a constant (see below for its value).*

Proof. If g is Einstein, then $\text{Ric} = \mu g$ for some constant μ . From (2.4) we have

$$(4.22) \quad \text{Hess } u = \frac{\mu - \lambda}{m} u g \quad \text{for some } u > 0.$$

If M is compact, then u (thus f) is constant. So if u is not constant, then M is noncompact and $\lambda \leq 0, \mu \leq 0$ and $\mu > \lambda$. So $\lambda < 0, \lambda < \mu \leq 0$ and u is a strictly convex function. Therefore M^n is diffeomorphic to \mathbb{R}^n . By (4.22) and Theorem 4.1, $u = ce^{\sqrt{\frac{\mu-\lambda}{m}}r}$, where c is some constant. And M is $\mathbb{R} \times N^{n-1}$ with the warped product metric

$$g = dr^2 + a^{-2}e^{2ar}g_0,$$

where $a = \sqrt{\frac{\mu-\lambda}{m}}$. Since g is Einstein we get $\mu = -(n-1)\frac{\mu-\lambda}{m} < 0$ and g_0 is Ricci flat. \square

Remark 4.3. *The Taub-NUT metric [7] is a Ricci flat metric on \mathbb{R}^4 which is not flat.*

Since a 2-dimensional Riemannian manifold satisfies $\text{Ric} = \frac{R}{2}g$, we get an immediate corollary of Theorem 4.1.

Corollary 4.4. *A two dimensional quasi-Einstein metric 2.4 is a warped product metric.*

Now we will prove Theorem 1.2.

Proof. Since M is compact, by [10], we only need to prove the theorem when $\lambda > 0$. From (3.21) we have

$$(4.23) \quad R \geq \frac{2}{m+1}\lambda.$$

So up to a cover we may assume M is diffeomorphic to S^2 . Since M is 2-dimensional, we have $\text{Ric} = \frac{R}{2}g$. Thus (3.12) becomes

$$(4.24) \quad \nabla R = \frac{m+1}{m} \left(R - \frac{2}{m+1}\lambda \right) \nabla f.$$

Now let $u = e^{-\frac{f}{m}}$, then from (2.4) $\text{Hess } u = \frac{u}{m} \left(\frac{R}{2} - \lambda \right) g$. In particular, ∇u is conformal. By the Kazdan-Warner identity [9], we have

$$\int_M \langle \nabla R, \nabla u \rangle dV = 0.$$

Thus

$$-\frac{1}{m} \int \langle \nabla R, \nabla f \rangle e^{-\frac{f}{m}} dV = 0.$$

Using (4.24), since $R \geq \frac{2}{m+1}\lambda$, we get $\nabla f = \nabla R = 0$. \square

5. KÄHLER QUASI-EINSTEIN METRICS

There are many nontrivial examples of shrinking Kähler-Ricci solitons [2, 17]. In contrast, we will show here that Kähler quasi-Einstein metrics with finite m are very rigid.

Proof of Theorem 1.3 First, since on a Kähler manifold, the metric and Ricci tensor are both compatible with the complex structure J , from (2.4) we have $\text{Hess } u(JU, JV) = \text{Hess } u(U, V)$ for all vector fields U, V . That implies

$$(5.25) \quad J\nabla_X \nabla u = \nabla_{JX} \nabla u,$$

and the ϕ defined by $\phi(U, V) = \text{Hess } u(JU, V)$ is an $(1,1)$ -form. Note that $2\text{Hess } u = L_{\nabla u} g$. By (5.25) $L_{\nabla u} JX = JL_{\nabla u} X$, so $2\phi = L_{\nabla u} \omega$, where ω is the Kähler form. Since $L_X d = dL_X$ and ω is closed we have ϕ is closed. Furthermore, since the Ricci form is closed, from (2.4) we get that the $(1,1)$ -form $\frac{\phi}{u}$ is also closed, so $du \wedge \phi = 0$. Now

$$(du \wedge \phi)(U, V, W) = Uu \cdot g(\nabla_{JV} \nabla u, W) + Vu \cdot g(\nabla_{JW} \nabla u, U) + Wu \cdot g(\nabla_{JU} \nabla u, V).$$

Let $U, V \perp \nabla u, W = \nabla u$, we have $\text{Hess } u(JU, V) = 0$ for all $U, V \perp \nabla u$. Hence

$$(5.26) \quad \nabla_X \nabla u \parallel J\nabla u \quad \text{for all } X \perp \nabla u.$$

Let $U = \nabla u, V = J\nabla u, W \perp \nabla u$ we get $\nabla_{\nabla u} \nabla u \parallel \nabla u$.

Now we consider the 2-dimensional distribution T_1 that is spanned by ∇u and $J\nabla u$ at those points where ∇u is nonzero. We will show that $T_1 = \text{Span}\{\nabla u, J\nabla u\}$ is invariant under parallel transport, i.e. if γ is a path in M , and U is a parallel field along γ , then

$$(5.27) \quad \nabla_{\gamma'} \left(\frac{g(U, \nabla u) \nabla u}{|\nabla u|^2} + \frac{g(U, J\nabla u) J\nabla u}{|\nabla u|^2} \right) = 0.$$

Since the covariant derivative is linear in γ' , we can prove this in three cases:

- (1) when $\gamma' \perp \nabla u, J\nabla u$: so $\gamma' \perp \nabla u$ and $J\gamma' \perp \nabla u$, by (5.26) $\nabla_{\gamma'} \nabla u = 0$. Since the complex structure J is parallel, $\nabla_{\gamma'} J\nabla u = J\nabla_{\gamma'} \nabla u = 0$. By assumption $\nabla_{\gamma'} U = 0$, hence (5.27) follows.
- (2) when $\gamma' = \nabla u$: Using $\nabla_{\nabla u} \nabla u \parallel \nabla u$, we have

$$\begin{aligned}
& \nabla_{\gamma'} \left(\frac{g(U, \nabla u) \nabla u}{|\nabla u|^2} + \frac{g(U, J\nabla u) J\nabla u}{|\nabla u|^2} \right) \\
&= \gamma' \left(\frac{g(U, \nabla u)}{|\nabla u|^2} \right) \nabla u + \frac{g(U, \nabla u)}{|\nabla u|^2} \nabla_{\gamma'} \nabla u + \gamma' \left(\frac{g(U, J\nabla u)}{|\nabla u|^2} \right) J\nabla u + \frac{g(U, J\nabla u)}{|\nabla u|^2} \nabla_{\gamma'} J\nabla u \\
&= \frac{g(U, \nabla_{\nabla u} \nabla u)}{|\nabla u|^2} \nabla u - \frac{2g(\nabla u, \nabla_{\nabla u} \nabla u)g(U, \nabla u)}{|\nabla u|^4} \nabla u + \frac{g(U, \nabla u)}{|\nabla u|^2} \frac{g(\nabla u, \nabla_{\nabla u} \nabla u)}{|\nabla u|^2} \nabla u \\
&+ \frac{g(U, \nabla_{\nabla u} J\nabla u)}{|\nabla u|^2} J\nabla u - \frac{2g(\nabla u, \nabla_{\nabla u} \nabla u)g(U, J\nabla u)}{|\nabla u|^4} J\nabla u + \frac{g(U, J\nabla u)}{|\nabla u|^2} \frac{g(J\nabla u, \nabla_{\nabla u} J\nabla u)}{|\nabla u|^2} J\nabla u \\
&= \frac{\nabla u}{|\nabla u|^2} \left(\text{Hess } u(U, \nabla u) - \frac{\text{Hess } u(\nabla u, \nabla u)}{|\nabla u|^2} g(U, \nabla u) \right) \\
&+ \frac{J\nabla u}{|\nabla u|^2} \left(-\text{Hess } u(JU, \nabla u) + \frac{\text{Hess } u(\nabla u, \nabla u)}{|\nabla u|^2} g(JU, \nabla u) \right) \\
&= \frac{\nabla u}{|\nabla u|^2} \text{Hess } u \left(U - \frac{g(U, \nabla u)}{|\nabla u|^2} \nabla u, \nabla u \right) - \frac{J\nabla u}{|\nabla u|^2} \text{Hess } u \left(JU - \frac{g(JU, \nabla u)}{|\nabla u|^2} \nabla u, \nabla u \right) \\
&= 0,
\end{aligned}$$

where the last equality follows from (5.26) and that $U - \frac{g(U, \nabla u)}{|\nabla u|^2} \nabla u \perp \nabla u$.

(3) $\gamma' = J\nabla u$: Using $J\nabla_X \nabla u = \nabla_{JX} \nabla u$ it reduces to the previous case.

Now we have an orthogonal decomposition of the tangent bundle $TM = T_1 \oplus T_1^\perp$ that is invariant under parallel transport. By DeRham's decomposition theorem on a simply-connected manifold [11, Page 187], M is a Riemannian product, and all the claims in the theorem follow. \square

Proof of Corollary 1.4 Since the manifold M is compact, by [10], we can assume $\lambda > 0$. Then, by [16], the universal cover \tilde{M} is also compact. Now the result follows from Theorem 1.3 and 1.2. \square

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