RIGIDITY OF MINIMAL SUBMANIFOLDS IN SPACE FORMS

HANG CHEN AND GUOFANG WEI

Abstract. In this paper, we consider the rigidity for an $n(\geq 4)$-dimensional submanifold M^n with parallel mean curvature in the space form $\mathbb{M}_c^{n+p}$ when the integral Ricci curvature of M has some bound. We prove that, if $c + H^2 > 0$ and $\|\operatorname{Ric}_-^\lambda\|_{n/2} < \epsilon(n, c, \lambda, H)$ for λ satisfying $\frac{n-2}{n-1}(c + H^2) < \lambda \leq c + H^2$, then M is the totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{c+H^2}})$. Here H is the norm of the parallel mean curvature of M, and $\epsilon(n, c, \lambda, H)$ is a positive constant depending only on n, c, λ and H. This extends part of the earlier work of [16] from pointwise Ricci curvature lower bound to integral Ricci curvature lower bound.

1. Introduction

There is a long history of studying rigidity phenomenon for submanifolds under certain curvature pinching conditions. A lot of a rigidity theorems for closed minimal submanifolds in a sphere were proved by Simons, Chern-do Carmo-Kobayashi, Lawson, Yau and others (see [3–5, 7–10, 12–14, 20, 21]). Let $\mathbb{S}^n(r)$ and $\mathbb{M}_c^{n+p}$ denote the n-dimensional sphere with radius r and the $(n + p)$-dimensional (simply-connected) space form with constant curvature c respectively, and we will omit the radius r and just denote $\mathbb{S}^n$ if $r = 1$ for simplicity. In 1979, Ejiri proved the following theorem.

Theorem 1.1 ([4]). *Let M be an n-dimensional $(n \geq 4)$ simply connected compact orientable minimal submanifold immersed in $\mathbb{S}^{n+p}$. If $\operatorname{Ric}^M \geq n - 2$, then M is either the totally geodesic submanifold $\mathbb{S}^n$, the Clifford torus $\mathbb{S}^m(\sqrt{1/2}) \times \mathbb{S}^m(\sqrt{1/2})$ in $\mathbb{S}^{n+1}$ with $n = 2m$, or $\mathbb{C}P_{4/3}^2$ in $\mathbb{S}^7$. Here $\mathbb{C}P_{4/3}^2$ denotes the 2-dimensional complex projective space minimally immersed in $\mathbb{S}^7$ with constant holomorphic sectional curvature $4/3$.*

In 1992, Shen [13] proved that any 3-dimensional compact orientable minimal submanifold M immersed in $\mathbb{S}^{3+p}$ with $\operatorname{Ric}^M > 1$ must be totally geodesic. Later, Li [10] improved the pinching constant in Ejiri's theorem for odd-dimensional cases. In 2011, Xu and Tian [17] pointed out the assumption that M is simply connected in Ejiri's theorem can be removed. In 2013, Xu and Gu proved the following generalized Ejiri rigidity for compact submanifolds with parallel mean curvature in space forms.

Theorem 1.2 (Theorem 3.3 in [16]). *Let M be an n-dimensional $(n \geq 3)$ compact orientable submanifold with parallel mean curvature in the space form $\mathbb{M}_c^{n+p}$ with $c + H^2 > 0$. Here H is the norm of the parallel mean curvature of M. If $\operatorname{Ric}^M \geq (n - 2)(c + H^2)$, then M is either a totally umbilical sphere $\mathbb{S}^n(\frac{1}{\sqrt{c+H^2}})$, the Clifford torus $\mathbb{S}^m(\frac{1}{\sqrt{2(c+H^2)}}) \times$*

$\mathbb{S}^m(\frac{1}{\sqrt{2(c+H^2)}})$ *in the totally umbilical sphere* $\mathbb{S}^{n+1}(\frac{1}{\sqrt{c+H^2}})$ *with* $n = 2m$, *or* $\mathbb{CP}^2_{\frac{4}{3}(c+H^2)}$ *in* $\mathbb{S}^7(\frac{1}{\sqrt{c+H^2}})$.

In particular, this gives

Corollary 1.3 (Corollary 3.4 in [16]). *Let* M *be an* $n(\geq 3)$-*dimensional oriented compact submanifold with parallel mean curvature in* $\mathbb{M}^{n+p}_c$ *with* $c + H^2 > 0$. *If* $Ric^M > (n-2)(c + H^2)$, *then* M *is the totally umbilical sphere* $\mathbb{S}^n(\frac{1}{\sqrt{c+H^2}})$.

Note that the curvature conditions in both original and generalized Ejiri theorems are pointwise lower Ricci curvature bounds. It is natural to ask that if we can improve the pinching condition. In odd-dimensional case, the pinching constant can be lowered down (see Li [10], Xu-Leng-Gu [18]'s results). In this paper, we will consider the integral Ricci curvature condition instead of the pointwise Ricci curvature condition.

For each $x \in M$, let $\rho(x)$ be the smallest eigenvalue of the Ricci tensor at x, and $\mathrm{Ric}^\lambda_-(x) = \max\{0, (n-1)\lambda - \rho(x)\}$ for $\lambda \in \mathbb{R}$. Define

$$\| \mathrm{Ric}^\lambda_- \|_q := \left(\int_M (\mathrm{Ric}^\lambda_-)^q \right)^{1/q},$$

which measures the amount of Ricci curvature lying below the given bound $(n-1)\lambda$. It is easy to see that $\| \mathrm{Ric}^\lambda_- \|_q = 0$ if and only if $\mathrm{Ric}^M \geq (n-1)\lambda$.

Now we can state our main theorems.

Theorem 1.4. *Let* M *be an* n-*dimensional* $(n \geq 4)$ *minimal closed submanifold in* $\mathbb{S}^{n+p}(r)$. *Given* λ *satisfying* $(n-2)/r^2 < (n-1)\lambda \leq (n-1)/r^2$, *if*

$$\| \mathrm{Ric}^\lambda_- \|_{n/2} < \epsilon_r(n, \lambda),$$

then M *is totally geodesic. Here* $\epsilon_r(n, \lambda)$ *is an explicit constant defined in* (3.8).

In [11] Petersen and the second author established the fundamental comparison tools, the Laplacian and Bishop-Gromov volume comparisons, for integral Ricci curvature lower bound when $q > \frac{n}{2}$. Here we only require smallness of the integral curvature for $q = \frac{n}{2}$ as the manifold is special.

Remark 1.5. *For a minimal submanifold* M *in* $\mathbb{S}^{n+p}(r)$, *the Ricci curvature of* M *has the upper bound* $(n-1)/r^2$ *from* (2.5) *in Section 2. That is why we limit the range of* λ *in Theorem* 1.4.

Theorem 1.4 is a special case of the following result.

Theorem 1.6. *Let* M *be an* n-*dimensional* $(n \geq 4)$ *closed submanifold in* $\mathbb{M}^{n+p}_c$ *with parallel mean curvature (PMC). Denote* H *the norm of the parallel mean curvature of* M. *Assume* $c + H^2 > 0$. *Given* λ *satisfying* $(n-2)(c + H^2) < (n-1)\lambda \leq (n-1)(c + H^2)$, *if*

$$\| \mathrm{Ric}^\lambda_- \|_{n/2} < \epsilon(n, c, \lambda, H),$$

then M *is the totally umbilical sphere* $\mathbb{S}^n(\frac{1}{\sqrt{c+H^2}})$. *Here* $\epsilon(n, c, \lambda, H)$ *is an explicit constant defined in* (4.3).

This generalizes Corollary 1.3 for $n \geq 4$.

Remark 1.7. *For any $q > n/2$, we have $\|\operatorname{Ric}_-^\lambda\|_{n/2} \leq (\operatorname{vol}(M))^{\frac{2}{n} - \frac{1}{q}} \|\operatorname{Ric}_-^\lambda\|_q$ by the Hölder inequality. Aubry [1] shows that $\operatorname{vol}(M)$ is bounded from the above by a quantity in terms of $\|\operatorname{Ric}_-^\lambda\|_q$ for $\lambda > 0$. So when $\|\operatorname{Ric}_-^\lambda\|_q$ is small enough, we have $\|\operatorname{Ric}_-^\lambda\|_{n/2} < \epsilon(n, c, \lambda, H)$. Hence, given $q > n/2$, the conclusion of Theorem 1.6 still holds if $\|\operatorname{Ric}_-^\lambda\|_q < \epsilon(n, q, c, \lambda, H)$ for some constant $\epsilon(n, q, c, \lambda, H)$.*

Remark 1.8. *Xu [15] proved that, for an $n(\geq 3)$-dimensional closed M with parallel mean curvature in the unit sphere $\mathbb{S}^{n+p}$, if $\|S - nH^2\|_{n/2} < C(n, p, H)$, then M is $\mathbb{S}^n(\frac{1}{\sqrt{1+H^2}})$. From (2.6), $S - nH^2 = n(n-1)(1+H^2) - R$. Here S is the norm of the second fundamental form and R is the scalar curvature. Hence Xu's result is an integral perturbation of scalar curvature while our result is an integral perturbation of Ricci curvature. On the other hand while $R \leq n(n-1)(1+H^2)$, it is not clear if $\operatorname{Ric} \leq (n-1)(1+H^2)$ when $H \neq 0$.*

The paper is organized as follows. In Section 2 we introduce the notations and recall a few results from [15] which we will need. In Section 3 we prove Theorem 1.4, the starting point is the Simons' identity. In Section 4 we prove Theorem 1.6 by first showing it is pseudo-umbilical, then reducing it to Theorem 1.4 with dimension reduction.

2. Preliminaries

In this paper, we will use the following convention on the ranges of indices except special declaration:

$$1 \leq A, B, C, \cdots \leq n + p; \quad 1 \leq i, j, k, \cdots \leq n; \quad n + 1 \leq \alpha, \beta, \gamma, \cdots \leq n + p.$$

Assume that M^n is immersed in N^{n+p}. We choose a local orthonormal frame $\{e_1, \cdots, e_{n+p}\}$ such that $\{e_1, \cdots, e_n\}$ are tangent to M and $\{e_{n+1}, \cdots, e_{n+p}\}$ are normal to M when restricted to M. Let $\{\omega_A\}$ be the dual coframe. Denote

$$h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$$

the second fundamental form of M immersed in N, and define

$$A_\alpha = (h_{ij}^\alpha), \quad H^\alpha = \frac{\operatorname{tr} A_\alpha}{n}, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \quad H = |\mathbf{H}| = \sqrt{\sum_\alpha (H^\alpha)^2}, \quad S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2.$$

It is well known that when $N = \mathbb{M}_c^{n+p}$, Gauss, Codazzi and Ricci equations are given by:

$$(2.1) \qquad R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$(2.2) \qquad h_{ijk}^\alpha = h_{ikj}^\alpha,$$

$$(2.3) \qquad R_{\alpha\beta ij}^\perp = \sum_k h_{ik}^\alpha h_{kj}^\beta - \sum_k h_{ik}^\beta h_{kj}^\alpha,$$

where R_{ijkl} and h^α_{ijk} are the components of Riemannian curvature of M and covariant derivative of h^α_{ij} under the orthonormal frame respectively. The Ricci identity shows that

$$(2.4) \qquad h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m R_{mikl} h^\alpha_{mj} + \sum_m R_{mjkl} h^\alpha_{im} + \sum_\beta R^\perp_{\beta\alpha kl} h^\beta_{ij}.$$

From (2.1), we can get the Ricci curvature and the scalar curvature respectively as follows:

$$(2.5) \qquad R_{ij} = c(n-1)\delta_{ij} + n\sum_\alpha H^\alpha h^\alpha_{ij} - \sum_{\alpha,k} h^\alpha_{ik} h^\alpha_{kj},$$

$$(2.6) \qquad R = cn(n-1) + n^2 H^2 - S.$$

Since $S \geq nH^2$, we have $R \leq n(n-1)(c+H^2)$. When $H = 0$, $\mathrm{Ric} \leq (n-1)c$.

Next, we recall some results which will be used to prove the main theorems. Using a Sobolev inequality in [6], Xu proved the following inequality.

Proposition 2.1 (cf. [15]). *Let $M^n(n \geq 3)$ be a closed submanifold in N^{n+p}. Suppose N is a simply connected and complete manifold with non-positive curvature. Then for all $t > 0$ and $f \in C^1(M), f \geq 0$, we have*

$$\int_M |\nabla f|^2 \geq \mathfrak{A}(n,t) \left(\int_M f^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} - \mathfrak{B}(n,t) \int_M H^2 f^2,$$

where
$$(2.7)$$
$$\mathfrak{A}(n,t) = \frac{(n-2)^2}{4(n-1)^2(1+t)} \frac{1}{C^2(n)}, \quad \mathfrak{B}(n,t) = \frac{(n-2)^2}{4(n-1)^2 t}, \quad C(n) = 2^n \frac{(n+1)^{1+1/n}}{(n-1)\omega_n^{1/n}},$$

and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Now we can prove the following lemma.

Lemma 2.2. *Let $M^n(n \geq 3)$ be a closed submanifold in $\mathbb{M}^{n+p}_c$. Then for all $t > 0$ and $f \in C^1(M), f \geq 0$, we have*

$$\int_M |\nabla f|^2 \geq \mathfrak{A}(n,t) \left(\int_M f^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} - \mathfrak{B}(n,t) \int_M (c_+ + H^2) f^2,$$

where

$$c_+ := \max\{c,0\} = \begin{cases} c, & \text{if } c \geq 0; \\ 0, & \text{if } c \leq 0. \end{cases}$$

Proof. When $c \leq 0$, it is directly from Proposition 2.1.

When $c > 0$, considering the composition of isometric immersions $M \to \mathbb{S}^{n+p}(1/\sqrt{c}) \to \mathbb{R}^{n+p+1}$, we obtain the conclusion from Proposition 2.1 (cf. [15, 19]). $\qquad\square$

3. Minimal Case

In this section, we prove Theorem 1.4.

Proof of Theorem 1.4. At first, we assume that $r = 1$. Since $\lambda > \frac{n-2}{n-1}$, we can set $\Lambda := (n-1)\lambda = (n-2) + \delta$ for some $\delta > 0$.

Gauss equation (2.6) gives $R = n(n-1) - S$. Since $R \geq n\rho$ (recall ρ is the smallest eigenvalue of the Ricci tensor), we have

$$(3.1) \qquad \frac{S-n}{n} \leq (n-2) - \rho.$$

By definition,

$$(3.2) \qquad (n-2) - \rho = -\delta + (\Lambda - \rho) \leq -\delta + \mathrm{Ric}_-^\lambda .$$

Using (2.1)–(2.4), after a direct computation, we can obtain the well-known Simons' identity for a minimal submanifold M in the unit sphere $\mathbb{S}^{n+p}$ (cf. [4, 14])

$$(3.3)$$
$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n\sum_{i,j,\alpha}(h_{ij}^\alpha)^2 - \sum_{i,j,k,l,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta h_{kl}^\alpha h_{kl}^\beta - \sum_{i,j,\alpha,\beta}\left(\sum_k h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta\right)^2$$
$$= |\nabla h|^2 + nS - \sum_{\alpha,\beta} N([A_\alpha, A_\beta]) - \sum_{\alpha,\beta}\sigma_{\alpha\beta}^2,$$

where $[A_\alpha, A_\beta] = A_\alpha A_\beta - A_\beta A_\alpha, \sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta$, and $N(\Omega) = \mathrm{tr}(\Omega^t\Omega)$ is the norm of matrix Ω.

We claim

$$(3.4) \qquad \sum_{\alpha,\beta} N([A_\alpha, A_\beta]) \leq 4[(n-1) - \rho]S - \frac{4}{n}\sum_\alpha (N(A_\alpha))^2,$$

$$(3.5) \qquad \sum_{\alpha,\beta}\sigma_{\alpha\beta}^2 = \sum_\alpha (N(A_\alpha))^2 \leq (\sum_\alpha N(A_\alpha))^2 = S^2.$$

(3.5) is obvious, and we use the same argument in [4] to prove (3.4).

For a fixed α, we choose $\{e_i\}$ such that A_α is diagonalized, $A_\alpha = \mathrm{diag}\{\lambda_1^\alpha, \cdots, \lambda_n^\alpha\}$, then (2.5) gives

$$\sum_j \sum_{\beta \neq \alpha}(h_{ij}^\beta)^2 \leq (n-1) - \rho - (\lambda_i^\alpha)^2 \quad \text{for each } i,$$

and

$$\sum_\beta N([A_\alpha, A_\beta]) = \sum_{\beta \neq \alpha} N([A_\alpha, A_\beta]) = \sum_{\beta \neq \alpha}\sum_{i,j}(h_{ij}^\beta)^2(\lambda_i^\alpha - \lambda_j^\alpha)^2$$
$$\leq 2\sum_{i,j}\sum_{\beta \neq \alpha}(h_{ij}^\beta)^2\left((\lambda_i^\alpha)^2 + (\lambda_j^\alpha)^2\right) = \sum_{i,j}\sum_{\beta \neq \alpha}4(h_{ij}^\beta)^2(\lambda_i^\alpha)^2$$
$$\leq 4\sum_i [(n-1) - \rho - (\lambda_i^\alpha)^2](\lambda_i^\alpha)^2$$
$$= 4[(n-1) - \rho]N(A_\alpha) - 4N(A_\alpha^2).$$

Now making summation over α, we have

$$\sum_{\alpha,\beta} N([A_\alpha, A_\beta]) \leq 4[(n-1) - \rho]S - \frac{4}{n}\sum_\alpha(N(A_\alpha))^2,$$

here we used the Cauchy-Schwarz inequality. This completes the proof of (3.4).

Therefore, from (3.3), (3.4) and (3.5), we have $\frac{1}{2}\Delta S \geq |\nabla h|^2 + Q$, where

$$
\begin{aligned}
Q :=& S\left[n - 4((n-1) - \rho) + \frac{4-n}{n}S\right] \\
=& \frac{n-S}{n}(n-4)S - 4((n-2) - \rho)S \\
\geq& -(n-4)((n-2) - \rho)S - 4((n-2) - \rho)S \\
=& -n((n-2) - \rho)S.
\end{aligned}
$$

Here we used (3.1) for the inequality.

From (3.2) we have

$$
\begin{aligned}
\int_M Q \geq & n\delta \int_M S - n\int_M \mathrm{Ric}^\lambda_- S \\
\geq & n\delta \int_M S - n\|\mathrm{Ric}^\lambda_-\|_{n/2}\|S\|_{n/(n-2)},
\end{aligned}
$$
(3.6)

here we used Hölder's inequality in (3.6).

To deal with the term $|\nabla h|^2$, we need the following Kato-type inequality.

Lemma 3.1 (cf. Lemma 1 in [15]). *Let M^n be a minimal submanifold in $\mathbb{S}^{n+p}$. Set $h_\epsilon = (S + np\epsilon^2)^{1/2}$ for any constant $\epsilon \neq 0 \in \mathbb{R}$.*

Then we have

$$
|\nabla h|^2 = \sum_{\alpha,i,j.k} (h^\alpha_{ijk})^2 \geq \frac{n+2}{n}|\nabla h_\epsilon|^2.
$$
(3.7)

Remark 3.2. *In fact, Lemma 3.1 remains true when the ambient space is $\mathbb{M}^{n+p}_c$.*

Here one adds the ϵ term to make sure the radicand is strictly positive, so one can apply Lemma 2.2 to the function h_ϵ.

Now we continue the proof. From (3.7) and Lemma 2.2, we have

$$
\begin{aligned}
\int_M |\nabla h|^2 \geq & \int_M \frac{n+2}{n}|\nabla h_\epsilon|^2 \\
\geq & \frac{n+2}{n}\mathfrak{A}(n,t)\|h_\epsilon^2\|_{n/(n-2)} - \frac{n+2}{n}\mathfrak{B}(n,t)\int_M h_\epsilon^2,
\end{aligned}
$$

where $\mathfrak{A}(n,t), \mathfrak{B}(n,t)$ are defined as in (2.7). Letting $\epsilon \to 0$, we have

$$
\int_M |\nabla h|^2 \geq \frac{n+2}{n}\mathfrak{A}(n,t)\|S\|_{n/(n-2)} - \frac{n+2}{n}\mathfrak{B}(n,t)\int_M S.
$$

Then choosing t_0 such that $\frac{n+2}{n}\mathfrak{B}(n,t_0) = n\delta$ and from above inequalities, we have

$$
0 = \int_M \frac{1}{2}\Delta S \geq \left(\frac{n+2}{n}\mathfrak{A}(n,t_0) - n\|\mathrm{Ric}^\lambda_-\|_{n/2}\right)\|S\|_{n/(n-2)} \geq 0
$$

provided $\|\mathrm{Ric}^\lambda_-\|_{n/2} < \frac{n+2}{n^2}\mathfrak{A}(n,t_0)$. Hence we have $S \equiv 0$, i.e. M is totally geodesic if we set $\epsilon(n,\lambda) = \frac{n+2}{n^2}\mathfrak{A}(n,t_0)$.

Now set $\epsilon_r(n,\lambda) = \epsilon(n,\lambda r^2)$, we can prove the theorem for arbitrary $r > 0$ by rescaling.
$\square$

Remark 3.3. *In fact,*

$$(3.8) \qquad \epsilon_r(n,\lambda) = \epsilon(n,\lambda r^2) = \frac{P_n}{1 + \frac{1/r^2}{(n-1)\lambda - (n-2)/r^2} P_n} \cdot \frac{1}{C^2(n)}.$$

where $P_n = \frac{(n+2)(n-2)^2}{4n^2(n-1)^2}$. *It is easy to see that* $\epsilon(n,\lambda) \to 0^+$ *as* $\lambda \to (\frac{n-2}{n-1})^+$.

4. Parallel Mean Curvature Case

In this section, we will prove Theorem 1.6. First we prove the following proposition.

Proposition 4.1. *Let M be an n-dimensional ($n \geq 3$) submanifold in $\mathbb{M}_c^{n+p}$ with parallel mean curvature. Assume $c + H^2 > 0$ and $H \neq 0$. For each λ satisfying $(n-2)(c+H^2) < (n-1)\lambda \leq (n-1)(c+H^2)$, if*

$$\| \operatorname{Ric}_-^\lambda \|_{n/2} < \epsilon(n,c,\lambda,H),$$

then M is pseudo-umbilical.

Remark 4.2. *(1) We recall that (cf. Page 43 in [2]) M is called pseudo-umbilical (resp. totally umbilical) if*

$$\langle h(X,Y), \mathbf{H} \rangle = H^2 \langle X, Y \rangle \ \text{(resp. } h(X,Y) = \langle X,Y \rangle \mathbf{H}\text{)}$$

for all tangent vector fields X, Y on M.

(2) When the codimension $p = 1$, "pseudo-umbilical" is just "totally umbilical".

(3) When $\mathbf{H}$ is nowhere zero, we always choose $e_{n+1} = \mathbf{H}/H$, and $\{e_i\}$ diagonalizing A_{n+1}, i.e. $h_{ij}^{n+1} = \lambda_i^{n+1} \delta_{ij}$. Denote

$$S_H = \sum_{i,j}(h_{ij}^{n+1})^2, \quad \mu_i^{n+1} = H - \lambda_i^{n+1}, \quad B_2 = S_H - nH^2 = \sum_i (\mu_i^{n+1})^2.$$

It is easy to check that M is pseudo-umbilical if and only if $B_2 = 0$; M is totally umbilical if and only if $B_2 = 0$ and $\sum_{i,j;\alpha \neq n+1}(h_{ij}^\alpha)^2 = 0$.

Proof. Set $\Lambda := (n-1)\lambda = (n-2)(c+H^2) + \delta$ for some $\delta > 0$. From Gauss equation we have

$$(4.1) \qquad S - nH^2 \leq n\big[(n-1)(c+H^2) - \rho\big] \leq n\big[-\delta + (c+H^2) + \operatorname{Ric}_-^\lambda\big].$$

By some direct computations (see (3.7) in [16] for details), we obtain the following estimate:

$$\frac{1}{2}\Delta S_H \geq \sum_{i,j,k}(h_{ijk}^{n+1})^2 + B_2 Q,$$

where

$$Q = n(c+H^2) - \frac{n-3}{n-2}(S - nH^2) - \frac{1}{n-2}n\big[(n-1)(c+H^2) - \rho\big].$$

There is an analogous version of Lemma 3.1 for submanifolds with parallel mean curvature.

Lemma 4.3 (cf. Lemma 1 in [15]). *Let M^n be a submanifold with parallel mean curvature in $\mathbb{M}_c^{n+p}$. Assume that $H \neq 0$. Set $f_\epsilon = (S_H - nH^2 + n\epsilon^2)^{1/2}$ for any constant $\epsilon \neq 0 \in \mathbb{R}$. Then*

$$(4.2) \qquad \sum_{i,j,k}(h_{ijk}^{n+1})^2 \geq \frac{n+2}{n}|\nabla f_\epsilon|^2.$$

Now similar as in the proof of Theorem 1.4, by using (4.1), (4.2) and Lemma 2.2, we have

$$\int_M B_2 Q \geq n\delta \int_M B_2 - n\|\operatorname{Ric}^\lambda_-\|_{n/2}\|B_2\|_{n/(n-2)}.$$

$$\int_M \sum_{i,j,k} (h_{ijk}^{n+1})^2 \geq \frac{n+2}{n}\mathfrak{A}(n,t)\|B_2\|_{n/(n-2)} - \frac{n+2}{n}\mathfrak{B}(n,t)(c_+ + H^2)\int_M B_2.$$

Choose t_1 such that $\frac{n+2}{n}\mathfrak{B}(n,t_1)(c_+ + H^2) = n\delta$, then

$$0 = \int_M \frac{1}{2}\Delta S_H \geq \left(\frac{n+2}{n}\mathfrak{A}(n,t_1) - n\|\operatorname{Ric}^\lambda_-\|_{n/2}\right)\|B_2\|_{n/(n-2)} \geq 0$$

provided $\|\operatorname{Ric}^\lambda_-\|_{n/2} < \frac{n+2}{n^2}\mathfrak{A}(n,t_1)$. Hence we have $B_2 \equiv 0$, i.e. M is pseudo-umbilical if we set $\epsilon(n,c,\lambda,H) = \frac{n+2}{n^2}\mathfrak{A}(n,t_1)$. □

Remark 4.4. *In fact,*

$$(4.3) \qquad \epsilon(n,c,\lambda,H) = \frac{P_n}{1 + \frac{c_+ + H^2}{(n-1)\lambda - (n-2)(c+H^2)}P_n} \cdot \frac{1}{C^2(n)},$$

where c_+ is defined as in Lemma 2.2.

We also have $\epsilon(n,c,\lambda,H) = \epsilon_{1/\sqrt{c+H^2}}(n,\lambda)$ for $c \geq 0$, and $\epsilon(n,c,\lambda,H) < \epsilon_{1/\sqrt{c+H^2}}(n,\lambda)$ for $c < 0$ from Remark 3.3.

Proof of Theorem 1.6. The proof is same as the proof of Theorem 3.3 in [16] and the following lemma on reduction of codimension due to Yau [20] will be used.

Lemma 4.5 (Theorem 1 in [20])**.** *Let N be a conformally flat manifold. Let N_1 be a sub-bundle of the normal bundle of M with fiber dimension k. Suppose M is umbilical with respect to N_1 and N_1 is parallel in the normal bundle. Then M lies in an $n + p - k$ dimensional umbilical submanifold N' of N such that the fiber of N_1 is everywhere perpendicular to N'.*

When $H = 0$, that is Theorem 1.4.

When $H \neq 0$. If $p = 1$, then the conclusion is from Proposition 4.1.

If $p \geq 2$, we can conclude M is a minimal submanifold in $\mathbb{S}^{n+p-1}(\frac{1}{\sqrt{c+H^2}})$. The detail can be found in [16], but we restate it briefly for convenience of the reader. From Lemma 4.5, M actually lies in $\mathbb{M}_{\tilde{c}}^{n+p-1}$. Then $\mathbf{H}$ is decomposed orthogonally into two parts

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2,$$

where $\mathbf{H}_1$ is the mean curvature of M in $\mathbb{M}_{\tilde{c}}^{n+p-1}$, and $\mathbf{H}_2$ is normal to $\mathbb{M}_{\tilde{c}}^{n+p-1}$ in $\mathbb{M}_c^{n+p}$. But $\mathbf{H} \perp \mathbf{H}_1$ from Lemma 4.5 again, we have $\mathbf{H}_1 = 0$, which means M is minimal in $\mathbb{M}_{\tilde{c}}^{n+p-1}$, and $H = |\mathbf{H}| = |\mathbf{H}_2|$. By Gauss equation we have $\tilde{c} = c + H^2$.

Since $\epsilon(n,c,\lambda,H) \leq \epsilon_{1/\sqrt{c+H^2}}(n,\lambda)$ from Remark 4.4, by applying Theorem 1.4, we conclude that $M = S^n(\frac{1}{\sqrt{c+H^2}})$, which is totally umbilical in $\mathbb{M}_c^{n+p}$. □

References

[1] Erwann Aubry, *Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 675–695. MR2191529

[2] Bang-yen Chen, *Geometry of submanifolds*, Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 22. MR0353212

[3] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968), 1970, pp. 59–75. MR0273546

[4] Norio Ejiri, *Compact minimal submanifolds of a sphere with positive Ricci curvature*, J. Math. Soc. Japan **31** (1979), no. 2, 251–256. MR527542

[5] Hillel Gauchman, *Minimal submanifolds of a sphere with bounded second fundamental form*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 779–791. MR860393

[6] David Hoffman and Joel Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727. MR0365424

[7] Takehiro Itoh, *On Veronese manifolds*, J. Math. Soc. Japan **27** (1975), no. 3, 497–506. MR0383309

[8] H. Blaine Lawson Jr., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. (2) **89** (1969), 187–197. MR0238229

[9] An-Min Li and Jimin Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math. (Basel) **58** (1992), no. 6, 582–594. MR1161925

[10] Haizhong Li, *Curvature pinching for odd-dimensional minimal submanifolds in a sphere*, Publ. Inst. Math. (Beograd) (N.S.) **53(67)** (1993), 122–132. MR1319765

[11] Peter Petersen and Guofang Wei, *Relative volume comparison with integral curvature bounds*, Geom. Funct. Anal. **7** (1997), no. 6, 1031–1045. MR1487753

[12] Chun-Li Shen, *A global pinching theorem of minimal hypersurfaces in the sphere*, Proc. Amer. Math. Soc. **105** (1989), no. 1, 192–198. MR973845

[13] Yi-Bing Shen, *Curvature pinching for three-dimensional minimal submanifolds in a sphere*, Proc. Amer. Math. Soc. **115** (1992), no. 3, 791–795. MR1093604

[14] James Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. MR0233295

[15] Hong-Wei Xu, $L_{n/2}$*-pinching theorems for submanifolds with parallel mean curvature in a sphere*, J. Math. Soc. Japan **46** (1994), no. 3, 503–515. MR1276835

[16] Hong-Wei Xu and Juan-Ru Gu, *Geometric, topological and differentiable rigidity of submanifolds in space forms*, Geom. Funct. Anal. **23** (2013), no. 5, 1684–1703. MR3102915

[17] Hong-Wei Xu and Ling Tian, *A new pinching theorem for closed hypersurfaces with constant mean curvature in S^{n+1}*, Asian J. Math. **15** (2011), no. 4, 611–630. MR2853651

[18] Hong-Wei Xu, Yan Leng, and Juan-Ru Gu, *Geometric and topological rigidity for compact submanifolds of odd dimension*, Sci. China Math. **57** (2014), no. 7, 1525–1538. MR3213887

[19] Hong-Wei Xu and Dengyun Yang, *The gap phenomenon for extremal submanifolds in a sphere*, Differential Geom. Appl. **29** (2011), no. 1, 26–34. MR2784286

[20] Shing-Tung Yau, *Submanifolds with constant mean curvature*, Amer. J. Math. **96** (1974), no. 2, 346–366. MR0370443

[21] ———, *Submanifolds with constant mean curvature II*, Amer. J. Math. **97** (1975), no. 1, 76–100. MR0370443

(Hang Chen) School of Mathematics and Statistics, Northwestern Polytechnical University, Xi' an 710129, P. R. China, email: chenhang86@nwpu.edu.cn

(Guofang Wei) Department of Mathematics, University of California, Santa Barbara CA 93106, email: wei@math.ucsb.edu