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NEGATIVE RICCI CURVATURE AND ISOMETRY GROUP

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1. Introduction. It is now known that negative Ricci curvature does not imply any topological restriction on the underlying manifold (cf. [L]). It does, however, impose geometric restriction by the classical result of Bochner [Bo], namely, the isometry group must be finite. In this paper we consider a quantitative version of Bochner's theorem.

Consider the class of Riemannian n-manifolds satisfying

$$-\Lambda \leqslant \mathrm{Ric} \leqslant -\lambda < 0, \qquad \mathrm{inj} \geqslant i_0, \qquad \mathrm{vol} \leqslant V. \tag{1}$$

(Here the upper bound on the volume is equivalent to an upper bound on the diameter.) By a result of M. Anderson [A] this class is precompact in $C^{1,\alpha}$ topology. We prove the following theorem.

THEOREM 1.1. *Let M_i be a sequence of n-manifolds satisfying (1) and $C^{1,\alpha}$ convergent to a $C^{1,\alpha}$ Riemannian manifold M. Then*
(a) $\#\{\mathrm{Iso}(M)\} < +\infty$.
(b) $\overline{\lim}_{i\to\infty} \#\{\mathrm{Iso}(M_i)\} \leqslant \#\{\mathrm{Iso}(M)\}$.

An immediate consequence of Theorem 1.1 is the following result, which was obtained by Katsuda [K], with an additional assumption that the sectional curvature is bounded from below.

COROLLARY 1.2. *There is a constant $N = N(n, \lambda, \Lambda, i_0, V)$ such that for any n-dimensional Riemannian manifold M satisfying (1), the order of the isometry group* $\mathrm{Iso}(M)$ *is smaller than N.*

Remark 1. This result can also be considered as a generalization of the Hurwicz Theorem for hyperbolic surfaces, which gives a (very explicit) bound on the order of the isomerty group in terms of the genus.

Remark 2. If one assumes the sectional curvature bound:

$$-\Lambda \leqslant K < 0,$$

then one can drop the injectivity radius lower bound; see [Ym]. Here one can use

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Gromov's uniform estimate on the volume of the fundamental domain of a discrete isometric action (not necessarily free) of such manifolds. Note that the underlying manifold may not be compact.

Remark 3. We refer to [AS], [H], [Im], [M], [K], [Ym] for the previous work.

The crucial step in proving Theorem 1.1 is the following critical gap theorem, similar to the main lemma in [K].

Let $d(\cdot, \cdot)$ be the distance on M induced by the Riemannian metric and $\delta_\phi(p) = d(p, \phi(p))$ be the displacement function of an isometry ϕ.

THEOREM 1.3. *Let M be a compact Riemannian n-manifold satisfying (1). There is a constant $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda, i_0, V)$ such that if an isometry ϕ of M satisfies $\delta_\phi(p) \leqslant \varepsilon_0$ for all p in M, then ϕ is the identity map.*

This result can be viewed as a discrete version of Bochner's theorem. In fact, our proof follows Bochner's original idea. Instead of the norm (square) of the Killing vector field, we consider the displacement function. Thus in Section 4 we derive a Bochner type formula for the displacement function. Besides the Ricci curvature term and a positive term (as in Bochner's formula), we also have an error term involving the curvature tensor and Jacobi fields. The whole point here then becomes the estimate of the error term. Part of this is based on Anderson's regularity theorem, which provides us with a uniform L^p bound on the curvature tensor for the class of manifolds satisfying (1). However there is still some subtlety in applying this result, and this is discussed in Section 6. In Section 5 we establish a uniform estimate for Jacobi fields, based on the recent result of R. Brocks [Br]. The actual estimate of the error term, which is shown to be uniformly small in L^p norm, is presented in Section 6. From this estimate Theorem 1.3 follows by using Sobolev inequality and Hölder inequality.

Remark. As in [K], Corollary 1.2 can also be obtained directly from Theorem 1.3 by covering methods (see [K] for details).

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2. Some general remarks on isometry group. In this section we will observe that the degree of symmetry (defined below) is lower semicontinuous in the Gromov-Hausdorff topology. For a compact Riemannian manifold M, its isometry group $\mathrm{Iso}(M)$ is a compact Lie group. We define the degree of symmetry by

$$\deg_s\colon \mathscr{M} \to \mathbb{N} \cup \{0\}$$

$$\deg_s(M, g) = \dim \mathrm{Iso}(M, g),$$

where dim Iso(M, g) is the dimension of Iso(M, g) as a manifold. Clearly $\deg_s(M, g)$ is equal to the maximal number of linearly independent Killing fields. Thus for a Riemannian manifold (M, g) with negative Ricci curvature, $\deg_s(M, g) = 0$ by Bochner's theorem.

Consider the class of Riemannian n-manifolds satisfying

$$|\mathrm{Ric}| \leqslant \lambda, \qquad \mathrm{inj} \geqslant i_0, \qquad \mathrm{vol} \leqslant V. \tag{2}$$

We have the following result of M. Anderson.

THEOREM 2.1 (M. Anderson [A]). *For all (M, g) in class (2), there exist constants $R_0 = R(n, \lambda, i_0)$ and $C = C(n, \lambda, i_0)$ such that for any $x \in M$ there is a harmonic coordinate system $\{u_j\}_{j=1}^n$ on $B(x, R_0)$ satisfying*

$$C^{-1}\delta_{ij} \leqslant g_{ij} \leqslant C\delta_{ij}$$

and $\|g_{ij}\|_{C^{1,\alpha}} \leqslant C$, where $g_{ij} = g(\nabla u_i, \nabla u_j)$ and the norm is taken with respect to the coordinates $\{u_j\}$ on $B(x, R_0)$.

As pointed out in [A], one can actually get a better uniform bound on the metric tensor via elliptic regularity. This is because with respect to the harmonic coordinates, the Ricci tensor takes a particularly simple form:

$$g^{ij}\frac{\partial g_{rs}}{\partial x_i \partial x_j} + Q\left(\frac{\partial g_{kl}}{\partial x_m}\right) = (\mathrm{Ric})_{rs}, \tag{3}$$

where Q is a quadratic term in the first derivatives of g. Equation (3) is a uniformly elliptic system of PDE for which we have, locally, uniform $C^{1,\alpha}$ bounds on the coefficients g^{ij}, $C^{0,\alpha}$ bounds on the quadratic term Q, and L^∞ bounds on the right-hand side. The elliptic regularity theory then gives a uniform bound on $\|g\|_{L^{2,p}}$, $\forall p < \infty$. (Of course the uniform bound may depend on p.) This fact is very important here, and we will be using it in an essential way.

Using this we prove the following lemma.

LEMMA 2.2. *Let $\{X^{(i)}\}$ be an L^2-bounded sequence of Killing fields of (M_i, g_i) with (M_i, g_i) satisfying (2) and converging to a $C^{1,\alpha}$ Riemannian manifold (M_∞, g_∞) in the $C^{1,\alpha}$ topology. Then the Killing fields $\{X^{(i)}\}$ sub-converge to a Killing field of (M_∞, g_∞).*

Proof. We first show a uniform $C^{1,\alpha}$ bound for Killing fields. Let ξ be the 1-form dual to a Killing field X of any manifold satisfying (2). Then Bochner's formula gives

$$\Delta\xi = 2\sum R_{ij}\xi^j\, dx^i.$$

By Theorem 2.1 (and the above discussion) R_{ij} has a uniform L^p bound ($\forall p < \infty$).

It follows from the elliptic regularity that we have a uniform $C^{1,\alpha}$ bound for ξ, hence for X. It follows then from Arzela-Ascoli's theorem that the Killing fields $\{X^{(i)}\}$ have a subsequence converging in $C^{1,\alpha'}$ topology to a $C^{1,\alpha}$ vector field X for any $\alpha' < \alpha$. That X is a Killing field follows from the fact that a vector field X is Killing if and only if it satisfies the following first order equation

$$\nabla_j \xi_i + \nabla_i \xi_j = 0. \qquad \blacksquare$$

PROPOSITION 2.3. *The function* $\deg_s(M, g)$ *is lower semicontinuous in the Gromov-Hausdorff topology. More precisely, for any sequence of manifolds* (M_i, g_i) *in the class (2) converging to* (M_∞, g_∞) *in the* $C^{1,\alpha}$ *topology,*

$$\overline{\lim}_{i\to\infty} \deg_s(M_i, g_i) \leqslant \deg_s(M_\infty, g_\infty). \tag{4}$$

Proof. If it is not true, we will have a subsequence (denoted by M_i too) with

$$\deg_s(M_i, g_i) \geqslant \deg_s(M_\infty, g_\infty) + 1$$

for i sufficiently large. That means we can find $N = \deg_s(M_\infty, g_\infty) + 1$ linearly independent Killing fields $X_1^{(i)}, \ldots, X_N^{(i)}$ of (M_i, g_i) for i sufficiently large. By normalizing them with respect to the L^2-norm induced by g_i we can assume $\{X_1^{(i)}, \ldots, X_N^{(i)}\}$ is orthonormal. By Lemma 2.2 the sequence $\{X_1^{(i)}\}$ has a subsequence converging to a Killing field X_1 of (M_∞, g_∞). By restricting to this subsequence we find a sub-subsequence of $\{X_1^{(i)}, X_2^{(i)}\}$ converging to a pair of Killing fields $\{X_1, X_2\}$. Continuing this process we see that by passing to a subsequence, $\{X_1^{(i)}, \ldots, X_N^{(i)}\}$ converges to $\{X_1, \ldots, X_N\}$, each one being a Killing field of (M_∞, g_∞). These will also be orthonormal, since by Theorem 2.1, one can take the $C^{1,\alpha}$ topology, under which the L^2-norm is clearly preserved. Thus we actually obtain $\deg_s(M_\infty, g_\infty) + 1$ linearly independent Killing fields of (M_∞, g_∞), a contradiction. This completes the proof of the proposition. $\blacksquare$

Remark. Intuitively this result says that the limit manifold will inherit all possible symmetry from the sequence.

3. Negative Ricci curvature. As we mentioned in the previous section, when (M, g) is negatively Ricci curved, we have $\deg_s(M, g) = 0$. In this case, the next quantity to measure "the degree of symmetry" will be the order of the isometry group $\mathrm{Iso}(M)$. The content of Theorem 1.1 is that this "secondary" measure of degree of symmetry is also lower semicontinuous.

We prove Theorem 1.1 in this section using Theorem 1.3. The proof of Theorem 1.3 will be presented in the remaining sections.

First we need some regularity result about isometry, due to Calabi-Hartman.

THEOREM 3.1 (Calabi-Hartman [CH]). *Let* (M, g) *be a connected Riemannian manifold and* $\phi \in \mathrm{Iso}(M, g)$. *Then the* $C^{k,\alpha}$*-norm of* ϕ *is bounded by the* $C^{k-1,\alpha}$*-norm of* g *for all* $k \geqslant 1, \alpha > 0$.

Proof of Theorem 1.1. Part (a). Without loss of generality, we can assume that $M_i = M$ for all i as smooth manifolds, and (instead) we have a sequence of smooth metrics g_i satisfying (1), $C^{1,\alpha'}$-converging to a $C^{1,\alpha}$-metric g for all $\alpha' < \alpha$. By the theorem of Calabi-Hartman, $\mathrm{Iso}(M, g)$ is a Lie group acting $C^{2,\alpha}$ on M. Now for a C^1 vector field X and a smooth metric h on M, an integral version of Bochner's formula states that (see [Y, page [41]])

$$\int_M \left\{ \mathrm{Ric}_h(X, X) + \frac{1}{2}|\mathscr{L}_X h|^2 - |\nabla^h X|^2 - (\mathrm{div}_h X)^2 \right\} dv = 0. \tag{5}$$

Let $h = g_i$ and apply (5) to a Killing field X of g. Since $\mathrm{Ric} \leqslant -\lambda$, we have

$$\int_{M_i} \left\{ \lambda |X|^2 + |\nabla^{g_i} X|^2 + (\mathrm{div}_{g_i} X)^2 - \frac{1}{2}|\mathscr{L}_X g_i|^2 \right\} dv \leqslant 0.$$

Passing to the limit yields

$$\int_M \{\lambda |X|^2 + |\nabla^g X|^2 + (\mathrm{div}_g X)^2\}\, dv \leqslant 0.$$

This implies that $X = 0$. Thus $\mathrm{Iso}(M, g)$ is finite.

Part (b). We follow the idea of the proof of Lemma 2.2 and Proposition 2.3. Namely (A) we show that every sequence of isometries of g_i has a subsequence converging to an isometry of g; (B) if none of the isometries in the sequence is the identity then neither is the limit.

Again, as in the proof of Lemma 2.2, (A) follows from the regularity result of Calabi-Hartman. The regularity result of g in class (1) again follows from Theorem 2.1 of M. Anderson.

Combining these two we see that the $C^{2,\alpha}$-norm of isometries ϕ for (M, g) in class (1) is uniformly bounded by a constant depending only on λ, Λ, i_0, V. Hence by Arzela-Ascoli's theorem any sequence of isometries has a $C^{2,\alpha'}$ ($\alpha' < \alpha$) convergent subsequence. Since isometries are characterized by the following first-order differential equation

$$\phi^* g = g,$$

it follows that the limit must also be an isometry for the limiting metric, proving (A).

(B) is an immediate consequence of Theorem 1.3 if we note that the displacement function is a continuous function with respect to the Gromov-Hausdorff topology in class (1). ∎

4. A Bochner type formula. We now turn to the proof of Theorem 1.3. As mentioned earlier, we follow Bochner's original idea, but replace the norm square of a Killing field by the displacement function δ_ϕ of an isometry. (The norm of a

Killing field can be thought of as an infinitesimal analogue of the displacement function.) Thus we would like to compute the Laplacian of δ_ϕ where $\delta_\phi(p) = d(p, \phi(p))$.

For a point $p \in M$ with $\phi(p) \neq p$ but $\delta_\phi(p) \leqslant i_0/2$, let $\gamma(t)$ be the unique minimal geodesic connecting p and $\phi(p)$, and $T = \gamma'(t)$. Take an orthonormal basis $\{V_1 = T, V_2, \ldots, V_n\}$ for T_pM. Then by definition

$$(\Delta\delta_\phi)(p) = \operatorname{tr} \operatorname{Hess} \delta_\phi = \sum_{i=1}^{n} \operatorname{Hess}_{\delta_\phi}(V_i, V_i).$$

Let $\{\tilde{V}_i(t)\}$ be the Jacobi fields along γ such that $\tilde{V}_i(0) = V_i$ and $\tilde{V}_i(\delta_\phi) = \phi_*(V_i)$. Then by the second variational formula [CE]

$$\operatorname{Hess}_{\delta_\phi}(V_i, V_i) = \int_0^{\delta_\phi(p)} \|\nabla_T \tilde{V}_i \wedge T\|^2 - \langle R(\tilde{V}_i, T)T, \tilde{V}_i\rangle,$$

where $\|\nabla_T \tilde{V}_i \wedge T\|^2 = \langle \nabla_T \tilde{V}_i, \nabla_T \tilde{V}_i\rangle - \langle \nabla_T \tilde{V}_i, T\rangle^2$, which is nonnegative by the Cauchy-Schwarz inequality. Therefore

$$\Delta\delta_\phi = \sum_{i=1}^{n} \int_0^{\delta_\phi(p)} \|\nabla_T \tilde{V}_i \wedge T\|^2 - \langle R(\tilde{V}_i, T)T, \tilde{V}_i\rangle.$$

Write

$$\tilde{V}_i(t) = V_i(t) + W_i(t), \tag{6}$$

where $V_i(t)$ is the parallel translation along $\gamma(t)$ of the vector V_i at p. Then $W_i(0) = 0$, and we obtain the following lemma.

LEMMA 4.1. *Let $\phi \in \operatorname{Iso}(M)$ and $\delta_\phi(p) \leqslant i_0/2$ for all $p \in M$. Then outside the fixed point set of ϕ,*

$$\Delta\delta_\phi(p) = -\int_0^{\delta_\phi} \operatorname{Ric}(T) + \sum_{i=1}^{n} \int_0^{\delta_\phi} \|\nabla_T \tilde{V}_i \wedge T\|^2 - F(R, W_i), \tag{7}$$

with

$$\begin{aligned} F(R, W_i) &= \langle R(W_i, T)T, W_i\rangle + 2\langle R(W_i, T)T, V_i\rangle \\ &= K(W_i, T)\|W_i \wedge T\|^2 + 2\sum_{j=1}^{n} \langle R(V_j, T)T, V_i\rangle \langle W_i, V_j\rangle. \end{aligned} \tag{9}$$

In the above formula, $\{W_i\}$ measures the deviation of the Jacobi fields $\{\tilde{V}_i\}$ from being orthonormal. We note that the error term $F(R, W_i)$ involves both the (shifted) curvature tensor and W_i. For the (unshifted) curvature tensor, Anderson's theorem provides us with an L^p bound. On the other hand, when $\max_{p \in M} \delta_\phi(p)$ is small, W_i should also be small. Thus one expects the error term $F(R, W_i)$ to be small relative to $\max \delta_\phi$. Indeed, we have the following.

LEMMA 4.2. *For M in the class (1) and any $\delta > 0$, there exists an $\varepsilon = \varepsilon(\delta, n, \lambda, \Lambda, i_0, V)$ such that if $\max_{p \in M} \delta_\phi(p) \leqslant \varepsilon$, then we have, in the sense of distribution*

$$\Delta\delta_\phi \geqslant \lambda\delta_\phi - \delta G\delta_\phi, \tag{10}$$

where G is a function on M with a uniform L^p bound

$$\|G\|_p \leqslant C_p(n, \lambda, \Lambda, i_0, V)$$

for all $1 \leqslant p < +\infty$.

The proof of this main lemma occupies the next two sections. Here we show Theorem 1.3, assuming Lemma 4.2.

Proof of Theorem 1.3. Without loss of generality, we may assume that $n \geqslant 3$. The Sobolev inequality states that

$$\|f\|^2_{2n/(n-2)} \leqslant C_0(\|\nabla f\|^2_2 + \|f\|^2_2), \tag{11}$$

with $C_0 = C_0(n, \Lambda, i_0, V)$ (see [Be]). Take $p = n/2$ and $\delta = (1/2)\min((\lambda/C_0 C_{n/2}(n, \lambda, \Lambda, i_0, V)), (8(n-2)/n^2)(1/C_0 C_{n/2}(n, \lambda, \Lambda, i_0, V)))$. By Lemma 4.2 there exists an $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda, i_0, V)$ such that (10) holds. Multiply it by $\delta_\phi^{(n-2)/2}$ and integrate by parts, and we obtain

$$\frac{8(n-2)}{n^2}\|\nabla\delta_\phi^{n/4}\|^2_2 + \lambda\|\delta_\phi^{n/4}\|^2_2 \leqslant \delta\int_M G\delta_\phi^{n/2} d\,\text{vol}. \tag{12}$$

Applying the Hölder inequality and the Sobolev inequality (11) to the right side of (12), one has

$$\begin{aligned}\delta\int_M G\delta_\phi^{n/2} d\,\text{vol} &\leqslant \delta\|G\|_{n/2}\|\delta_\phi^{n/4}\|^2_{2n/(n-2)} \\ &\leqslant C_1\delta(\|\nabla\delta_\phi^{n/4}\|^2_2 + \|\delta_\phi^{n/4}\|^2_2)\end{aligned} \tag{13}$$

with $C_1 = C_0 C_{n/2}(n, \lambda, \Lambda, i_0, V)$. It follows from (12) and (13) that

$$(\lambda - C_1\delta)\|\delta_\phi^{n/4}\|^2_2 \leqslant \left(C_1\delta - \frac{8(n-2)}{n^2}\right)\|\nabla\delta_\phi^{n/4}\|^2_2.$$

Clearly by our choice of δ, we must have $\delta_\phi \equiv 0$. This completes the proof. ■

5. Uniform estimates of Jacobi fields. In this section, we assume that the manifolds M^n are in the class

$$\mathrm{Ric} \geqslant -\Lambda, \qquad \mathrm{inj} \geqslant i_0 . \tag{14}$$

In the presence of sectional curvature bound, the (uniform) bound on Jacobi fields is an immediate consequence of the Rauch comparison theorem. What is a little surprising here is that one also obtains a uniform bound on Jacobi fields in terms of a lower bound on Ricci curvature and a lower bound on the injectivity radius.

PROPOSITION 5.1. *For the Jacobi fields $\tilde{V}_i(t)$ defined in Section 4, if $\delta_\phi(p) \leqslant i_0/2$ we have the following uniform bound: (we are suppressing the subscript i)*

$$\|\tilde{V}\| \leqslant C(n, \Lambda, i_0).$$

In particular,

$$\|W\| \leqslant C(n, \Lambda, i_0).$$

This proposition follows from some more general estimate for Jacobi fields vanishing at a point. First we need to introduce some further notions. Let $p_0 \in M$ be a fixed point, and $r(p) = d(p_0, p)$ the distance function from p_0. The Hessian of r is also the second fundamental form of its level surfaces, i.e., the geodesic spheres. We denote $A = \mathrm{Hess}\, r$. Now if $c(t)$ is a minimal geodesic starting at p_0 and $J(t)$ is a Jacobi field along c such that $J(0) = 0$, $\langle J'(0), T\rangle = 0$. Then

$$J' = AJ.$$

Write $A(t) = B(t) + (1/t)I$. (Note that $A(t)$ has a singularity at $t = 0$. In fact $B(t)$ is exactly the regular part; cf. [Br].) First we prove the following estimate for Jacobi fields.

LEMMA 5.2. *For Jacobi fields constructed above, we have*

$$\|J(t)\| \leqslant \left(\frac{t}{t_0}\right) e^{\int_t^{t_0} \|B(s)\| ds} \|J(t_0)\|, \qquad 0 \leqslant t \leqslant t_0 \tag{15}$$

for any fixed $0 < t_0 < i_0$.

Proof. Write $J(t) = tU(t)$. Then for fixed t_0, $U(t)$ satisfies the following equations:

$$U'(t) = B(t)U(t)$$

$$U(0) = J'(0), \qquad U(t_0) = \frac{J(t_0)}{t_0}.$$

Let $\bar{U}(t) = U(t_0 - t)$. We have

$$\bar{U}'(t) = -B(t_0 - t)\bar{U}(t)$$

$$\bar{U}(0) = \frac{J(t_0)}{t_0}.$$

Then

$$\|\bar{U}'(t)\| \leqslant \|B(t - t_0)\| \, \|\bar{U}(t)\|,$$

which gives

$$\|\bar{U}(t)\| \leqslant e^{\int_0^t \|B(t_0 - s)\| ds} \frac{\|J(t_0)\|}{t_0}.$$

Thus

$$\|U(t)\| \leqslant e^{\int_t^{t_0} \|B(s)\| ds} \frac{\|J(t_0)\|}{t_0},$$

and

$$\|J(t)\| \leqslant \left(\frac{t}{t_0}\right) e^{\int_t^{t_0} \|B(s)\| ds} \|J(t_0)\|. \qquad \blacksquare$$

The estimate of $B(s)$ involved in the above lemma is based on the following sharp lower-bound estimate of the Laplacian of the distance function, due to R. Brocks [Br].

PROPOSITION 5.3 (R. Brocks). *There exists a constant $C(n, \Lambda, i_0)$ such that*

$$0 \leqslant ct_\Lambda(t) - \frac{\operatorname{tr} A}{n-1} \leqslant C(n, \Lambda, i_0) \qquad \textit{for } 0 \leqslant t \leqslant \frac{i_0}{2},$$

where $ct_\Lambda(t) = \sqrt{\Lambda} \cosh \sqrt{\Lambda} t / \sinh \sqrt{\Lambda} t$.

We refer to [Br, Satz 5.6] for the proof. The first inequality is of course well known. (Cf. also [AC] for a weaker lower-bound estimate on the Laplacian of the distance function.)

For our purpose, we rewrite it as

$$C_1(n, \Lambda, i_0) \leqslant \frac{\operatorname{tr} A}{n-1} - \frac{1}{t} \leqslant C_2(n, \Lambda, i_0) \qquad \text{for } 0 \leqslant t \leqslant \frac{i_0}{2}.$$

Now we can prove Proposition 5.1.

Proof of Proposition 5.1. For a Jacobi field J vanishing at $t = 0$ and perpendicular to $c(t)$, we have $J' = (B + (1/t)I)J$. From the Jacobi equation

$$J'' + R(J, T)T = 0,$$

we have

$$B' + B^2 + \frac{2}{t}B + R = 0,$$

where $R = R(\cdot, T)T$. Therefore

$$\operatorname{tr} B' + \|B\|^2 + \frac{2}{t}\operatorname{tr} B + \operatorname{Ric}(\dot{c}) = 0. \tag{16}$$

Multiply (16) by a factor $t^{1/2}$ and integrate along $c(t)$:

$$\begin{aligned}\int_0^{t_0} t^{1/2}\|B(t)\|^2\,dt &= -t_0^{1/2}\operatorname{tr} B(t_0) - \frac{3}{2}\int_t^{t_0} t^{-1/2}\operatorname{tr} B(t)\,dt - \int_0^{t_0} t^{1/2}\operatorname{Ric}(\dot{c})\,dt\\ &\leqslant C(n, \Lambda, i_0).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{t_0}\|B(t)\|\,dt &\leqslant \left(\int_0^{t_0} t^{-1/2}\,dt\right)^{1/2}\left(\int_0^{t_0} t^{1/2}\|B(t)\|^2\,dt\right)^{1/2}\\ &\leqslant C(n, \Lambda, i_0).\end{aligned} \tag{17}$$

Combining (15), we have

$$\|J(t)\| \leqslant C(n, \Lambda, i_0)\frac{t}{t_0}\|J(t_0)\| \leqslant C(n, \lambda, i_0)\|J(t_0)\|, \qquad 0 \leqslant t \leqslant t_0 \leqslant \frac{i_0}{2}. \tag{18}$$

In fact, this estimate holds for any Jacobi field which vanishes at the starting point. This can be easily seen by decomposing the Jacobi field into tangential and perpendicular components and noting that the tangential component is linear in t. But $\tilde{V}(t)$ can be written as the sum, $\tilde{V}(t) = \tilde{V}_1(t) + \tilde{V}_2(t)$, where $\tilde{V}_i(t)$ are Jacobi fields along γ with $\tilde{V}_1(0) = 0$, $\tilde{V}_1(\delta_\phi) = \phi_*(V)$, $\tilde{V}_2(0) = V$, $\tilde{V}_2(\delta_\phi) = 0$. Applying the above estimates to $\tilde{V}_i$ we obtain Proposition 5.1. ∎

Remark. Proposition 5.3 can also be used to derive a Toponogov-type comparison estimate for Ricci curvature; for more detail, see [DW].

It turns out that this estimate is still not good enough for our purpose (especially for applying Anderson's theorem). We now improve the estimate to the following.

PROPOSITION 5.4. *Using the same notation, we have*

$$\|\tilde{V} - V\| = \|W\| \leqslant C(n, \Lambda, i_0)(\delta_\phi^{1/2} + \|W(\delta_\phi)\|)$$

provided $\delta_\phi \leqslant i_0/4$.

We divide the proof into two lemmas.

LEMMA 5.5. *Let* $\bar{V}$ *be the Jacobi field along* γ *such that* $\bar{V}(0) = V$, $\bar{V}(-i_0/4) = 0$. *Then*

$$\|\bar{V} - V\| \leqslant C(n, \Lambda, i_0)\delta_\phi^{1/2}, \qquad \text{for } t \in [0, \delta_\phi],$$

provided $\delta_\phi \leqslant i_0/4$.

LEMMA 5.6. *We also have*

$$\|\tilde{V} - \bar{V}\| \leqslant C(n, \Lambda, i_0)(\delta_\phi^{1/2} + \|W(\delta_\phi)\|), \qquad \text{for } t \in [0, \delta_\phi],$$

provided $\delta_\phi \leqslant i_0/4$.

Proof of Lemma 5.5. Let $r_1(p) = d(p, \gamma(-i_0/4))$ be the distance function from $\gamma(-i_0/4)$. Let A be its Hessian, and $A(t) = A(\gamma(t))$ for $t \in [-i_0/4, i_0/4]$. Then

$$\bar{V}' = A\bar{V}.$$

Therefore

$$\|\bar{V}(t)\| \leqslant \|\bar{V}(0)\| e^{\int_0^{\delta_\phi} \|A(\tau)\| d\tau} \leqslant C(n, \Lambda, i_0), \qquad \text{for } t \in [0, \delta_\phi].$$

This will give the estimate we want, since

$$\begin{aligned} |\bar{V}(t) - \bar{V}(0)\| &\leqslant \int_0^t \|A(\tau)\| \, \|\bar{V}(\tau)\| \, d\tau \\ &\leqslant \left(\int_0^t \|A(\tau)\|^2 \, d\tau\right)^{1/2} \left(\int_0^t \|\bar{V}(\tau)\|^2 \, d\tau\right)^{1/2} \\ &\leqslant C(n, \Lambda, i_0) t^{1/2} \qquad \text{for all } t \in [0, \delta_\phi]. \end{aligned}$$

This finishes the proof of Lemma 5.5. ∎

Proof of Lemma 5.6. $\underline{V}(t) = \tilde{V}(t) - \bar{V}(t)$ is still a Jacobi field. Moreover, $\underline{V}(0) = 0$, $\underline{V}(\delta_\phi) = \phi_*(V) - \bar{V}(\delta_\phi)$. Note that

$$\|\underline{V}(\delta_\phi)\| \leqslant \|\phi_*(V) - V(\delta_\phi)\| + \|\bar{V}(\delta_\phi) - V(\delta_\phi)\|.$$

Now

$$\|\phi_*(V) - V(\delta_\phi)\| = \|W(\delta_\phi)\|. \tag{19}$$

By Lemma 5.5,

$$\|\bar{V}(\delta_\phi) - V(\delta_\phi)\| \leqslant C(n, \Lambda, i_0)\delta_\phi^{1/2}. \tag{20}$$

Combining (19) and (20), we have

$$\|\underline{V}(\delta_\phi)\| \leqslant C(n, \Lambda, i_0)(\delta_\phi^{1/2} + \|W(\delta_\phi)\|).$$

Now the estimate (18) yields

$$\|\underline{V}(t)\| \leqslant C(n, \Lambda, i_0)(\delta_\phi^{1/2} + \|W(\delta_\phi)\|). \qquad \blacksquare$$

6. Estimate of the error term. Our error term is

$$G(R, W) = \sum_{i=1}^{n} \int_0^{\delta_\phi} K(W_i, T)\|W_i \wedge T\|^2\, dt + \sum_{i,j=1}^{n} \int_0^{\delta_\phi} 2\langle R(V_j, T)T, V_i\rangle\langle W_i, V_j\rangle\, dt.$$

We have shown that W_i is small if δ_ϕ is small. In the light of Anderson's theorem, Lemma 4.2 seems immediate. Unfortunately we cannot apply Anderson's result directly here, as we explain now.

Anderson's theorem gives

$$\int_M \max_{\|X_i\|=1} |\langle R_p(X_1, X_2)X_3, X_4\rangle|^q d\,\mathrm{vol}(p) \leqslant C_q(n, \lambda, \Lambda, i_0, V), \qquad \forall q < \infty.$$

Here R_p denotes the curvature tensor at the point p.

On the other hand, for fixed t, we can define

$$f_t\colon M \to M$$
$$p \to \exp_p t \exp_p^{-1} \phi(p).$$

This is a well-defined C^∞-map if $\delta_\phi \leqslant i_0/2$. We note that $f_0 = \mathrm{Id}$, $f_1 = \phi$.

By a change of variable, we have

$$G(R, W) = \delta_\phi\biggl(\sum_{i=1}^{n} \int_0^1 K(W_i(t\delta_\phi), T(t\delta_\phi)) \, \|W_i(t\delta_\phi) \wedge T(t\delta_\phi)\|^2 \, dt$$

$$+ \sum_{i,j=1}^{n} \int_0^1 2\langle R(V_j(t\delta_\phi), T(t\delta_\phi))T(t\delta_\phi), V_i(t\delta_\phi)\rangle \langle W_i(t\delta_\phi), V_j(t\delta_\phi)\rangle \, dt\biggr).$$

From Proposition 5.4,

$$\|W\| \leqslant C(n, \Lambda, i_0)(\delta_\phi^{1/2} + \|W(\delta_\phi)\|).$$

The following lemma shows that $\|W(\delta_\phi)\|$ is also controlled by $\max_{p \in M} \delta_\phi(p)$.

LEMMA 6.1. *For all manifolds in class (2) and any $\delta > 0$, there exists a constant $\varepsilon = \varepsilon(\delta, n, \lambda, i_0, V)$ such that if* $\max \delta_\phi < \varepsilon$, *then* $\|W(\delta_\phi)\| < \delta$.

Proof. It follows from the fact that ϕ is an isometry of M, and the class (2) is $C^{1,\alpha}$ compact. ■

Thus, if $\max \delta_\phi < \varepsilon\ (<\delta^2)$,

$$|G(R, W)(p)| \leqslant \delta_\phi(p) C(n, \lambda, \Lambda, i_0, V)\delta \int_0^1 |R_{\max}(f_t(p))| \, dt;$$

here we have used the shorthand

$$|R_{\max}(f_t(p))| = \max_{\|X_i\|=1} |\langle R_{f_t(p)}(X_1, X_2)X_3, X_4\rangle|.$$

Let $G(p) = C(n, \Lambda, i_0) \int_0^1 |R_{\max}(f_t(p))| \, dt$. We want to show that

$$\|G\|_q \leqslant C_q(n, \lambda, \Lambda, i_0, V).$$

We first establish the following lemma.

LEMMA 6.2. *Let $f(x, t)$ be a nonnegative function and*

$$F(x) = \left(\int_0^1 f^p(x, t) \, dt\right)^N.$$

Then

$$\|F\|_q \leqslant \left(\int_0^1 \|f(\cdot, t)\|_{Npq}^p \, dt\right)^N \leqslant \max_{0 \leqslant t \leqslant 1} \|f(\cdot, t)\|_{Npq}^{pN}.$$

Proof. This follows from Minkowski's inequality for integrals, which states

$$\left(\int_Y \left(\int_X |f(x, y)|\, dx\right)^p dy\right)^{1/p} \leqslant \int_X \left(\int_Y |f(x, y)|^p\, dy\right)^{1/p} dx. \qquad \blacksquare$$

Thus it remains to show that

$$\max_{0 \leqslant t \leqslant 1} \int |R_{\max}(f_t(p))|^q d\, \mathrm{vol}(p) \leqslant C_q(n, \lambda, \Lambda, i_0, V).$$

The left-hand side can be rewritten as

$$\max_{0 \leqslant t \leqslant 1} \int |R_{\max}(f_t(p))|^q \frac{1}{\mathrm{Jac}(f_t)} d\, \mathrm{vol}(f_t(p)).$$

Thus the desired estimate would follow from the change of variable formula and Anderson's theorem if we show the following.

THEOREM 6.3. *We have if* $\max \delta_\phi \leqslant \varepsilon$,

$$1 - C(n, \lambda, \Lambda, i_0, V)\delta \leqslant \mathrm{Jac}(f_t) \leqslant 1 + C(n, \lambda, \Lambda, i_0, V)\delta.$$

In particular, $f_t (0 \leqslant t \leqslant 1)$ *is a diffeomorphism when we take* $\delta < 1/2C(n, \lambda, \Lambda)$.

Proof. We note that

$$(f_t)_*(V) = \tilde{V}(t\delta_\phi).$$

Thus the lemma follows from the estimate of Proposition 5.4 and Lemma 6.1. $\blacksquare$

Combining the above discussions, we have shown that for any $\delta > 0$ sufficiently small, there exists an $\varepsilon = \varepsilon(\delta, n, \lambda, \Lambda, i_0, V)$ such that if $\max \delta_\phi < \varepsilon$,

$$\Delta\delta_\phi(p) \geqslant \lambda\delta_\phi(p) - \delta G\delta_\phi(p), \tag{21}$$

for all p such that $\delta_\phi(p) > 0$.

In order to finish the proof of Lemma 4.2, we now show that (21) holds everywhere in the sense of distribution, i.e.,

$$\int_M (\nabla\delta_\phi\nabla\psi + \lambda\delta_\phi\psi - \delta G\delta_\phi\psi)\, dv \leqslant 0, \tag{22}$$

for all $\psi \in C^\infty(M)$. Since the set $A := \{\delta_\phi(x) = 0\}$ is the set of fixed points of ϕ, A is a totally geodesic submanifold of M, which of course has zero measure. Thus it suffices to prove the following

$$\int_M \Delta\delta_\phi\psi\, dv = -\int_M \nabla\delta_\phi\nabla\psi\, dv \tag{23}$$

for all $\psi \in C^\infty(M)$. Let A_ε denote the ε-neighborhood of A in M. One has

$$\int_{M\backslash A_\varepsilon} \Delta\delta_\phi\psi\, dv = \int_{\partial A_\varepsilon} \psi\frac{\partial\delta_\phi}{\partial n}\, ds - \int_{\partial A_\varepsilon} \nabla\delta_\phi\nabla\psi\, dv. \tag{24}$$

Since δ_ϕ is a Lipschitz function with Lipschitz constant 2, the first term on the right of (24) goes to zero as $\varepsilon \to 0$. Then one obtains (23). This completes the proof of (22).

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