SINGULAR METRICS WITH NONNEGATIVE SCALAR CURVATURE AND RCD

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Abstract. We show that a uniformly Euclidean metric with isolated singularity on closed $M^n = T^n \# M_0$, where $4 \le n \le 7$ or $n \ge 4$, M_0 spin, and nonnegative scalar curvature on the smooth part is flat and extends smoothly over the singularity. This confirms Schoen's Conjecture in these cases. The novel approach here, which is the key to the proof, is to show that the space has nonnegative synthetic Ricci curvature, i.e., an $RCD(0, n)$ space. Our result also holds when the singular set consists of a finite union of submanifolds (of possibly different dimensions) intersecting transversally under additional assumption on the co-dimension and the location of the singular set.

Contents

1. Introduction

The existence problem of Riemannian metrics with positive scalar curvature is a fundamental topic in Riemannian geometry. In the smooth setting, it has been well studied and many important results have been established. Kazdan-Warner [26] and Schoen [36] proved that on a closed manifold, there exists a smooth metric with positive scalar curvature iff its Yamabe constant (aka σ-constant or Schoen constant) is positive. Moreover, their rigidity result says that on a closed manifold with a nonpositive Yamabe constant,

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any metric with nonnegative scalar curvature must be Ricci flat. Recall that for a closed manifold M, its Yamabe constant $\sigma(M)$ is defined as

$$\sigma(M) := \sup\{Y(M,[g_0]) \mid [g_0] \text{ is a conformal class of smooth metrics on } M\},$$

where

$$Y(M,[g_0]) := \inf\left\{\int_M \mathrm{Sc}_g d\mathrm{vol}_g \mid g \in [g_0], \quad \mathrm{Vol}_g(M) = 1\right\}$$

and Sc_g denotes the scalar curvature of a smooth metric g. The Yamabe constant $\sigma(M)$ is a diffeomorphism invariant of M. For example, for a closed M^n, $\sigma(T^n\#M) \leq 0$ as a consequence of the Geroch conjecture.

Motivated by general relativity, Geroch's conjecture says that there is no smooth metric of positive scalar curvature in $T^n\#M$ (M closed). This was proved by Schoen-Yau [37, 38] using the minimal surface method and by Gromov-Lawson [18, 19] for spin manifolds using the Dirac operator method. Moreover, one has the rigidity result which says that the metric with nonnegative scalar curvature on such manifolds must be flat. Recently, these results have been generalized to allow M to be non-compact by Chodosh-Li [10] for dimensions ≤ 7, Wang-Zhang [45] for spin manifolds of arbitrary dimensions (see also [42]), and also Lesourd-Unger-Yau [29] in dimension three. The work of Chodosh-Li and Wang-Zhang will be instrumental in our paper.

For simply connected manifolds of dimension ≥ 5, the problem of what closed manifolds should admit metrics of positive scalar curvature is fully understood in terms of the second Stiefel–Whitney class (which determins whether the manifold is spin or not) and α-invariant [17, 41].

Inspired by the investigation of weak notions of nonnegative scalar curvature, e.g., Gromov's polyhedral comparison theory [20], and the positive mass theorem for singular metrics, it is natural to study the forementioned Kazdan-Warner result and Geroch conjecture for singular metrics. A natural class of such singular metrics is that of uniformly Euclidean metrics. Following Li-Mantoulidis [30], we say that a measurable symmetric 2-tensor g on a closed manifold M is a *uniformly Euclidean metric* (also called a L^∞ metric), if

$$\Lambda^{-1}\bar{g} \leq g \leq \Lambda\bar{g} \tag{1.1}$$

holds a.e. on M for some smooth metric $\bar{g}$ on M and constant $\Lambda > 0$. A point is a smooth point for g if there is an open neighborhood in which g is smooth. Otherwise, we will call it a singular point. The singular set is the collection of singular points; it is closed by definition as its complement, called the smooth part of g, is an open set. We say g has positive (nonnegative resp.) scalar curvature if its scalar curvature on the smooth part is positive (nonnegative resp.).

There are two fundamental questions in this direction. First of all, just like the smooth case, there is the existence/non-existence problem of singular uniformly Euclidean metrics with positive scalar curvature on a closed manifold with nonpositive Yamabe constant. In other words, the question is whether the singularity will affect the Yamabe type. Secondly, related to the rigidity part, a new challenge here is the smooth extension/removable singularity problem of uniformly Euclidean metrics with nonnegative scalar curvature. Note that once the metric can be smoothly extended over the singular set, the rigidity result follows directly from the Kazdan-Warner result.

For uniformly Euclidean metrics with conical singularities along a submanifold of codimension 2, Li-Mantoulidis [30] obtained an affirmative answer to both problems, provided the angle of the cone $\leq 2\pi$. They also provided a counterexample when the angle of the cone was $> 2\pi$. In the case of a singular set having codimension ≥ 3, Schoen has the following conjecture.

Conjecture 1.1 (Conjecture 1.5 in [30])**.** Suppose M is a closed smooth manifold whose Yamabe constant is nonpositive, and g is a uniformly Euclidean metric on M that is smooth away from a closed embedded submanifold $S \subset M$ with codimension ≥ 3. Then g extends smoothly to M, and hence is Ricci flat. In other words,

$$\begin{aligned}&\sigma(M) \leq 0, \quad \mathrm{Sc}_g \geq 0 \ \text{ on } \ M \setminus S, \text{ and } \mathrm{codim}(S) \geq 3 \\ &\Longrightarrow g \ \text{ extends smoothly to } \ M \ \text{ and } \ \mathrm{Ric}_g \equiv 0.\end{aligned}$$

The main result of this paper is a partial affirmative solution of this conjecture for manifolds that are connected sums with the torus.

Theorem 1.2. *Let $M^n = T^n \# M_0$, $n \geq 4$, be a connected closed manifold. Assume that $4 \leq n \leq 7$ or M_0 is spin. Suppose*
1). g is a uniformly Euclidean metric on M smooth away from a closed, embedded submanifold S with codimension $\geq \max(4, \frac{n}{2}+1)$.
2). There is a $(n-1)$-sphere that divides $M = T^n \# M_0$ into the T^n-part and M_0-part, so that each component of S lies either in the T^n-part or in the M_0-part. In addition, if a component of S lies in the torus part, we assume that it is contained in a topological ball.
3). The scalar curvature $\mathrm{Sc}_g \geq 0$ on $M \setminus S$.
Then g extends smoothly to M (possibly with a change of differential structure) and (M^n, g) is isometric to a flat torus.

In particular, this confirms Schoen's Conjecture for isolated singularity when $M^n = T^n \# M_0$, $4 \leq n \leq 7$ or $n \geq 4$ and M_0 is spin.

Li-Mantoulidis [30] proved Conjecture 1.1 for 3-dimensional M. They used a removable singularity result of Smith-Yang [40] for 3-dimensional Einstein metrics. On the other hand, very recently, Cecchini-Frenck-Zeidler [5] constructed counterexamples to Conjecture 1.1 on a simply connected closed manifold of dimension ≥ 8. In contrast, the conjecture could still be true on manifolds with certain topology. Dai-Sun-Wang [12] proved Conjecture 1.1 for metrics with isolated conical singularities on $T^n \# M_0$, including a smooth extension and rigidity result, provided that $3 \leq n \leq 7$ or M_0 is spin. For continuous metrics, Lee-Tam [28] proved a more general version of Schoen's conjecture by using the Ricci flow method.

Many authors have proved the non-existence and Ricci flatness on the smooth part for uniformly Euclidean metrics under various conditions. These include Kazaras [24] for four-dimensional enlargeable closed smooth manifolds, Cecchini-Frenck-Zeidler [5] for closed spin manifolds with the fundamental group satisfying a certain index theoretic condition, Wang-Xie [44] for manifolds admitting a Lipschitz map to T^n with nonzero degree, and Shi-Wang-Wu-Zhu [39] for closed Schoen-Yau-Schick manifolds.

Remark 1.3. As indicated in [5], one may need to change the differential structure when extending the metric. More precisely, here a smooth extension of a uniformly Euclidean metric g means that there exists a smooth Riemannian metric h on M and a homeomorphism $\psi : M \to M$, such that $\psi : (M, d_g) \to (M, d_h)$ is an isometry as metric spaces, and ψ is a diffeomorphism from $M \setminus S$ onto its image and $g = \psi^* h$ on $M \setminus S$. This will be the meaning of smooth extension in our general discussion. In Theorem 1.2, for isolated singularity, it can be strengthened further so that ψ can be taken to be the identity on the smooth part. This is because we prove that g is flat on the smooth part. We can further apply Theorem 3.1 in [40].

Remark 1.4. One may think of Theorem 1.2 as a removable singularity theorem. Unlike the well-known removable singularity results for Einstein metrics or Yang-Mills instantons, our result is not local but rely on the global topological structure. Our result also holds for singular sets that consist of a finite union of submanifolds (of possibly different dimensions) intersecting transversally.

A novel and crucial ingredient in the proof of Theorem 1.2 is the following result, which is of independent interest. It says that a uniformly Euclidean metric on a closed manifold smooth away from a closed singular set of codimension 4 or higher and with Ricci curvature bounded below by K on the smooth part is RCD(K, n), i.e., its Ricci curvature is globally bounded below by K in the weak sense.

Theorem 1.5. *Let M^n be a connected closed manifold of dimension $n \geq 4$, and g a uniformly Euclidean metric on M that is smooth away from a closed set S with codimension ≥ 4. If, for some constant K, the Ricci curvature $\mathrm{Ric}_g \geq K$ on the smooth part, then (M^n, g) is a* RCD(K, n) *space.*

As another interesting application, we obtain the following removable singularity result. In contrast to the local nature of the previous results of this type, our result is global, depending crucially on the global topological assumption.

Theorem 1.6. *Let M^n be a connected closed manifold of dimension $n \geq 4$ with infinite fundamental group. If g is a uniformly Euclidean metric on M with isolated singularity and nonnegative Ricci curvature on the smooth part, then (M^n, g) has a smooth extension in the sense as explained in Remark 1.3. Moreover, a finite cover $\hat{M}$ of M is homeomorphic to $T^k \times X$, for some $1 \leq k \leq n$ and X a closed simply connected manifold. Furthermore, there is a locally isometrically trivial fibration $\hat{M} \longrightarrow T^k$ with fiber X.*

The main technical component of the proof of Theorem 1.5 is a gradient estimate for eigenfunctions. It consists of two steps. First, using De Giorgi-Nash-Moser estimate and Cheng-Yau gradient estimate, we obtain an initial gradient estimate for eigenfunctions, which blows up at the singular part of g. Nevertheless, this enables us to establish an L^2 bound on the Hessian of the eigenfunctions, using the uniform Euclidean and codimension assumptions. Then, by Bochner formula and Kato's inequality, a differential inequality for the norm of the gradient of an eigenfunction is shown to hold over the whole manifold in the weak sense, thanks to the L^2 Hessian bound. The final gradient estimate then follows by the Moser iteration. This is similar in spirit to the strategy used in [33, 2] for stratified spaces. Most recently, this circle of ideas has been used by Honda-Sun [22] in more general settings.

This paper is organized as follows. In Section 2, we prove Theorem 1.5 and then apply it to prove Theorems 1.2. More precisely, in Subsection 2.1, we recall a definition of RCD space and describe the metric measure structure induced by a uniformly Euclidean metric as in Conjecture 1.1. We also prove that such metric measure spaces satisfy the infinitesimal Hilbertian condition and Sobolev-to-Lipschitz property. In Subsection 2.2, we derive a gradient estimate for the eigenfunctions of the Laplace operator, which plays an important role in the proof of the Bakry-Émery (BE, for short) condition for uniformly Euclidean metrics. Based on these, in Subsection 2.3, we prove Theorem 1.5, which follows from [21, Theorem 3.7]; see also Bertrand-Ketterer-Mondello-Richard's work on stratified spaces [2], as well as related work of Jiang-Sheng-Zhang [23] and Szekelyhidi [43]. By applying Theorem 1.5, we then prove Theorems 1.2, and give several interesting examples. These examples illustrate how naturally our approach of using metric measure spaces deals with the possible change of differential structures, as well as situations outside Theorem 1.5 for which our results/methods still apply. Finally, in Appendix A, we provide a detailed proof of Theorem 2.6.

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2. Removable singularity via RCD

2.1. **RCD spaces and uniformly Euclidean metric.** We recall a definition of the $\mathrm{RCD}(K,N)$ condition for metric measure spaces, see [1, 13, 15].

Definition 2.1. *A metric measure space (X,d,m) is said to be an $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$ and $N\in[1,\infty]$, if*

(1) *there exists* $x_0 \in X$ *and a constant* $C > 0$ *such that* $\int_X e^{-Cd(x,x_0)} dm < +\infty$,
(2) (Infinitesimally Hilbertian condition) *the Sobolev space* $W^{1,2}(X, d, m)$ *is a Hilbert space,*
(3) (Sobolev-to-Lipschitz property) *for any* $f \in W^{1,2}(X, d, m)$ *satisfying* $|\nabla f| \in L^\infty(X)$, *it has a Lipschitz representative* $\tilde{f}$ *with* $\mathrm{Lip}(\tilde{f}) \leq \|\nabla f\|_{L^\infty(X)}$,
(4) (BE(K, N) condition) *for any* $f \in D(\Delta)$, *hence* $\Delta f \in W^{1,2}(X, d, m)$, *and any test function* $\varphi \in L^\infty(X) \cap D(\Delta)$ *with* $\varphi \geq 0$ *and* $\Delta\varphi \in L^\infty(X)$, *the weak Bochner inequality*

$$\frac{1}{2}\int_X |\nabla f|^2 \Delta\varphi dm \geq \int_X \varphi \left(\tfrac{1}{N}(\Delta f)^2 + \langle \nabla f, \nabla \Delta f\rangle + K|\nabla f|^2\right) dm \tag{2.1}$$

holds.

In this paper, we focus on a closed connected smooth manifold M^n with a uniformly Euclidean metric g as in Conjecture 1.1, that is, g is smooth away from a closed singular set S with codimension ≥ 3. We will denote by $\Omega := M \setminus S$ the smooth part of the metric and by $k := \dim(S)$.

We describe a metric measure structure on such (M^n, g) as follows. We say that a piecewise C^1 curve $\gamma : [a, b] \to M$ is *admissible* if its image $\gamma([a, b])$ intersects with S at only finitely many points. For such a curve, its length is defined as $L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt$. Then, the distance between any $x, y \in M$ is defined as

$$d_g(x, y) := \inf\{L_g(\gamma) \mid \gamma : [a, b] \to M \ \text{ is admissible curve with } \ \gamma(a) = x, \gamma(b) = y\}.$$

The volume measure $d\mu_g$ is defined in the usual way. A direct consequence of the uniformly Euclidean condition is

$$\Lambda^{-\frac{n}{2}} d\mu_{\bar{g}} \leq d\mu_g \leq \Lambda^{\frac{n}{2}} d\mu_{\bar{g}}, \tag{2.2}$$

where $d\mu_{\bar{g}}$ is the volume measure of the smooth metric $\bar{g}$ on M. As a result, on a sufficiently small tubular neighborhood $T_\epsilon(S)$ of the singular set S, we have that locally,

$$d\mu_g \sim r^{n-k-1} dr d\theta^{n-k-1} d\mu_{\bar{g}_S}, \tag{2.3}$$

where $r = r(x) := d(x, S)$, $d\theta^{n-k-1}$ is the volume measure on $\mathbb{S}^{n-k-1}$, and $d\mu_{\bar{g}_S}$ is the volume measure on S with the restricted metric $\bar{g}_S$. Here $\sim$ means bounded above and below by the right hand side (not precise asymptotic). By applying this volume measure estimate, we prove the following zero capacity property of S.

Lemma 2.2 (Zero capacity property)**.** *Let* M^n, $n \geq 3$, *be a closed manifold, and* g *a uniformly Euclidean metric on* M *with singular set a closed embedded submanifold of codimension* ≥ 3. *There exists a family of functions* $\psi_\delta \in C_0^\infty(\Omega)$, $\delta \in (0, \delta_0)$ *for some* $\delta_0 > 0$, *such that the following two properties hold:*

(1) *for any compact subset* $A \subset \Omega := M \setminus S$, $\psi_\delta|_A \equiv 1$ *hold for all sufficiently small* δ,

(2) $0 \le \psi_\delta \le 1$ *for all* δ *and* $|\nabla\psi_\delta| \le \frac{C}{\delta}$. *Moreover,*

$$\int_\Omega |\nabla\psi_\delta|^2 d\mu_g \to 0, \quad as \ \ \delta \to 0. \tag{2.4}$$

Proof. Take a smooth cut-off function $\eta : [0,\infty) \to [0,1]$ such that

$$\eta = \begin{cases} 0, & \text{in} \ \ [0, \frac{1}{2}], \\ 1, & \text{in} \ \ [1,\infty), \end{cases} \tag{2.5}$$

and $|\eta'| \le C$ on $[0,\infty)$ for some constant C. Let $r(x) := d(x,S)$, and $\psi_\delta(x) := \eta\left(\frac{r}{\delta}\right)$. Clearly, $|\nabla\psi_\delta| \le \frac{C}{\delta}$ and $\text{supp}(\nabla\psi_\delta) \subset A_{\frac{\delta}{2},\delta}(p) := \{x \in M \mid \frac{\delta}{2} \le d(x,S) \le \delta\}$. By combining with the estimate of the volume measure in (2.3), this implies

$$\int_\Omega |\nabla\psi_\delta|^2 d\mu_g \le \frac{C^2}{\delta^2}\int_{A_{\frac{\delta}{2},\delta}(p)} d\mu_g \le \frac{C'}{\delta^2}\int_{\frac{\delta}{2}}^{\delta} r^{n-k-1}dr \le C''\delta^{n-k-2} \to 0, \tag{2.6}$$

as $\delta \to 0$, since $n-k \ge 3$. This completes the proof. □

By using Lemma 2.2, one can show that the Sobolev space $H^{1,2}(M)$ defined as the $H^{1,2}$-completion of $C_0^\infty(M \setminus S)$ coincides with the Sobolev space $W^{1,2}(M, d_g, d\mu_g)$ on the metric measure space $(M, d_g, d\mu_g)$, see Remark 3.2 and Proposition 3.3 in [21], for details. In particular, $(M, d_g, d\mu_g)$ satisfies the infinitesimally Hilbertian condition in Definition 2.1.

For $f \in W^{1,2}(M)$, if there exists a function $u \in L^2(M)$ such that

$$\int_M uv d\mu_g = -\int_M \langle df, dv\rangle_g d\mu_g$$

holds for any test function $v \in C_0^\infty(M \setminus S)$, then $\Delta f := u$ is said to be the Laplace of f, and the domain of the Laplace operator is

$$D(\Delta) := \{f \in W^{1,2}(M) \mid \Delta f \in L^2(M)\}.$$

For proving the Sobolev-to-Lipschitz property for uniformly Euclidean metrics, we need the following result about the length of admissible curves.

Lemma 2.3. *Let M^n, $n \ge 3$, be a closed connected manifold, and g a uniformly Euclidean metric on M as in Conjecture 1.1. Let $\gamma : [a,b] \to M$ be an admissible curve on M. For any $\epsilon > 0$, there exists an admissible curve γ_ϵ with the same endpoints as γ such that the image of γ_ϵ is contained in the regular part $M \setminus S$ probably except the endpoints, and $L_g(\gamma_\epsilon) \le L_g(\gamma) + \epsilon$.*

Proof. Because γ intersects with S at only finitely many points and the length is additive, without loss of generality, we assume that γ intersects with S only at an interior point p. Then we only need to modify the curve γ near p. For that, we take a small ball $B_\delta(p)$ centered at p with radius δ with respect to the smooth metric $\bar{g}$. Let $\gamma(t_1)$ be the first point on γ that intersects $\partial B_\delta(p)$, and $\gamma(t_2)$ be the last such point, where $[t_1, t_2] \subset (a,b)$.

For sufficiently small $\delta > 0$, we can take a smooth curve c in $B_\delta(p)$ that connects $\gamma(t_1)$ and $\gamma(t_2)$, has length $< \frac{\epsilon}{\sqrt{\Lambda}}$ with respect to $\bar{g}$, and does not intersect with S. Now, define γ_ϵ to be the concatenation of $\gamma|_{[a,t_0]}$, c and $\gamma|_{[t_1,b]}$. Then the image of γ_ϵ intersects with S at most at the endpoints. Moreover, $L_g(\gamma_\epsilon) \le L_g(\gamma) + L_g(c) \le L_g(\gamma) + \epsilon$. □

Lemma 2.4. *Let M^n, $n \ge 3$, be a closed connected manifold, and g a uniformly Euclidean metric on M as in Conjecture 1.1. Any $f \in W^{1,2}(M)$ with gradient $|\nabla f| \in L^\infty(M)$ has a Lipschitz representative $\tilde{f}$ whose Lipschitz constant $\le \|\nabla f\|_{L^\infty(M)}$.*

Proof. First note that the smooth part $\Omega := M \setminus S$ is path connected, since the singular set S has codimension ≥ 3. Take any two points $x, y \in \Omega$. By Lemma 2.3, for any $\epsilon > 0$, there exists a piecewise smooth curve $\gamma_\epsilon \subset \Omega$ from x to y such that the length $L_g(\gamma_\epsilon)$ of γ with respect to g

$$L_g(\gamma) \le (1+\epsilon) d_g(x, y). \tag{2.7}$$

Then by locally Lipschitz property of f in Ω with Lipschitz constant $\le \|\nabla f\|_{L^\infty(M)}$, and applying the fundamental theorem of calculus to $u \circ \gamma$, we have

$$|f(x) - f(y)| \le (1+\epsilon)\|\nabla f\|_{L^\infty(M)} d_g(x, y).$$

Thus, by letting $\epsilon \to 0$, we have that u is uniformly Lipschitz on Ω with Lipschitz constant $\le \|\nabla f\|_{L^\infty(M)}$. Finally, because Ω is dense in M, u has a unique Lipschitz continuous extension $\tilde{f}$ on M, which equals f almost everywhere on M. □

The uniformly Euclidean property directly implies the following L^2-strong compactness property.

Lemma 2.5 (L^2-strong compactness)**.** *Let M^n, $n \ge 3$, be a closed manifold, and g a uniformly Euclidean metric on M. The inclusion map $i : W^{1,2}(M, g) \hookrightarrow L^2(M, g)$ is compact.*

Proof. By the definition, the Sobolev space $W^{1,2}(M, g)$ and $L^2(M, g)$ depend only on the quasi-isometry class of g and hence is equivalent to $W^{1,2}(M, \bar{g})$ for a smooth metric $\bar{g}$. Clearly $W^{1,2}(M, \bar{g})$ embeds compactly in $L^2(M, \bar{g})$ on a compact manifold. □

Using the L^2-strong compactness in Lemma 2.5 and the standard argument in functional analysis, one obtains that the spectrum of the Laplace operator Δ_g consists of discrete eigenvalues with finite multiplicities, and there are orthonormal bases of $L^2(M, g)$ by normalized eigenfunctions.

2.2. **Gradient estimate of eigenfunctions.** In order to prove the BE(K, n) condition for uniformly Euclidean metrics, we expand a Sobolev function in terms of eigenfunctions of Δ_g, as in Theorem 3.7 in [21]. The gradient estimate of eigenfunctions plays a crucial role in the proof. In this subsection, we derive estimates for the gradient and Hessian of the eigenfunctions.

Let M^n be a closed manifold, and g a uniformly Euclidean metric on M that is smooth away from a closed embedded submanifold S. As M is a closed manifold and g is uniformly Euclidean, for any $q \in \Omega$, $0 < 2r < d(q, S)$, the gradient estimate of Cheng-Yau [9] still applies on $B_r(q)$, provided one has a Ricci curvature lower bound on $B_{2r}(q)$. The following combines the versions in [6] and [31].

Theorem 2.6. *Let (M^n, g) be as above and $\Omega := M \setminus S$ the smooth part of g. If u is a positive function defined on the closed ball $B(q, 2r) \subset \Omega$ that satisfies $\Delta u = K(u)$ and $\mathrm{Ric}_g \geq -(n-1)H^2$ $(H \geq 0)$ on $B(q, 2r)$. Then on $B(q, r)$,*

$$\frac{|\nabla u(x)|}{u} \leq \max \Bigg\{ \sup_{B(q,2r)} \left(\tfrac{|K(u)|}{u}\right)^{\frac{1}{2}}, \\ (n-1)H + C_1(n)r^{-1} + C_2(n,H)r^{-\frac{1}{2}} + \sup_{B(q,2r)} \left[(n-1)\left|K' - \tfrac{n+1}{n-1}\tfrac{K}{u}\right|\right]^{\frac{1}{2}} \Bigg\}.$$

Since this version is not in the literature, for completeness we provide a proof in the appendix.

Here, as an application, we deduce the following crucial estimate.

Lemma 2.7. *Let (M^n, g) be as above.*

(1) *There exists $0 < \alpha = \alpha(n, \Lambda) < 1$, such that the $W^{1,2}$-eigenfunctions u of the Laplace operator Δ_g satisfy $u \in C^\alpha(M)$. Moreover, if λ is the eigenvalue, one has*

$$[u]_{C^\alpha(M)} \leq C(n, \Lambda, \lambda, \bar{g})\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)}. \tag{2.8}$$

(2) *Assume that $Ric_g \geq K$ on Ω for some constant K. For all $x \in \Omega$ with $r(x) = d(x, S) \leq 1$, we have the gradient estimate*

$$|\nabla u|(x) \leq \frac{C(n, \Lambda, \lambda, \bar{g}, K)}{r(x)^{1-\alpha}}\|u\|_{L^2(M)}, \tag{2.9}$$

for the same α in (2.8).

Proof. Since M is uniformly Euclidean the first statement (1) is the De Giorgi-Nash-Moser estimate, see Chapter 8 in [16], [32] for example.

For (2), in order to apply Theorem 2.6 we do a shift to get a positive solution. Namely fix any $y \in \Omega$ and let $r(y) = 2s > 0$. Then $v = u - m + s \geq s$ with $m = \inf\{u : B_{2s}(y)\}$ is a positive solution of

$$-\Delta v = \lambda v + \lambda(m - s) \tag{2.10}$$

in the ball $B_{2s}(y) \subset \Omega$. Since $\mathrm{Ric} \geq K$ on Ω, by Theorem 2.6, we have on $B_s(y)$

$$|\nabla u| = |\nabla v| \leq v\left((n-1)K + C(n)s^{-1} + C(n,K)s^{-\frac{1}{2}} + \left[(n-1)(2\lambda + \lambda\frac{|m| + s}{s})\right]^{1/2}\right)$$

$$= v\, s^{-1} \left((n-1)Ks + C(n) + C(n,K)\sqrt{s} + \sqrt{(n-1)s\lambda}\,(3s+|m|)^{1/2} \right).$$

By definition, there is a $q \in \overline{B_{2s}(y)}$ such that

$$|v(y)| \le |u(y) - u(q)| + s.$$

Now by (2.8),

$$|u(y) - u(q)| \le Cs^{\alpha}\, \|u\|_{L^2(M)}.$$

Hence, for $s \le 1$,

$$|\nabla u(y)| \le C(n, \Lambda, \lambda, \bar{g}, K) s^{\alpha} s^{-1}\, \|u\|_{L^2(M)}.$$

Our estimate follows since $y \in \Omega$ is arbitrary. □

Remark 2.8. When the metric is continuous, the Hölder exponent α in (1) of Lemma 2.7 can be chosen arbitrarily close to 1 [3]. Therefore, in this case, from the proof below one can see that Theorem 1.5 also holds for continuous metrics with singular set of codimension > 2.

By applying the gradient estimate in Lemma 2.7, we prove the L^2-integrability of the Hessian of eigenfunctions as in the following lemma.

Lemma 2.9. *Let M^n be a closed manifold, and g a uniformly Euclidean metric on M that is smooth away from a closed embedded submanifold S with codimension $n-k \ge 4$, and $Ric_g \ge K$ on Ω. For any $W^{1,2}$-eigenfunction u of Δ_g, $|\mathrm{Hess}\, u| \in L^2(M)$.*

Proof. Let u be a $W^{1,2}$-eigenfunction of Δ with eigenvalue λ, i.e. $-\Delta u = \lambda u$ in the weak sense. By standard elliptic regularity theory, $u \in C^\infty(\Omega)$. Bochner formula and $\mathrm{Ric}_g \ge K$ on Ω imply

$$\frac{1}{2}\Delta|\nabla u|^2 \ge |\mathrm{Hess}\, u|^2 + (K-\lambda)|\nabla u|^2 \quad \text{on} \quad \Omega. \tag{2.11}$$

Multiplying the inequality (2.11) by ψ_δ^2 in Lemma 2.2, and then integrating it over M gives

$$\frac{1}{2}\int_M |\nabla u|^2 \Delta\psi_\delta^2 d\mu_g \ge \int_M \psi_\delta^2 \left(|\mathrm{Hess}\, u|^2 + (K-\lambda)|\nabla u|^2\right) d\mu_g. \tag{2.12}$$

For the left hand side,

$$\begin{aligned}\int_M |\nabla u|^2 \Delta\psi_\delta^2 d\mu_g &= -4\int_M |\nabla u|\psi_\delta \langle \nabla|\nabla u|, \nabla\psi_\delta\rangle d\mu_g \\ &\le \int_M \psi_\delta^2 |\mathrm{Hess}\, u|^2 d\mu_g + 4\int_M |\nabla u|^2 |\nabla\psi_\delta|^2 d\mu_g.\end{aligned}$$

Plugging this into (2.12) and rearranging the equation give

$$\frac{1}{2}\int_M \psi_\delta^2 |\mathrm{Hess}\, u|^2 d\mu_g \le \int_M 2|\nabla u|^2|\nabla\psi_\delta|^2 d\mu_g - (K-\lambda))\int_M \psi_\delta^2|\nabla u|^2 d\mu_g.$$

Then by applying estimates in Lemmas 2.2 and 2.7 and equation (2.3) for the integrand in the first term on the right hand side, we obtain

$$\begin{aligned}\frac{1}{2}\int_M \psi_\delta^2|\mathrm{Hess}\, u|^2 d\mu_g &\leq C\int_{\frac{\delta}{2}}^{\delta} r^{2\alpha+n-k-5}dr-(K-\lambda)\int_M \psi_\delta^2|\nabla u|^2 d\mu_g\\ &\leq \tilde{C}\delta^{2\alpha-4+n-k}-(K-\lambda)\int_M \psi_\delta^2|\nabla u|^2 d\mu_g.\end{aligned}$$

Now because $n-k\geq 4$, by letting $\delta\to 0$, we obtain

$$\int_M |\mathrm{Hess}\, u|^2 d\mu_g \leq 2(\lambda-K)\int_M |\nabla u|^2 d\mu_g < +\infty. \tag{2.13}$$

□

By using Lemma 2.9, similar as in [33, p.56] for eigenfunctions on stratified spaces, we prove

Lemma 2.10. *Let (M^n,g) as in Lemma 2.9. For any $W^{1,2}$-eigenfunction u of Δ_g, $|\nabla u|\in L^\infty(M)$.*

Proof. Let u be a $W^{1,2}$-eigenfunction of Δ with eigenvalue λ. Then $u\in C^\infty(\Omega)$. For simplicity of notation, we denote $v:=|\nabla u|$. By (2.11), Kato's inequality $|\nabla|\nabla u||\leq |\mathrm{Hess}\, u|$, one can show that there exists a positive constant c such that

$$-\Delta v \leq cv \quad \text{in the weak sense on} \quad \Omega. \tag{2.14}$$

Indeed, we can show this by using a sequence of smooth barriers. Let $v_\epsilon=\sqrt{|\nabla u|^2+\epsilon^2}-\epsilon$. By the Kato's inequality, one has

$$|\nabla v_\epsilon|^2=\frac{|\nabla u|^2\cdot|\nabla|\nabla u||^2}{|\nabla u|^2+\epsilon^2}\leq \frac{|\nabla u|^2\cdot|\mathrm{Hess}\, u|^2}{|\nabla u|^2+\epsilon^2}\leq |\mathrm{Hess}\, u|^2. \tag{2.15}$$

Moreover, putting $c:=\max\{\lambda-K,1\}>0$, and using (2.11) and (2.15), one has

$$\begin{aligned}\frac{1}{2}\Delta(v_\epsilon+\epsilon)^2=\frac{1}{2}\Delta|\nabla u|^2 &\geq |\mathrm{Hess}\, u|^2-(\lambda-K)|\nabla u|^2\\ &\geq |\mathrm{Hess}\, u|^2-c((v_\epsilon+\epsilon)^2-\epsilon^2)\\ &\geq |\nabla v_\epsilon|^2-c(v_\epsilon+\epsilon)^2 \quad \text{on} \quad \Omega.\end{aligned}$$

Thus,

$$(v_\epsilon+\epsilon)\Delta(v_\epsilon+\epsilon)+|\nabla v_\epsilon|^2\geq |\nabla v_\epsilon|^2-c(v_\epsilon+\epsilon)^2.$$

Consequently, $\Delta v_\epsilon\geq -cv_\epsilon-c\epsilon$ for any $\epsilon>0$.

If $x\in\Omega$ and $v(x)\neq 0$, then v is smooth at x. The calculation above with $\epsilon=0$ shows that $\Delta v\geq -cv$.

If $v(x_0)=0$, then $v_\epsilon(x_0)=0=v(x_0)$ and $v_\epsilon\leq v$. Hence, $\Delta v\geq -cv$ in the weak sense at x_0.

We now claim that the inequality in (2.14) holds in the weak sense on the whole manifold M. Then, with the help of the Sobolev inequality in Lemma 2.5, a Moser iteration argument implies $v = |\nabla u| \in L^\infty(M)$.

In the rest of the proof, we show that the inequality in (2.14) holds in the weak sense on M, that is, for any $\varphi \in W^{1,2}(M)$, $\varphi \geq 0$,

$$\int_M \langle \nabla v, \nabla \varphi \rangle d\mu_g \leq c \int_M v\varphi d\mu_g. \tag{2.16}$$

Because $C^\infty(M)$ is dense in $W^{1,2}(M)$, we only need to establish the inequality (2.16) for $\varphi \in C^\infty(M)$, $\varphi \geq 0$. For such φ and ψ_δ in Lemma 2.2, we multiply the inequality in (2.14) by $\psi_\delta \varphi$, and integrate over M. Then by integrating by parts and rearranging the inequality, we obtain

$$\int_M \psi_\delta \langle \nabla v, \nabla \varphi \rangle d\mu_g \leq c \int_M \psi_\delta \varphi v d\mu_g - \int_M \varphi \langle \nabla v, \nabla \psi_\delta \rangle d\mu_g. \tag{2.17}$$

For the second term on the right hand side, because the smooth function φ on the compact manifold M is bounded, we have

$$\left| \int_M \varphi \langle \nabla v, \nabla \psi_\delta \rangle d\mu_g \right| \leq C \left| \int_M \langle \nabla v, \nabla \psi_\delta \rangle d\mu_g \right|.$$

Then by applying Cauchy-Schwarz inequality and the Kato's inequality, we obtain

$$\left| \int_M \varphi \langle \nabla v, \nabla \psi_\delta \rangle d\mu_g \right| \leq C \|\nabla v\|_{L^2(M)} \|\nabla \psi_\delta\|_{L^2(M)} \leq C \||\mathrm{Hess}\, u|\|_{L^2(M)} \|\nabla \psi_\delta\|_{L^2(M)} \to 0,$$

as $\delta \to 0$, since $\||\mathrm{Hess}\, u|\|_{L^2(M)}$ is finite by Lemma 2.9 and $\|\nabla \psi_\delta\|_{L^2(M)} \to 0$ by Lemma 2.2. Thus, by letting $\delta \to 0$ in (2.17), we prove the inequality (2.16) for nonnegative smooth function φ, so for nonnegative test functions $\varphi \in W^{1,2}(M)$. □

2.3. **Proof of Theorems 1.2 and 1.5.** With our work from the previous subsection, Theorem 1.5 follows from [21, Theorem 3.7], see also [2, 23]. For completeness, we give a detailed proof here.

Proof of Theorem 1.5. It suffices to check four conditions in Definition 2.1. Condition (1) follows from the compactness of M and the uniform Euclidean property of g. Condition (2) is also clear. Condition (3) follows from Lemma 2.4.

It remains to check the BE(K, n) condition in (4). Let $\Omega := M \setminus S$, $\{u_i\}_{i=1}^{+\infty}$ be the orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Δ. For any $f \in W^{1,2}(M)$ and $N \in \mathbb{N}$, let $f_N := \sum_{i=1}^N a_i u_i$, where $a_i := \int_M f u_i d\mu_g$. As before, $k = \dim S$, and by assumption $n - k \geq 4$.

Bochner formula and $\mathrm{Ric}_g \geq K$ on Ω imply

$$\frac{1}{2}\Delta |\nabla f_N|^2 \geq |\mathrm{Hess} f_N|^2 + \langle \nabla \Delta f_N, \nabla f_N \rangle + K |\nabla f_N|^2 \quad \text{on } \ \Omega. \tag{2.18}$$

By Lemma 2.9 and Lemma 2.10, for any eigenfunction u_i, we have

$$\int_M |\nabla|\nabla u_i|^2|^2 d\mu_g = 4\int_M |\nabla u_i|^2 |\nabla|\nabla u_i||^2 d\mu_g \le 4\int_M |\nabla u_i|^2 |\mathrm{Hess}\, u_i|^2 d\mu_g < +\infty.$$

Thus, $|\nabla u_i|^2 \in W^{1,2}(M)$ for all $i \in \mathbb{N}$, so $|\nabla f_N|^2 \in W^{1,2}(M)$. Also $|\mathrm{Hess} f_N| \in L^2(M)$ from Lemma 2.9.

For any test function $\varphi \in L^\infty(M) \cap D(\Delta)$ with $\varphi \ge 0$ and $\Delta\varphi \in L^\infty(M)$, by multiplying the inequality (2.18) by $\varphi\psi_\delta$, and integration by parts, we have

$$\begin{aligned} & -\frac{1}{2}\int_M \langle \nabla(\varphi\psi_\delta), \nabla|\nabla f_N|^2\rangle d\mu_g \\ \text{(2.19)} \qquad & \ge \int_M \varphi\psi_\delta \left(|\mathrm{Hess} f_N|^2 + \langle \nabla\Delta f_N, \nabla f_N\rangle + K|\nabla f_N|^2\right) d\mu_g. \end{aligned}$$

By Lemma 2.2, $\psi_\delta \to 1$ in $L^2(M)$ and $\nabla\psi_\delta \rightharpoonup 0$ in $L^2(M)$ as $\delta \to 0$, and since $|\nabla f_N|^2 \in W^{1,2}(M)$, we have

$$-\frac{1}{2}\int_M \langle \nabla(\varphi\psi_\delta), \nabla|\nabla f_N|^2\rangle \to -\frac{1}{2}\int_M \langle \nabla\varphi, \nabla|\nabla f_N|^2\rangle d\mu_g = \frac{1}{2}\int_M |\nabla f_N|^2 \Delta\varphi d\mu_g,$$

as $\delta \to 0$. Moreover, by the dominated convergence theorem, we have

$$\begin{aligned} & \int_M \varphi\psi_\delta \left(|\mathrm{Hess} f_N|^2 d\mu_g + \langle \nabla\Delta f_N, \nabla f_N\rangle + K|\nabla f_N|^2\right) d\mu_g \\ \to & \int_M \varphi \left(|\mathrm{Hess} f_N|^2 + \langle \nabla\Delta f_N, \nabla f_N\rangle + K|\nabla f_N|^2\right) d\mu_g. \end{aligned}$$

Thus, by letting $\delta \to 0$ in (2.19), we have

$$\begin{aligned} \frac{1}{2}\int_M |\nabla f_N|^2 \Delta\varphi d\mu_g & \ge \int_M \varphi \left(|\mathrm{Hess} f_N|^2 + \langle \nabla\Delta f_N, \nabla f_N\rangle + K|\nabla f_N|^2\right) d\mu_g \\ & \ge \int_M \varphi \left(\frac{(\Delta f_N)^2}{n} + \langle \nabla\Delta f_N, \nabla f_N\rangle + K|\nabla f_N|^2\right) d\mu_g. \end{aligned}$$

Finally, letting $N \to \infty$, we obtain that

$$\frac{1}{2}\int_M |\nabla f|^2 \Delta\varphi d\mu_g \ge \int_M \varphi \left(\frac{(\Delta f)^2}{n} + \langle \nabla\Delta f, \nabla f\rangle + K|\nabla f|^2\right) d\mu_g$$

for any test function $\varphi \in L^\infty(M) \cap D(\Delta)$ with $\varphi \ge 0$ and $\Delta\varphi \in L^\infty(M)$. This completes the proof. ☐

Theorem 1.5 provides the crucial ingredient to the proof of Theorem 1.2.

Proof of Theorem 1.2. First of all, we note that, under the assumptions for M_0 in Theorem 1.2, there is no uniformly Euclidean metric g on $M := T^n \# M_0$ smooth away from a closed embedded submanifold S with co-dimension $\ge \frac{n}{2} + 1$, which has nonnegative scalar curvature on the smooth part and positive scalar curvature at a point. This non-existence result can be established by using the conformal blow up technique dating back to Schoen's final resolution of Yamabe problem [35]. It reduces the problem to the non-existence results of Chodosh-Li [10] and Wang-Zhang [45] for complete metrics. This was recently used in studies of uniformly Euclidean metric with positive scalar curvature by Li-Mantoulidis [30], Kazaras [24], Wang-Xie [44] and others, and also in studies of positive mass theorem and positive scalar curvature with conical singularity by Dai-Sun-Wang [11, 12]. Since the method carries over in a similar way, we refer to these references for details. Roughly speaking, one uses the Green's function of a modified conformal Laplacian where the scalar curvature is suitably truncated to conformally blow up the singular metric. This produces a complete metric with positive scalar curvature on $(T^n \# M_0) \setminus S$, leading to a contradiction with non-existence results of Chodosh-Li [10] and Wang-Zhang [45]. We only emphasize a few points here. Firstly, for uniformly Euclidean metrics, the Green's function can be solved by using Littman-Stampacchia-Weinberger [32] and Schoen-Yau [37]. Also, the condition that the co-dmension of $S \geq \frac{n}{2} + 1$ guarantees the completeness of the blow up metric, see Proposition 2.4 in Wang-Xie [44]. And finally, the assumption on the location and size of the singular set, Condition 2), makes sure that $M \setminus S$ can again be written as $T^n \# M_1$ for suitable M_1, and hence the results of Chodosh-Li [10] and Wang-Zhang [45] can be applied.

Then by Theorem B in Kazdan [25], the metric g must be Ricci flat. Therefore, Theorem 1.5 implies that $(T^n \# M_0, d_g, d\mu_g)$ is a $\mathrm{RCD}(0, n)$ space. Consequently, we conclude that $(T^n \# M_0, d_g)$ is isometric to a flat torus T^n by Corollary 1.4 in Mondino-Wei [34]. Thus, we have an isometry $\psi : (M, d_g) \longrightarrow (T^n, d_h)$, for a flat metric h on the torus, which is necessarily a homeomorphism. By [4], ψ is smooth on the smooth part of g and hence a diffeomorphism on the smooth part, and $g = \psi^* h$. This completes the proof. □

If g has only isolated singularity, Theorem 1.2 can be further strengthened so that ψ can be taken to be the identity on the smooth part. This is because we prove that g is flat on the smooth part. We can then apply Theorem 3.1 in [40].

Proof of Theorem 1.6. By Theorem 1.5, (M, g) is $\mathrm{RCD}(0, n)$ and the splitting theorem of Gigli [14] applies. Since $\pi_1(M)$ is infinite, the universal cover $\tilde{M}$ splits as $\mathbb{R} \times \tilde{X}$, where $\tilde{X}$ is a level set of the Busemann function. Because the Busemann function is harmonic, it is smooth on the smooth part of $(\tilde{M}, \tilde{g})$, where $\tilde{g}$ is the lifted L^∞ metric. Moreover, since g has isolated singularity on the closed manifold M, $\tilde{g}$ can only have an at most countable singular set. In particular, the image of singular set under the Busemann function is an at most countable set as well. As a result, we can choose a level set $\tilde{X}$ of the Busemann function such that $\tilde{g}$ is smooth on X. Therefore, $(\mathbb{R} \times \tilde{X}, dt^2 + \tilde{g}|_{\tilde{X}})$ is a smooth Riemannian manifold and is isometric to $\tilde{M}$ as a metric measure space. Consequently, we obtain a smooth extension as stated in Theorem 1.6. Now if $\tilde{X}$ still contains geodesic lines, we can further split $\tilde{X}$ as $\mathbb{R}^{k-1} \times X$, where X is a closed simply connected manifold. Finally, we obtain that a finite cover $\hat{M}$ of M is homeomorphic to $T^k \times X$, see [8, Theorem 9.2] and [7] for more details. $\square$

We will end this section with several examples.

Example 1: This example is motivated by Cecchini-Frenck-Zeidler [5]; see [5] for related discussion. Let $M = T^n \# \Sigma$, where Σ is an exotic sphere of dimension n. Consider g a uniformly Euclidean metric on M with an isolated singularity at a point in Σ and nonnegative scalar curvature on the smooth part. Then the metric measure space (M, g) induced by g does not remember the original smooth structure, and hence it is isometric to the flat torus T^n, which can be thought as the connected sum of the torus with the standard sphere.

Example 2: Let $M = T^n$ and $T^k \subset M$ a linear sub-torus where $n - k \geq 4$. If g is a uniformly Euclidean metric on M with singular set $S = T^k$ and nonnegative scalar curvature on the smooth part. Then by Gromov-Lawson [17], $M \setminus S$ is Λ^2-enlargeable, and hence cannot carry complete metrics of positive scalar metric. It follows by the conformal blowup method as in the proof of Theorem 1.2, g must be Ricci flat on the smooth part. Then by Theorem 1.5 and [34], g extends smoothly over S. This example illustrates the situation where our result still applies even if the singular set is not contained in a topological ball. See [17, Example 6.9] for more examples.

Example 3: Consider a uniformly Euclidean metric g with isolated singularity on a closed enlargeable manifold M. By Cecchini-Frenck-Zeidler [5, Theorem C and Remark 1.9], it must be Ricci flat on the smooth part. Again by Theorem 1.5, (M, g) is $\mathrm{RCD}(0, n)$. Hence by a straightford generalization of a result of Gromov-Lawson [18] (see also [27, Proposition 5.8 in Chapter IV]) using Gigli's splitting theorem [14], the singularity is removable, and in fact (M, g) must be flat.

Example 4: Consider a closed Schoen-Yau-Schick (SYS) manifold M of dimension $4 \leq n \leq 7$ (here we use the definition given in [39]). By the very recent work of Shi-Wang-Wu-Zhu [39, Corollary 1.9], such a closed SYS manifold with a finite number of points removed does not admit complete metrics of positive scalar curvature. Therefore, by the conformal blowup method, a uniformly Euclidean metric with isolated singularity on M and nonnegative scalar curvature on the smooth part must be Ricci flat on the smooth part. On the other hand, a SYS manifold has infinite fundamental group. By Theorem 1.6, the metric extends smoothly, in other words, isolated singularity is removable.

Appendix A. Proof of Theorem 2.6

The main ingredients are Bochner's formula, the maximal principle, cut-off functions and the Laplacian comparison.

Proof of Theorem 2.6. : Let $h = \log u$. Then $\nabla h = \frac{\nabla u}{u}, \Delta h = \frac{\Delta u}{u} - \left|\frac{\nabla u}{u}\right|^2 = \frac{K(u)}{u} - |\nabla h|^2$. By the Bochner formula we have

$$\begin{aligned} \frac{1}{2}\Delta|\nabla h|^2 &= |\mathrm{Hess}\, h|^2 + \langle \nabla h, \nabla(\Delta h)\rangle + \mathrm{Ric}(\nabla h, \nabla h) \\ &\geq |\mathrm{Hess}\, h|^2 + \langle \nabla h, \nabla \tfrac{K(u)}{u}\rangle - \langle \nabla h, \nabla(|\nabla h|^2)\rangle - (n-1)H^2|\nabla h|^2. \end{aligned} \tag{A.1}$$

For the Hessian term one could use the Schwarz inequality

$$|\mathrm{Hess}\, h|^2 \geq \frac{(\Delta h)^2}{n}. \tag{A.2}$$

Indeed, when $H = 0$, this estimate is enough. For $H > 0$, using this estimate one would get $(n(n-1))^{1/2}$ instead of $n-1$ for the coefficient of H in (2.8). To get the best constant for $H > 0$, note that (A.2) is only optimal when Hessian at all directions are same. Harmonic functions in the model spaces are radial functions, so their Hessian along the radial direction would be different from the spherical directions. Therefore one computes the norm by separating the radial direction and uses Schwarz inequality in the spherical directions. Let $\{e_i\}$ be an orthonormal basis with $e_1 = \frac{\nabla h}{|\nabla h|}$ (we only need to prove the case $|\nabla h| \neq 0$), the potential radial direction, denote $h_{ij} = \mathrm{Hess}\, h(e_i, e_j)$. We compute,

$$\begin{aligned} |\mathrm{Hess}\, h|^2 &= h_{11}^2 + 2\sum_{j=2}^{n} h_{1j}^2 + \sum_{i,j\geq 2} h_{ij}^2 \\ &\geq h_{11}^2 + 2\sum_{j=2}^{n} h_{1j}^2 + \frac{(\Delta h - h_{11})^2}{n-1} \\ &= h_{11}^2 + 2\sum_{j=2}^{n} h_{1j}^2 + \frac{(|\nabla h|^2 + h_{11} - \frac{K(u)}{u})^2}{n-1} \\ &\geq \frac{n}{n-1}\sum_{j=1}^{n} h_{1j}^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}h_{11}|\nabla h|^2 - \frac{2}{n-1}\frac{K(u)}{u}\left(h_{11} + |\nabla h|^2\right). \end{aligned}$$

Now $h_{1j} = \frac{1}{|\nabla h|}\langle \nabla_{e_j}\nabla h, \nabla h\rangle = \frac{1}{2|\nabla h|}e_j(|\nabla h|^2)$ and $h_{11} = \frac{1}{2|\nabla h|^2}\langle \nabla|\nabla h|^2, \nabla h\rangle$. Therefore

$$|\mathrm{Hess}\, h|^2 \geq \frac{n}{4(n-1)}\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{|\nabla h|^4 + \langle \nabla|\nabla h|^2, \nabla h\rangle}{n-1} - \frac{2}{n-1}\frac{K(u)}{u}\left(h_{11} + |\nabla h|^2\right). \tag{A.3}$$

Note that when $K(u) = 0$, i.e. u is harmonic, then equality holds if and only if Hess h are same on the level set of h, i.e.

$$\mathrm{Hess}\, h = -\frac{|\nabla h|^2}{n-1}\left(g - \frac{1}{|\nabla h|^2}dh \otimes dh\right).$$

By plugging (A.3) into (A.1) we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \quad &\geq \quad \frac{n}{4(n-1)}\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2}+\frac{|\nabla h|^4}{n-1}-\frac{n-2+\frac{K(u)}{u|\nabla h|^2}}{n-1}\langle\nabla|\nabla h|^2,\nabla h\rangle \\ &\quad +(K'-\tfrac{n+1}{n-1}\tfrac{K}{u})|\nabla h|^2-(n-1)H^2|\nabla h|^2. \end{aligned} \tag{A.4}$$

If $|\nabla h|^2$ achieves a local maximum inside $B(q,2R)$ then we are done. Assume $|\nabla h|(q_0)$ is the maximum for some $q_0 \in B(q,2R)$, then $\nabla|\nabla h|^2(q_0)=0, \Delta|\nabla h|^2(q_0)\leq 0$. Plug these into (A.4) gives

$$0 \geq \frac{|\nabla h|^4}{n-1}+(K'-\tfrac{n+1}{n-1}\tfrac{K}{u})|\nabla h|^2-(n-1)H^2|\nabla h|^2.$$

Hence $|\nabla h|^2 \leq (n-1)^2H^2+(n-1)\sup_{B(q,2R)}\left|K'-\frac{n+1}{n-1}\frac{K}{u}\right|$.

In general the maximum could occur at the boundary and one has to use a cut-off function to force the maximum is achieved in the interior. Let $f:[0,2R]\to[0,1]$ be a smooth function with

$$f|_{[0,R]}\equiv 1, \ \mathrm{supp} f \subset [0,2R), \tag{A.5}$$

$$-cR^{-1}f^{1/2}\leq f' \leq 0, \tag{A.6}$$

$$|f''|\leq cR^{-2}, \tag{A.7}$$

where $c>0$ is a universal constant. Let $\phi:\overline{B(q,2R)}\to[0,1]$ with $\phi(x)=f(r(x))$, where $r(x)=d(x,q)$ is the distance function. Set $G=\phi|\nabla h|^2$. Then G is nonnegative on M and has compact support in $B(q,2R)$. Therefore it achieves its maximum at some point $q_0\in B(q,2R)$. We can assume $r(q_0)\in[R,2R)$ since if $q_0\in B(q,R)$, then $|\nabla h|^2$ achieves maximal at q_0 on $B(q,R)$ and previous argument applies.

If q_0 is not a cut point of q then ϕ is smooth at q_0 and we have

$$\Delta G(q_0)\leq 0, \quad \nabla G(q_0)=0. \tag{A.8}$$

At the smooth point of r, $\nabla G=\nabla\phi|\nabla h|^2+\phi\nabla|\nabla h|^2$,

$$\Delta G=\Delta\phi|\nabla h|^2+2\langle\nabla\phi,\nabla|\nabla h|^2\rangle+\phi\Delta|\nabla h|^2.$$

Using (A.4), (A.8) and express $|\nabla h|^2, \nabla|\nabla h|^2$ in terms of $G,\nabla G$ we get, At the maximal point q_0 of G,

$$\begin{aligned}0\geq\Delta G(q_0) \quad &\geq \quad \frac{\Delta\phi}{\phi}G-2\frac{|\nabla\phi|^2}{\phi^2}G+\frac{n}{2(n-1)}\frac{|\nabla\phi|^2}{\phi^2}G+\frac{2}{n-1}\frac{G^2}{\phi} \\ &\quad +\frac{2(n-2+\frac{K(u)}{u|\nabla h|^2})}{n-1}\langle\nabla h,\nabla\phi\rangle\frac{G}{\phi}+2(K'-\tfrac{n+1}{n-1}\tfrac{K}{u})G-2(n-1)H^2G.\end{aligned} \tag{A.9}$$

Since $\phi(x) = f(r(x))$ is a radial function, $|\nabla\phi| = |f'|$, by (A.6),

$$\langle \nabla h, \nabla \phi\rangle \geq -|\nabla h||\nabla\phi| = -G^{\frac{1}{2}}\frac{|\nabla\phi|}{\phi^{1/2}} \geq -G^{\frac{1}{2}}\frac{c}{R}.$$

Also $\Delta\phi = f'\Delta r + f''$. Since $f' \leq 0$, by the Laplacian comparison theorem and (A.7)

$$\Delta\phi \geq f'\Delta_H r - cR^{-2},$$

where $\Delta_H r = (n-1)H\coth(Hr)$, which is $\leq (n-1)H\coth(HR)$ on $[R, 2R]$. Hence

$$\begin{aligned}\Delta\phi &\geq -cR^{-1}\left((n-1)H\coth(HR) + R^{-1}\right)\\ &\geq -cR^{-1}\left[(n-1)(2R^{-1}+4H) + R^{-1}\right]\end{aligned}$$

Multiply (A.9) by $\frac{(n-1)\phi}{G}$ and plug these in, we get

$$\begin{aligned}0 \geq\ & 2G - 2(n-2+|\tfrac{K(u)}{u|\nabla h|^2}|)\frac{c}{R}G^{\frac{1}{2}}\\ &-\frac{c}{R}\left(\frac{(n-1)(2n-1)}{R} + 4(n-1)^2H + \left(\frac{3n}{2}-2\right)\frac{c}{R}\right)\\ &-2(n-1)\left|K' - \tfrac{n+1}{n-1}\tfrac{K}{u}\right| - 2(n-1)^2H^2.\end{aligned}$$

Now if $|\nabla h|^2 \leq \frac{|K(u)|}{u}$, we are done. So we can assume $-\frac{|K(u)|}{u|\nabla h|^2} \geq -1$. In this case we have

$$\begin{aligned}0 \geq\ & 2G - 2(n-1)\frac{c}{R}G^{\frac{1}{2}} - \frac{c}{R}\left(\frac{(n-1)(2n-1)}{R} + 4(n-1)^2H + \left(\frac{3n}{2}-2\right)\frac{c}{R}\right)\\ &-2(n-1)\left|K' - \tfrac{n+1}{n-1}\tfrac{K}{u}\right| - 2(n-1)^2H^2.\end{aligned}$$

Solving this quadratic inequality gives

$$(G(q_0))^{\frac{1}{2}} \leq (n-1)H + C_1(n)R^{-1} + C_2(n,H)R^{-1/2} + \left((n-1)\left|K' - \tfrac{n+1}{n-1}\tfrac{K}{u}\right|\right)^{1/2}.$$

Therefore

$$\begin{aligned}\sup_{B(q,R)}|\nabla h| = \sup_{B(q,R)} G^{1/2} \leq \sup_{B(q,2R)} G^{1/2}\\ \leq (n-1)H + C_1(n)R^{-1} + C_2(n,H)R^{-1/2} + \left[(n-1)\left|K' - \tfrac{n+1}{n-1}\tfrac{K}{u}\right|\right]^{1/2}.\end{aligned}$$

If q_0 is in the cut locus of q, use the upper barrier $r_{q_0,\epsilon}(x)$ for $r(x)$ and let $\epsilon \to 0$ gives the same estimate. □

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