16

Universal Covers of Ricci Limit and RCD Spaces

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Abstract

By Gromov's precompactness theorem, any sequence of n-dimensional manifolds with uniform Ricci curvature lower bound has a convergent subsequence. The limit spaces are referred to as Ricci limit spaces. Cheeger–Colding–Naber developed great regularity and geometric properties for Ricci limit spaces. However, unlike Alexandrov spaces, these spaces could locally have infinite topological type. Sormani and Wei [44, 46] gave the first topological result by showing that the universal cover of any Ricci limit space exists. Here, the universal cover is in the sense of the universal covering map (which does not need to be simply connected). This is extended to RCD spaces by Mondino–Wei [35]. Recently Pan–Wei [38] showed that the non-collapsing Ricci limit spaces are semi-locally simply connected and therefore the universal covers are simply connected. We give a survey of these results and pose some questions.

16.1 Introduction

In 1981 Gromov proved a precompactness result (see Theorem 16.2.3) for Gromov–Hausdorff topology, which revolutionized the field of Riemannian geometry. The Gromov–Hausdorff distance between two metric spaces, roughly speaking, is a coarse measure of their alikeness (see Definition 16.2.2). It provides a platform to study the set of all manifolds with curvature bounds and other geometric conditions (for example, diameter, volume) as a whole. For a Gromov–Hausdorff convergent sequence $M_i \xrightarrow{GH} X$, understanding the structure of the limit space X and the links between the geometrical and topological structures of X and M_i are crucial. When the sequence satisfies a lower sectional curvature bound, the limit spaces are the so called Alexandrov spaces [7]. Alexandrov spaces are locally contractible. On the other hand, the Ricci limit space may have infinite second homology even in the non-collapsing case, see [33]. Around 2000, Cheeger and Colding developed a rich theory on Ricci limit spaces [10, 11, 12, 13]. In recent years, Cheeger, Colding, and Naber further deepened this theory [18, 14, 15]. These results offer powerful tools in understanding Ricci curvature. On the other hand, very little is known about the topology of the Ricci limit spaces. Since Ricci curvature has control on the fundamental groups, it is natural to ask the following question:

Question 16.1.1 Is any Ricci limit space always semi-locally simply connected?

As a first step, Sormani and Wei proved the following.

Theorem 16.1.2 ([44, 46]) *If X is the Gromov–Hausdorff limit of a sequence of complete Riemannian manifolds M_i^n with Ricci curvature $\geq K$, then X has a universal cover.*

In [20, 21] descriptions of the universal cover in terms of the universal covers of the sequence are also given for Ricci limit spaces. Recently Theorem 16.1.2 has been generalized to RCD spaces.

Theorem 16.1.3 ([35]) *Any* $\mathsf{RCD}^*(K,N)$ *space* $(X,\mathsf{d},\mathfrak{m})$ *admits a universal cover* $(\widetilde{X},\widetilde{\mathsf{d}},\widetilde{\mathfrak{m}})$*, which is itself* $\mathsf{RCD}^*(K,N)$*, where* $K\in\mathbf{R}$*, $N\in(1,+\infty)$.*

Note that the universal cover is not assumed to be simply connected, see Definition 16.3.2. Very recently, the authors were able to answer Question 16.1.1 positively in the non-collapsing case.

Theorem 16.1.4 ([38]) *Any non-collapsing Ricci limit space is semi-locally simply connected. Therefore the universal cover, is simply connected.*

With the existence of the universal cover, many results about the fundamental groups of manifolds with Ricci curvature bounded from below can be extended to the deck transformation groups on the universal of the Ricci limit and RCD spaces, see [44, 46, 35]. For the non-collapsing Ricci limit spaces, the results are extended to the fundamental group (see [38]).

In this chapter we present the ideas of these developments. In Section 16.2, we review some basic properties of Ricci limit and RCD spaces. In Section 16.3, we introduce the tools and sketch the ideas used in proving Theorems 16.1.2 and 16.1.3. In Section 16.4, we sketch the idea of the proof for Theorem 16.1.4.

Question 16.1.1 remains open for general Ricci limit spaces. One can also ask the same question for $\mathsf{RCD}(K,N)$ spaces. Some more questions are presented in Sections 16.3 and 16.4.

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16.2 Some Properties of Ricci Limit and RCD Spaces

In this section we review some basic properties for Ricci limit spaces developed by Cheeger–Colding [10, 11, 12, 13], many of which have been extended to RCD spaces by various authors.

First we recall Hausdorff and Gromov–Hausdorff distance, see [40, Chap. 11] for further details.

Definition 16.2.1 Let X and Y be two compact subsets in a metric space Z. We define the Hausdorff distance between X and Y as

$$d_H(X,Y) = \inf\{\varepsilon > 0 | X \subset B_\varepsilon(Y), Y \subset B_\varepsilon(X)\},$$

where $B_\varepsilon(X)$ is the ε-neighborhood of X in Z.

Definition 16.2.2 Let X and Y be two compact metric spaces. We define

$$d_{GH}(X,Y) = \inf_{Z,f,g}\{d_H(f(X),g(Y)) \mid f : X \to Z \text{ and } g : Y \to Z \text{ are isometric embeddings}\},$$

where the infimum is taken over all metric spaces Z and all isometric embeddings f, g.

So, d_{GH} defines a distance function on the set of all isometric classes of compact metric spaces. We say that a sequence of compact metric spaces X_i converges in the Gromov–Hausdorff topology to a limit space X, denoted as $X_i \xrightarrow{GH} X$, if $d_{GH}(X_i, X) \to 0$. For noncompact spaces, we say that a sequence of pointed metric spaces (X_i, x_i) converges to (X,x) in the pointed Gromov–Hausdorff sense if $B(x_i,R) \xrightarrow{GH} B(x,R)$

for all $R > 0$, where $B(x, R)$ is the closed ball centered at x with radius R with the restricted metric.

The starting point for this subject is Gromov's precompactness theorem (see [28]).

Theorem 16.2.3 *Let $\{(M_i, x_i)\}_i$ be a sequence of complete Riemannian n-manifolds of*

$$\mathrm{Ric}_{M_i} \geq -(n-1),$$

then $\{(M_i, x_i)\}_i$ has a convergent subsequence with respect to the pointed Gromov–Hausdorff distance.

The Gromov–Hausdorff limits, referred as Ricci limit spaces, are length spaces. Recall that a length space is a metric space such that the distance between each pair of points equals the infimum of the length of the curves joining the points.

The Ricci limit space has the same Ricci lower bound as in the sequence in a weak sense. In fact they are $\mathsf{RCD}(-(n-1), n)$-spaces (see below for the definition). Note that the sequence may collapse and that $d\mathrm{vol}_{M_i}$ goes to zero. Considering the renormalized measure $\mu_i = \frac{d\mathrm{vol}_{M_i}}{\mathrm{vol}_{B_1(x_i)}}$, this converges uniformly to a limit measure μ_∞, called the renormalized limit measure. See [10].

$\mathsf{RCD}^*(K, N)$-spaces are the Ricci curvature analog of the celebrated Alexandrov spaces – the generalization of manifolds with Ricci curvature bounded from below by K and dimension bounded above by N to the (Riemannian) metric measure space. In order to define the notion of $\mathsf{RCD}^*(K, N)$-space we first recall the curvature dimension condition, the $\mathsf{CD}(K, N)$ condition, introduced in [49, 50, 31] using tools of optimal transport.

Definition 16.2.4 A metric measure space (X, d, μ) satisfies the (Ricci) curvature dimension condition $\mathsf{CD}(K, N)(K, N \in \mathbb{R}, N \geq 1)$ if the Renyi entropy H_N is K*-convex* on the space of probability measures.

Here the Renyi entropy, $H_{N,\mu} : P_2(X, \mu) = \{\nu | \nu = \rho\mu\} \to \mathbb{R}$, is

$$H_{N,\mu} = -\int_X \rho^{1-1/N} d\mu.$$

When $K = 0$, "K-convex" means the usual convex function.

There is also the notion of the $\mathsf{CD}^*(K, N)$ condition introduced in [4]. The $\mathsf{CD}^*(K, N)$ condition is a priori weaker than the $\mathsf{CD}(K, N)$ condition, and the two coincide for $K = 0$. The $\mathsf{CD}^*(K, N)$ condition is also equivalent to $\mathsf{CD}_{loc}(K,N)$, and (non-branching) $\mathsf{CD}^*(K,N)$ satisfies the local-to-global property. Recently, Cavalletti–Milman have shown the equivalence of the CD and CD^* conditions when the space is essentially non-branching and has finite measure [8, Cor. 13.6].

A natural version of the Bishop–Gromov volume growth estimate holds on $\mathsf{CD}^*(K,N)$-spaces (see [4, Thm. 6.2] and [9]). Here we only state a weaker version, which is enough for most applications:

Theorem 16.2.5 *Let $K \in \mathbf{R}$, $N \geq 1$, be fixed. Then there exists a function $\Lambda_{K,N}(\cdot,\cdot) : \mathbf{R}_{>0} \times \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that if $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}^*(K,N)$-space, for some $K \in \mathbf{R}$, $N \geq 1$, the following holds:*

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} \geq \Lambda_{K,N}(r,R), \quad \forall 0 < r \leq R < \infty. \tag{16.1}$$

For $K = 0$ the following more explicit bound holds:

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} \geq \left(\frac{r}{R}\right)^N, \quad \forall 0 < r \leq R < \infty. \tag{16.2}$$

Key features of both the CD and CD^* conditions are the compatibility with the smooth Riemannian case and the stability under measured Gromov–Hausdorff convergence of metric measure space. In particular, these classes include Ricci limit spaces. On the other hand, they also include Finsler manifolds. In order to rule out Finsler structures while retaining the crucial stability properties of Lott–Sturm–Villani spaces, the following stricter condition is introduced in [24].

Definition 16.2.6 For a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ one says that the $\mathsf{RCD}(K,N)$ condition ($\mathsf{RCD}^*(K,N)$ condition) holds if it satisfies the $\mathsf{CD}(K,N)$ condition ($\mathsf{CD}^*(K,N)$ condition) and the Sobolev space $W^{1,2}(X, \mathfrak{m})$ is a Hilbert space.

The $\mathsf{RCD}^*(K,N)$ condition is also stable under convergence (see also [2], [26], and [22] for other important properties); therefore the class of $\mathsf{RCD}^*(K,N)$-spaces includes Ricci limit spaces (whether they are collapsed or not). Moreover, the class of $\mathsf{RCD}^*(K,N)$-spaces contains weighted manifolds satisfying Bakry–Émery lower curvature bounds as well as their non-smooth limits, cones, warped products, and Alexandrov spaces.

As $\mathsf{RCD}^*(K,N)$-spaces are essentially non-branching (see [41, Cor. 1.2], in particular under the assumption of finite measure, $\mathsf{RCD}(K,N)$ is equivalent to $\mathsf{RCD}^*(K,N)$. It is expected that $\mathsf{RCD}(K,N)$ is equivalent to $\mathsf{RCD}^*(K, N)$ without any further assumptions. As a result we sometimes abuse notation and do not distinguish between writing RCD and RCD^*.

The subject has been developed tremendously in the last few years. The Bochner inequality holds in a weak sense for $\mathsf{RCD}(K, N)$. As a result, many geometric and analytical results for manifolds with Ricci curvature bounded below, and Cheeger–Colding theory for Ricci limit spaces, have been generalized to $\mathsf{RCD}^*(K, N)$-spaces. Here we state a few fundamental properties of $\mathsf{RCD}^*(K, N)$-spaces.

A fundamental property of $\mathsf{RCD}^*(0, N)$-spaces is the extension of the celebrated Cheeger–Gromoll splitting theorem proved in [23]. For Ricci limit spaces, the splitting theorem was established in [10].

Theorem 16.2.7 (Splitting) *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(0, N)$-space with $1 \leq N < \infty$. Suppose that X contains a line. Then $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to $(X' \times \mathbf{R}, \mathsf{d}' \times \mathsf{d}_E, \mathfrak{m}' \times \mathsf{L}_1)$, where d_E is the Euclidean distance, L_1 is the Lebesgue measure, and $(X', \mathsf{d}', \mathfrak{m}')$ is an $\mathsf{RCD}^*(0, N-1)$-space if $N \geq 2$ and a singleton if $N < 2$.*

Definition 16.2.8 $x \in X$ is called a k-regular point if there exists an integer $k = k(x) \in [1, N] \cap \mathbb{N}$ such that, for any sequence $r_i \to \infty$, the rescaled pointed metric spaces $(X, r_i \mathsf{d}, x)$ converge in the pointed Gromov–Hausdorff sense to the pointed Euclidean space $(\mathbf{R}^k, \mathsf{d}_E, 0)$. Otherwise x is called a singular point. We denote $\mathcal{R}^k$ as the set of k-regular points in X, and $\mathcal{R} = \cup_k \mathcal{R}^k$ as the set of all regular points.

Note that the singular set could be dense in X.

Theorem 16.2.9 (Infinitesimal regularity of $\mathsf{RCD}^*(K, N)$-spaces, [34]) *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(K, N)$-space for some $K \in \mathbf{R}, N \in (1, \infty)$. Then $\mathfrak{m}$-almost everywhere $x \in X$ is a regular point. In fact (X, μ) is μ-rectifiable; namely, up to measure zero, it is the union of sets which are bi-Lipschitz equivalent to subsets of Euclidean spaces.*

Some further regularity properties that follow include the existence of a unique integer k such that $\mathfrak{m}(\mathcal{R} \setminus \mathcal{R}_k) = 0$, which gives $\mathfrak{m}(X \setminus \mathcal{R}_k) = 0$; see [18] for Ricci limit spaces, [6] for RCD spaces. The isometry group of X is a Lie group. For Riemannian manifolds this goes back to [36]; for Ricci limit spaces, see [12, 18], and see [47] for RCD spaces.

A natural version of the Abresch–Gromoll inequality [1] holds on $\mathsf{RCD}^*(K, N)$-spaces (see [25]), which plays an important role in proving Theorems 16.1.2 and 16.1.3. This Abresch–Gromoll estimate has been recently improved by Mondino–Naber [34]. We will use the following simpler version, which is a particular case of [34, Cor. 3.8].

Theorem 16.2.10 (Abresch–Gromoll inequality) *Given $K \in \mathbf{R}$ and $N \in (1, +\infty)$ there exist $\alpha(N) \in (0, 1)$ and $C(K, N) > 0$ with the following properties. Given $(X, \mathsf{d}, \mathfrak{m})$ an $\mathsf{RCD}^*(K, N)$-space, fix $p, q \in X$ with $\mathsf{d}_{p,q} := \mathsf{d}(p, q) \leq 1$ and let γ be a constant speed minimizing geodesic from p to q. Then*

$$e_{p,q}(x) \leq C(K, N) r^{1+\alpha(N)} \mathsf{d}_{p,q}, \quad \forall x \in B_{r\,\mathsf{d}_{p,q}}(\gamma(1/2)), \tag{16.3}$$

where $e_{p,q}(x) := \mathsf{d}(p, x) + \mathsf{d}(x, q) - \mathsf{d}(p, q)$ is the so called excess function associated to p, q.

For non-collapsing Ricci limit spaces, we need the following volume convergence result to prove Theorem 16.1.4.

Theorem 16.2.11 [17, 11] *Let (M_i, p_i) be a sequence of complete n-manifolds converging to (X, p) in the pointed Gromov–Hausdorff topology. Suppose that*

$$\mathrm{Ric} \geq -(n-1), \quad \mathrm{vol}(B_1(p_i)) \geq v > 0.$$

Then, for a sequence $q_i \in M_i$ converging to $q \in X$ and any $r > 0$, we have

$$\mathrm{vol}(B_r(q_i)) \to \mathcal{H}^n(B_r(q)),$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure of X.

A non-collapsing Ricci limit space (X, d, μ) has more regularity. For example, μ is a multiple of $\mathcal{H}^n$, the n-dim Hausdorff measure; the set of singular points has $\dim \mathcal{S} \leq n-2$; every tangent cone of X is a metric cone $C(Y)$, Y is a length space with diam $(Y) \leq \pi$, see [10, 11]. Non-collapsing RCD spaces are introduced in [19], and these results are mostly extended.

Definition 16.2.12 (Non-collapsed RCD spaces) For $K \in \mathbb{R}$ and $N \in (1, +\infty)$, $(X, \mathsf{d}, \mathfrak{m})$ is called a non-collapsed $\mathsf{RCD}(K, N)$-space if it is an $\mathsf{RCD}(K, N)$-space and $m = \mathcal{H}^N$.

See [14, 16] for more structures on the singular set of non-collapsing Ricci limit spaces, and [15, 30] for the wonderful structure when the sequence is non-collapsing and has a two-sided Ricci curvature bound.

16.3 Universal and δ-Covers

In this section we first recall the definitions for various covering spaces and then indicate the proof of the existence of the universal covers for RCD spaces, namely Theorem 16.1.3.

Definition 16.3.1 We say that $\bar{X}$ is a *covering space* of X if there is a continuous map $\pi : \bar{X} \to X$ such that for all $x \in X$ there is an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets of $\bar{X}$, each of which is mapped homeomorphically onto U by π (we say U is evenly covered by π).

When (X, d) is a locally compact length space, there is a unique length metric on any covering space $\bar{X}$ such that the covering map $\pi : \bar{X} \to X$ is distance nonincreasing and a local isometry. This is done by lifting the length structure on X to $\bar{X}$. See [42] for the study of the geometry and topology of length spaces. When X has a measure, we can also equip the covering space with a measure so that the covering map is locally measure preserving.

The universal cover is often defined as the simply connected cover. Here we do not assume it is simply connected, rather that the cover is universal in the sense of being the cover of all covers.

Definition 16.3.2 ([48, p. 82]) We say that $\widetilde{X}$ is a universal cover of a path connected space X if $\widetilde{X}$ is a cover of X such that, for any other cover $\bar{X}$ of X, there is a commutative triangle formed by a covering map $f : \widetilde{X} \to \bar{X}$ and the two covering projections:

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr] & & \bar{X} \arrow[dl] \\ & X & \end{tikzcd}$$

The universal cover may not exist, as can be seen by the Hawaiian earring (Figure 16.1). However, if it exists, then it is unique. Furthermore, if a space is locally path connected and semi-locally simply connected, then it has a universal cover and that cover is simply connected. On the other hand, the universal covering space of a locally path connected space may not be simply connected, as shown by the Griffiths twin cone (Figure 16.2) [27].[1]

[1] Figures reprinted with permission from J. Brazas' blog "Wild Topology" https://wildtopology.wordpress.com/2014/06/28/the-griffiths-twin-cone.

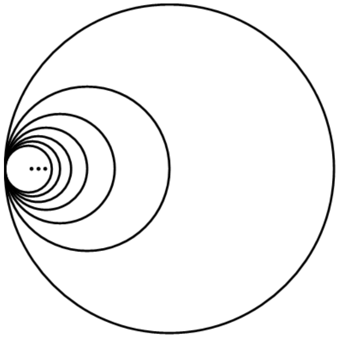

Figure 16.1 Hawaiian earring.

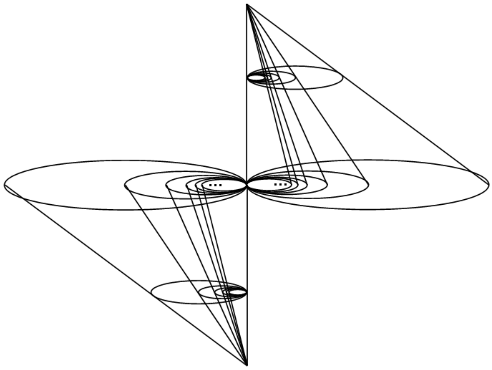

Figure 16.2 Griffiths twin cone.

Given $x \in X$, we denote the fundamental group of X based at x by $\pi_1(X, x)$. Recall that a locally path connected topological space X is said to be *semi-locally simply connected* (or semi-locally one connected) if for all $x \in X$ there is a neighborhood U_x of x such that any curve in U_x is contractible in X, i.e. $\pi_1(U_x, x) \to \pi_1(X, x)$ is trivial (see [48, p. 78], [32, p. 142]). This is weaker than saying that U_x is simply connected.

Let $\mathcal{U}$ be any open covering of X. For any $x \in X$, by [48, p. 81], there is a covering space $\widetilde{X}_{\mathcal{U}}$ of X with covering group $\pi_1(X, \mathcal{U}, p)$, where $\pi_1(X, \mathcal{U}, x)$ is a normal subgroup of $\pi_1(X, p)$ generated by homotopy classes of closed paths having a representative of the form $\alpha^{-1} \circ \beta \circ \alpha$, where β is a closed path lying in some element of $\mathcal{U}$ and α is a path from x to $\beta(0)$.

Now we recall the notion of δ-covers introduced in [44], which plays an important role in studying the existence of the universal cover.

Definition 16.3.3 Given $\delta > 0$, the δ-cover, denoted $\widetilde{X}^{\delta}$, of a length space X is defined to be $\widetilde{X}_{\mathcal{U}_\delta}$, where $\mathcal{U}_\delta$ is the open covering of X consisting of all balls of radius δ.

Intuitively, a δ-cover is the result of unwrapping all but the loops generated by small loops in X. Clearly $\widetilde{X}^{\delta_1}$ covers $\widetilde{X}^{\delta_2}$ when $\delta_1 \le \delta_2$.

Note that the δ-cover is a regular or Galois cover. That is, the lift of any closed loop in X is either always closed or always open in the δ-cover.

Example 16.3.4 Let $T^2 = S^1_1 \times S^1_{1/2}$, the product of circles with radius 1 and $\frac{1}{2}$. Then the δ-cover of T^2 is the cylinder when $\pi \le \delta < 2\pi$ and is $\mathbb{R}^2$ when $\delta < \pi$.

The basic properties of the δ-cover are well studied in the series of joint works of Sormani and Wei [44, 46, 45]. First we focus on the compact length spaces. The following theorem [44, Thm. 3.7] gives a way of proving the existence of the universal cover without showing that the space is semi-locally simply connected. We need only to show that the δ-covers eventually stabilize for sufficiently small δ.

Theorem 16.3.5 *For a compact metric space X, if there exists $\delta_0 > 0$ sufficiently small such that the covering map from $\widetilde{X}^{\delta}$ maps isometrically to $\widetilde{X}^{\delta_0}$ for all $0 < \delta \le \delta_0$, then $\widetilde{X}^{\delta_0}$ is the universal cover of X.*

Unlike universal covers, δ-covers behave very well under the Gromov–Hausdorff convergence. The following are proved in [44, Thm. 3.6] and [45, Prop. 7.3].

- If a sequence of compact length spaces X_i converge to a compact length space X in the Gromov–Hausdorff topology, then for any $\delta > 0$ there is a subsequence of X_i such that their δ-covers also converge in the pointed Gromov–Hausdorff topology.
- If compact length spaces X_i converge in the Gromov–Hausdorff topology to a compact space X, and the δ-covering of X_i, $(\widetilde{X}_i^{\delta}, \widetilde{p}_i)$, converges in the pointed Gromov–Hausdorff topology to $(X^{\delta}, \widetilde{p}_\infty)$, then $(X^{\delta}, \widetilde{p}_\infty)$ is a covering space of X, which is almost the δ-cover of X. Namely, for all $\delta_1 > \delta$ we have the covering mapping

$$\widetilde{X}^{\delta} \to X^{\delta} \to \widetilde{X}^{\delta_1} \to X.$$

(Both results are not true in general for universal covers.)

Remark 16.3.6 X^δ may not be the δ-cover: take a sequence of flat tori of side lengths 1 by $(n-1)/(2n)$. Let $\delta = 1/2$. Then, δ-covers of these tori are cylinders since all loops of length $< \delta$ are not unraveled. However, they converge to a torus of side lengths 1 by $1/2$, whose δ-cover is Euclidean space.

Now we sketch the proof of Theorem 16.1.3 in the compact case. The proof is divided into two steps. The first step is to prove the stability of δ-covers for small balls at good points.

Theorem 16.3.7 *If X is an $\mathsf{RCD}^*(K,N)$-space and $x \in X$ is a regular point, then there exists $r_x > 0$ such that $B(x, r_x)$ lifts isometrically to $\widetilde{X}^\delta$ for all $\delta > 0$.*

Intuitively, if this is not true, there are shorter and shorter closed based geodesic loops in X shrinking towards x, so we can find corresponding closed curves in the tangent cone $\mathbb{R}^k$ which are "almost closed based geodesic loops." However, since $\mathbb{R}^k$ has no closed based geodesic loops, we get a contradiction. In order to make a rigorous proof, quantitative estimates are needed. Assuming the statement of Theorem 16.3.7 is false, then the orbit of $\widetilde{x}$ is getting closer and closer to $\widetilde{x}$ as $\delta \to 0$, where $\widetilde{x}$ in the δ-cover projects to $x \in X$. On each δ-cover, the segment between $\widetilde{x}$ and the closed orbit point projects to a geodesic loop based at x. The midpoint m of this geodesic loop is a cut point of x. In particular, for $y \in X$ with $d(x,y) > D$, where $D = d(x,m)$, we have

$$D + d(y,m) > d(x,y).$$

Using the Abresch–Gromoll inequality Theorem 16.2.10 on δ-covers, one can establish a quantitative version of this inequality: for all $D \leq \frac{1}{2}$, y with $d(x,y) \geq D + S_n D$, we have

$$D + d(y,m) - S_n D \geq d(x,y),$$

where $S_n > 0$ is a small constant. This estimate is called the *uniform cut* lemma [43]. Recall that there is a sequence of these geodesic loops based at x, thus we can pass the distance estimate to the tangent cone at x, which is $\mathbb{R}^k$. However, the uniform distance gap fails on $\mathbb{R}^k$, and this leads to a contradiction.

Theorem 16.2.9 states that the regular points have full measure. We need only one regular point. Then the second step is to use Theorem 16.3.7 at one regular point combined with *Bishop–Gromov's relative volume comparison theorem* on $\widetilde{X}^\delta$ and a *packing argument* to show that the $\widetilde{X}^\delta$ stabilizes everywhere.

When X is not compact, Theorem 16.3.5 is not true. A simple example is a cylinder with one side pinched to a cusp. For the noncompact case, a natural way is to consider bigger and bigger balls. In fact when (X, d) is semi-locally simply connected, the universal cover of X can be obtained as the Gromov–Hausdorff limit of the universal cover of larger and larger balls, see [21, Prop. 1.2]. Since we do not have this extra hypothesis, a different argument is required. Also, the universal cover of a ball may not exist, even if the universal cover of the whole space exists, and one would prefer to avoid the boundaries of balls. For this purpose, Wei and Sormani introduced the relative δ-covers in [46].

We use the following convention: open balls are denoted $B_R(x)$ and closed balls are denoted $B(x, R)$, all with intrinsic metric.

Definition 16.3.8 (Relative δ-cover) Suppose X is a length space, $x \in X$ and $0 < r < R$. Let

$$\pi^\delta : \widetilde{B}_R(x)^\delta \to B_R(x)$$

be the δ-cover of the open ball $B_R(x)$. A connected component of

$$(\pi^\delta)^{-1}(B(x, r)),$$

where $B(x, r)$ is a closed ball, is called a relative δ-cover of $B(x, r)$ and is denoted $\widetilde{B}(x, r, R)^\delta$.

Instead of Theorem 16.3.5, which was the key to proving the existence of the universal cover for a compact length space, for noncompact spaces the key role will be played by the following result [46, Thm. 2.5].

Theorem 16.3.9 *Let (X, d) be a length space and assume that there is $x \in X$ with the following property: for all $r > 0$, there exists $R \geq r$, such that $\widetilde{B}(x, r, R)^\delta$ stabilizes for all δ sufficiently small. Then (X, d) admits a universal cover $\widetilde{X}$.*

The two steps in the proof of stabilizing the δ-cover for the compact case can be adapted to prove the stabilization of the relative δ-cover for the noncompact case. One subtle thing to note is that relative δ-covers are no longer RCD spaces, but volume comparison still holds on balls of controlled size. See [35] for details.

With the existence of the universal cover, results concerning the fundamental group of manifolds with Ricci curvature bounded from below can now be extended to the revised fundamental group for RCD spaces. Here the revised fundamental group is the group of deck transformations of the universal cover.

Note that the Abresch–Gromoll inequality (Theorem 16.2.10) plays an important role in the proof of the existence of universal cover here. The Abresch–Gromoll inequality does not hold for $\mathsf{CD}(K,N)$-spaces. Therefore it is natural to ask the following question:

Question 16.3.10 Does the universal cover of a $\mathsf{CD}(K,N)$-space always exist?

16.4 Non-Collapsing Ricci Limit Spaces

We outline a very recent work by the authors in this section, proving that non-collapsing Ricci limit spaces are semi-locally simply connected (Theorem 16.1.4). We point out that the techniques needed to prove Theorem 16.1.4 are independent of the material in Section 16.3.

Note that, by the regularity theory of non-collapsing Ricci limit spaces, local semi-simple connectedness holds at almost all points. In fact, any regular point has almost maximal local volume and the regular points form a subset of full measure in X; by the proof in [39], for a regular point $x \in X$ there exists $r > 0$ such that $B_r(x)$ is contractible in $B_{2r}(x)$. In particular, loops in $B_r(x)$ are contractible in $B_{2r}(x)$. However, this is far from the goal that X is semi-locally simply connected, since it is possible for a loop to go through or even be contained entirely in the singular set.

To attack the problem we quantify the points and make the following definition:

Definition 16.4.1 Let X be a metric space and let $x \in X$. We define the 1-contractibility radius at x as

$$\rho(t,x) = \inf\{\infty, \rho \geq t |\text{ any loop in } B_t(x) \text{ is contractible in } B_\rho(x)\}.$$

Note that X is semi-locally simply connected if for any $x \in X$ there is $T > 0$ such that $\rho(T,x) < \infty$; X is locally simply connected if for any $x \in X$ there is $t_i \to 0$ such that $\rho(t_i,x) = t_i$. For a Riemannian manifold M and $x \in M$, $\rho(t,x) = t$ for t smaller than the injective radius at x.

We state the local version of Theorem 16.1.4 with an estimate on the 1-contractibility radius.

Theorem 16.4.2 *Let (M_i, p_i) be a sequence of Riemannian n-manifolds (not necessarily complete) converging to (X, p) such that, for all i,*

(1) $B_2(p_i) \cap \partial M_i = \emptyset$ and the closure of $B_2(p_i)$ is compact;

(2) $\mathrm{Ric} \geq -(n-1)$ *on* $B_2(p_i)$*,* $\mathrm{vol}(B_1(p_i)) \geq v > 0$*.*

Then $\lim_{t\to 0} \rho(t,x)/t = 1$ *holds for any* $x \in B_1(p)$*.*

The key step in proving Theorem 16.4.2 is showing that $\lim \rho(t,x) = 0$, which is sometimes called *essentially locally simply connected.* After obtaining this, we can further improve the result to $\lim \rho(t,x)/t = 1$ by using the structure of tangent cones and Sormani's uniform cut techniques.

We classify the points in X according to the local 1-contractibility radius on manifolds:

Definition 16.4.3 For $x \in X$, let x_i in M_i converge to x. We say

- x is of *type I* if there exists $r > 0$ such that the family of functions $\{\rho(q,t) | q \in B_r(x_i), i \in \mathbb{N}\}$ is equi-continuous at $t = 0$;
- x is of *type II* if $\{\rho(x_i,t)\}_{i\in\mathbb{N}}$ is not equi-continuous at $t = 0$;
- x is of *type III* if it is neither of type I nor type II.

Definition 16.4.3 is well defined: the type of x does not depend on the choice of x_i. Type I requires control at every point around x_i, not just at x_i. Type II points may exist due to a positive Ricci curvature example by Otsu [37, p. 262, Rem. 2] (see also the Eguchi–Hanson metric on the tangent bundle of $\mathbb{R}P^2$ for a Ricci flat example). By definition, x being of type III implies that $\{\rho(x_i,t)\}_i$ is equi-continuous at $t = 0$.

For a family of functions $\{\rho_\alpha(t)\}_{\alpha\in A}$ with $\rho_\alpha(0) = 0$, the family is equi-continuous at $t = 0$ if and only if there is a continuous function $\lambda(t)$ defined on $[0,T]$ with $\lambda(0) = 0$ such that $\rho_\alpha(t) \leq \lambda(t)$ for all $t \in [0,T]$ and all $\alpha \in A$. For type I points, we can pass the local 1-contractibility control on local balls around x_i to that around x in the limit space:

Theorem 16.4.4 *Let* (X_i, x_i) *be a sequence of length spaces converging to* (X,x) *satisfying the following conditions:*

(1) the closure of $B_1(x_i)$ *is compact;*
(2) there exists a continuous function λ *on* $[0,T)$ *with* $\lambda(0) = 0$ *such that, for all* i *and all* $q \in B_2(x_i)$*,* $\rho(t,q) \leq \lambda(t) < 1/2$ *holds on* $[0,T)$*.*

Then $\rho(t,q) \leq \lambda(t)$ *for all* $t \in [0,T)$ *and all* $q \in B_{1/2}(x)$*. In particular,* $\lim_{t\to 0} \rho(t,q) = 0$*.*

Note that Theorem 16.4.4 does not require any curvature conditions (X_i may not even be manifolds). We briefly explain two different proofs of Theorem 16.4.4. For a loop c in the limit space, we can find a sequence of loops c_i in M_i that converges uniformly to c. For c in a sufficiently small ball, we can contract c_i for all sufficiently large i. The first proof is related to the method given in [5]: we transfer the null homotopy of c_i along the sequence and pass it to the limit space by uniform convergence; in this method, we need to control the distance between null homotopies to assure uniform convergence. The second proof constructs the null homotopy gradually in the limit space through a sequence of refining skeletons (see Figure 16.3). By controlling the extensions on the new skeletons at each step, these maps on the skeletons converge uniformly to a continuous map defined on the disk.

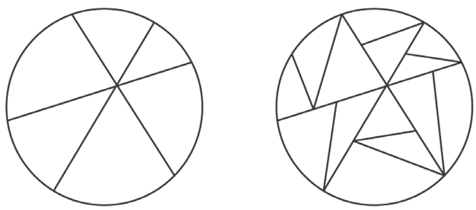

Figure 16.3 Constructing a null homotopy by refining 1-skeletons.

For a type II point x, there are $\varepsilon > 0$ and $t_i \to 0$ such that $\rho(t_i, x_i) \geq \varepsilon$. Let U_i be a small ball centered at y_i ($\pi_i(y_i) = x_i$) in the universal covering space of $B_\varepsilon(x_i)$. The local fundamental group Γ_i has a subgroup generated by these non-contractible short loops H_i. By a result of [3], each H_i is a finite group whose order is controlled by some constant $C(n, v)$. We consider the equivariant Gromov–Hausdorff convergence:

$$\begin{tikzcd}
(U_i, y_i, \Gamma_i, H_i) \arrow[r, "GH"] \arrow[d, "\pi_i"] & (Y, y, G, H) \arrow[d, "\pi"] \\
(B_\varepsilon(x_i), x_i) \arrow[r, "GH"] & (B_\varepsilon(x) = Y/G, x),
\end{tikzcd}$$

Figure 16.4 Equivariant Gromov–Hausdorff convergence.

Roughly speaking, the covering group H_i increases the volume around y_i when compared with the volume around x_i. Together with volume convergence (see Theorem 16.2.11) the local volume around y is at least twice of that around x: $\mathcal{H}^n(B_s(y)) \geq 2 \cdot \mathcal{H}^n(B_s(x))$. Recall that volume comparison states that $\mathcal{H}^n(B_s(y)) \leq \mathrm{vol}(B^n_s(-1))$, where $B^n_s(-1)$ is the s-ball in the n-dimensional space form of constant curvature -1. Therefore, if x has $\mathcal{H}^n(B_s(x)) > \mathrm{vol}(B^n_s(-1))/2$, then Figure 16.4 cannot occur. In fact, this implies that such a point x must be of type I, and we have a linear estimate of the 1-contractibility radius at these points.

Theorem 16.4.5 *Given $n \geq 2$, $\kappa \geq 0$, and $\omega > 1/2$, there exist positive constants $\varepsilon(n,\kappa,\omega)$ and $C(n,\kappa,\omega)$ such that the following holds.*
Let (M,p) be a complete n-manifold satisfying

$$\mathrm{Ric} \geq -(n-1)\kappa, \quad \mathrm{vol}(B_1(p)) \geq \omega \cdot \mathrm{vol}(B^n_1(-\kappa)).$$

Then every loop in $B_r(p)$ is contractible in $B_{Cr}(p)$, where $r \in [0,\varepsilon)$.

Note that by Otsu's example [37], the half volume lower bound cannot be weakened to a non-collapsing condition. One may compare the 1-contractibility radius estimate with Theorem 16.4.6 below on sectional curvature and volume lower bounds from [29]:

Theorem 16.4.6 (Grove–Petersen) *Given $n \geq 2, \kappa \geq 0$, and $v > 0$, there exist positive constants $\varepsilon(n,\kappa,v)$ and $C(n,\kappa,v)$ such that, for any complete n-manifold (M,p) of*

$$\sec_M \geq -\kappa, \quad \mathrm{vol}(B_1(p)) \geq v,$$

$B_r(p)$ is contractible in $B_{Cr}(p)$, where $r \in [0,\varepsilon)$.

If $B_s(x)$ does not have a half volume lower bound, then the volume of $B_s(y)$ in Y will be doubled. This inspires us to use an induction argument on the local volume.

We define $\omega(x) = \lim_{r\to 0} \frac{\mathcal{H}^n(B_r(x))}{\mathrm{vol}(B^n_r(0))}$. Note that, by relative volume comparison, the limit always exists and $0 < \omega(x) \leq 1$; the equality holds if and only if x is a regular point. Also, $\omega(x)$ has a uniform lower bound for all $x \in B_1(p)$.

If $\omega(x) > 1/2$ for all $x \in B_1(p)$, then every point $x \in B_1(p)$ is of type I and we can apply Theorem 16.4.4 directly. Next we consider the case that $\omega(x) > 1/4$ for all $x \in B_1(p)$. If x is type I, then we again apply Theorem 16.4.4. If x is of type II, we can take the convergence of the local universal covers (U_i, y_i), then the corresponding limit point y has $\omega(y) > 1/2$. For s small, we can lift the loop in $B_s(x)$ to a loop in

$B_s(y)$, which we know is contractible. Projecting the homotopy down, we obtain the desired homotopy in X. We need the following technical result for type III points.

Theorem 16.4.7 *Let (X_i, x_i) be a sequence of length spaces converging to (X, x) with the closure of each $B_2(x_i)$ being compact. Suppose that $\lim_{t\to 0} \rho(t, y) = 0$ holds for all points y of type II in $B_{3/2}(x)$, then it holds for all points of type III in $B_1(x)$. Consequently, $\lim_{t\to 0} \rho(t, y) = 0$ holds for all $y \in B_1(x)$.*

Similar to the statement of Theorem 16.4.4, Theorem 16.4.7 does not require any curvature conditions. Assuming Theorem 16.4.7, we clear the step $\omega(x) > 1/4$ and continue the induction argument.

We briefly illustrate the proof of Theorem 16.4.7. We observe that, with the assumptions of Theorem 16.4.7, for any $y \in B_1(x)$, either $\{\rho(t, y_i)\}$ is equi-continuous at $t = 0$ (case y is type I or III), or $\lim \rho(t, y) = 0$ (case y is type I or II). The starting step is the same as the second proof of Theorem 16.4.4, using the homotopy on a manifold from the sequence M_i to extend the loop on a 1-skeleton. Then we apply two different procedures to the sub-triangles. Roughly speaking, if a sub-triangle is displaced from a point of type III, we can directly contract this sub-triangle; if not, we will use the local 1-contractibility from the sequence to extend the map on a finer 1-skeleton (see Figure 16.5). The actual proof is quite technical since we need to assure uniform convergence and control the size of the eventual null homotopy.

Unlike Theorem 16.1.2, which is generalized to RCD spaces, a generalization of Theorem 16.1.4 to RCD spaces seems very difficult. Note that the proof relies heavily on the sequence, which consists of manifolds. More specifically, investigating the convergence of universal covers of local balls $B_\varepsilon(x_i)$ is a key step in the study of type II points. On the other hand, if we are assuming that a space is locally essentially simply connected, then we can always take the universal cover of any local ball. Using the same arguments, it can be shown that for a convergent sequence of uniformly non-collapsing RCD spaces (see Definition 16.2.12), if each space of the sequence is essentially locally simply connected, then the limit space is as well.

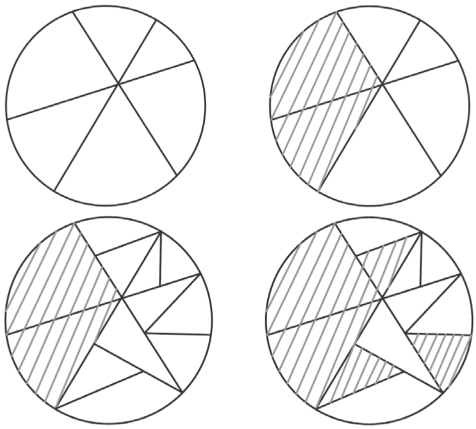

Figure 16.5 Refining 1-skeletons with partial extensions.

In closing, we ask a question about the stability of fundamental groups.

Question 16.4.8 Is the following statement true?

Given $n, D, v > 0$, there is a constant $\varepsilon(n, D, v) > 0$ such that for any two compact Riemannian n-manifolds M_i $(i = 1, 2)$ satisfying:

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{diam}(M_i) \leq D, \quad \mathrm{vol}(M_i) \geq v, \quad d_{GH}(M_1, M_2) \leq \varepsilon,$$

then $\pi_1(M_1)$ and $\pi_1(M_2)$ must be isomorphic.

From [3] it is known that the fundamental groups of the spaces in the class above have only finitely many isomorphism types. It is known that the fundamental group of X may be different from that of M_i for all i: in Otsu's example, a sequence of Riemannian metrics on $S^3 \times \mathbb{R}P^2$ converges to a simply connected Ricci limit space.

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