SINGULAR WEYL'S LAW WITH RICCI CURVATURE BOUNDED BELOW

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Abstract. We establish two surprising types of Weyl's laws for some compact $\mathrm{RCD}(K, N)$/Ricci limit spaces. The first type could have power growth of any order (bigger than one). The other one has an order corrected by logarithm similar to some fractals even though the space is 2-dimensional. Moreover the limits in both types can be written in terms of the singular sets of null capacities, instead of the regular sets. These are the first examples with such features for $\mathrm{RCD}(K, N)$ spaces. Our results depend crucially on analyzing and developing important properties of the examples constructed in [61], showing them isometric to the α-Grushin halfplanes. Of independent interest, this also allows us to provide counterexamples to conjectures in [20, 47].

1. Introduction

Weyl's law describes the asymptotic behavior of eigenvalues of the Laplace-Beltrami operator. Its study has a long rich history extended over a century, and plays an important role both in mathematics and physics. See for instance [42] for a survey.

For a closed Riemannian manifold $M^n = (M^n, g)$, the minus Laplacian $-\Delta = -\Delta^g$ has discrete unbounded spectrum as follows:

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty, \tag{1.1}$$

where λ_i is counted with multiplicities. It satisfies the following well-known Weyl's law

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathrm{vol}(M^n), \tag{1.2}$$

where $\mathrm{vol} = \mathrm{vol}^g$ denotes the Riemannian volume measure,

$$N(\lambda) = \sharp \{ i \in \mathbb{N} \mid \lambda_i \le \lambda \}, \tag{1.3}$$

and ω_n is the volume of the n-dimensional Euclidean unit ball.

In this paper we discuss the validity of similar Weyl's law for singular spaces with Ricci curvature bounded below, the so-called Ricci limit spaces, and more generally we deal with $\mathrm{RCD}(K, N)$ spaces.

Roughly speaking, an $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space with Ricci curvature bounded below by K and dimension bounded above by N in

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Key words and phrases. Weyl's law, RCD space, Ricci limit space, heat kernel asymptotic.

X. Dai is partially supported by the Simons Foundation Grant 635115.

S. Honda is partially supported by the Grant-in-Aid for Scientific Research (B) of 20H01799, the Grant-in-Aid for Scientific Research (B) of 21H00977 and Grant-in-Aid for Transformative Research Areas (A) of 22H05105.

G. Wei is partially supported by NSF DMS 2104704.

a synthetic sense. Moreover the heat flow h_t on X is linear. The linearity of the heat flow allows us to define the heat kernel H on X and the short time behavior of H should be related to the validity of expected Weyl's law on X as in the case of closed Riemannian manifolds. Let us emphasize that the reference measure $\mathfrak{m}$ is not necessarily a Hausdorff measure. See §2 for the preliminaries on this topic.

An $\mathrm{RCD}(K,N)$ space is called a Ricci limit space if it can be approximated by smooth Riemannian manifolds with Ricci curvature bounded below and dimension bounded above. See for instance [19, 20, 21, 22] for the structure theory on Ricci limit spaces. An important example of Ricci limit spaces is a closed weighted Riemannian manifold $(M^n, g, e^{-f}d\mathrm{vol})$ for some $f \in C^\infty(M^n)$ (see [52]). In this case the corresponding (metric measure) Laplacian coincides with the Witten Laplacian $\Delta^g - g(\nabla f, \nabla \cdot)$ and (1.2) is still satisfied.

In order to state our results, here we recall the following. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and some finite $N \in [1, \infty)$. Then it is proved in [14, Theorem 0.1] (after [24] for Ricci limit spaces) that there exists a unique $n \in \mathbb{N} \cap [1, N]$ such that the n-dimensional regular set $\mathcal{R}_n$ has $\mathfrak{m}$-full measure. This n is called the rectifiable dimension or the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$. A point $x \in X$ is called n-regular if

$$\left(X, \frac{1}{r}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_r(x))}, x\right) \stackrel{\mathrm{pmGH}}{\to} \left(\mathbb{R}^n, \mathsf{d}_{\mathbb{R}^n}, \frac{\mathcal{H}^n}{\omega_n}, 0_n\right), \quad \text{as } r \to 0^+. \tag{1.4}$$

It is called singular if it is not k-regular for any integer k.

For a compact $\mathrm{RCD}(K,N)$ space $(X, \mathsf{d}, \mathfrak{m})$, the canonical inclusion

$$\iota : H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \hookrightarrow L^2(X, \mathfrak{m})$$

is a compact operator. Hence the eigenvalues of the minus Laplacian still satisfies (1.1). Its Weyl's law has been studied in [5, 69], see the earlier work [31] for Ricci limit spaces. In these works, in particular in [5], the asymptotic is related to the Hausdorff (not reference) measure of the reduced regular set $\mathcal{R}_n^*$ of the rectifiable dimension n under the assumption that

$$\lim_{r \to 0^+} \int_X \frac{r^n}{\mathfrak{m}(B_r(x))} d\mathfrak{m} = \int_X \lim_{r \to 0^+} \frac{r^n}{\mathfrak{m}(B_r(x))} d\mathfrak{m} < \infty. \tag{1.5}$$

Moreover (1.5) is equivalent to the validity of such Weyl's law:

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(\mathcal{R}_n^*) < \infty. \tag{1.6}$$

See Theorem 5.5 for the precise statement. In this paper we call (1.6) *regular* Weyl's laws. It seems to us that experts may believe the validity of (1.5) for any compact $\mathrm{RCD}(K,N)$ space. It is worth mentioning that if $(X, \mathsf{d}, \mathfrak{m})$ is non-collapsed, then (1.5) is easily satisfied, thus (1.6) holds. See Remark 5.8.

In this paper we show Weyl's law relating to the *singular* set, giving the first example with this feature. This is an unexpected result, since in particular (1.5) is not satisfied.

Theorem 1.1 (Corollary 4.8)**.** *For any $\beta \in (2, \infty)$, there is a compact* $\mathrm{RCD}(-1, N)$ *space $(X, \mathsf{d}, \mathfrak{m})$ for N big depending on β, whose rectifiable dimension is* 2*, such that the following limit exists*

$$0 < \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{\beta/2}} < \infty.$$

Moreover the limit coincides with the Hausdorff measure of the singular set $\mathcal{S}$ of X up to multiplication by a "canonical" constant.

Note that β may not be an integer, unlike all earlier results, and that the canonical constant stated above is determined by the short time behavior of the heat kernel $\mathsf{H}(x,x,s^2)$ on the diagonal near the singular set. See also Theorem 4.7. Moreover, it is worth mentioning that the regular set $\mathcal{R}_2$ of the example above is a smooth weighted 2-dimensional (incomplete) Riemannian manifold and that the singular set $\mathcal{S}=X\setminus\mathcal{R}_2$ has null 2-capacity with respect to $\mathfrak{m}$, in particular the eigenvalues on X coincides with that of the Dirichlet Laplacian on $\mathcal{R}_2$. See Proposition 3.14.

In another direction we also show the surprising case that the power law $N(\lambda)\sim\lambda^\beta$ is not satisfied for any $\beta\geq 0$ as $\lambda\to\infty$. Namely,

Theorem 1.2 (Theorem 4.9)**.** *There is a compact* $\mathrm{RCD}(-1,10)$ *space* $(X,\mathsf{d},\mathfrak{m})$ *with both the Hausdorff and rectifiable dimensions* 2 *such that*

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda\log\lambda}=\frac{1}{4\pi}.$$

Note that the metric structure of any examples above is not an Alexandrov spaces though they are topologically 2-dimensional. This fact should be compared with [49, 54]. Moreover the value $\frac{1}{4\pi}$ also coincides with the 2-dimensional Hausdorff measure of the singular set $\mathcal{S}$ up to multiplication by a "canonical" constant; see Theorem 4.9. It is worth mentioning that any neighborhood of $\mathcal{S}$ has infinite 2-dimensional Hausdorff measure, a very different situation from Theorem 1.1 where the space has a finite β-dimensional Hausdorff measure; see Remark 3.7. In particular this also provides a counterexample to [20, Conjecture 1.34]; see Remark 3.8.

Remark 1.3. There are many studies of Weyl's law for singular, or subRiemannian, or almost Riemannian incomplete metrics with various measures, see e.g. [51, 12, 23, 30]. The techniques do not apply to our case though. In [12, 23], the measure blows up at the singular set while our measure degenerates at the singular set. In [30] the measure is regular (i.e., neither blows up nor degenerates at the singularity). Moreover in these three papers the spaces are not RCD, and the power of the asymptotic growth is always rational (integer or half integer) while our power could be irrational. With a change of variable as in Remark 3.4, a weak form of our results (i.e. only the rate of growth instead of precise asymptotics) can be derived from [51, 5.3]. But the method in [51] does not allow one to obtain the precise asymptotics, as examples in that paper show. Indeed, without curvature condition, precise asymptotics do not necessarily exist in that case (note that such a behavior is known in the fractal setting [8, Corollary 7.2]). Our focus is to find the precise asymptotics as well as the exact leading order term. Moreover we can write down the limit of $N(\lambda)$ as $\lambda\to\infty$ in terms of the singular set explicitly. The surprising points in the examples above are; the spaces have lower bounds of Ricci curvature in a synthetic sense and the limits can be written by sets of null-capacity (note that any set of null 2-capacity can be removable from the point of view of potential theory). Such behaviors seem to be unknown even in the fractal setting.

In another direction, if one considers Schrödinger operators on complete noncompact manifolds, then Weyl's law could also involve log term, see [26, Theorem 1.1 and Remark 1.2]. For related results on the small time heat kernel expansion of Witten Laplacian, see [25, Theorem 1.1]. Such expansion implies a Weyl law by the Karamata-Tauberian type theorem [26, Theorem 2.4]

Crucial to our construction we further develop the examples constructed by the last two authors in [61]. These examples are the first Ricci limit spaces whose Hausdorff dimensions may not be integer and moreover, the Hausdorff dimension of the singular set is bigger than that of the regular set, see Theorem 3.1. Thus the examples obtained above are actually Ricci limit spaces and we call the above asymptotic behaviors of eigenvalues *singular* Weyl's laws.

Let us explain how to achieve the results above. By Karamata theorem [32, Page 445, Theorem 2]

(1.7)
$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\alpha/2}L(\lambda^{-1})}=C\in[0,\infty)\Longleftrightarrow\lim_{t\to 0^+}\frac{\sqrt{t}^{\alpha}}{L(t)}\sum_i e^{-\lambda_i t}=\Gamma(\alpha+1)C\in[0,\infty)$$

for any slowly varying function $L(t):\mathbb{R}_+\to\mathbb{R}_+$ as $t\to 0^+$ (i.e. $\lim_{t\to 0^+}\frac{L(at)}{L(t)}=1$ for any fixed $a>0$). We will apply this to L a constant function for Theorem 1.1 and L the log function for Theorem 1.2. Hence the asymptotic of the eigenvalues is reduced to the short time behavior of the heat trace. To understand the heat kernel, we first derive a complete and explicit description of the metric and measure of the Pan-Wei examples Y in §3.1, 3.2. Surprisingly it turns out that Y is isometric to the 2α-Grushin halfplane. The Grushin plane is a well known space in sub-Riemannian geometry [9, Section 3.1]; see [60] for further connection of Ricci limit space and sub-Riemannian geometry. In particular the space Y has important metric dilations as follows.

Theorem 1.4. *For each $\lambda>0$, there is a homeomorphism $F_\lambda:Y\to Y$ such that*

$$\mathsf{d}(F_\lambda(y_1),F_\lambda(y_2))=\lambda\cdot\mathsf{d}(y_1,y_2)$$

for all $y_1,y_2\in Y$.

See Theorem 3.5 for more detail.

For Riemannian manifolds, only the standard Euclidean space $\mathbb{R}^n$ has such dilation property. For Alexandrov spaces with metric dilations, they must be metric cones. The Pan-Wei examples are the first Ricci limit spaces with dilations that are not metric cones. Note that in [56] a (nonpolar) Ricci limit space is constructed as a tangent cone at infinity which is invariant under certain scales, but not all scales. We also mention that the Carnot groups admit dilations, but these spaces are not RCD spaces; they even cannot be $\mathrm{CD}(K,N)$ for any $K\in\mathbb{R}$, $N\geq 1$ [46].

Even with the explicit metric and measure, computing the heat kernel is still a challenge. By Li-Yau's estimate (2.9), the study of the asymptotic rate (although not the precise asymptotic) can be reduced to that of the measure of small balls instead of the heat kernel. Still, the measure of balls cannot be computed explicitly; instead we give good effective estimates with the help of explicit description of the metric and measure. To get a grip on the small time asymptotic of the heat kernel, we consider the ratios of the measure of balls to suitable rectangular shaped domains whose measures can be computed explicitly. With sophisticated use of the dilation property we obtain formulas for integrals of the heat kernel in terms of certain functions, some explicit and some not, but with good control of asymptotics. These allow us to estimate and compute several limits playing important role for later study of asymptotic behaviors; see Theorem 3.19. These are carried out in §3.3-3.5.

The space Y above is a noncompact $\mathrm{RCD}(0,N)$ space. To construct the compact $\mathrm{RCD}(-1,N)$ space X, first we study the heat kernel of quotient $\bar{Y} = Y/\mathbb{Z}$ by extending the relation of heat kernels between covering spaces for manifolds in [50] to RCD space, see Proposition 4.1. For $\alpha = \frac{1}{2}$, the eigenvalues and eigenfunctions of $\bar{Y}$ with respect to the unweighted measure have been computed explicitly in [12]. Amazingly in this case the eigenvalues for weighted measure with $n = 9$ turns out to be exactly the same as in the unweighted case, see Subsection 4.2. In this case $\bar{Y}$ is $\mathrm{RCD}(0,10)$ (10 is the smallest to have RCD 0 lower bound. See Remark 4.3). Then we do a perturbation of $\bar{Y}$, and a doubling to obtain X. By keeping track of the heat kernels in each step, we show that the small time asymptotic of the heat kernel of X is essentially determined by that of the heat kernel of Y, see Theorem 4.6.

Applying the above while changing the geometric parameters appeared in the process, we finally establish the results above.

Naively, for a metric measure space, the short time behavior of the heat kernel depends more on the distance/metric as compared to the large time behavior, where the measure character shows up more. Therefore it is natural to make the following conjecture.

Conjecture 1.5. *For a compact* $\mathrm{RCD}(K,N)$ *space* $(X,\mathsf{d},\mathfrak{m})$ *with Hausdorff dimension* β*, the limit* $\lim_{t\to 0} t^{\beta/2}\int_X \mathsf{H}(x,x,t)d\mathfrak{m}(x)$ *is the Hausdorff measure of* X *up to multiplication by a "canonical" constant.*

Note that the conjecture includes the case when $\mathcal{H}^\beta(X) = \infty$. Actually for the example $(X,\mathsf{d},\mathfrak{m})$ in Theorem 1.2, the regular part of any open ball at any singular point has infinite 2-dimensional Hausdorff measure; see Remark 3.7. This property has an independent interest and of course, gives supporting evidence for the conjecture. We plan to show the valid of the conjecture for a large class of RCD spaces in a future paper.

Finally in connection with the validity of Weyl's law, we also deal with the asymptotic behavior of eigenfunctions in §5.2. In particular, after introducing a negative answer to a question by Ding in [31], we provide related Weyl type asymptotics. See Theorem 5.6 and Proposition 5.9.

Acknowledgments. The first named author thanks Junrong Yan for useful discussions. The second named author wishes to thank Atsushi Katsuda and Tatsuya Tate for explaining him on the quantum ergodicity, in particular [43]. Moreover he is very grateful to Naotaka Kajino and Takashi Kumagai for suggestions on Theorem 1.2 which help improve the result. The third named author would like to thank Richard Montgomery for introducing him the Grushin planes. Finally we wish to thank the referee for valuable suggestions for the revision.

2. Preliminaries: Heat Kernels on $\mathrm{RCD}(K,N)$ spaces

All terminologies for singular spaces we will discuss in the sequel are prepared from the theory of metric measure spaces with Ricci curvature bounded below in a synthetic sense (namely $\mathrm{RCD}(K,N)$ spaces). Here we give a quick introduction on the terminology and results that will be needed later. See for instance [1] for a nice survey.

A triple $(X,\mathsf{d},\mathfrak{m})$ is said to be a metric measure space if (X,d) is a complete separable metric space and $\mathfrak{m}$ is a Borel measure on X, where this is finite on each bounded subset and the support coincides with X. Roughly speaking a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be an RCD(K, N) space for some $K \in \mathbb{R}$ and some $N \in [1, \infty]$ if the Ricci curvature is bounded below by K and the dimension is bounded above by N in a synthetic sense, moreover the $H^{1,2}$-Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is a Hilbert space.

The precise definition is as follows. Define the Cheeger energy $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, \infty)$ by

$$\mathrm{Ch}(f) := \inf_{\{f_i\}_i} \left\{ \liminf_{i \to \infty} \int_X (\mathrm{Lip} f_i)^2 d\mathfrak{m} \right\}, \tag{2.1}$$

where the infimum of the right hand side above runs over all bounded Lipschitz functions $f_i \in L^2(X, \mathfrak{m})$ L^2-converging to f on X and $\mathrm{Lip} f(x)$ denotes the local slope of f at x (see also [18]). Then the $H^{1,2}$-Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is defined by the finiteness domain of Ch. We denote by $|\nabla f| \in L^2(X, \mathfrak{m})$ the minimal object as in the right hand side of (2.1), so-called the minimal relaxed slope of f, then we know the following equality

$$\mathrm{Ch}(f) = \int_X |\nabla f|^2 d\mathfrak{m}. \tag{2.2}$$

Note that in general $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ equipped with the norm

$$\|f\|_{H^{1,2}} = \left(\|f\|_{L^2}^2 + \mathrm{Ch}(f) \right)^{1/2} \tag{2.3}$$

is a Banach space. We say that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian (IH) if $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is Hilbert. In the sequel let us assume that $(X, \mathsf{d}, \mathfrak{m})$ is IH.

Then for all $f_i \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) (i = 1, 2)$,

$$\langle \nabla f_1, \nabla f_2 \rangle = \lim_{\epsilon \to 0} \frac{|\nabla (f_1 + \epsilon f_2)|^2 - |\nabla f_1|^2}{2\epsilon} \in L^1(X, \mathfrak{m}). \tag{2.4}$$

is well-defined. The domain $D(\Delta)$ of the Laplacian Δ of $(X, \mathsf{d}, \mathfrak{m})$ is defined by the set of all $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ such that there exists a unique $h \in L^2(X, \mathfrak{m})$, denoted by Δf, with

$$\int_X \langle \nabla f, \nabla \phi \rangle d\mathfrak{m} = -\int_X h \phi d\mathfrak{m}, \quad \text{for any } \phi \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}). \tag{2.5}$$

We are ready to introduce the heat flow $\mathsf{h}_t : L^2(X, \mathfrak{m}) \to L^2(X, \mathfrak{m})$. It is defined by a continuous curve $t \mapsto \mathsf{h}_t f \in L^2(X, \mathfrak{m})$ on $[0, \infty)$, which is locally absolutely continuous on $(0, \infty)$ (more strongly smooth, namely in $C^\infty((0, \infty), H^{1,2}(X, \mathsf{d}, \mathfrak{m}))$, see [36, Proposition 5.2.12]), with $\mathsf{h}_t f \in D(\Delta)$ for $t > 0$, $\mathsf{h}_0 f = f$ and

$$\frac{d}{dt} \mathsf{h}_t f = \Delta \mathsf{h}_t f, \quad \text{for } \mathcal{L}^1\text{-a.e. (or equivalently for any) } t > 0. \tag{2.6}$$

This will play the core role to define a main target of the paper, so-called the heat kernel below.

We are now in a position to introduce the RCD(K, N) condition. An IH metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be an RCD(K, N) space for some $K \in \mathbb{R}$ and some $N \in [1, \infty]$ if the following three conditions are satisfied.

- (Volume growth) There exist $C > 1$ and $x \in X$ such that $\mathfrak{m}(B_r(x)) \le Ce^{Cr^2}$ holds for any $r > 1$.
- (Sobolev-to-Lipschitz) If $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ satisfies $|\nabla f|(x) \le 1$ for $\mathfrak{m}$-a.e. $x \in X$, then f has a 1-Lipschitz representative.

- (Bochner inequality) It holds that

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{(\Delta f)^2}{N} + \langle \nabla \Delta f, \nabla f\rangle + K|\nabla f|^2 \tag{2.7}$$

in a weak sense.

See for instance [2, 3, 6, 15, 35, 33] for the details above. We often say that X is an $\mathrm{RCD}(K,N)$ space, without noting the distance and measure, for short.

By Bishop-Gromov inequality ([53, Theorem 5.31] and [66, Theorem 2.3]), and a $(1,1)$-Poincaré inequality ([63, Theorem 1]), it is known from [64, Proposition 2.3] and [65, Corollary 3.3] that there exists a unique continuous function $\mathsf{H} : X \times X \times (0,\infty) \to (0,\infty)$, so-called the heat kernel of $(X, \mathsf{d}, \mathfrak{m})$, such that

$$\mathsf{h}_t f(x) = \int_X f(y)\mathsf{H}(x,y,t) d\mathfrak{m}(y) \tag{2.8}$$

for any $f \in L^2(X, \mathfrak{m})$. Moreover by [45, Theorem 1.2], Li-Yau inequality is satisfied in this setting, namely we know the following Gaussian estimates for H: for any $\epsilon \in (0,1)$

$$\begin{aligned} \frac{C_1^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\exp\left(-\frac{\mathsf{d}(x,y)^2}{(4-\epsilon)t} - C_2 t\right) &\leq \mathsf{H}(x,y,t) \\ &\leq \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(x))}\exp\left(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + C_2 t\right) \end{aligned} \tag{2.9}$$

for some $C_1 = C_1(K,N,\epsilon) > 1$ and some $C_2 = C_2(K,N,\epsilon) \geq 0$. Moreover $C_2 = 0$ when $K = 0$.

Furthermore, with [44, Theorem 1.2] and [27, Theorem 4] we have

$$|\nabla_x \mathsf{H}(x,y,t)| \leq \frac{C_1}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(x))}\exp\left(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + C_2 t\right), \tag{2.10}$$

which shows that H is locally Lipschitz, and

$$|\Delta_x \mathsf{H}(x,y,t)| = \left|\frac{d}{dt}\mathsf{H}(x,y,t)\right| \leq \frac{C_1}{t\mathfrak{m}(B_{\sqrt{t}}(x))}\left(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + C_2 t\right). \tag{2.11}$$

It is worth pointing out that if $K = 0$, then C_2 can be chosen as 0 (see [45] for the details) and that the heat flow can be extended to $L^p(X, \mathfrak{m})$ for any $1 \leq p \leq \infty$ by (2.8) based on the estimates above (or by density).

Note that under rescaling $(X, a\mathsf{d}, b\mathfrak{m})$ for some $a > 0$ and some $b > 0$, the space is an $\mathrm{RCD}(a^{-2}K, N)$ space and we have the following:

$$\mathsf{H}_{(X,a\mathsf{d},b\mathfrak{m})}(x,y,t) = b^{-1}\mathsf{H}_{(X,\mathsf{d},\mathfrak{m})}(x,y,a^{-2}t). \tag{2.12}$$

When $(X, \mathsf{d}, \mathfrak{m})$ is compact, denoting by f_i a corresponding eigenfunction of λ_i with $\|f_i\|_{L^2} = 1$, the standard functional analysis allows us to show that $\{f_i\}_i$ (can be chosen to) form an L^2-orthonormal basis of $L^2(X, \mathfrak{m})$ and that

$$\mathsf{H}(x,y,t) = \sum_i e^{-\lambda_i t} f_i(x) f_i(y), \quad \text{in } C(X \times X) \tag{2.13}$$

is satisfied.

Finally let us mention that the all objects above are preserved by isomorphisms of metric measure spaces. In particular if $T : (X_1, \mathsf{d}_1, \mathfrak{m}_1) \to (X_2, \mathsf{d}_2, \mathfrak{m}_2)$ is an isomorphism, namely isometry and measure preserving, between $\mathrm{RCD}(K,N)$ spaces, then

$$\mathsf{H}_{(X_2,\mathsf{d}_2,\mathfrak{m}_2)}(T(x),T(y),t)=\mathsf{H}_{(X_1,\mathsf{d}_1,\mathfrak{m}_1)}(x,y,t). \tag{2.14}$$

Remark 2.1. The local isomorphism preserves the gradient operators and the Laplacians because of their localities, however it does not preserve the heat kernel in general as discussed in §4.1.

3. Geometry of Pan-Wei's examples

First we recall the construction of the Ricci limit space Y in [61]. Consider $M=M^{n+1}=[0,\infty)\times_f\mathbb{S}^{n-1}\times_h\mathbb{S}^1$, with

$$f(r)=r(1+r^2)^{-\frac{1}{4}}\sim\sqrt{r},\quad h(r)=(1+r^2)^{-\alpha}\sim r^{-2\alpha}\ \text{ as } r\to\infty,$$

where $\alpha>0$. Then M is a Riemannian manifold diffeomorphic to $\mathbb{R}^n\times\mathbb{S}^1$. When integer $n\geq\max\{4\alpha+3,16\alpha^2+8\alpha+1\}$, M has positive Ricci curvature.

Let $p\in M$ be a point in $\{r=0\}$, $(\widetilde{M},\tilde{p})$ be the universal cover of (M,p). Let Y be the asymptotic cone of $(\widetilde{M},\tilde{p})$. Then Y is clearly an $\mathrm{RCD}(0,N)$ space with $N=n+1$.

Theorem 3.1 (Pan-Wei)**.** *When $\alpha\geq 1/2$, the Hausdorff dimension of the Ricci limit space Y is $1+2\alpha$, which can be arbitrarily large by raising α and n, while the rectifiable dimension of Y is 2. One can also construct a compact $\mathrm{RCD}(K,n+1)$ space X with this feature for some negative K.*

For more details, see [61, Theorem A]; for construction of a compact X, see [61, Remark 1.8] and §4.4. Also see [62], where similar constructions give examples whose Busemann functions at a point are not proper.

In order to study the Weyl's law of X, we derive precise descriptions of the distance, metric, measure of Y in the subsections below, extending the distance estimate in [61].

3.1. **Metric and dilations of Y.**

Let S be a surface of revolution $[0,\infty)\times_h\mathbb{S}^1$ with boundary and let $\widetilde{S}$ be its universal cover. Let $q\in S$ at $r=0$ and let $\tilde{q}\in\widetilde{S}$ be a lift of q.

Lemma 3.2. *Let $r_i\to\infty$ be a sequence. Then the two sequences $(r_i^{-1}\widetilde{M},\tilde{p})$ and $(r_i^{-1}\widetilde{S},\tilde{q})$ are equivalent in the pointed Gromov-Hausdorff topology, namely, pointed Gromov-Hausdorff distances between $(r_i^{-1}\widetilde{M},\tilde{p})$ and $(r_i^{-1}\widetilde{S},\tilde{q})$ go to 0 as $i\to\infty$.*

Proof. We write (r,v) as a point in S, where $r\in[0,\infty)$ and $v\in\mathbb{S}^1=[-1/2,1/2]/\sim$. This also naturally defines an (r,v)-coordinate on $\widetilde{S}$, where $r\in[0,\infty)$ and $v\in\mathbb{R}$, such that the preimage of $(r,0)\in S$ is $\{(r,v)|v\in\mathbb{Z}\}$. Recall that the Riemannian metric on S is given by $g_S=dr^2+h(r)^2dv^2$; this expression also gives the Riemannian metric on $\widetilde{S}$.

On M, we can write each point in the form of (r,x,v), where $r\in[0,\infty)$, $x\in\mathbb{S}^{n-1}$, and $v\in\mathbb{S}^1$. From g on M, it is clear that

$$\mathsf{d}^g((r,x,v),(r,x',v))\leq f(r)\mathsf{d}_{n-1}(x,x'),$$

where d_{n-1} is the standard distance on the unit sphere $\mathbb{S}^{n-1}$. We fix a point $x_0\in\mathbb{S}^{n-1}$ and define an immersion

$$\phi:S\to M,\quad (r,v)\mapsto(r,x_0,v).$$

Note that ϕ is indeed a Riemannian immersion and $\phi(S)$ is convex. For $r>0$, we write

$$S_r = \{(r,v)|v\in\mathbb{S}^1\},\quad M_r=\{(r,x,v)|x\in S^{n-1}, v\in S^1\}.$$

$\phi(S_r)$ is $f(r)$-dense in M_r.

Similar to $\widetilde{S}$, we can naturally assign an (r,x,v)-coordinate on $\widetilde{M}$. We also write

$$\widetilde{S}_r=\{(r,v)\in\widetilde{S}|v\in\mathbb{R}\},\quad \widetilde{M}_r=\{(r,x,v)\in\widetilde{M}|x\in\mathbb{S}^{n-1},v\in\mathbb{R}\}.$$

The map ϕ lifts to a Riemannian immersion

$$\tilde{\phi}:\widetilde{S}\to\widetilde{M},\quad (r,v)\mapsto(r,x_0,v)$$

with a convex image. Moreover, $\tilde{\phi}(\widetilde{S}_r)$ is $f(r)$-dense in $\widetilde{M}_r$. Now the result follows from the fact that $f(r)\sim\sqrt{r}$ as $r\to\infty$. □

Lemma 3.3. *We write $\widetilde{S}_+=\{(r,v)\in\widetilde{S}|r>0,v\in\mathbb{R}\}$ and set a reference point $z=(1,0)\in\widetilde{S}_+$. Then as $\lambda\to\infty$, $(\widetilde{S}_+,\lambda^{-2}g_{\widetilde{S}},(1,0))$ converges (smoothly on each compact subset of $(0,\infty)\times\mathbb{R}$) to a limit (incomplete) Riemannian metric*

$$g_\infty = dr^2+r^{-4\alpha}dv^2 \tag{3.1}$$

defined on $(0,\infty)\times\mathbb{R}$; moreover, the reference point z converges to $(0,0)$, a point in the metric completion of g_∞.

Proof. For convenience, we write $\lambda^{-2}g_{\widetilde{S}}=g_\lambda$ below. We apply a change of variables $s=\lambda^{-1}r$ and $w=\lambda^{-1-2\alpha}v$. Then

$$\begin{aligned}g_\lambda &= \lambda^{-2}(dr^2+(1+r^2)^{-2\alpha}dv^2)\\ &= ds^2+\left(\frac{1+\lambda^2s^2}{\lambda^2}\right)^{-2\alpha}dw^2.\end{aligned}$$

In other words, the space $(\widetilde{S}_+,g_\lambda,z)$ is isometric to

$$((0,\infty)\times\mathbb{R}, ds^2+((1+\lambda^2s^2)/\lambda^2)^{-2\alpha}dw^2,(\lambda^{-1},0)).$$

Let $\lambda\to\infty$, then the result follows. □

Remark 3.4. The coordinate change $y=(\frac{r}{1+2\alpha})^{1+2\alpha}, x=\frac{v}{(1+2\alpha)^{2\alpha}}$ turns the metric into the form

$$\frac{dx^2+dy^2}{y^{2c}},\qquad c=\frac{2\alpha}{1+2\alpha}.$$

Note that the hyperbolic metric corresponds to $c=1$.

Theorem 3.5. *$\widetilde{M}$ has a unique asymptotic cone (Y,y) with the following properties:*
(1) Y has an (r,v)-coordinate, where $(r,v)\in[0,\infty)\times\mathbb{R}$, and $y=(0,0)$;
(2) the distance function d on Y is induced by the incomplete Riemannian metric $g_\infty=dr^2+r^{-4\alpha}dv^2$ defined on $(0,\infty)\times\mathbb{R}$;
(3) for each $\lambda>0$, the map $F_\lambda:(r,v)\mapsto(\lambda r,\lambda^{1+2\alpha}v)$ is a metric dilation of Y with scale λ, that is,

$$\mathsf{d}(F_\lambda(y_1),F_\lambda(y_2))=\lambda\cdot\mathsf{d}(y_1,y_2)$$

for all $y_1,y_2\in Y$;
(4) $\mathsf{d}((0,v),(0,0))=C|v|^{\frac{1}{1+2\alpha}}$ for all $v\in\mathbb{R}$, where $C>0$ is a constant.

Proof. (1) and (2) follow directly from Lemmas 3.2 and 3.3.

We prove (3). It is direct to check that F_λ satisfies

$$F_\lambda^\star g_\infty = \lambda^2 g_\infty.$$

Therefore, F_λ is a metric dilation with scale λ on $(0,\infty)\times\mathbb{R}$. We extend F_λ to Y by continuity. The result follows.

For (4), we assume $v>0$ without lose of generality. Applying the metric dilation F_λ with $\lambda = v^{\frac{1}{1+2\alpha}}$, we have

$$\mathsf{d}((0,v),(0,0)) = \mathsf{d}(F_\lambda(0,1),(0,0)) = \lambda \mathsf{d}((0,1),(0,0)).$$

□

We make some remarks related to Theorem 3.5.

Remark 3.6. It follows from the metric dilations in Theorem 3.5(3) that Y is scaling invariant, that is, (sY,y) is isometric to (Y,y) for all $s>0$. In particular, the tangent cone of Y at y and the one at infinity are both unique and isometric to Y.

Remark 3.7. Take a box $B=(0,1]\times[0,1]\subseteq\mathcal{R}$. Then

$$\mathcal{H}^2(B) = \int_B r^{-2\alpha}drdv = \int_0^1 r^{-2\alpha}dr.$$

The indefinite integral is finite if and only if $\alpha<1/2$. Recall that $\dim_H(\mathcal{S}) = 1+2\alpha$ and $\dim_H(\mathcal{R})=2$. Thus in these examples, we always have

$$\dim_H(\mathcal{S}) < \dim_H(\mathcal{R}) \iff \mathcal{H}^2(B)<\infty.$$

Remark 3.8. When $\alpha = 1/2$, Y has both Hausdorff dimension and rectifiable dimension 2. Based on Remark 3.7, we can see that $(Y,\mathsf{d},\mathcal{H}^2)$ does not satisfy the doubling condition. In fact, let $x=(1,0)\in Y$ expressed in the rv-coordinate and let $r=2/3$. Then by triangle inequality, $B_{2r}(x)$ contains the set $D=[0,1/3]\times[-(1/3C)^2,(1/3C)^2]$, where C is the constant in Theorem 3.5(4). D, thus $B_{2r}(x)$, has infinite $\mathcal{H}^2$ measure from Remark 3.7. On the other hand, $\mathcal{H}^2(B_r(x))$ is finite. Therefore, there is no upper bound for the ratio

$$\frac{\mathcal{H}^2(B_{2r}(x))}{\mathcal{H}^2(B_r(x))}.$$

This answers [20, Conjecture 1.34] in the negative even in the case when the space is 2-dimensional with the full support $\mathcal{H}^2$.

Remark 3.9. The Grushin plane is $\mathbb{R}^2$ with a subRiemannian metric (see [9, Section 3.1]). Under the (x,y)-coordinate, its distribution is generated by $X=\partial_x$ and $Y=|x|\partial_y$. Setting $\{X,Y\}$ orthonormal defines a subRiemannian metric on $\mathbb{R}^2$. Note that Y degenerates only along the y-axis. Thus the distribution has full rank almost everywhere; a sub-Riemannian manifold with this property are called almost-Riemannian. One can also define the α-Grushin planes by alternatively setting $Y=|x|^\alpha\partial_y$, where $\alpha>0$. Outside the y-axis, the sub-Riemannian metric becomes Riemannian and equals

$$g = dx^2 + |x|^{-2\alpha}dy^2.$$

By Theorem 3.5, our example Y is isometric to the 2α-Grushin halfplane.

Remark 3.10. It is conjectured in [47] that any $\mathrm{RCD}(K,N)$ space with an upper curvature bound in the sense of Alexandrov, namely $\mathrm{CAT}(\kappa)$ space, satisfies that the topological dimension coincides with the Hausdorff dimension. See [47, Theorem 3.15] and a sentence afterwords. Since the metric structure of Y is CAT(0) (the proof is the same to that of a standard fact that any Hadamard manifold is a CAT(0) space, because the smooth part of Y has nonpositive sectional curvature and all two points in Y can be joined by a unique minimal geodesic), Y gives a counterexample to this conjecture.

Let

$$\gamma : Y \to Y, \quad (r,v) \mapsto (r, v+1),$$

which is an isometry of Y. We prove the following distance estimate which will be used later.

Lemma 3.11. *There is a function $C(r) > 0$ such that*

$$C(r)l^{\frac{1}{1+2\alpha}} \le \mathsf{d}(\gamma^l x, x) \le 3l^{\frac{1}{1+2\alpha}}$$

holds for all $x = (r,v) \in Y$ and all $l \in \mathbb{Z}_+$.

Proof. Without lose of generality, we assume that $x = (r,0)$. When $r = 0$, we have

$$\mathsf{d}(\gamma^l x, x) = \mathsf{d}((0,l),(0,0)) = Cl^{\frac{1}{1+2\alpha}}$$

from (4) of Theorem 3.5. Below we assume $r > 0$.

Let $s > 0$ and we consider a path σ_s from x to $\gamma^l(x) = (r,l)$ as follows: σ_s first go through a horizontal segment from $(r,0)$ to $(r+s,0)$, then go through a v-curve to $(r+s,l)$, then go back to $(r+s,0)$ via a horizontal segment. We have

$$\mathsf{d}(\gamma^l x, x) \le \mathrm{length}(\sigma_s) = 2s + (r_0+s)^{-2\alpha} l$$

for all $s \ge 0$. Using $s = l^{\frac{1}{1+2\alpha}}$, we have

$$\mathsf{d}(\gamma^l x, x) \le 2l^{\frac{1}{1+2\alpha}} + \left(r + l^{\frac{1}{1+2\alpha}}\right)^{-2\alpha} l \le 3l^{\frac{1}{1+2\alpha}}.$$

Let c be a minimal geodesic from $(r,0)$ to (r,l). We set

$$R := \max\{t - r | (t,v) \in c\} \ge 0.$$

Then

$$\mathsf{d}(\gamma^l x, x) = \mathrm{length}(c) \ge (r+R)^{-2\alpha} l.$$

Together with the upper bound, they yield

$$(r+R)^{-2\alpha} l \le 3l^{\frac{1}{1+2\alpha}}.$$

Thus

$$R \ge C_1 l^{\frac{1}{1+2\alpha}} - r,$$

where $C_1 = 3^{-1/(2\alpha)}$. When $2r \le C_1 l^{\frac{1}{1+2\alpha}}$, we have

$$\mathsf{d}(\gamma^l x, x) \ge 2R \ge 2C_1 l^{\frac{1}{1+2\alpha}} - 2r \ge C_1 l^{\frac{1}{1+2\alpha}}.$$

Next, we consider the case $2r > C_1 l^{\frac{1}{1+2\alpha}}$, that is, $r^{-2\alpha} l < (2/C_1)^{1+2\alpha} r$. Note that

$$R \le \frac{1}{2}\mathsf{d}(\gamma^l x, x) \le \frac{1}{2} r^{-2\alpha} l.$$

Thus

$$\mathsf{d}(\gamma^l x, x) \geq (r+R)^{-2\alpha} l \geq \left(r + \frac{1}{2} r^{-2\alpha} l\right)^{-2\alpha} l$$
$$\geq \left(r + \frac{1}{2}\left(\frac{2}{C_1}\right)^{1+2\alpha} r\right)^{-2\alpha} l$$
$$= C_2 r^{-2\alpha} l \geq C_2 r^{-2\alpha} l^{\frac{1}{1+2\alpha}},$$

where $C_2 = (1 + 2^{2\alpha}/C_1^{1+2\alpha})^{-2\alpha}$. The last inequality holds because $l \geq 1$.

In conclusion, we choose $C(r) = \min\{C_1, C_2 r^{-2\alpha}\}$. Then the desired inequality holds. □

3.2. **Limit measure.** In this subsection we derive the limit measure of Y.

Lemma 3.12. *Let $\mathfrak{m}$ be a limit measure on Y. Then*

$$d\mathfrak{m} = c_{\mathfrak{m}} r^{\frac{n-1}{2} - 2\alpha} dr dv \tag{3.2}$$

for some constant $c_{\mathfrak{m}} \in [C_1, C_2]$, where $C_1, C_2 > 0$ are constants depending only on n and α.

Proof. We fix a sequence $\lambda_i \to \infty$. We first estimate the volume of $B_{\lambda_i}(\tilde{p})$. We continue to use the (r, x, v)-coordinate on $\widetilde{M}$, introduced in the proof of Lemma 3.2. By [61, Lemma 1.1], we have distance estimate

$$C_1 |l|^{\frac{1}{1+2\alpha}} - 2 \leq \mathsf{d}((0,x,0),(0,x,l)) \leq C_2 |l|^{\frac{1}{1+2\alpha}}$$

for all $l \in \mathbb{Z}$ large. We denote

$$D_i = \{(r,x,v) | 0 \leq r \leq \lambda_i, x \in \mathbb{S}^{n-1}, |v| \leq C_3 \lambda_i^{1+2\alpha}\} \subseteq \widetilde{M},$$

where $C_3 = (2C_1)^{-1-2\alpha}$. D_i has volume

$$\mathrm{vol}(D_i) = 2C_3 \lambda_i^{1+2\alpha} \int_0^{\lambda_i} f(r)^{n-1} h(r) dr.$$

Note that the boundary ∂D_i equals

$$\left([0,\lambda_i] \times \mathbb{S}^{n-1} \times \{\pm C_3 \lambda_i^{1+2\alpha}\}\right) \cup \left(\{\lambda_i\} \times \mathbb{S}^{n-1} \times [-C_3\lambda_i^{1+2\alpha}, C_3 \lambda_i^{1+2\alpha}]\right)$$

By triangle inequality, for any $(r,x,v) \in D_i$,

$$\mathsf{d}(\tilde{p}, (r,x,v)) \leq C_2 |v|^{\frac{1}{1+2\alpha}} + 1 + r \leq (C_2 C_3^{\frac{1}{1+2\alpha}} + 1)\lambda_i + 1.$$

Also, when $(r,x,v) \in \{\lambda_i\} \times \mathbb{S}^{k-1} \times [-C_3\lambda_i^{1+2\alpha}, C_3\lambda_i^{1+2\alpha}]$,

$$\mathsf{d}(\tilde{p}, (r,x,v)) \geq \lambda_i;$$

when $(r,x,v) \in [0,\lambda_i] \times \mathbb{S}^{k-1} \times \{\pm C_3 \lambda_i^{1+2\alpha}\}$,

$$\mathsf{d}(\tilde{p},(r,x,v)) \geq \mathsf{d}(\tilde{p},(0,x,v)) - \mathsf{d}((0,x,v),(r,x,v)) \geq C_1 |v|^{\frac{1}{1+2\alpha}} - \lambda_i \geq \lambda_i.$$

Therefore,

$$B_{\lambda_i}(\tilde{p}) \subseteq D_i \subseteq B_{C_4 \lambda_i}(\tilde{p})$$

holds for all i large, where $C_4 = C_2 C_3^{\frac{1}{1+2\alpha}} + 2$. By Bishop-Gromov inequality, there is a constant $C_5 > 0$, depending on n and α, such that

$$C_5 \mathrm{vol}(D_i) \leq \mathrm{vol}(B_{\lambda_i}(\tilde{p})) \leq \mathrm{vol}(D_i).$$

Passing to a subsequence of λ_i, we assume that

$$\frac{\mathrm{vol}(B_{\lambda_i}(\tilde{p}))}{\mathrm{vol}(D_i)} \to c_0 \in [C_5, 1]$$

as $i \to \infty$.

To check that the limit measure $\mathfrak{m}$ on Y has the stated expression, it suffices to check $\mathfrak{m}$ on each rectangle in Y. Let

$$R = [a, b] \times [c, d] \subseteq Y$$

written in (r, v)-coordinate of Y, where $0 \leq a < b$ and $c < d$. Let

$$R_i = \{(r, x, v) | r \in [\lambda_i a, \lambda_i b], x \in \mathbb{S}^{n-1}, v \in [\lambda_i^{1+2\alpha} c, \lambda_i^{1+2\alpha} d]\} \subseteq \widetilde{M}.$$

By Lemma 3.2 and the proof of Lemma 3.3, R_i converges to R with respect to the convergent sequence

$$(\widetilde{M}, \lambda_i^{-2} g_{\widetilde{M}}, \tilde{p}) \xrightarrow{GH} (Y, \mathsf{d}, y).$$

R_i has volume

$$\mathrm{vol}(R_i) = \lambda_i^{1+2\alpha}(c-d) \int_{\lambda_i a}^{\lambda_i b} f(r)^{n-1} h(r) dr.$$

Then

$$\begin{aligned}
\mathfrak{m}(R) &= \lim_{i\to\infty} \frac{\mathrm{vol}(R_i)}{\mathrm{vol}(B_{\lambda_i}(\tilde{p}))} = \lim_{i\to\infty} \frac{\mathrm{vol}(R_i)}{c_0 \mathrm{vol}(D_i)} \\
&= c_0^{-1} \lim_{i\to\infty} \frac{\lambda_i^{1+2\alpha}(c-d) \int_{\lambda_i a}^{\lambda_i b} f(r)^{n-1} h(r) dr}{2C_3 \lambda_i^{1+2\alpha} \int_0^{\lambda_i} f(r)^{n-1} h(r) dr} \\
&= (2c_0 C_3)^{-1} \cdot (c-d) \cdot \left(b^{\frac{n+1}{2}-2\alpha} - a^{\frac{n+1}{2}-2\alpha}\right) \\
&= \int_R c_{\mathfrak{m}} r^{\frac{n-1}{2}-2\alpha} dr dv,
\end{aligned}$$

where $c_{\mathfrak{m}} = (2c_0 C_3)^{-1} \cdot (\frac{n+1}{2} - 2\alpha)^{-1}$. Here in order to realize the limit formula for $\mathfrak{m}(R)$, we used a fact that the boundary of R is $\mathfrak{m}$-negligible which is justified by, for instance, [22, Theorem 4.6]. $\square$

Remark 3.13. One can show directly that our $Y = [0, \infty) \times \mathbb{R}$ with the metric $dr^2 + r^{-4\alpha} dv^2 (\alpha > 0)$, and measure $r^{\frac{n-1}{2}-2\alpha} dr dv$ is $\mathrm{RCD}(0, n+1)$ when $n \geq \max\{4\alpha + 3, 16\alpha^2 + 8\alpha + 1\}$ by checking the n-Bakry-Èmery Ricci curvature is nonnegative under the condition in the interior and that the second fundamental form of the boundary of $Y_{r_0} = [r_0, \infty) \times \mathbb{R}$ is strictly positive for $r_0 > 0$, and let $r_0 \to 0$.

Let us introduce another immediate consequence of the lemma above. For any open subset U, $H_0^{1,2}(U, \mathsf{d}, \mathfrak{m})$ denotes the $H^{1,2}$-closure of $\mathrm{Lip}_c(U, \mathsf{d})$. See for instance [10] for the definition of the capacity.

Proposition 3.14. *The singular set $\mathcal{S}$ has null 2-capacity with respect to $\mathfrak{m}$, namely $H^{1,2}(Y, \mathsf{d}, \mathfrak{m}) = H_0^{1,2}(\mathcal{R}, \mathsf{d}, \mathfrak{m})$. In particular $H^{1,2}(Y, \mathsf{d}, \mathfrak{m})$ coincides with the $H^{1,2}$-closure of $C_c^\infty(\mathcal{R})$.*

Proof. It is enough to check that for any $f \in \mathrm{Lip}_c(X, \mathsf{d})$ there exists a sequence $f_i \in \mathrm{Lip}_c(\mathcal{R}, \mathsf{d})$ such that $f_i \to f$ in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. First the lemma above tells us that for any bounded open subset U

$$\int_U \frac{1}{\mathsf{d}_{\mathcal{S}}^2} d\mathfrak{m} < \infty \tag{3.3}$$

holds, where $\mathsf{d}_{\mathcal{S}}$ is the distance function from $\mathcal{S}$. Take cut-off functions ϕ_r with $\phi_r = 0$ on $B_r(\mathcal{S})$, $\phi_r = 1$ on $Y \setminus B_{2r}(\mathcal{S})$ and $|\nabla \phi_r| \le r^{-1}$. Then the sequence $\phi_r f \in \mathrm{Lip}_c(\mathcal{R}, \mathsf{d})$ satisfies

$$\int_Y |\nabla(\phi_r f)|^2 d\mathfrak{m} \le 2 \sup |f|^2 \int_Y |\nabla \phi_r|^2 d\mathfrak{m} + 2 \int_Y |\nabla f|^2 d\mathfrak{m} \le C < \infty \tag{3.4}$$

because of (3.3), namely $\phi_r f$ is a bounded sequence in $H^{1,2}(Y, \mathsf{d}, \mathfrak{m})$. Thus $\phi_r f$ $H^{1,2}$-weakly converge to f as $r \to 0^+$. Thus Mazur's lemma completes the proof. □

Remark 3.15. We can prove that if

$$\int_U \frac{1}{\mathsf{d}_{\mathcal{S}}^4} d\mathfrak{m} < \infty \tag{3.5}$$

for any bounded open subset $U \subset Y$, then $D(\Delta)$ coincides with the closure of $C_c^\infty(\mathcal{R})$, where the norm of $D(\Delta)$ is $(\|f\|_{H^{1,2}}^2 + \|\Delta f\|_{L^2}^2)^{1/2}$. In particular the Laplacian of any $f \in D(\Delta)$ can be approximated in L^2 by the Witten Laplacians of smooth functions with compact supports in $\mathcal{R}$. Since this is not directly related to our works below, let us only give a sketch of the proof as follows.

Let $f \in D(\Delta)$. Considering the heat flow $\mathsf{h}_t f$ (which converge to f in $D(\Delta)$ as $t \to 0^+$), it is enough to consider the case when f is smooth on $\mathcal{R}$ (c.f. [38, Theorem 7.20]). Recalling the proof of the existence of good cut-off functions in [57, Lemma 3.1], applying such cut-off functions, with no loss of generality, we can assume that f has bounded support and that for any $0 < r < 1$ we can find a smooth cut-off function $\phi_r \in C^\infty(\mathcal{R})$ with $\phi_r = 0$ on $B_r(\mathcal{S})$, $\phi_r = 1$ on $Y \setminus B_{2r}(\mathcal{S})$, and $r|\nabla \phi_r| + r^2 |\Delta \phi_r| \le C$. Then a similar argument as in the proof of Proposition 3.14 with (3.5) allows us to conclude that a sequence of finite convex combinations of $\phi_{r_i} f$ converge to f in $D(\Delta)$, where $r_i \to 0^+$. Thus we get the desired result.

It is worth pointing out that (3.5) is satisfied if n is large.

3.3. **Heat Kernel of Y.** In the rest of this subsection we denote by $\mathsf{H}(x, y, t)$ the heat kernel of the Ricci limit space $(Y, \mathsf{d}, \mathfrak{m})$.

Since $F_\lambda : (Y, \lambda \mathsf{d}, \mathfrak{m}) \to (Y, \mathsf{d}, (F_\lambda)_* \mathfrak{m})$ is an isomorphism, and

$$(F_\lambda)_* \mathfrak{m} = \lambda^{-\frac{n+3}{2}} \mathfrak{m},$$

by (2.14) and (2.12) we have

$$\mathsf{H}(x, y, t) = \lambda^{\frac{n+3}{2}} \mathsf{H}(F_\lambda(x), F_\lambda(y), \lambda^2 t). \tag{3.6}$$

Denote $s = \sqrt{t}$, and put

$$q(x, y, s) = \mathfrak{m}(B_s(x)) \cdot \mathsf{H}(x, y, s^2).$$

Since $\mathfrak{m}(B_s(x)) = s^{\frac{n+3}{2}} \mathfrak{m}(B_1(F_{1/s}(x)))$, combining with (3.6), we have

$$q(x, y, s) = q(F_\lambda(x), F_\lambda(y), \lambda s). \tag{3.7}$$

For $x = (r, v) \in Y$, $q(x, x, s)$ only depends on r, s. Denote $h(r, s) = q(x, x, s)$.

Lemma 3.16. *h is invariant under the space-time scaling, i.e. for all $\lambda > 0,\ s > 0,\ r \geq 0$*

$$h(\lambda r, \lambda s) = h(r, s).$$

In particular $h(0, s) = h(0, 1)$. Moreover for some $C = C(n) > 1$

$$C^{-1} \leq h(r, s) \leq C.$$

Proof. The space-time scaling invariance is a reformulation of (3.7) by setting $x = y = (r, 0)$. The inequality follows from that Y is an $\mathrm{RCD}(0, n+1)$ space and the Li-Yau estimate (2.9) when $K = 0$. □

3.4. **Measure of balls in Y.** First we give the following estimate on $\mathfrak{m}(B_s(x))$ when the ball is away from the singular set of Y.

Lemma 3.17. *For $x = (r_0, v_0) \in Y$ and $r_0 > s > 0$, we have*

$$c_{\mathfrak{m}} \pi s^2 (r_0 - s)^{\frac{n-1}{2}} \leq \mathfrak{m}(B_s(x)) \leq c_{\mathfrak{m}} \pi s^2 (r_0 + s)^{\frac{n-1}{2}}. \tag{3.8}$$

In particular, $\mathfrak{m}(B_s(x)) = c_{\mathfrak{m}} \pi s^2 r_0^{\frac{n-1}{2}} [1 + O(r_0^{-1} s)]$ when $s \to 0$ or $r_0 >> s$.

Proof. Since the distance on the regular set of Y is induced by the Riemannian metric

$$g = dr^2 + r^{-4\alpha} dv^2,$$

we have $B_s(x) \subseteq \{(r, v) \,|\, r_0 - s < r < r_0 + s\}$ and

$$B_s^{g_2}(x) \subseteq B_s(x) \subseteq B_s^{g_1}(x),$$

where $g_1 = dr^2 + (r_0 + s)^{-4\alpha} dv^2,\ g_2 = dr^2 + (r_0 - s)^{-4\alpha} dv^2$ are Euclidean metrics on regular set of Y. Hence

$$\begin{aligned} \mathfrak{m}(B_s(x)) &\leq \int_{B_s^{g_1}(x)} d\mathfrak{m} = c_{\mathfrak{m}} \int_{B_s^{g_1}(x)} r^{\frac{n-1}{2} - 2\alpha} dr dv \\ &\leq c_{\mathfrak{m}} \int_{B_s^{g_1}(x)} (r_0 + s)^{\frac{n-1}{2} - 2\alpha} dr dv = (r_0 + s)^{\frac{n-1}{2}} \mathrm{vol}^{g_1}(B_s^{g_1}(x)) \\ &= c_{\mathfrak{m}} (r_0 + s)^{\frac{n-1}{2}} \pi s^2. \end{aligned}$$

Similarly we have the lower bound. □

In general, to get a handle on $\mathfrak{m}(B_s(x))$, we compare it to a "rectangular" domain which is easier to compute, as follows.

Note $B_s(x) = F_s(B_1(x_s))$, where $x_s = F_s^{-1}(x) = (r_0 s^{-1}, v_0 s^{-2\alpha - 1})$ for $x = (r_0, v_0) \in Y$. Denote the "unit rectangle" at x_s by

$$C_1(x_s) = \{(r, v) \,|\, (r_0 s^{-1} - 1)_+ < r < r_0 s^{-1} + 1, -1 < v < 1\}.$$

Then

$$\frac{\mathfrak{m}(B_s(x))}{\mathfrak{m}(F_s(C_1(x_s)))} = \frac{\mathfrak{m}(B_1(x_s))}{\mathfrak{m}(C_1(x_s))} \tag{3.9}$$

Since $F_s(C_1(x_s)) = \{(r,v)\,|\,(r_0-s)_+ < r < r_0+s,\ -s^{2\alpha+1} < v < s^{2\alpha+1}\}$, we compute, as $n+1 > 4\alpha$,

$$\begin{aligned}
&\mathfrak{m}(F_s(C_1(x_s)))\\
&= c_{\mathfrak{m}} \int_{-s^{1+2\alpha}}^{s^{1+2\alpha}} \int_{(r_0-s)_+}^{r_0+s} r^{\frac{n-1}{2}-2\alpha} dr dv\\
&= c_{\mathfrak{m}} \frac{2}{\frac{n+1}{2}-2\alpha} s^{1+2\alpha} \left((r_0+s)^{\frac{n+1}{2}-2\alpha} - (r_0-s)_+^{\frac{n+1}{2}-2\alpha}\right)\\
&= \frac{4c_{\mathfrak{m}}}{n+1-4\alpha} s^{1+2\alpha} r_0^{\frac{n+1}{2}-2\alpha} \left((1+sr_0^{-1})^{\frac{n+1}{2}-2\alpha} - (1-sr_0^{-1})_+^{\frac{n+1}{2}-2\alpha}\right)\\
&= 4c_{\mathfrak{m}} s^{2+2\alpha} r_0^{\frac{n-1}{2}-2\alpha}[1+O(r_0^{-1}s)] \text{ when } \ r_0 >> s.
\end{aligned}$$

Combining the above with Lemma 3.17, we have for $r_0 >> s$,

$$\frac{\mathfrak{m}(F_s(C_1(x_s)))}{\mathfrak{m}(B_s(x))} = \frac{4}{\pi}(sr_0^{-1})^{2\alpha}[1+O(r_0^{-1}s)].$$

Since $x_s = (r_0 s^{-1}, v_0 s^{-2\alpha-1})$ and the measure is independent of the v-coordinate, $\frac{\mathfrak{m}(C_1(x_s))}{\mathfrak{m}(B_1(x_s))}$ is a continuous function of $r_0 s^{-1}$ only. Denote

$$\tau = sr_0^{-1}, \qquad \frac{\mathfrak{m}(C_1(x_s))}{\mathfrak{m}(B_1(x_s))} = f(\tau^{-1}).$$

In summary we have

Proposition 3.18. *Let* $C = \frac{4c_{\mathfrak{m}}}{n+1-4\alpha}$*. For* $x = (r_0, v_0) \in Y$*, when* $r_0 > 0$*,*

$$\frac{1}{\mathfrak{m}(B_s(x))} = C^{-1} s^{-1-2\alpha} r_0^{-\frac{n+1}{2}+2\alpha} G(\tau), \tag{3.10}$$

where

$$G(\tau) = \left((1+\tau)^{\frac{n+1}{2}-2\alpha} - (1-\tau)_+^{\frac{n+1}{2}-2\alpha}\right)^{-1} f(\tau^{-1}). \tag{3.11}$$

Moreover, f *is a continuous function with* $f(0) > 0$ *and* $f(\tau^{-1}) = \frac{4}{\pi}\tau^{2\alpha}[1+O(\tau)]$ *when* $\tau \to 0$*.*
When $r_0 = 0$*,*

$$\mathfrak{m}(B_s(x)) = \frac{C}{f(0)}\, s^{\frac{n+3}{2}}. \tag{3.12}$$

3.5. **Heat Trace Integral.** Using the estimates from the above two subsections, we can compute the integral of the heat kernel on any rectangular coordinate region $\{(r,v)|0 \le r_1 \le r \le r_2 \le \infty,\ -\infty < v_1 \le v \le v_2 < \infty\} \subseteq Y$.

Recall that, for $x = (r, v)$ (we also say that $r = r(x)$),

$$\mathsf{H}(x,x,s^2) = \frac{1}{\mathfrak{m}(B_s(x))} q(x,x,s) = \frac{1}{\mathfrak{m}(B_s(x))} h(r,s).$$

Set $\tau = \frac{s}{r}$. Using (3.10) and Lemma 3.16 (also denoting $C' = \frac{n+1-4\alpha}{4}$)

$$
\begin{aligned}
(3.13)\qquad & \int_{v_1}^{v_2}\int_{r_1}^{r_2} \mathsf{H}(x,x,s^2)d\mathfrak{m} \\
&= C's^{-1-2\alpha}\int_{v_1}^{v_2}\int_{r_1}^{r_2} h(r,s)r^{-\frac{n+1}{2}+2\alpha}G(\tfrac{s}{r})r^{\frac{n-1}{2}-2\alpha}drdv \\
&= C's^{-1-2\alpha}(v_2-v_1)\int_{r_1}^{r_2} h(1,\tfrac{s}{r})r^{-1}G(\tfrac{s}{r})dr \\
(3.14)\qquad &= C's^{-1-2\alpha}(v_2-v_1)\int_{s/r_2}^{s/r_1} h(1,\tau)\tau^{-1}G(\tau)d\tau.
\end{aligned}
$$

Since h is bounded (Lemma 3.16) and $n-1>4\alpha$, by (3.11), the integral is convergent at $\tau=\infty$. When $\tau\to 0$, since $f(\tau^{-1})\sim\tau^{2\alpha}$, the integrand is $\sim\tau^{2\alpha-2}$. Thus the integral is convergent at $\tau=0$ when $\alpha>\frac{1}{2}$; diverges when $\alpha\le\frac{1}{2}$. Therefore, by taking various r_1, r_2, we obtain

Theorem 3.19. *For $\alpha>\frac{1}{2}$, $k=2\alpha+1$,*

$$0<\lim_{s\to 0}s^k\int_{v_1}^{v_2}\int_0^\infty \mathsf{H}(x,x,s^2)d\mathfrak{m}<\infty,$$

$$\lim_{s\to 0}s^k\int_{v_1}^{v_2}\int_{r_1}^\infty \mathsf{H}(x,x,s^2)d\mathfrak{m}=0 \quad \textit{for any} \quad r_1>0,$$

$$\lim_{L\to\infty}\lim_{s\to 0}s^k\int_{v_1}^{v_2}\int_{sL}^\infty \mathsf{H}(x,x,s^2)d\mathfrak{m}=0.$$

For $\alpha=\frac{1}{2}$, $k>2$,

$$\lim_{s\to 0}s^k\int_{v_1}^{v_2}\int_0^{r_2} \mathsf{H}(x,x,s^2)d\mathfrak{m}=0.$$

For $0<\alpha<\frac{1}{2}$, $k=2$,

$$0<\lim_{s\to 0}s^2\int_{v_1}^{v_2}\int_0^{r_2} \mathsf{H}(x,x,s^2)d\mathfrak{m}<\infty.$$

When $\alpha<\frac{1}{2}$, $k=2$, it is in the setting studied in [5, Theorem 4.3], and the condition [5, (4.10)] (namely (5.9)) is satisfied.

For the critical case $\alpha=\frac{1}{2}$, when $r_1=0$, we can show the integral in (3.14) diverges slowly like log.

Lemma 3.20. *For any fixed $r_2>0$, let $\tilde{L}(s)=\int_{s/r_2}^\infty h(1,\tau)\tau^{-1}G(\tau)d\tau$. Then for $\alpha=\frac{1}{2}$, $s\in(0,s_0]$ and s_0 finite,*

$$\tilde{L}(s)=\frac{1}{(n-1)\pi}(-\log s+\log r_2)+\tilde{C}_0+O(s)$$

for some constant $\tilde{C}_0$ given explicitly in the proof. In particular $\tilde{L}$ is slowly varying as $s\to 0^+$, i.e. for any $a>0$ fixed,

$$\lim_{s\to 0^+}\frac{\tilde{L}(as)}{\tilde{L}(s)}=1.$$

Proof. By Lemma 3.16, $h(1,\tau)$ is bounded between two positive constant. Since $(1,0) \in Y$ is a regular point with tangent cone $\mathbb{R}^2$, in fact a smooth point contained in a smooth neighborhood a definite distance away from the singular set, by the standard small time asymptotic expansion of the heat kernel, for small τ,

$$h(1,\tau) = \frac{1}{4} + O(\tau).$$

When $\alpha = \frac{1}{2}$, by Proposition 3.18 $G(\tau)$ is bounded on $(0,\infty)$ and, for small τ,

$$G(\tau) = \frac{4}{\pi} \cdot \frac{1}{n-1} + O(\tau).$$

Recall the integral $\int h(1,\tau)\tau^{-1}G(\tau)d\tau$ is always convergent at infinity. Denote $C_0 = \int_1^\infty h(1,\tau)\tau^{-1}G(\tau)d\tau$. Then

$$\begin{aligned}
\tilde{L}(s) &= \int_{s/r_2}^1 h(1,\tau)\tau^{-1}G(\tau)d\tau + C_0 \\
&= \int_{s/r_2}^1 \tau^{-1}\left(h(1,\tau)G(\tau) - \frac{1}{(n-1)\pi}\right)d\tau + \int_{s/r_2}^1 \frac{\tau^{-1}}{(n-1)\pi}d\tau + C_0 \\
&= \frac{-1}{(n-1)\pi}(\log(s/r_2)) + C_0 + \int_{s/r_2}^1 \tau^{-1}\left(h(1,\tau)G(\tau) - \frac{1}{(n-1)\pi}\right)d\tau.
\end{aligned}$$

The last integral is

$$\int_0^1 \tau^{-1}\left(h(1,\tau)G(\tau) - \frac{1}{(n-1)\pi}\right)d\tau - \int_0^{s/r_2} \tau^{-1}\left(h(1,\tau)G(\tau) - \frac{1}{(n-1)\pi}\right)d\tau.$$

Since $h(1,\tau)G(\tau) - \frac{1}{(n-1)\pi} = O(\tau)$, the first integral is convergent at $\tau = 0$ and the second one is $O(s)$. □

As a consequence, for the critical case,

Theorem 3.21. *For $\alpha = \frac{1}{2}$, $k = 2$, and $s \in (0, s_0]$, s_0 finite,*

$$s^2 \int_{v_1}^{v_2}\int_0^{r_2} \mathsf{H}(x,x,s^2)d\mathfrak{m} = \frac{v_2 - v_1}{4\pi}(-\log s + \log r_2) + \tilde{C} + O(s),$$

where $\tilde{C} = \frac{(n-1)(v_2-v_1)}{4}\tilde{C}_0$. And

$$0 < \lim_{s\to 0} s^2 \int_{v_1}^{v_2}\int_{r_1}^{r_2} \mathsf{H}(x,x,s^2)d\mathfrak{m} < \infty \quad \text{for any } r_1 > 0.$$

Remark 3.22. Similarly in above theorems we can show that for any continuous function $\varphi(x)$ with compact support, the same results hold for

$$\lim_{s\to 0} s^k \int \varphi(x)\mathsf{H}(x,x,s^2)d\mathfrak{m}.$$

For the statements about the nonzero limit, one needs to make sure that the support of φ contains the singular set $\mathcal{S} = \{r = 0\}$.

4. Heat Kernel and Weyl's Law of X

4.1. **Heat Kernel and Covering Spaces.** Let $\bar{Y} = Y/\mathbb{Z}$ with the quotient distance $\bar{\mathsf{d}}$ and the measure $\bar{\mathfrak{m}}$, and $\bar{H}(x, y, t)$ be its heat kernel, where the $\mathbb{Z}$-action is generated by the translation isomorphism $\gamma : Y \to Y,\ (r, v) \mapsto (r, v + 1)$. This is also an RCD$(0, n + 1)$ space (see [61, Remark 1.8]). In this subsection we study the heat kernel $\bar{H}$ of $\bar{Y}$.

First we prove the following relation of heat kernels for covering spaces of RCD spaces which is of independent interest.

Proposition 4.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K, N) *space for some $K \in \mathbb{R}$ and some $N \in [1, \infty)$, $\tilde{X}$ a connected covering space with lifted metric $\tilde{\mathsf{d}}$ and measure $\tilde{\mathfrak{m}}$ and deck transformation Γ (then $(\widetilde{X}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}})$ is also an* RCD(K, N) *space because of the same reason as in* [59, Lemma 2.8]*). Assume that there exists a fundamental domain D such that $\tilde{\mathfrak{m}}(\partial D) = 0$. Then for all $\tilde{x},\ \tilde{y} \in \tilde{X}$, and $x = \pi(\tilde{x}),\ y = \pi(\tilde{y})$, we have*

$$\mathsf{H}_X(x, y, t) = \sum_{\gamma\in\Gamma} \mathsf{H}_{\widetilde{X}}(\tilde{x}, \gamma\tilde{y}, t). \tag{4.1}$$

We call D as above a metric measure fundamental domain (Definition A.4). See also Remarks A.5 and A.7.

When $(X, \mathsf{d}, \mathfrak{m})$ is an unweighted Riemnnian manifold with Ricci curvature bounded from below, Proposition 4.1 is proved in [50, Pages 182-186] (see also [11, Proposition 2.12]). The same strategy works in the setting of Proposition 4.1, but it is more technical. We write the proof in the appendix §A.2. From now on we will abuse notation and simply denote $[x]$ by x.

Clearly our $\bar{Y}$ has a fundamental domain $D = (0, \infty) \times (0, 1)$ with $\mathfrak{m}(\partial D) = 0$, thus we can apply Proposition 4.1 to $\bar{Y}$. Combining with Lemma 3.11, we can then quickly get

Proposition 4.2. *For $x = (r, v) \in \bar{Y}$, as $s \to 0$,*

$$\bar{\mathfrak{m}}(B_s^{\bar{Y}}(x))\bar{\mathsf{H}}(x, x, s^2) = h(r, s) + O(e^{-\frac{C(r_0)}{6s^2}}).$$

Moreover

$$\lim_{s\to 0} s^k \int_0^1 \int_0^{r_0} \bar{\mathsf{H}}(x, x, s^2) d\bar{\mathfrak{m}} = \lim_{s\to 0} s^k \int_0^1 \int_0^{r_0} \mathsf{H}(x, x, s^2) d\mathfrak{m} \tag{4.2}$$

for any $0 < r_0 < \infty$ and all cases of α, k in Theorem 3.19.

Proof. When $s << 1$, $\bar{\mathfrak{m}}(B_s^{\bar{Y}}(x)) = \mathfrak{m}(B_s^Y(x))$, therefore

$$\begin{aligned}\bar{\mathfrak{m}}(B_s^{\bar{Y}}(x))\bar{H}(x, x, s^2) &= \mathfrak{m}(B_s^Y(x))\left(\mathsf{H}(x, x, s^2) + \sum_{l\in\mathbb{Z}, l\neq 0} \mathsf{H}(x, \gamma^l x, s^2)\right) \\ &= q(x, x, s) + \sum_{l\in\mathbb{Z}, l\neq 0} q(x, \gamma^l x, s).\end{aligned}$$

By the Li-Yau estimate (2.9)

$$q(x, \gamma^l x, s) \le C(n) \exp(-\tfrac{\bar{\mathsf{d}}^2(x,\ \gamma^l x)}{6s^2}).$$

By Lemma 3.11,

$$\mathsf{d}(x,\ \gamma^l x) \ge C(r_0)|l|^{\frac{1}{1+2\alpha}}$$

for all $x = (r, v)$ with $0 \le r \le r_0$ and all $l \in \mathbb{Z}$. Therefore

$$\sum_{l\in\mathbb{Z}, l\neq 0} q(x, \gamma^l x, s) \le C(p, \alpha) e^{-\frac{C(r_0)}{6s^2}} \quad \text{for} \quad s \in (0, 1)$$

is exponentially decaying as $s \to 0$. This proves the first equation.

The second equation follows from the first since $\frac{1}{\bar{\mathfrak{m}}(B_s^{\bar{Y}}(x))}$ grows at most polynomially of degree $\frac{n+3}{2}$. □

4.2. **Eigenvalues of $\bar{Y}$ when $\alpha = \frac{1}{2}$.** We use separation of variables (see e.g. [16, Page 41]) to study the eigenvalues of $\bar{Y} = [0, \infty) \times \mathbb{S}^1$. First we do the computation with general α. Recall $\bar{Y}$ equipped with the metric $g = dr^2 + r^{-4\alpha} d\theta^2$ (Theorem 3.5), the weighted measure $d\bar{\mathfrak{m}} = c r^{\frac{n-1}{2} - 2\alpha} dr d\theta$ (Lemma 3.12). Therefore the weighted Laplacian is

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\left(\frac{n-1}{2} - 2\alpha\right)\frac{\partial}{\partial r} + r^{4\alpha}\frac{\partial^2}{\partial \theta^2}.$$

Write the eigenfunction $u = \varphi(r)\psi(\theta)$, then we have

$$\psi'' = -k^2\psi, \quad k \in \mathbb{Z}$$

and

$$\varphi''(r) + \left(\tfrac{n-1}{2} - 2\alpha\right)\frac{1}{r}\varphi'(r) - k^2 r^{4\alpha}\varphi = -\lambda\varphi.$$

The change of variable $y(r) = \varphi(r) r^{\frac{n-1}{4} - \alpha}$ gives

$$y''(r) = \left[\left(\tfrac{n-1}{4} - \alpha\right)\left(\tfrac{n-1}{4} - \alpha - 1\right)\frac{1}{r^2} + k^2 r^{4\alpha} - \lambda\right] y.$$

When $n = 1$ or 9 and $\alpha = \frac{1}{2}$, this becomes

$$y''(r) = \left[\tfrac{3}{4}\tfrac{1}{r^2} + k^2 r^2 - \lambda\right] y.$$

For $n = 1$, it corresponds to the Grushin half cylinder with unweighted measure studied in [12]. Surprisingly for $n = 9$ it gives the same constant $\frac{3}{4}$ in above. For $k \neq 0$, with the change of variable $z = |k| r^2$, and divide by $4k^2 r^2$, we have [1]

$$\frac{\partial^2 y}{\partial z^2} + \frac{z}{2}\frac{\partial y}{\partial z} = \left(\frac{3}{16 z^2} + \frac{1}{4} - \frac{\lambda}{4|k|z}\right) y.$$

Letting $\phi(z) = z^{\frac{1}{4}} y(z)$ gives

$$\phi'' = \left(\frac{1}{4} - \frac{\lambda}{4|k|z}\right)\phi,$$

which is the Whittaker equation. Following [12], the eigenvalues are $\lambda_{n,k} = 4|k|n$ for $n \in \mathbb{N}$, $k \in \mathbb{Z}/\{0\}$ ($k = 0$ leads to the continuous spectrum).

Remark 4.3. If $(\bar{Y}, \bar{\mathsf{d}}, \bar{\mathfrak{m}})$ is an $\mathrm{RCD}(K, N)$ space, then the N-Bakry-Èmery Ricci curvature on the regular set of $\bar{Y}$ is bounded below by K. For our $\bar{Y}$ with $\alpha = \frac{1}{2}$, the N-Bakry-Èmery Ricci curvature is nonnegative if and only if $N \ge 10$. In particular the optimal dimension to have nonnegative Ricci curvature in the RCD sense is indeed 10, i.e., $n = 9$. The same observation is valid for Y.

[1]In [12] the sentence after equation (3.2) is not exactly correct; we would like to thank Prof. Dario Prandi for communicating this and explaining the correction.

Therefore this allows us to conclude that the optimal dimension to have a lower bound in the RCD sense for our 2-(Hausdorff) dimensional spaces is also equal to 10 because of the following two facts;

- any tangent cone at any point of any $\mathrm{RCD}(K,N)$ space is an $\mathrm{RCD}(0,N)$ space;
- any tangent cone at any singular point of such spaces is isometric to Y.

4.3. **Heat Kernel of Compact Space X.** We consider the compact space X obtained by perturbing the warping function of $\bar{Y}$ to $\tilde{h} = \mathit{constant}$ for $r \geq 2$, and then double. Precisely let $\widetilde{Y}$ be the finite cylinder $[0,3]\times\mathbb{S}^1$ with the singular metric $dr^2 + \tilde{h}(r)^2 dv^2$, where

$$\tilde{h}(r) = \begin{cases} r^{-2\alpha} & 0 < r \leq 1 \\ \frac{1}{2} & 2 \leq r \leq 3 \end{cases}$$

and connect smoothly between $r = 1$ and $r = 2$ with decreasing and convex function. Correspondingly, the measure $d\tilde{\mathfrak{m}} = \tilde{h}^{-\frac{n-1}{4\alpha}+1} dr\, dv$. Let $X = \widetilde{Y} \cup \widetilde{Y}$ by gluing at the smooth end $r = 3$. Then $\widetilde{Y}$ and X are $\mathrm{RCD}(K, n+1)$ spaces with K some negative number.

Remark 4.4. The above X can be constructed as the limit of a sequence of closed manifolds N_i with $\mathrm{Ric} \geq K$, where $K < 0$ (see [61, Remark 1.8]); then $\widetilde{Y} \subseteq X$ is the limit of a sequence of open convex subsets in N_i. We also point out that nonnegative Ricci curvature cannot hold on N_i; otherwise, the universal cover $\widetilde{N_i}$ would split off a line isometrically, resulting in X being isometric to a standard bounded cylinder.

Since $\widetilde{Y}$ is a smooth weighted Riemannian manifold near the end $r = 3$, each eigenfunction on $\widetilde{Y}$ is smooth near the smooth end and it satisfies the Neumann boundary value condition because of the elliptic regularity theorem. In particular the heat kernel $\mathsf{H}_{\widetilde{Y}}$ of $\widetilde{Y}$ is smooth near the smooth end with the same boundary value condition by (2.13).

By the symmetry of X, the heat kernel H_X of X can be identified with the heat kernel of $\widetilde{Y}$ with the Neumann boundary condition at the smooth end. Now the heat kernel of $\tilde{Y}$ can be constructed by gluing the heat kernel of $\bar{Y}$, with the Neumann heat kernel of cylinder $\hat{Y} = [1/2, 3] \times \mathbb{S}^1$ as a smooth weighted manifold. The warping function $\hat{h}$ of $\hat{Y}$ is defined as $\hat{h} = \tilde{h}$ for $r \in [\frac{1}{2}, 3]$ and the measure $d\hat{\mathfrak{m}} = \hat{h}^{-\frac{n-1}{4\alpha}+1} dr\, dv$.

Following [7], let $\rho(a,b) \in C^\infty(\mathbb{R})$ be a smooth increasing cut-off function such that $\rho \equiv 0$ when $r \leq a$ and $\rho \equiv 1$ when $r \geq b$.

Define $\phi_1, \phi_2, \psi_1, \psi_2$ as follows.

$$\psi_1 = 1 - \psi_2, \quad \psi_2 = \rho(1 + \tfrac{1}{4}, 2 - \tfrac{1}{4}), \quad \phi_1 = 1 - \rho(2,3), \quad \phi_2 = \rho(\tfrac{1}{2}, 1).$$

Then

$$\psi_1 + \psi_2 \equiv 1, \quad \phi_i \equiv 1 \text{ on } \operatorname{supp} \psi_i, \quad \mathsf{d}_{\mathbb{R}}(\operatorname{supp} \phi_i', \operatorname{supp} \psi_i) \geq \tfrac{1}{4}.$$

It follows that

$$\tilde{\mathsf{d}}(\operatorname{supp} \phi_i', \operatorname{supp} \psi_i) \geq \tfrac{1}{4}. \tag{4.3}$$

Here $\tilde{\mathsf{d}}$ is the distance of $\tilde{Y}$.

Let

$$\mathsf{K}(x,y,t) = \phi_1(r(x))\mathsf{H}_{\bar{Y}}(x,y,t)\psi_1(r(y)) + \phi_2(r(x))\mathsf{H}_{\hat{Y}}(x,y,t)\psi_2(r(y)),$$

where $H_{\hat{Y}}(x,y,t)$ is the heat kernel of the smooth weighted manifold $\hat{Y}$ with Neumann boundary condition. Then K is a parametrix of the heat equation on $\tilde{Y}$. Indeed

$$(4.4)\quad (\partial_t-\Delta_x)\mathsf{K}(x,y,t) = -\phi_1''\mathsf{H}_{\bar{Y}}\psi_1-\langle\nabla\phi_1,\nabla\mathsf{H}_{\bar{Y}}\rangle\psi_1-\phi_2''\mathsf{H}_{\widetilde{Y}}\psi_2-\langle\nabla\phi_2,\nabla\mathsf{H}_{\widetilde{Y}}\rangle\psi_2,$$

where Δ_x means the Laplacian acting on the x-variable. Similarly the gradient ∇ is with respect to the x-variable as well. Note that by our choice of the cut-off functions, the right hand side of (4.4) is away from the singular set in the x-variable, and, by the regularity theory, the differentiations can be taken in the usual sense. Let

$$\mathsf{Q}(x,y,t) = (\partial_t-\Delta_x)\mathsf{K}(x,y,t).$$

By (4.4), $\mathsf{Q}(x,y,t)$ is nonzero only when $r(x)\in\operatorname{supp}\phi_1'$ and $r(y)\in\operatorname{supp}\psi_1$ or when $r(x)\in\operatorname{supp}\phi_2'$ and $r(y)\in\operatorname{supp}\psi_2$. By the Li-Yau estimates (2.9), (2.10) and (4.3), we deduce for $0\le s<t$ and $r(z)\in\operatorname{supp}\phi_1'\cup\operatorname{supp}\phi_2'\subset[1/2,1]\cup[2,3]$,

$$\begin{aligned}
|\mathsf{Q}(z,y,t-s)| &\le \frac{C_1}{(t-s)\mathfrak{m}(B_{\sqrt{t-s}}(z))}\exp\left(-\frac{1}{90(t-s)}+C_2(t-s)\right)\\
&\le \frac{C_3}{(t-s)}\exp\left(-\frac{1}{90(t-s)}+C_2(t-s)\right)\\
&\le C_4\exp\left(-\frac{1}{180(t-s)}+C_2(t-s)\right)\le C_5\exp\left(-\frac{1}{180t}\right) \qquad (4.5)
\end{aligned}$$

for $0\le s<t\le 1$. Duhamel Principle expresses the heat kernel $\mathsf{H}_{\tilde{Y}}(x,y,t)$ in terms of the parametrix $\mathsf{K}(x,y,t)$ (i.e. the approximate solution) and an error term.

Lemma 4.5 (Duhamel Principle)**.** *We have*

$$(4.6)\qquad \mathsf{H}_{\widetilde{Y}}(x,y,t) = \mathsf{K}(x,y,t) - \int_0^t\int_{\widetilde{Y}}\mathsf{H}_{\tilde{Y}}(x,z,s)\mathsf{Q}(z,y,t-s)d\tilde{\mathfrak{m}}(z)ds.$$

This is well known in the smooth case; see [17] for the most general formulation. The proof of the lemma goes by directly checking the heat equation. Namely, it follows from a direct calculation that the right hand side of (4.6) solves the heat equation. Then the lemma follows by a similar argument as in Proposition 4.1. See §A.3 for the detail.

Plugging (4.5) into (4.6), we obtain

$$(4.7)\qquad \mathsf{H}_{\widetilde{Y}}(x,y,t) = \mathsf{K}(x,y,t) + O(e^{-C/t})$$

for $C=\frac{1}{180}$. From here, we obtain by using Theorems 3.19 and 3.21

Theorem 4.6. *For $\alpha>\frac{1}{2}$, $k=2\alpha+1$, the following limit exists and*

$$(4.8)\qquad 0<\lim_{s\to 0}s^k\int_X\mathsf{H}_X(x,x,s^2)d\mathfrak{m} = 2\lim_{s\to 0}s^k\int_0^1\int_0^3\mathsf{H}_Y(x,x,s^2)d\mathfrak{m}<\infty.$$

Here v is from 0 to 1, r is from 0 to 3, and 3 can be replaced by any positive number.

For $\alpha=\frac{1}{2}$, $k=2$, and s small,

$$(4.9)\qquad s^2\int_X\mathsf{H}_X(x,x,s^2)d\mathfrak{m} = 2s^2\int_0^1\int_0^3\mathsf{H}_Y(x,x,s^2)d\mathfrak{m} = \frac{1}{2\pi}(-\log s)+O(1).$$

For $\alpha = \frac{1}{2},\ k > 2,$

$$\lim_{s\to 0} s^k \int_X \mathsf{H}_X(x,x,s^2)d\mathfrak{m} = 0.$$

For $0 < \alpha < \frac{1}{2},\ k = 2,$

$$0 < \lim_{s\to 0} s^2 \int_X \mathsf{H}_X(x,x,s^2)d\mathfrak{m} < \infty.$$

Proof. By the symmetry of X and (4.7) we have

$$\begin{aligned} s^k \int_X \mathsf{H}_X(x,x,s^2)d\mathfrak{m} &= 2s^k \int_{\tilde{Y}} \mathsf{H}_{\tilde{Y}}(x,x,s^2)d\mathfrak{m} \\ &= 2s^k \int_{\tilde{Y}} \mathsf{H}_{\tilde{Y}}(x,x,s^2)\psi_1(r(x))d\mathfrak{m} \\ &\quad +2s^k \int_{\hat{Y}} \mathsf{H}_{\hat{Y}}(x,x,s^2)\psi_2(r(x))d\mathfrak{m} + O(e^{-C'/s^2}) \\ &= 2s^k \int_0^1 \int_0^3 \mathsf{H}_Y(x,x,s^2)\psi_1(r(x))d\mathfrak{m} \\ &\quad +2s^k \int_{\hat{Y}} \mathsf{H}_{\hat{Y}}(x,x,s^2)\psi_2(r(x))d\mathfrak{m} + O(e^{-C'/s^2}), \end{aligned} \tag{4.10}$$

where we have made use of (4.2) in the last identity. The first term can be dealt with by Theorems 3.19 and 3.21 (more specifically Remark 3.22), while the second term is classical (or we can use [5]). The desired results follow. ☐

4.4. **Weyl's Law of X.** We are now in a position to introduce our main results, singular Weyl's laws on compact Ricci limit, thus $\mathrm{RCD}(K,N)$ spaces $(X,\mathsf{d},\mathfrak{m})$ constructed in the previous subsection with the parameter α. In order to introduce a surprising result (Corollary 4.8), we prepare the following.

Theorem 4.7. *Let* $\alpha > \frac{1}{2}$ *and* $k = 2\alpha + 1$. *Then*

$$s^k \mathsf{H}(x,x,s^2)d\mathfrak{m}(x) \to c \cdot 1_{\mathcal{S}}\, d\mathcal{H}^k, \quad \text{weakly as } s \to 0^+ \tag{4.11}$$

for some constant $c > 0$, *where* $\mathcal{S} = X \setminus \mathcal{R}_2$ *is the singular set. Moreover the support of* $1_{\mathcal{S}}\, d\mathcal{H}^k$ *coincides with* $\mathcal{S}$.

Proof. By Theorem 4.6,

$$0 < \lim_{s\to 0^+} s^k \int_X \mathsf{H}_X(x,x,s^2)d\mathfrak{m} < \infty.$$

Therefore for any sequence $s_i \to 0^+$, there is a subsequence such that the left hand side of (4.11) converges to a Radon measure weakly. Take a sequence $s_i \to 0^+$ and a Radon measure ν on X such that

$$(s_i)^k \mathsf{H}(x,x,(s_i)^2)d\mathfrak{m}(x) \to \nu, \quad \text{weakly as } i \to \infty. \tag{4.12}$$

For any $z = (r_0, v_0) \in \tilde{Y}$, denote $\mathrm{Box}_{(r_1,v_1)}(z) = \{(r,v) | 0 \le r_0 < r < r_1 \le 3,\ 0 \le v_0 < v < v_1 \le 1\}$. Then the weak convergence gives

$$\liminf_{i\to\infty} \int_{\mathrm{Box}_{(r_1,v_1)}(z)} (s_i)^k \mathsf{H}(x,x,(s_i)^2)d\mathfrak{m}(x) \ge \nu(\mathrm{Box}_{(r_1,v_1)}(z)) \tag{4.13}$$

and

$$\limsup_{i\to\infty}\int_{\overline{\mathrm{Box}_{(r_1,v_1)}(z)}}(s_i)^k\mathsf{H}(x,x,(s_i)^2)d\mathfrak{m}(x)\le \nu(\overline{\mathrm{Box}_{(r_1,v_1)}(z)}). \tag{4.14}$$

On the other hand, by (4.10) and (3.14), we know

$$\lim_{\epsilon\to 0^+}\left(\limsup_{s\to 0^+}\int_{\mathrm{Box}_{(r_1,v_1)}(z)\setminus\mathrm{Box}_{(r_1,v_1-\epsilon)}(z)}s^k\mathsf{H}(x,x,s^2)d\mathfrak{m}(x)\right)=0. \tag{4.15}$$

Combining (4.13), (4.14) with (4.15) shows

$$\lim_{i\to\infty}\int_{\mathrm{Box}_{(r_1,v_1)}(z)}(s_i)^k\mathsf{H}(x,x,(s_i)^2)d\mathfrak{m}(x)=\nu(\mathrm{Box}_{(r_1,v_1)}(z)), \tag{4.16}$$

and

$$\nu(\overline{\mathrm{Box}_{(r_1,v_1)}(z)})=\nu(\mathrm{Box}_{(r_1,v_1)}(z)). \tag{4.17}$$

Next let us prove that the support of ν coincides with $\mathcal{S}$. Theorem 3.19 and (4.10) allow us to conclude that for any compact subset A in $\mathcal{R}$,

$$\int_A s^k\mathsf{H}(x,x,s^2)d\mathfrak{m}\to 0,\quad \text{as } s\to 0^+. \tag{4.18}$$

This implies that the support of ν is included in $\mathcal{S}$. Theorem 3.19 and (4.10) also imply $\nu(\mathrm{Box}_{(r_1,v_1)}(z))>0$ whenever $z\in\mathcal{S}$. Hence the support of ν coincides with $\mathcal{S}$.

Moreover Theorem 3.5 (4) with (3.14) and (4.16) shows that there exist $C>0$ and $\delta>0$ such that $\nu(B_r(z))=Cr^k$ holds for all $z\in\mathcal{S}$ and $0<r<\delta$. Combining this with the observation above easily implies $\nu=c1_{\mathcal{S}}d\mathcal{H}^k$ for some $c>0$. Finally since we have

$$c=\mathcal{H}^k(\mathcal{S})^{-1}\cdot\lim_{s\to 0^+}s^k\int_X\mathsf{H}_X(x,x,s^2)d\mathfrak{m}$$

which does not depend on the sequence $s_i\to 0^+$, we conclude because s_i is arbitrary. $\square$

We are now in a position to introduce a main result of this subsection. Let us emphasize that k is not necessarily an integer.

Corollary 4.8. *Let $\alpha>\frac{1}{2}$ and $k=2\alpha+1$. Then*

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{k/2}}=\frac{c}{\Gamma(k+1)}\mathcal{H}^k(\mathcal{S})\in(0,\infty), \tag{4.19}$$

where c is given in (4.11).

Proof. This is a direct consequence of Theorem 4.7 and (1.7) with $L=1$ $\square$

The following is another surprising result which is an immediate consequence of Theorem 4.6 with the same techniques as in the proof above. It in particular shows that for general compact $\mathrm{RCD}(K,N)$ spaces, the eigenvalue counting function $N(\lambda)$ defined in (1.3) does not satisfy the following asymptotic formula:

$$N(\lambda)\sim\lambda^\beta,\quad \text{for some } \beta\ge 0 \text{ as } \lambda\to\infty.$$

In connection with (4.20) below, we recall that any neighborhood of any singular point has infinite 2-dimensional Hausdorff measure; see Remark 3.7.

Theorem 4.9. *Let $\alpha = \frac{1}{2}$. Then*

$$\frac{s^2}{-\log s}\mathsf{H}(x,x,s^2)d\mathfrak{m}(x) \to c \cdot 1_{\mathcal{S}}\, d\mathcal{H}^2, \quad \textit{weakly as } s \to 0^+ \tag{4.20}$$

for some constant $c > 0$. In particular

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda \log \lambda} = \frac{c}{\Gamma(3)}\mathcal{H}^2(\mathcal{S}) = \frac{1}{4\pi}. \tag{4.21}$$

Proof. Applying a similar argument as in the proof of Theorem 4.7 with Theorem 3.21 shows (4.20). Then the first equality in (4.21) follows from (1.7) with $L = \log$, and the second equality comes from (4.9).

□

5. Asymptotics of eigenfunctions

5.1. **Questions and negative answer.** Let us start by recalling a question raised by Ding in [31, page 511].

Question 5.1 (Ding)**.** *For any integer $n \geq 2$, does there exist a positive small number $\epsilon_n > 0$ such that*

$$\int_{M^n\setminus A} f^2 \mathrm{dvol}^g > \frac{1}{2}\int_{M^n} f^2 \mathrm{dvol}^g \tag{5.1}$$

holds for any n-dimensional closed Riemannian manifold (M^n, g) with nonnegative Ricci curvature, any eigenfunction f of $-\Delta^g$ on M^n and any Borel subset $A \subset M^n$ with $\mathrm{vol}^g A < \epsilon_n \mathrm{vol}^g M^n$?

It is easy to see from [43, Theorem 1.1] that the question has a negative answer even for the n-dimensional canonical unit sphere $\mathbb{S}^n$. Actually for any great circle c in $\mathbb{S}^n$, we can find an L^2-orthonormal basis $\{f_j\}_j$ consisting of j-th eigenfunctions f_j of $\mathbb{S}^n$ and a subsequence $i(j)$ satisfying that

$$f_{i(j)}^2 \mathrm{dvol}^g \to \nu_c, \quad \text{weakly}, \tag{5.2}$$

where ν_c denotes the canonical probability 1-dimensional measure whose support coincides with c. Then it is easily checked that the sequence $f_{i(j)}$ provides a counterexample to Question 5.1. Namely the question above is related to the quantum ergodicity (see also [68]).

Since it is also well-known that it is really hard to determine all limit measures appeared as in (5.2), let us switch the question above to a weaker one;

Question 5.2. *How many eigenfunctions satisfy (5.1)?*

This new question should be related to Weyl's law.

The main purpose of this section is to give positive answers to this new question for more general spaces, namely RCD(K,N) spaces. The obtained results later will be related to the validity of "regular" Weyl's law which is completely different from the results in the previous sections. Thus let us start the next subsection by introducing the regular Weyl's law for compact RCD(K,N) spaces.

5.2. **Regular Weyl's law and its consequence.** We start this subsection by giving the following elementary lemma.

Lemma 5.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$ *and some* $N \in [1, \infty)$*. Then for any* $l \geq 0$ *and any Borel subset* $A \subset X$*, we have*

$$\mathcal{H}^l(A) \leq C(N, l) \liminf_{s \to 0^+} \int_{B_s(A)} \frac{s^l}{\mathfrak{m}(B_s(x))} d\mathfrak{m}. \tag{5.3}$$

Proof. This is essentially the same as [31, Lemma 7.11]. Let us take a maximal s-separated set $\{x_i\}_i$ of A. Then

$$\sum_i s^l \leq C(N) \sum_i \int_{B_s(x_i)} \frac{s^l}{\mathfrak{m}(B_s(x))} d\mathfrak{m}(x) \leq C(N) \int_{B_s(A)} \frac{s^l}{\mathfrak{m}(B_s(x))} d\mathfrak{m}(x), \tag{5.4}$$

where in the first inequality in (5.4), we used

$$\mathfrak{m}(B_s(x_i)) \geq c(N)\mathfrak{m}(B_{2s}(x_i)) \geq c(N)\mathfrak{m}(B_s(x)) \tag{5.5}$$

on $B_s(x_i)$. Note that the first inequality in (5.5) is a direct consequence of Bishop-Gromov inequality (for small $s > 0$) and that the second inequality follows from the inclusion $B_s(x) \subset B_{2s}(x_i)$. Then letting $s \to 0^+$ in (5.4) implies (5.3). □

Remark 5.4. The dependence on l in the constant in the lemma above comes from the definition of the l-dimensional Hausdorff measure;

$$\mathcal{H}^l(A) = \lim_{\delta \to 0^+} \mathcal{H}^l_\delta(A), \tag{5.6}$$

where

$$\mathcal{H}^l_\delta(A) = \inf \left\{ \sum_i c_l r_i^l \middle| A \subset \bigcup_i B_{r_i}(x_i),\ r_i < \delta \right\} \tag{5.7}$$

for some constant $c_l > 0$. When l is an integer, we always choose c_l as $\omega_l = \mathcal{H}^l(B_1(0_l))$.

Let us introduce the following result, where $\mathcal{R}_n^*$ denotes the reduced n-regular set of $(X, \mathsf{d}, \mathfrak{m})$ defined by the set of all n-dimensional regular points $x \in X$ with the existence of a finite positive limit:

$$\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{r^n} \in (0, \infty). \tag{5.8}$$

Note that $\mathfrak{m}(X \setminus \mathcal{R}_n^*) = 0$ and that $\mathfrak{m}$ and $\mathcal{H}^n$ are absolutely continuous to each other on $\mathcal{R}_n^*$ (see [5, Theorem 4.1]).

Theorem 5.5 (Regular Weyl's law)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$ *and some* $N \in [1, \infty)$ *and let* n *be the rectifiable dimension. Then*

$$\lim_{r \to 0^+} \int_X \frac{r^n}{\mathfrak{m}(B_r(x))} d\mathfrak{m} = \int_X \lim_{r \to 0^+} \frac{r^n}{\mathfrak{m}(B_r(x))} d\mathfrak{m} < \infty \tag{5.9}$$

holds if and only if Weyl's law is satisfied in a regular sense, namely

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(\mathcal{R}_n^*) < \infty. \tag{5.10}$$

Moreover if (5.9) is valid, then $\mathcal{H}^n(X) < \infty$*, in particular the Hausdorff dimension of* X *is equal to* n*.*

Proof. The desired equivalence is already obtained in [5, Theorem 4.3]. The remaining one comes from Lemma 5.3 □

In the sequel, we always fix an L^2-orthonormal basis $\{f_j\}_j$ consisting of j-th eigenfunctions f_j of $L^2(X,\mathfrak{m})$. The following is a main result of this subsection, which also gives a Weyl type asymptotics.

Theorem 5.6. *Let $(X,\mathsf{d},\mathfrak{m})$ be a compact $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$ and some finite $N\in[1,\infty)$ whose rectifiable dimension is equal to n. If (5.9) is satisfied, then for all $0<\delta<\epsilon<1$ and any Borel subset $A\subset X$ with $\mathcal{H}^n(A\cap\mathcal{R}_n^*)\le\delta\mathcal{H}^n(\mathcal{R}_n^*)$, letting*

$$m_{A,\epsilon}(\lambda)=\sharp\left\{j\in\mathbb{N}\,\middle|\,\int_{X\setminus A}f_j^2d\mathfrak{m}\ge(1-\epsilon)\int_X f_j^2d\mathfrak{m}\ \text{and}\ \lambda_j\le\lambda\right\}\ \text{for any}\ \lambda>0,\tag{5.11}$$

we have

$$m_{A,\epsilon}(\lambda)\sim\lambda^{n/2},\quad \text{as}\ \lambda\to\infty.\tag{5.12}$$

In particular

$$\int_{X\setminus A}f_j^2d\mathfrak{m}\ge(1-\epsilon)\int_X f_j^2d\mathfrak{m}\tag{5.13}$$

is satisfied for infinitely many eigenfunctions f_j of $-\Delta$ on X.

Proof. Since $m_{A,\epsilon}(\lambda)\le N(\lambda)$, thanks to the validity of (5.10) with [5, Proposition 5.6], it is enough to prove that

$$\liminf_{s\to0^+}s^n\sum_{i\in\mathcal{M}}e^{-\lambda_i s^2}>0,\tag{5.14}$$

where

$$\mathcal{M}=\left\{j\in\mathbb{N}\,\middle|\,\int_{X\setminus A}f_j^2d\mathfrak{m}\ge(1-\epsilon)\int_X f_j^2d\mathfrak{m}\right\}.\tag{5.15}$$

Note

$$\begin{aligned}
&\int_X s^n\mathsf{H}(x,x,s^2)d\mathfrak{m}\\
&=s^n\sum_{j\in\mathcal{M}}e^{-\lambda_j s^2}+s^n\sum_{j\notin\mathcal{M}}e^{-\lambda_j s^2}\\
&\ge s^n\sum_{j\in\mathcal{M}}e^{-\lambda_j s^2}+\frac{s^n}{1-\epsilon}\sum_{j\notin\mathcal{M}}e^{-\lambda_j s^2}\int_{X\setminus A}f_j^2d\mathfrak{m}\\
&\ge s^n\sum_{j\in\mathcal{M}}e^{-\lambda_j s^2}+\frac{1}{1-\epsilon}\int_{X\setminus A}s^n\mathsf{H}(x,x,s^2)d\mathfrak{m}-\frac{s^n}{1-\epsilon}\sum_{j\in\mathcal{M}}e^{-\lambda_j s^2},
\end{aligned}\tag{5.16}$$

namely

$$s^n\sum_{j\in\mathcal{M}}e^{-\lambda_j s^2}\ge\frac{1}{\epsilon}\int_{X\setminus A}s^n\mathsf{H}(x,x,s^2)d\mathfrak{m}-\frac{1-\epsilon}{\epsilon}\int_X s^n\mathsf{H}(x,x,s^2)d\mathfrak{m}.\tag{5.17}$$

Letting $s \to 0^+$ shows

$$\liminf_{s\to 0^+} s^n \sum_{j\in\mathcal{M}} e^{-\lambda_j s^2} \ge \frac{1}{(4\pi)^{n/2}\epsilon}\mathcal{H}^n(\mathcal{R}_n^*\setminus A) - \frac{1-\epsilon}{(4\pi)^{n/2}\epsilon}\mathcal{H}^n(\mathcal{R}_n^*)$$

$$\ge \frac{1-\delta}{(4\pi)^{n/2}\epsilon}\mathcal{H}^n(\mathcal{R}_n^*) - \frac{1-\epsilon}{(4\pi)^{n/2}\epsilon}\mathcal{H}^n(\mathcal{R}_n^*)$$

$$= \frac{\epsilon-\delta}{(4\pi)^{n/2}\epsilon}\mathcal{H}^n(\mathcal{R}_n^*) > 0 \tag{5.18}$$

because it is known from the proof of [5, Theorem 4.3] that (5.9) implies the L^1-strong convergence of $s^n\mathsf{H}(x,x,s^2)$ to $(4\pi)^{-n/2}1_{\mathcal{R}_n^*}$ as $s \to 0^+$. Thus we conclude. □

Note that the proposition above can be applied to a metric measure space $([0,1], \mathsf{d}_{\mathbb{R}}, \sin^{N-1} t dt)$, see [5, Example 4.5].

Remark 5.7. In connection with the theorem above, it is worth mentioning that under the assumption that $\mathcal{H}^n(\mathcal{R}_n^*) < \infty$ (in particular it is satisfied if (5.9) holds), for any $\epsilon > 0$ there exists $\delta > 0$ such that if a Borel subset $A \subset X$ satisfies $\mathfrak{m}(A) \le \delta$, then $\mathcal{H}^n(A \cap \mathcal{R}_n^*) \le \epsilon$. The proof of this fact is as follows.

Denoting

$$d\mathcal{H}^n = \phi d\mathfrak{m}, \quad \text{for some } \phi : \mathcal{R}_n^* \to [0,\infty) \tag{5.19}$$

on $\mathcal{R}_n^*$, since

$$\int_{\mathcal{R}_n^*} \phi d\mathfrak{m} = \int_{\mathcal{R}_n^*} d\mathcal{H}^n = \mathcal{H}^n(\mathcal{R}_n^*) < \infty, \tag{5.20}$$

we have $\phi \in L^1(\mathcal{R}_n^*, \mathfrak{m})$. In particular for any $\epsilon > 0$ there exists $\delta > 0$ such that if a Borel subset $B \subset \mathcal{R}_n^*$ satisfies $\mathfrak{m}(B) < \delta$, then

$$\int_B \phi d\mathfrak{m} < \epsilon. \tag{5.21}$$

which proves the desired statement.

Remark 5.8. Introduced in [29, Definition 1.1], an $\mathrm{RCD}(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$ is said to be non-collapsed if $\mathfrak{m} = \mathcal{H}^N$, which is a synthetic counterpart of volume non-collapsed Ricci limit spaces. It is known from [29, 48] that non-collapsed $\mathrm{RCD}(K,N)$ spaces have finer properties than that of general $\mathrm{RCD}(K,N)$ spaces. Note that any noncollapsed $\mathrm{RCD}(K,N)$ space has the rectifiable dimension N and that (5.9) is satisfied as $n = N$ by Bishop-Gromov inequality if it is compact.

In connection with this, we point out that the following three conditions for a compact $\mathrm{RCD}(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$ are equivalent.

(1) We have
$$\liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{N/2}} > 0.$$
(2) N is an integer with $\mathfrak{m} = c\mathcal{H}^N$ for some $c > 0$.
(3) N is an integer with
$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{N/2}} = \frac{\omega_N}{(2\pi)^{N/2}}\mathcal{H}^N(X) \in (0,\infty).$$

The nontrivial part is the implication from (1) to (2). If (1) holds, then [5, (5.5)] with (2.9) implies

$$0 < \liminf_{s\to 0^+} s^N \int_X p(x,x,s^2)d\mathfrak{m} \le C \liminf_{r\to 0^+} \int_X \frac{r^N}{\mathfrak{m}(B_r(x))} d\mathfrak{m} = C \int_X \lim_{r\to 0^+} \frac{r^N}{\mathfrak{m}(B_r(x))} d\mathfrak{m},$$

where the final equality comes from Bishop-Gromov inequality and the dominated convergence theorem. In particular

$$\mathfrak{m}\left(\left\{x \in X \middle| \lim_{r\to 0^+} \frac{r^N}{\mathfrak{m}(B_r(x))} > 0\right\}\right) > 0.$$

Thus we can apply [13, Theorem 1.3] or [41, Corollary 1.3] to prove that (2) holds.

Finally let us give an answer to a similar question for X as in Theorem 4.7, namely in the case of singular Weyl's law.

Proposition 5.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be as in Theorem 4.7. For all $0 < \delta < \epsilon < 1$ and any closed subset $A \subset X$ with $\mathcal{H}^k(\partial A \cap \mathcal{S}) = 0$ and $\mathcal{H}^k(A \cap \mathcal{S}) \le \delta \mathcal{H}^k(\mathcal{S})$, letting*

$$m_{A,\epsilon}(\lambda) = \sharp \left\{ j \in \mathbb{N} \middle| \int_{X\setminus A} f_j^2 d\mathfrak{m} \ge (1-\epsilon) \int_X f_j^2 d\mathfrak{m} \text{ and } \lambda_j \le \lambda \right\} \text{ for any } \lambda > 0, \tag{5.22}$$

we have

$$m_{A,\epsilon}(\lambda) \sim \lambda^{k/2}, \quad \text{as } \lambda \to \infty. \tag{5.23}$$

In particular

$$\int_{X\setminus A} f_j^2 d\mathfrak{m} \ge (1-\epsilon) \int_X f_j^2 d\mathfrak{m} \tag{5.24}$$

is satisfied for infinitely many eigenfunctions f_j of $-\Delta$ on X.

Proof. The proof is similar to that of Theorem 5.6 via

$$\int_{X\setminus A} s^k \mathsf{H}(x,x,s^2) d\mathfrak{m} \to c\mathcal{H}^k(\mathcal{S} \setminus A). \tag{5.25}$$

□

Remark 5.10. It is a direct consequence of the proposition above that for any $0 < \epsilon < 1$ and any compact subset $A \subset \mathcal{R}_2$, we have

$$m_{A,\epsilon}(\lambda) \sim \lambda^{k/2} \tag{5.26}$$

which should be compared with [43, Theorem 1.1] referred at the beginning of the subsection.

We are also able to disucuss similar asymptotic behaviors of the eigenfunctions for X as in Theorem 4.9.

Proposition 5.11. *Let $(X, \mathsf{d}, \mathfrak{m})$ be as in Theorem 4.9. For all $0 < \delta < \epsilon < 1$ and any closed subset $A \subset X$ with $\mathcal{H}^2(\partial A \cap \mathcal{S}) = 0$ and $\mathcal{H}^2(A \cap \mathcal{S}) \le \delta \mathcal{H}^2(\mathcal{S})$, letting*

$$m_{A,\epsilon}(\lambda) = \sharp \left\{ j \in \mathbb{N} \middle| \int_{X\setminus A} f_j^2 d\mathfrak{m} \ge (1-\epsilon) \int_X f_j^2 d\mathfrak{m} \text{ and } \lambda_j \le \lambda \right\} \text{ for any } \lambda > 0, \tag{5.27}$$

we have

$$m_{A,\epsilon}(\lambda) \sim \lambda \log \lambda, \quad \text{as } \lambda \to \infty. \tag{5.28}$$

In particular

$$\int_{X\setminus A} f_j^2 d\mathfrak{m} \geq (1-\epsilon)\int_X f_j^2 d\mathfrak{m} \tag{5.29}$$

is satisfied for infinitely many eigenfunctions f_j of $-\Delta$ on X.

Proof. The proof is essentially same to that of Proposition 5.9. The only difference is to find the "$t/(-\log t), \lambda \log \lambda$" version of [5, Propositions 5.5 and 5.6] instead of "t^γ, λ^γ", respectively. Since the proofs of [5, Propositions 5.5 and 5.6] still work in this setting, we omit it. □

Appendix A. Proofs of Proposition 4.1 and Lemma 4.5

A.1. **Fundamental domain.** let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and some $N \in [1,\infty)$, and let $\pi : \widetilde{X} \to X$ be a connected covering space with the lifted metric $\tilde{\mathsf{d}}$ and measure $\tilde{\mathfrak{m}}$ and the deck transformation Γ. Then $(\widetilde{X}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}})$ be an $\mathrm{RCD}(K,N)$ space because of the same reason as in [59, Lemma 2.8]. Let us begin with recalling the following standard notion;

Definition A.1 (Fundamental domain)**.** An open connected subset D of $\widetilde{X}$ is said to be a *fundamental domain* (of the covering $\pi : \widetilde{X} \to X$) if
(1) $D \cap \gamma D = \emptyset$ holds for any $\gamma \in \Gamma$ with $\gamma \neq 1$, and
(2) $\widetilde{X} = \bigcup_{\gamma\in\Gamma} \gamma\overline{D}$.

To obtain a fundamental domain, we follow the standard construction of Dirichlet domain. We fix a point $\tilde{x} \in \widetilde{X}$. For every $\gamma \neq 1$ in Γ, we put

$$H_\gamma(\tilde{x}) = \{y \in \widetilde{X} | \tilde{\mathsf{d}}(y,\tilde{x}) < \tilde{\mathsf{d}}(y,\gamma x)\}.$$

Proposition A.2 (Dirichlet domain)**.** *For any $\tilde{x} \in \tilde{X}$, we define*

$$D(\tilde{x}) := \bigcap_{1\neq\gamma\in\Gamma} H_\gamma(\tilde{x}), \tag{A.1}$$

and call it the Dirichlet domain centered at $\tilde{x}$. Then $D(\tilde{x})$ is a fundamental domain.

The proof is essentially the same as the Riemannian case. We include the proof for readers' convenience.

Lemma A.3. *Let $\gamma \neq 1$ and let $\sigma : [0,L] \to \widetilde{X}$ be a minimal geodesic from $\tilde{x}$. If $\sigma(t_0)$ satisfies $\tilde{\mathsf{d}}(\sigma(t_0),\tilde{x}) = \tilde{\mathsf{d}}(\sigma(t_0),\gamma\tilde{x})$ for some $t_0 \in [0,L]$, then $\sigma(t) \in H_\gamma(\tilde{x})$ for all $t \in [0,t_0)$ and $\sigma(t) \in \widetilde{X} - H_\gamma(\tilde{x})$ for all $t \in (t_0,L]$.*

Proof. Let us prove the first statement by contradiction. If not, then there exists $s \in [0,t_0)$ such that $\tilde{\mathsf{d}}(\sigma(s),\tilde{x}) = \tilde{\mathsf{d}}(\sigma(s),\gamma\tilde{x})$. Then

$$\begin{aligned}\tilde{\mathsf{d}}(\gamma\tilde{x},\sigma(t_0)) = \tilde{\mathsf{d}}(\tilde{x},\sigma(t_0)) &= \tilde{\mathsf{d}}(\tilde{x},\sigma(s)) + \tilde{\mathsf{d}}(\sigma(s),\sigma(t_0)) \\ &= \tilde{\mathsf{d}}(\gamma\tilde{x},\sigma(s)) + \tilde{\mathsf{d}}(\sigma(s),\sigma(t_0)),\end{aligned}$$

which in particular proves that a minimal geodesic from $\sigma(t_0)$ to $\gamma\tilde{x}$ can be branched at $\sigma(s)$. This contradicts a fact that any minimal geodesic in X is nonbranching [28, Theorem 1.3].

Next, let us prove the second statement by contradiction again. If not, then there exists $s' \in (t_0, L]$ such that $\tilde{\mathsf{d}}(\sigma(s'), \tilde{x}) \le \tilde{\mathsf{d}}(\sigma(s'), \gamma\tilde{x})$. If the equality holds, then we have a contradiction by the first statement. Thus we have

$$\begin{aligned}\tilde{\mathsf{d}}(\sigma(s'), \gamma\tilde{x}) > \tilde{\mathsf{d}}(\sigma(s'), \tilde{x}) &= \tilde{\mathsf{d}}(\sigma(s'), \sigma(t_0)) + \tilde{\mathsf{d}}(\sigma(t_0), \tilde{x}) \\ &= \tilde{\mathsf{d}}(\sigma(s'), \sigma(t_0)) + \tilde{\mathsf{d}}(\sigma(t_0), \gamma\tilde{x}) \ge \tilde{\mathsf{d}}(\sigma(s'), \gamma\tilde{x}),\end{aligned}$$

which is also a contradiction. □

Proof of Proposition A.2. We first prove the claim below.

Claim 1: The family of closed sets $\{\widetilde{X} - H_\gamma(\tilde{x})\}_{\gamma\neq 1}$ is locally finite. In fact, let $y \in \widetilde{X}$ and let $r = d(y, \tilde{x})$. Note that for all $\gamma \neq 1$ and all $z \in (\widetilde{X} - H_\gamma(\tilde{x})) \cap B_1(y)$,

$$\begin{aligned}\tilde{\mathsf{d}}(z, \gamma y) \le& \tilde{\mathsf{d}}(z, y) + \mathsf{d}(y, \gamma\tilde{x}) + \tilde{\mathsf{d}}(\gamma\tilde{x}, \gamma y) \\ <& 1 + \tilde{\mathsf{d}}(y, \tilde{x}) + \tilde{\mathsf{d}}(\tilde{x}, y) = 1 + 2r.\end{aligned}$$

This proves that

$$(\widetilde{X} - H_\gamma(\tilde{x})) \cap B_1(y) \subset B_{1+2r}(\gamma y) \cap B_1(y).$$

Because there are only finitely many γy in $B_{2+2r}(y)$, the right hand of the above inclusion relation is non-empty only for finitely many γ. Claim 1 follows.

It follows from Claim 1 that $D(\tilde{x})$ is an open subset of $\widetilde{X}$.

By Lemma A.3, for every $y \in D(\tilde{x})$, there is a minimal geodesic σ from $\tilde{x}$ to y such that σ is contained in $H_\gamma(\tilde{x})$ for all $\gamma \neq 1$, thus in $D(\tilde{x})$. This shows that $D(\tilde{x})$ is connected.

Next, we define a set of representative points, denoted as F: for each orbit Γy, we choose a point in the orbit such that it is the closest to $\tilde{x}$.

Claim 2: $D(\tilde{x}) \subset F \subset \overline{D(\tilde{x})}$. In fact, by the definition of $D(\tilde{x})$, any $y \in D(\tilde{x})$ satisfies

$$\tilde{\mathsf{d}}(y, \tilde{x}) < \tilde{\mathsf{d}}(y, \gamma\tilde{x}) = \tilde{\mathsf{d}}(\gamma^{-1}y, \tilde{x})$$

for all $\gamma \neq 1$. Thus $y \in F$ and we see that $D(\tilde{x}) \subset F$. We prove the other inclusion. Let $y \in F$ and let $\sigma : [0, L] \to \widetilde{X}$ be a minimal geodesic from $\tilde{x}$ to y. Note that

$$\tilde{\mathsf{d}}(y, \tilde{x}) \le \tilde{\mathsf{d}}(\gamma^{-1}y, \tilde{x}) = \tilde{\mathsf{d}}(y, \gamma\tilde{x})$$

for all $\gamma \neq 1$. By Lemma A.3, the image of $\sigma|_{[0,L)}$ is contained in $H_\gamma(\tilde{x})$ for all $\gamma \neq 1$, thus in $D(\tilde{x})$. Hence $y = \sigma(L) \in \overline{D(\tilde{x})}$. This proves Claim 2.

With Claim 2, now we prove that $D(\tilde{x})$ is a fundamental domain. To prove (1) in Definition A.1, suppose that there is $\gamma \neq 1$ and $y_1, y_2 \in D(\tilde{x})$ such that $\gamma y_1 = y_2$. By Claim 2, both y_1, y_2 belong to F. Since the orbit Γy_1 only has one point in F, we must have $y_1 = y_2$, that is, $\gamma y_1 = y_1$. This contradicts the facts that Γ-action is free and $\gamma \neq 1$. To prove (2) in Definition A.1, note that

$$\widetilde{X} = \bigcup_{\gamma\in\Gamma} \gamma F \subset \bigcup_{\gamma\in\Gamma} \gamma\overline{D(\tilde{x})}.$$

□

Definition A.4. A fundamental domain D of the covering $\pi : (\widetilde{X}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}}) \to (X, d, \mathfrak{m})$ is said to be a *metric measure fundamental domain* if $\tilde{\mathfrak{m}}(\partial D) = 0$.

Since $\overline{\gamma D} \cap \overline{D} \subset \partial D$ holds for any $\gamma \neq 1$ and any fundamental domain D, we have

$$\int_X f d\mathfrak{m} = \int_{\widetilde{X}} \tilde{f} d\tilde{\mathfrak{m}} \tag{A.2}$$

for any nonnegative function $\tilde{f} : \widetilde{X} \to [0, \infty]$ if D is a metric measure fundamental domain, where $f(x) = \sum_{\gamma \in \Gamma} \tilde{f}(\gamma \tilde{x})$ and $\pi(\tilde{x}) = x$. The proof of (A.2) is as follows.

Fistly, since π is a local isomorphism as metric measure spaces, we have $\mathfrak{m}(X \setminus \pi(D)) = 0$. Then the monotone convergence theorem allows us to compute

$$\begin{aligned}\int_X f d\mathfrak{m} = \int_{\pi(D)} f d\mathfrak{m} = \int_{\pi(D)} \sum_{\gamma \in \Gamma} \tilde{f}(\gamma \tilde{x}) d\tilde{\mathfrak{m}}(\tilde{x}) &= \sum_{\gamma \in \Gamma} \int_D \tilde{f}(\gamma \tilde{x}) d\tilde{\mathfrak{m}}(\tilde{x}) \\ &= \sum_{\gamma \in \Gamma} \int_{\gamma D} \tilde{f}(\tilde{x}) d\tilde{\mathfrak{m}}(\tilde{x}) = \int_{\widetilde{X}} \tilde{f} d\tilde{\mathfrak{m}}.\end{aligned} \tag{A.3}$$

This argument will play a role in the next subsection.

Remark A.5. A minimal geodesic $\sigma : [-\epsilon, \epsilon] \to \widetilde{X}$ in $\widetilde{X}$ is said to be *strongly nonbranching* if a minimal geodesic $\eta : [-\delta, \delta] \to X$ for some $\delta < \epsilon$ satisfy $\sigma(0) = \eta(0)$ and $\angle \dot{\sigma}\dot{\eta}(0) = 0$, then $\sigma = \eta$ on $[-\delta, \delta]$ (see [39, 40] for the definition of angles). Then if any minimal geodesic in $\widetilde{X}$ is strongly nonbranching, then a Dirichlet domain is a metric measure fundamental domain. The proof of this fact is as follows.

Since the boundary of $D(\tilde{x})$ is included in

$$\bigcup_{1 \neq \gamma \in \Gamma} \left\{ z \in \widetilde{X} \middle| \tilde{\mathsf{d}}(z, \tilde{x}) = \tilde{\mathsf{d}}(z, \gamma \tilde{x}) \right\}, \tag{A.4}$$

it is enough to prove that the set (A.4) is $\tilde{\mathfrak{m}}$-negligible. If not, then there exists $\gamma \neq 1$ such that

$$\tilde{\mathfrak{m}} \left(\left\{ z \in \widetilde{X} \middle| \tilde{\mathsf{d}}(z, \tilde{x}) = \tilde{\mathsf{d}}(z, \gamma \tilde{x}) \right\} \setminus (\mathrm{Cut}(\tilde{x}) \cup \mathrm{Cut}(\gamma \tilde{x})) \right) > 0 \tag{A.5}$$

because the proof of [37, Lemma 3.1] allows us to conclude that the cut locus $\mathrm{Cut}(\tilde{z})$ of any point $\tilde{z}$ in $\tilde{X}$ is $\tilde{\mathfrak{m}}$-negligible. On the other hand the locality of the minimal relaxed slope yields

$$|\nabla(\tilde{\mathsf{d}}_{\tilde{x}} - \tilde{\mathsf{d}}_{\gamma \tilde{x}})|^2 = 0, \quad \text{for } \tilde{\mathfrak{m}}\text{-a.e. } \tilde{z} \in \left\{ z \in \widetilde{X} \middle| \tilde{\mathsf{d}}(z, \tilde{x}) = \tilde{\mathsf{d}}(z, \gamma \tilde{x}) \right\}, \tag{A.6}$$

where $\tilde{\mathsf{d}}_{\tilde{z}}$ denotes the distance function from $\tilde{z}$. In particular (A.5) and (A.6) imply that there exists $\widetilde{z} \in X \setminus (\mathrm{Cut}(\tilde{x}) \cup \mathrm{Cut}(\gamma \tilde{x}))$ such that all two minimal geodesics σ, η from $\tilde{z}$ to $\tilde{x}$, $\gamma \tilde{x}$ respectively satisfy $\angle \dot{\sigma}\dot{\eta}(0) = 0$ which contradicts the strongly nonbranching property. Thus $\tilde{\mathfrak{m}}(\partial D(\tilde{x})) = 0$.

Note that this strongly nonbranching property does not follow the nonbranching property established in [28, Theorem 1.3].

Remark A.6. We can easily see that Proposition A.2 can be generalized to geodesic metric spaces with nonbranching property. It is worth mentioning that any Alexandrov space satisfies the strongly nonbranching property by definition. Thus in particular all of the above discussion can be applied to Alexandrov spaces with any Borel measure.

Remark A.7. A result in a very recent preprint [67] proves that the right hand side of (A.4) is actually $\mathfrak{m}$-negligible without assuming strongly nonbranching property, thus the Dirichlet domain centered at any point is actually a metric measure fundamental domain. See [67, Theorem 4].

A.2. **Proof of Proposition 4.1; the heat kernel on a covering space.** We continue our discussion with the same setting as in the previouse subsection.

For the simplicity of notation we give a proof of Proposition 4.1 only in the case when $K = 0$, however it is easy to generalize it to the general case when $K < 0$ via the Li-Yau inequalities (2.9), (2.10) and (2.11). The proof is divided into several lemmas as follows.

For any $y \in X$, take (a sufficiently small) $r(y) > 0$ satisfying that the preimage $\pi^{-1}(B_s(y))$ consists of the pairwise disjoint union of $\{B_s(\gamma\tilde{y})\}_{\gamma\in\Gamma}$ for any $s \in (0, r(y)]$, where $\pi(\tilde{y}) = y$. Without loss of generality we can assume

$$\inf_{y\in A} r(y) > 0, \quad \text{for any bounded subset } A \subset X. \tag{A.7}$$

Let us recall our main target; for any $t > 0$, define a symmetric function on $X \times X$;

$$\mathsf{H}_1(x, y, t) := \sum_{\gamma\in\Gamma} \mathsf{H}_{\widetilde{X}}(\tilde{x}, \gamma\tilde{y}, t) \in (0, \infty], \tag{A.8}$$

where $x, y \in X$ and $\tilde{x}, \tilde{y} \in \widetilde{X}$ with $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$. Note that the right hand side of (A.8) does not depend on the choices of $\tilde{x}, \tilde{y}$, thus it is well-defined.

Lemma A.8. H_1 *is finite. More precisely for any* $s \in (0, r(y)]$, *we have*

$$\begin{aligned}\mathfrak{m}(B_s(y))\mathsf{H}_1(x, y, t) &\le C(N)\exp\left(\frac{s^2}{4t}\right)\int_{\widetilde{X}} \mathsf{H}_{\widetilde{X}}(\tilde{x}, \tilde{z}, 2t)\,d\tilde{\mathfrak{m}}(\tilde{z}) \\ &\le C(N)\exp\left(\frac{s^2}{4t}\right)\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x})).\end{aligned} \tag{A.9}$$

Proof. Applying the parabolic Harnack inequality proved in [34, Theorem 1.4] and in [44, Theorem 1.3] implies

$$\tilde{\mathfrak{m}}(B_s(\tilde{y}))\mathsf{H}_{\widetilde{X}}(\tilde{x}, \tilde{y}, t) \le C(N)\exp\left(\frac{s^2}{4t}\right)\int_{B_s(\tilde{y})} \mathsf{H}_{\widetilde{X}}(\tilde{x}, \tilde{z}, 2t)\,d\tilde{\mathfrak{m}}(\tilde{z}).$$

Thus

$$\begin{aligned}\mathfrak{m}(B_s(y))\mathsf{H}_1(x, y, t) &= \sum_{\gamma\in\Gamma} \tilde{\mathfrak{m}}(B_s(\gamma\tilde{y}))\mathsf{H}_{\widetilde{X}}(\tilde{x}, \gamma\tilde{y}, t) \\ &\le C(N)\exp\left(\frac{s^2}{4t}\right)\int_{\widetilde{X}} \mathsf{H}_{\widetilde{X}}(\tilde{x}, \tilde{z}, 2t)\,d\tilde{\mathfrak{m}}(\tilde{z}).\end{aligned}$$

The final inequality in (A.9) comes from the Li-Yau inequality (2.9), Bishop-Gromov inequality and [13, Lemma 2.7]. □

Corollary A.9. *If* $s \in (0, r(y)]$, *then*

$$\sum_{\gamma\in\Gamma} \exp\left(-\frac{\tilde{\mathsf{d}}(\tilde{x}, \gamma\tilde{y})^2}{3t}\right) \le C(N)\exp\left(\frac{s^2}{4t}\right)\frac{\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x}))^2}{\mathfrak{m}(B_s(y))}. \tag{A.10}$$

Proof. Since the Li-Yau inequality (2.9) implies

$$\mathsf{H}_1(x,y,t)=\sum_{\gamma\in\Gamma}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)\geq\frac{C(N)}{\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x}))}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\tilde{\mathsf{d}}(\tilde{x},\gamma\tilde{y})^2}{3t}\right),$$

we have (A.10) because of (A.9). □

In order to continue the proof of Proposition 4.1, let us recall local notions of Sobolev functions and Laplacian.

Definition A.10 (Local Sobolev and Laplacian)**.** Let us denote by $H^{1,2}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ the set of all $f\in L^2_{\mathrm{loc}}(X,\mathfrak{m})$ satisfying that $\phi f\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ for any $\phi\in\mathrm{Lip}_c(X,\mathsf{d})$ and that $|\nabla f|\in L^2_{\mathrm{loc}}(X,\mathfrak{m})$, where $|\nabla f|$ makes sense on X because of the locality of the minimal relazed slope, where $\mathrm{Lip}_c(X,\mathsf{d})$ denotes the set of all Lipschitz functions on X with compact supports. Moreover we denote by $D_{\mathrm{loc}}(\Delta)$ the set of all $f\in H^{1,2}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ satisfying that there exists a (unique) $\psi\in L^2_{\mathrm{loc}}(X,\mathfrak{m})$, denoted by $\psi=\Delta f$ if no confusion, such that

$$\int_X\langle\nabla f,\nabla\phi\rangle d\mathfrak{m}=-\int_X\psi\phi d\mathfrak{m},\quad\text{for any }\phi\in\mathrm{Lip}_c(X,\mathsf{d}).\tag{A.11}$$

The stability properties of local Sobolev functions and of local Laplacian can be found in [4].

Lemma A.11. H_1 *is locally Lipschitz continuous. Moreover for all* $y\in X$ *and* $t>0$, $\mathsf{H}_1(x,y,t)$ *viewed as a function of* x *on* X*, is in* $D_{\mathrm{loc}}(\Delta)$ *with*

$$\Delta_x\mathsf{H}_1(x,y,t)=\sum_{\gamma\in\Gamma}\Delta_x\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma y,t)=\sum_{\gamma\in\Gamma}\frac{d}{dt}\mathsf{H}_{\widetilde{X}}(x,\gamma y,t),\quad\text{in }L^2(X,\mathfrak{m}).\tag{A.12}$$

Proof. Let A be a finite subset of Γ. Then for any $s\in(0,r(y)]$ by (2.10) and Corollary A.9 we have

$$\begin{aligned}\left|\nabla_{\tilde{x}}\sum_{\gamma\in A}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)\right|&\leq\sum_{\gamma\in\Gamma}|\nabla_{\tilde{x}}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)|\\&\leq\frac{C(N)}{\sqrt{t}\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x}))}\sum_{\gamma\in\Gamma}\exp\left(-\frac{\tilde{\mathsf{d}}(\tilde{x},\gamma\tilde{y})^2}{5t}\right)\\&\leq\frac{C(N)}{\sqrt{t}\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x}))}\cdot\exp\left(\frac{s^2}{t}\right)\cdot\frac{\tilde{\mathfrak{m}}(B_{\sqrt{5t/3}}(\tilde{x}))^2}{\mathfrak{m}(B_s(y))}\\&\leq C(N)\exp\left(\frac{s^2}{t}\right)\frac{\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x}))}{\sqrt{t}\mathfrak{m}(B_s(y))}\end{aligned}\tag{A.13}$$

which gives an equi-local Lipschitz continuity of $\{\sum_{\gamma\in A}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)\}_{A\subseteq\Gamma}$ on $\tilde{X}\times\tilde{X}\times(0,\infty)$ because of (A.7). Thus letting $A\uparrow\Gamma$ completes the proof of the desired local Lipschitz continuity.

The remaining statements come from the dominated convergence theorem, the stability of the Laplacian and an estimate;

$$\left|\frac{d}{dt}\sum_{\gamma\in A}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)\right|\leq C(N)\exp\left(\frac{s^2}{t}\right)\cdot\frac{\tilde{\mathfrak{m}}(B_{\sqrt{t}}(\tilde{x}))}{t\mathfrak{m}(B_s(y))}$$

which is proved by a similar way as in (A.13) via (2.11). □

Lemma A.12. *For all $x \in X$ and $t > 0$, $\mathsf{H}_1(x, y, t)$ viewed as a function of y on X, is in $D(\Delta)$ if some fundamental domain D of X has the $\mathfrak{m}$-negligible topological boundary.*

Proof. Since

$$\begin{aligned}
\int_X \mathsf{H}_1(x,y,t)^2 d\mathfrak{m}(y) &= \sum_{\gamma_1,\gamma_2\in\Gamma}\int_D \mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_1\tilde{y},t)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y}) \\
&= \sum_{\gamma_1,\gamma_2\in\Gamma}\int_D \mathsf{H}_{\widetilde{X}}(\gamma_1^{-1}\tilde{x},\tilde{y},t)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y}) \\
&= \sum_{\gamma_1,\gamma_2\in\Gamma}\int_D \mathsf{H}_{\widetilde{X}}(\gamma_2\gamma_1^{-1}\tilde{x},\gamma_2\tilde{y},t)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y}) \\
&= \sum_{\gamma_2\in\Gamma}\left(\int_D\left(\sum_{\gamma_1\in\Gamma}\mathsf{H}_{\widetilde{X}}(\gamma_2\gamma_1^{-1}\tilde{x},\gamma_2\tilde{y},t)\right)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y})\right) \\
&= \sum_{\gamma_2\in\Gamma}\left(\int_D\left(\sum_{\gamma\in\Gamma}\mathsf{H}_{\widetilde{X}}(\gamma\tilde{x},\gamma_2\tilde{y},t)\right)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y})\right) \\
&= \sum_{\gamma\in\Gamma}\left(\sum_{\gamma_2\in\Gamma}\int_{X_1}\mathsf{H}_{\widetilde{X}}(\gamma\tilde{x},\gamma_2\tilde{y},t)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y})\right) \\
&= \sum_{\gamma\in\Gamma}\int_{\widetilde{X}}\mathsf{H}_{\widetilde{X}}(\gamma\tilde{x},\tilde{y},t)\mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y}) \\
&= \mathsf{H}_1(\tilde{x},\tilde{x},2t) < \infty, \qquad \text{(A.14)}
\end{aligned}$$

we have $\mathsf{H}_1 \in L^2(X, \mathfrak{m})$. On the other hand,

$$\begin{aligned}
\int_X |\nabla_y \mathsf{H}_1|^2(x,y,t)d\mathfrak{m}(y) &= \int_D \left\langle \nabla_{\tilde{y}}\sum_{\gamma\in\Gamma}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t), \nabla_{\tilde{y}}\sum_{\gamma\in\Gamma}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)\right\rangle d\tilde{\mathfrak{m}}(\tilde{y}) \\
&\le \sum_{\gamma_1,\gamma_2\in\Gamma}\int_D \left|\nabla_{\tilde{y}}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_1\tilde{y},t)\right|\cdot\left|\nabla_{\tilde{y}}\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},t)\right| d\tilde{\mathfrak{m}}(\tilde{y}). \qquad \text{(A.15)}
\end{aligned}$$

Since the Li-Yau inequalities (2.9) and (2.10) show for $\tilde{x}, \tilde{y} \in \tilde{X}$

$$|\nabla_{\tilde{y}}\mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},t)| \le \frac{C(N)}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(\tilde{y}))}\exp\left(-\frac{\tilde{\mathsf{d}}(\tilde{x},\tilde{y})^2}{5t}\right) \le \frac{C(N)}{\sqrt{t}}\mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},2t),$$

the right hand side of (A.15) is bounded above by

$$\frac{C(N)}{t}\sum_{\gamma_1,\gamma_2\in\Gamma}\int_D \mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_1\tilde{y},2t)\mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma_2\tilde{y},2t)d\tilde{\mathfrak{m}}(\tilde{y})$$

which is also bounded above by $C(N)t^{-1}\mathsf{H}_1(x,x,4t)$ because of (A.14). Thus $\mathsf{H}_1 \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Similarly, using Lemma A.11 with

$$|\Delta_{\tilde{y}}\mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},t)| \le C(N)t^{-1}\mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},2t)$$

which is justified by the Li-Yau inequalities (2.9) and (2.11) (or [27, Theorem 3]), we have

$$\int_X (\Delta_y \mathsf{H}_1(x,y,t))^2 \, d\mathfrak{m}(y) \le \frac{C(N)}{t^2}\mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{x},4t) < \infty.$$

Thus $\mathsf{H}_1 \in D(\Delta)$. □

From now on we assume that some fundamental domain D has the $\mathfrak{m}$-negligible topological boundary (see also Remark A.7). Note that

$$\int_X \mathsf{H}_1(x,y,t)d\mathfrak{m}(\tilde{y}) = \sum_{\gamma\in\Gamma}\int_D \mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y}) = \int_{\widetilde{X}} \mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},t)d\tilde{\mathfrak{m}}(\tilde{y}) = 1.$$

Then thanks to Lemma A.12, for all $f \in L^2(X,\mathfrak{m}), x \in X$ and $t > 0$,

$$\mathsf{h}_t^1 f(x) := \int_X \mathsf{H}_1(x,y,t)f(y)d\mathfrak{m}(y)$$

is well-defined.

Lemma A.13. *Let $f \in C_c(X)$. Then we have the following;*

(1) *$\mathsf{h}_t^1 f(x) = \mathsf{h}_t(f\circ\pi)(x)$ with $\mathsf{h}_t^1 f(x) \to f(x)$ as $t \to 0^+$ for any $x \in X$. In particular $\mathsf{h}_t^1 f$ L^2-locally strongly converge to f as $t \to 0^+$;*
(2) *for any $t > 0$, $\mathsf{h}_t^1 f \in D(\Delta)$ with $\frac{d}{dt}\mathsf{h}_t^1 f = \Delta \mathsf{h}_t^1 f$ in $L^2(X,\mathfrak{m})$.*

Proof. Since (2) is a direct consequence of Lemma A.11, let us check (1). By definition of $\tilde{X}$, we have

$$\begin{aligned} \mathsf{h}_t(f\circ\pi)(\tilde{x}) &= \int_{\widetilde{X}} \mathsf{H}_{\widetilde{X}}(\tilde{x},\tilde{y},t)f\circ\pi(\tilde{y})d\tilde{\mathfrak{m}}(\tilde{y}) \\ &= \sum_{\gamma\in\Gamma}\int_D \mathsf{H}_{\widetilde{X}}(\tilde{x},\gamma\tilde{y},t)f\circ\pi(\tilde{y})d\tilde{\mathfrak{m}}(\tilde{y}) \\ &= \int_X \mathsf{H}_1(x,y,t)f(y)d\mathfrak{m}(y) = \mathsf{h}_t^1 f(x). \end{aligned} \tag{A.16}$$

On the other hand since $f\circ\pi \in C_b(Y)$, thanks to [58, Lemma 2.54], we know $\mathsf{h}_t(f\circ\pi)(x)$ converges to $f\circ\pi(x)$, namely $\mathsf{h}_t^1 f(x) \to f(x)$. The final statement of (1) comes from this with the dominated convergence theorem and a fact $|\mathsf{h}_t^1 f(x)| \le \sup|f|$. □

As an immediate consequence of Lemma A.13 (1), we have $\mathsf{h}_{t_i}^1 f_i(x) \to f(x)$ for all $x \in X, t_i \to 0^+$ and uniform convergent sequence $f_i \to f$ in $C_c(X)$. We are now in a position to prove Proposition 4.1.

Lemma A.14. *We have $\mathsf{H}_1 = \mathsf{H}_X$. Namely Proposition 4.1 holds.*

Proof. Fix $w \in X$. For any $R \ge 1$, take a Lipschitz cut-off $\phi_R : X \to [0,1]$ with $\phi_R \equiv 1$ on $B_R(w)$, $\operatorname{supp}\phi_R \subset B_{2R}(w)$ and $|\nabla\phi_R| \le 2R^{-1}$. Then Lemma A.13 (1) with the continuity of H_1 allows us to show

$$\mathsf{H}(x,y,t) - \mathsf{H}_1(x,y,t) = \int_0^t \frac{d}{ds}\int_X \phi_R(z)^2 \mathsf{H}_1(x,z,t-s)\mathsf{H}_X(z,y,s)d\mathfrak{m}(z)ds. \tag{A.17}$$

Moreover, Lemma A.13 (2) also allows us to prove that the right hand side of (A.17) is equal to

$$
\begin{aligned}
&-\int_0^t\int_X \phi_R(z)^2\Delta_z\mathsf{H}_1(x,z,t-s)\cdot\mathsf{H}_X(z,y,s)d\mathfrak{m}(z)ds\\
&+\int_0^t\int_X \phi_R(z)^2H_1(x,z,t-s)\Delta_z\mathsf{H}_X(z,y,s)d\mathfrak{m}(z)ds, \qquad \text{(A.18)}
\end{aligned}
$$

namely, $\mathsf{H}-\mathsf{H}_1$ is equal to

$$
\begin{aligned}
&2\int_0^t\int_X \phi_R(z)\mathsf{H}_X(z,y,s)\left\langle\nabla_z\mathsf{H}_1(x,z,t-s),\nabla\phi_R(z)\right\rangle d\mathfrak{m}(z)ds\\
&-2\int_0^t\int_X \phi_R(z)H_1(x,z,t-s)\left\langle\nabla_z\mathsf{H}_X(z,y,s),\nabla\phi_R(z)\right\rangle d\mathfrak{m}(z)ds. \qquad \text{(A.19)}
\end{aligned}
$$

On the other hand the first term of (A.19) can be estimated as;

$$
\begin{aligned}
&2\left|\int_0^t\int_X \phi_R(z)\mathsf{H}_X(z,y,s)\left\langle\nabla_z\mathsf{H}_1(x,z,t-s),\nabla\phi_R(z)\right\rangle d\mathfrak{m}(z)ds\right|\\
&\le\frac{4}{R}\int_0^t\left(\int_{B_{2R}(w)\setminus B_R(w)}\mathsf{H}_X^2(z,y,s)d\mathfrak{m}(z)\right)^{\frac{1}{2}}\cdot\left(\int_{B_{2R}(w)\setminus B_R(w)}|\nabla_z\mathsf{H}_1(x,z,t-s)|^2d\mathfrak{m}(z)\right)^{\frac{1}{2}}ds\\
&\text{(A.20)}\\
&\le\frac{4}{R}\int_0^t\left(\int_{X\setminus B_R(w)}\mathsf{H}_X^2(z,y,s)d\mathfrak{m}(z)\right)^{\frac{1}{2}}\cdot\left(\int_{X\setminus B_R(w)}|\nabla_z\mathsf{H}_1(x,z,t-s)|^2d\mathfrak{m}(z)\right)^{\frac{1}{2}}ds.
\end{aligned}
$$

Thus letting $R\to\infty$ with the monotone convergence theorem and Lemma A.12 shows that the first term of (A.19) converges to 0. Similarly we see that under the same limit of $R\to\infty$, the second term of (A.19) also converges to 0. Thus letting $R\to\infty$ in (A.17), we have $\mathsf{H}=\mathsf{H}_1$. □

A.3. **Proof of Lemma 4.5; Duhamel principle.** Since the proof is essentially similar to the discussions in the previous subsection, let us provide only an outline of the proof because this situation is simpler than the previous one since the space $\widetilde{Y}$ is compact. Firstly let us denote by H_2 the right hand side of (4.6). Then it is trivial that H_2 is locally Lipschitz on $\widetilde{Y}\times\widetilde{Y}\times(0,\infty)$ because $\mathsf{H}_{\bar{Y}}$ and $\mathsf{H}_{\widetilde{Y}}$ are locally Lipschitz. Thus in particular for all $f\in L^2(\widetilde{Y},\tilde{\mathfrak{m}}), x\in X$ and $t>0$,

$$
\mathsf{h}_t^2f(x):=\int_{\widetilde{Y}}\mathsf{H}_2(x,y,t)f(y)d\tilde{\mathfrak{m}}(y) \qquad \text{(A.21)}
$$

is well-defined. Recall (4.5);

$$
|\mathsf{Q}(z,y,t)|\le\exp\left(-\frac{C}{t}\right). \qquad \text{(A.22)}
$$

In particular for any $f\in C(\tilde{Y})$ we have

$$
\lim_{s\to0^+}\int_{\widetilde{Y}}f(z)\mathsf{Q}(z,y,s)d\tilde{\mathfrak{m}}(z)=0 \qquad \text{(A.23)}
$$

and

$$
\lim_{t\to0^+}\mathsf{h}_t^2f(x)=\int_{\widetilde{Y}}\mathsf{K}(x,y,t)f(y)d\tilde{\mathfrak{m}}(y)=f(x), \qquad \text{(A.24)}
$$

where the final equality in (A.24) comes from the definition of K.

Then recalling a formula;

$$\frac{d}{dt}\int_0^t \phi(t,s)ds = \int_0^t \frac{d}{dt}\phi(t,s)ds + \phi(t,s), \tag{A.25}$$

we have

$$\begin{aligned}
\frac{d}{dt}\mathsf{H}_2(x,y,t) &= \frac{d}{dt}\mathsf{K}(x,y,t) - \int_0^t \int_{\widetilde{Y}} \mathsf{H}_{\widetilde{Y}}(x,z,s)\frac{d}{dt}\mathsf{Q}(z,y,t-s)d\tilde{\mathfrak{m}}(z)ds \\
&\quad - \lim_{u\to 0^+}\int_{\widetilde{Y}} \mathsf{H}_{\widetilde{Y}}(x,z,t-u)\mathsf{Q}(z,y,u)d\tilde{\mathfrak{m}}(z) \\
&= \frac{d}{dt}\mathsf{K}(x,y,t) + \int_0^t \int_{\widetilde{Y}} \mathsf{H}_{\widetilde{Y}}(x,z,s)\frac{d}{ds}\mathsf{Q}(z,y,t-s)d\tilde{\mathfrak{m}}(z)ds \\
&= \frac{d}{dt}\mathsf{K}(x,y,t) + \lim_{u\to 0}\int_{\widetilde{Y}} \mathsf{H}_{\widetilde{Y}}(x,z,t-u)\mathsf{Q}(z,y,u)d\tilde{\mathfrak{m}}(z) - \mathsf{Q}(x,y,t) \\
&\quad - \int_{\widetilde{Y}}\int_0^t \frac{d}{ds}\mathsf{H}_{\widetilde{Y}}(x,z,s)\cdot \mathsf{Q}(z,y,t-s)dsd\tilde{\mathfrak{m}}(z) \\
&= \frac{d}{dt}\mathsf{K}(x,y,t) - \mathsf{Q}(x,y,t) - \int_{\widetilde{Y}}\int_0^t \frac{d}{ds}\mathsf{H}_{\widetilde{Y}}(x,z,s)\cdot \mathsf{Q}(z,y,t-s)dsd\tilde{\mathfrak{m}}(z) \\
&= \Delta_x\mathsf{K}(x,y,t) - \int_{\widetilde{Y}}\int_0^t \Delta_x\mathsf{H}_{\widetilde{Y}}(x,z,s)\cdot \mathsf{Q}(z,y,t-s)dsd\tilde{\mathfrak{m}}(z) \\
&= \Delta_x\mathsf{H}_2(x,y,t). \qquad\qquad \text{(A.26)}
\end{aligned}$$

In particular combining (A.26) with (A.24) shows that for any $f \in C(\widetilde{Y})$, we have $\frac{d}{dt}\mathsf{h}^2_t f = \Delta \mathsf{h}^2_t f$ and $\mathsf{h}^2_t f \to f$ as $t \to 0^+$ in $L^2(\widetilde{Y}, \tilde{\mathfrak{m}})$ because of the dominated convergence theorem with $|\mathsf{h}^2_t f| \le \sup|f|$ by definition. In particular $\mathsf{h}^2_t f$ coincides with the heat flow $\tilde{\mathsf{h}}_t f$ on $\widetilde{Y}$.

In order to conclude that $\mathsf{H}_2 = \mathsf{H}_{\widetilde{Y}}$, it is enough to check $\mathsf{h}^2_t f = \tilde{\mathsf{h}}_t f$ for any $f \in L^2(\widetilde{Y}, \tilde{\mathfrak{m}})$ because of the continuity of H_2. Fix $f \in L^2(\widetilde{Y}, \tilde{\mathfrak{m}})$ and take a sequence $f_i \in C(\widetilde{Y})$ with $f_i \to f$ in $L^2(\widetilde{Y}, \tilde{\mathfrak{m}})$. It is known that $\tilde{\mathsf{h}}_t f_i \to \tilde{\mathsf{h}}_t f$ in $L^2(\widetilde{Y}, \tilde{\mathfrak{m}})$ in general. In particular after passing to a subsequence, $\tilde{\mathsf{h}}_t f_i(x) \to \tilde{\mathsf{h}}_t f(x)$ as $i \to \infty$ for $\tilde{\mathfrak{m}}$-a.e. $x \in \widetilde{Y}$. Finally since

$$\begin{aligned}
\left|\mathsf{h}^2_t f_i(x) - \mathsf{h}^2_t f(x)\right| &\le \int_{\widetilde{Y}} \mathsf{H}_2(x,y,t)\,|f_i(y) - f(y)|\,d\tilde{\mathfrak{m}}(y) \\
&\le \left(\int_{\bar{Y}} \mathsf{H}_2(x,y,t)^2 d\tilde{\mathfrak{m}}(y)\right)^{1/2} \cdot \|f_i - f\|_{L^2} \to 0, \qquad \text{(A.27)}
\end{aligned}$$

we have $\mathsf{h}^2_t f(x) = \tilde{\mathsf{h}}_t f(x)$ for $\tilde{\mathfrak{m}}$-a.e. $x \in \widetilde{Y}$ because of $\mathsf{h}^2_t f_i(x) = \tilde{\mathsf{h}}_t f_i(x)$. Thus we have Lemma 4.5.

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