EXAMPLES OF RICCI LIMIT SPACES WITH NON-INTEGER HAUSDORFF DIMENSION

JIAYIN PAN AND GUOFANG WEI

Abstract. We give the first examples of collapsing Ricci limit spaces on which the Hausdorff dimension of the singular set exceeds that of the regular set; moreover, the Hausdorff dimension of these spaces can be non-integers. This answers a question of Cheeger-Colding [4, Page 15] about collapsing Ricci limit spaces.

Recall that for any sequence of complete n-dimensional Riemannian manifolds (M_i, p_i) with Ricci curvature uniformly bounded from below, there is always a subsequence Gromov-Hausdorff converging to a metric space (X, p). X is called a Ricci limit space. The regularity theory of Ricci limit spaces has been studied extensively by Cheeger, Colding, and Naber. A point $x \in X$ is called k-regular if every tangent cone at x is isometric to $\mathbb{R}^k$, where k is an integer; otherwise, it is called singular. By the work of Cheeger-Colding [3, 5] and Colding-Naber [6], X has a unique integer $k \in [0, n]$ such that $\mathcal{R}^k$ has full measure and X is k-rectifiable with respect to any limit measure of X, where $\mathcal{R}^k$ is the set of k-regular points of X. This integer k is called the rectifiable dimension of X.

When the sequence is non-collapsing $\mathrm{vol}(B_1(p_i)) \geq v > 0$, the limit space X has rectifiable dimension n, which is also the Hausdorff dimension of X; moreover, the singular set has Hausdorff dimension at most $n-2$ [3]. When the sequence is collapsing $\mathrm{vol}(B_1(p_i)) \to 0$, as pointed out in [4, Page 15], an important question is whether the Hausdorff dimension of the singular set could exceed that of the regular set (also see the discussion in [3, Page 408]). Equivalently, one can ask if the rectifiable dimension of X equals the Hausdorff dimension of X [11, Open Problem 3.2]. In this paper, we give answers to these open questions by providing examples of Ricci limit spaces on which the Hausdorff dimension of the singular set is larger than that of the regular set; therefore, the Hausdorff dimension of these examples is larger than their rectifiable dimension. Moreover, their Hausdorff dimension can be non-integers.

The example is an asymptotic cone of an open manifold N of positive Ricci curvature. Recall that an asymptotic cone (Y, y) of N is the Gromov-Hausdorff limit of a sequence $(r_i^{-1}N, x)$, where $r_i \to \infty$ and $x \in N$. In general, (Y, y) may depend on the choice of r_i and thus may not be unique.

Theorem A. *Given any $\beta > 0$, for n sufficiently large (depending on β), there is an open n-manifold N with $\mathrm{Ric} > 0$, such that any asymptotic cone (Y, y) of N satisfies the followings:*
(1) $Y = \mathcal{S} \cup \mathcal{R}^2$, where $\mathcal{S}$ is the singular set and $\mathcal{R}^2$ is the set of 2-regular points;

J. Pan was partially supported by AMS Simons travel grant during the preparation of this paper, and is currently supported by Fields Postdoc Fellowship.

G. Wei is partially supported by NSF DMS 1811558, 2104704.

(2) $\mathcal{S}$ has Hausdorff dimension $1+\beta$.
When $\beta > 1$, Y has Hausdorff dimension $1+\beta$, larger than its rectifiable dimension.

By [5], the above $\mathcal{R}^2$ has Hausdorff dimension 2 and Y has rectifiable dimension 2. Thus for $\beta > 1$, the Hausdorff dimension of $\mathcal{S}$ is larger than that of the regular set; also, Y has Hausdorff dimension $1+\beta$, which is not an integer when $\beta \notin \mathbb{N}$. Note that with respect to any limit measure, the regular set always has full measure [3]. Our examples show that for a collapsing Ricci limit space, its limit measure and any dimensional Hausdorff measure may not be mutually absolutely continuous.

We point out that Y is non-polar at y; see Remark 1.5. The non-polarity of (Y,y) is consistent with [4, Theorem 1.38]: if any iterated tangent cone of Y at y is polar, then the Hausdorff dimension of Y is an integer. The first examples of non-polar asymptotic cones are constructed by Menguy [10].

We can also use the examples in Theorem A to construct compact Ricci limit spaces with non-integer Hausdorff dimension; see Remark 1.8.

Our examples are indeed closely related to some known examples [12, 1, 15]. Nabonnand showed that for suitable functions $f(r)$ and $h(r)$, the doubly warped product $[0,\infty)\times_f S^2 \times_h S^1$ has positive Ricci curvature [12]. Our construction starts with a doubly warped product $M = [0,\infty)\times_f S^{p-1}\times_h S^1$, where warping functions $f(r), h(r)$ are the ones in [15] and $h(r)$ has decay rate $r^{-\beta}$; then we take the Riemannian universal cover N of M as our example in Theorem A.

One key feature of the universal cover N with $\pi_1(M,p)$-action is that, $d(\gamma^l\tilde{p},\tilde{p})$ is comparable to $l^{\frac{1}{1+\beta}}$, where $p\in M$ at $r=0$, $\tilde{p}\in N$ is a lift of p, and γ is a generator of $\pi_1(M,p)\cong\mathbb{Z}$. With this, after passing to the asymptotic cone (Y,y), the limit orbit at y has Hausdorff dimension $1+\beta$ (see Section 1.3).

We would like to thank Jeff Cheeger, Aaron Naber, and the anonymous referees for very helpful comments.

1. The examples and their geometry

1.1. Riemannian metric on M. On $[0,\infty)\times S^{p-1}\times S^1$, where $p\geq 2$, we define a doubly warped product metric as

$$g = dr^2 + f(r)^2 ds_{p-1}^2 + h(r)^2 ds_1^2,$$

where ds_k^2 is the standard metric on the unit sphere S^k. At $r=0$, we require that

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad h(0)>0,\quad h'(0)=0.$$

Then (M,g) is a Riemannian manifold diffeomorpic to $\mathbb{R}^p\times S^1$.

Let $H=\partial/\partial r$, U a unit vector tangent to S^{p-1}, and V a unit vector tangent to S^1. The metric has Ricci curvature

$$\begin{aligned}
\mathrm{Ric}(H,H) &= -\frac{h''}{h} - (p-1)\frac{f''}{f},\\
\mathrm{Ric}(U,U) &= -\frac{f''}{f} + \frac{p-2}{f^2}\left[1-(f')^2\right] - \frac{f'h'}{fh},\\
\mathrm{Ric}(V,V) &= -\frac{h''}{h} - (p-1)\frac{f'h'}{fh}.
\end{aligned}$$

We use

$$f(r) = r(1+r^2)^{-1/4},\quad h(r) = (1+r^2)^{-\alpha},$$

where $\alpha > 0$ [15]. On $(0, \infty)$, f and h satisfy

$$f' > 0, \quad 1 - (f')^2 > 0, \quad f'' < 0, \quad h' < 0.$$

Then $\mathrm{Ric}(U, U) > 0$ always holds. Also,

$$\mathrm{Ric}(H, H) > \frac{r^2}{(1+r^2)^2}\left[\frac{p-1}{4} - (2\alpha + 4\alpha^2)\right]$$
$$\mathrm{Ric}(V, V) > \frac{\alpha r^2}{(1+r^2)^2}\left[p - (3 + 4\alpha)\right]$$

When integer $p \geq \max\{4\alpha + 3, 16\alpha^2 + 8\alpha + 1\}$, (M, g) has positive Ricci curvature. From now on, we denote this open Riemannian manifold as (M, g_α).

1.2. **Distance estimate on Riemannian universal cover** N**.** Let (M, g_α) as constructed above and let $(N, \tilde{g_\alpha})$ be the Riemannian universal cover of (M, g_α). Let $S^1(r)$ be a copy of S^1 in M at distance r; it has length

$$L(r) = 2\pi(1 + r^2)^{-\alpha}.$$

As the length of $S^1(r)$ decreases a lot, one expects that the length of minimal geodesics representing γ^l grows much smaller than l, for γ representing S^1 at $r = 0$. Namely,

Lemma 1.1. *Let $p \in M$ be a point with $r = 0$ and let γ be a generator of $\pi_1(M, p) \cong \mathbb{Z}$. Then for all positive integers $l \geq 9^{1+\frac{1}{2\alpha}}$,*

$$C \cdot l^{\frac{1}{1+2\alpha}} - 2 \leq d(\gamma^l \tilde{p}, \tilde{p}) \leq 9 \cdot l^{\frac{1}{1+2\alpha}},$$

where $C = 2 \cdot 9^{-\frac{1}{2\alpha}}$ and $\tilde{p} \in N$ is a lift of p.

Proof. Let c_l be the minimal geodesic loop based at p that represents $\gamma^l \in \pi_1(M, p)$. We estimate the length of c_l.

Upper bound: We consider a loop σ_l at p constructed as follows. First it moves along a ray from p to distance r_l, where r_l to be determined later; then it goes around $S^1(r_l)$ circle l times; then it moves back to p along the ray. Then

$$d(\gamma^l \tilde{p}, \tilde{p}) = \mathrm{length}(c_l) \leq \mathrm{length}(\sigma_l) \leq 2r_l + l \cdot L(r_l).$$

We choose $r_l = l^{\frac{1}{1+2\alpha}}$. Then

$$d(\gamma^l \tilde{p}, \tilde{p}) \leq (2 + 2\pi) \cdot l^{\frac{1}{1+2\alpha}}.$$

Lower bound: Let $R_l > 0$ be the size of c_l, that is, the smallest number so that c_l is contained in the closed ball $\overline{B_{R_l}(p)}$. Since $h(r)$ is decreasing in r, we have

$$l \cdot L(R_l) \leq \mathrm{length}(c_l) \leq 9 \cdot l^{\frac{1}{1+2\alpha}}.$$

This yields

$$R_l \geq 9^{-\frac{1}{2\alpha}} l^{\frac{1}{1+2\alpha}} - 1.$$

The desired lower bound follows from $d(\gamma^l \tilde{p}, \tilde{p}) \geq 2R_l$. $\square$

On the other hand, we show that for s large and suitable l, length of c_l at $r = s$ is still comparable to $l \cdot L(s)$.

Lemma 1.2. *Given any $\epsilon \in (0,1)$, $s > 0$, let l be any integer between $\frac{1}{2}\epsilon s(1+s^2)^\alpha$ and $\epsilon s(1+s^2)^\alpha$ (which always exists when s is large). We denote c_l the minimal geodesic loop based at q representing $\gamma^l \in \pi_1(M,q)$, where $q \in M$ satisfies $r = s$ and γ is a generator of $\pi_1(M,q) \cong \mathbb{Z}$. Then*
(1) $\epsilon \cdot Cs \leq \mathrm{length}(c_l) \leq \epsilon \cdot 2\pi s$, where $C > 0$ only depends on α.
(2) $\lim_{\epsilon\to 0}\left(\limsup_{s\to\infty} \dfrac{\mathrm{size}(c_l)}{\mathrm{length}(c_l)}\right) = 0$, where $\mathrm{size}(c_l)$ is the smallest number so that c_l is contained in $\overline{B_{d_l}}(q)$.

Proof. (1) It is clear that

$$\mathrm{length}(c_l) \leq l \cdot 2\pi(1+s^2)^{-\alpha} \leq \epsilon \cdot 2\pi s.$$

Let d_l be the size of c_l. Clearly, d_l has a rough upper bound

$$d_l \leq \frac{1}{2}\mathrm{length}(c_l) \leq \epsilon \cdot \pi s.$$

Then

$$\begin{aligned}\mathrm{length}(c_l) &\geq l \cdot 2\pi(1+(s+d_l)^2)^{-\alpha}\\ &\geq \epsilon\pi s\left(\frac{1+s^2}{1+(s+\epsilon\pi s)^2}\right)^\alpha\\ &\geq \epsilon\pi s \cdot \left(\frac{1}{1+\epsilon\pi}\right)^{2\alpha} \geq \frac{\pi}{(1+\pi)^{2\alpha}}\epsilon s.\end{aligned}$$

(2) To find a better estimate for d_l, we consider a bounded cylinder

$$W = [0, s+d_l] \times S^1(s+d_l)$$

with product metric. The map

$$F : \overline{B_{s+d_l}}(p) \to W, \quad (r,u,v) \mapsto (r,v)$$

is distance non-increasing, where (r,u,v) corresponds to the coordinate $[0,\infty)\times_f S^{p-1}\times_h S^1$ and $\overline{B_{s+d_l}}(p)$ is endowed with intrinsic metric. It is clear that $F(c_l)$ is contained in $[s, s+d_l] \times S^1(s+d_l)$. Note that the loop $F(c_l)$ has winding number l and touches both boundaries $\{s\} \times S^1(s+d_l)$ and $\{s+d_l\} \times S^1(s+d_l)$. Thus

$$\mathrm{length}(c_l) \geq \mathrm{length}(F(c_l)) \geq 2\left[d_l^2 + l^2\pi^2(1+(s+d_l)^2)^{-2\alpha}\right]^{1/2}.$$

It follows from the upper bound of $\mathrm{length}(c_l)$ that

$$\begin{gathered}4d_l^2 + 4\pi^2 l^2(1+(s+d_l)^2)^{-2\alpha} \leq 4\pi^2 l^2(1+s^2)^{-2\alpha},\\ \frac{d_l^2}{s^2} \leq \epsilon^2\pi^2\left[1 - \left(\frac{1+s^2}{1+(s+d_l)^2}\right)^{2\alpha}\right].\end{gathered}$$

For any sequence $s_i \to \infty$ and a sequence of integers l_i between $\frac{1}{2}\epsilon s_i(1+s_i^2)^\alpha$ and $\epsilon s_i(1+s_i^2)^\alpha$ such that $\lim\limits_{i\to\infty} d_{l_i}/s_i = \delta \in [0, \epsilon\pi]$, then

$$\delta^2 \leq \epsilon^2\pi^2\left[1 - \frac{1}{(1+\delta)^{4\alpha}}\right].$$

Thus as $\epsilon \to 0$, we see that $\delta/\epsilon \to 0$. Together with (1), we conclude that

$$\lim_{i\to\infty}\frac{\mathrm{size}(c_{l_i})}{\mathrm{length}(c_{l_i})} \leq \lim_{i\to\infty}\frac{d_{l_i}}{C\epsilon s_i} = \frac{\delta}{C\epsilon} \to 0, \text{ as } \epsilon \to 0.$$

This proves (2). ∎

1.3. **Structure of asymptotic cones of N.** To better understand the structure of asymptotic cones of N, we will take the isometric Γ-action on N into account, where $\Gamma = \pi_1(M,p)$. We use the notion of pointed equivariant Gromov-Hausdorff convergence introduced by Fukaya (see [8] and [9, Section 3] for reference). Let $r_i \to \infty$, passing to a subsequence if necessary, we have the following equivariant Gromov-Hausdorff convergence

$$\begin{array}{ccc} (r_i^{-1}N, \tilde{p}, \Gamma) & \xrightarrow{GH} & (Y, y, G) \\ \downarrow \pi & & \downarrow \pi \\ (r_i^{-1}M, p) & \xrightarrow{GH} & (X, x). \end{array}$$

One should view the limit orbit $G \cdot \tilde{z}$ as the limit of $\Gamma \cdot \tilde{q_i}$, where $\tilde{z} \in Y$ and $\tilde{q_i} \in N$ such that

$$(r_i^{-1}N, \tilde{p}, \tilde{q_i}) \xrightarrow{GH} (Y, y, \tilde{z}).$$

Also by [8, Theorem 2.1] (also see [9, Lemma 3.4]), (X, x) is isometric to the quotient space $(Y/G, \bar{y})$, where $\bar{y}$ is the image of y under the quotient map.

The first author observed that the limit orbit Gy is not geodesic in Y [13]. Here, we give more detailed description of (Y, y, G) and prove Theorem A.

From the warping functions f and h, we see that (X, x) is a half-line $[0, \infty)$ with $x = 0$. Since Y/G is isometric to $X = [0, \infty)$, we can view Y as attaching the limit orbit $\pi^{-1}(z)$ to each $z \in [0, \infty)$. By construction of M, the limit orbit $G\tilde{z}$ is homeomorphic to $\mathbb{R}$ for any $\tilde{z} \in Y$. Thus Y is homeomorphic to $\mathbb{R} \times [0, \infty)$; $y = (0, 0)$ and $\mathbb{R} \times \{z\}$ corresponds to the orbit $\pi^{-1}(z)$, where $z \in [0, \infty)$. Because the limit orbit Gy is noncompact and connected, we can choose $\gamma_\infty \in G$ such that $d(\gamma_\infty y, y) = 1$. We label the point $\gamma_\infty y$ as $(1, 0) \in \mathbb{R} \times [0, \infty) = Y$. Let $l_i \to \infty$ be a sequence of integers such that

$$(r_i^{-1}N, \tilde{p}, \gamma^{l_i}) \xrightarrow{GH} (Y, y, \gamma_\infty).$$

Then for any rational number $b = m/k$,

$$(r_i^{-1}N, \tilde{p}, \gamma^{\lfloor bl_i \rfloor}) \xrightarrow{GH} (Y, y, g)$$

with g satisfying $g^k = \gamma_\infty^m$. We label the point gy as $(b, 0)$. By Lemma 1.1, there are constants $C_1, C_2 > 0$, only depending on α such that

$$C_1 |b|^{\frac{1}{1+2\alpha}} \leq \lim_{i \to \infty} r_i^{-1} d(\gamma^{\lfloor bl_i \rfloor}\tilde{p}, \tilde{p}) \leq C_2 |b|^{\frac{1}{1+2\alpha}}.$$

Therefore, we can define a coordinate $(b, 0)$, where $b \in \mathbb{R}$, for any point in Gy. Moreover, the coordinate satisfies

$$d((b_1 - b_2, 0), (0, 0)) = d((b_1, 0), (b_2, 0))$$

for all $b_1, b_2 \in \mathbb{R}$. This proves the estimate below.

Lemma 1.3. *The distance on $Gy = \mathbb{R} \times \{0\}$ satisfies*

$$C_1 \cdot |b_1 - b_2|^{\frac{1}{1+2\alpha}} \leq d((b_1, 0), (b_2, 0)) \leq C_2 \cdot |b_1 - b_2|^{\frac{1}{1+2\alpha}}$$

for all $b_1, b_2 \in \mathbb{R}$, where $C_1, C_2 > 0$ only depend on α.

Proof of Theorem A. Let $\alpha = \beta/2$. Let (M, g_α) be the open manifold with $\operatorname{Ric} > 0$ constructed in Section 1.1 and let $(N, \tilde{g_\alpha})$ be its Riemannian universal cover.

(1) First note that y is singular. In fact, let (Y', y') be a tangent cone of Y at y. By a standard diagonal argument, (Y', y') is an asymptotic cone of N; in other words, we can find $s_i \to \infty$ such that

$$(s_i^{-1}N, \tilde{p}, \Gamma) \xrightarrow{GH} (Y', y', G').$$

The orbit $G'y'$ satisfies the distance estimate in Lemma 1.3. Therefore, (Y', y') is not isometric to any $\mathbb{R}^k$.

Next we apply Lemma 1.2 to show that any point outside Gy is 2-regular. Let $z \in X \setminus \{x\}$ and let $\tilde{z} \in Y$ be a lift of z such that $d(\tilde{z}, y) = d(z, x)$. Since any point outside Gy is some $g\tilde{z}$, it suffices to show that $\tilde{z}$ is 2-regular.

We choose a sequence of points $\tilde{q}_i$ such that $d(\tilde{p}, \tilde{q}_i) = r_i t$, where $t = d(z, x)$, and

$$(r_i^{-1}N, \tilde{p}, \tilde{q}_i) \xrightarrow{GH} (Y, y, \tilde{z}).$$

We write $s_i = r_i t$. Let $\epsilon \in (0,1)$ and let $l_i \to \infty$ be a sequence of integers between $\frac{1}{2}\epsilon s_i (1+s_i^2)^\alpha$ and $\epsilon s_i (1+s_i^2)^\alpha$. By Lemma 1.2(1),

$$(r_i^{-1}N, \tilde{q}_i, \gamma^{l_i}) \xrightarrow{GH} (Y, \tilde{z}, g)$$

with $C\epsilon t \le d(g\tilde{z}, \tilde{z}) \le \epsilon t$, where $C \in (0,1)$ only depends on α. Let σ_i be a minimal geodesic from $\tilde{q}_i$ to $\gamma^{l_i}\tilde{q}_i$. After blowing down, σ_i converges to a minimal geodesic σ from $\tilde{z}$ to $g\tilde{z}$. Let m be the midpoint of σ and let $h\tilde{z} \in G\tilde{z}$ be a closest point in the orbit to m. We write $\delta = d(m, h\tilde{z})$. Then $\tau := h^{-1}\sigma$ is a segment whose midpoint has distance δ to $\tilde{z}$. By our construction,

$$\frac{\delta}{\epsilon} \le t \cdot \frac{d(m, G\tilde{z})}{d(g\tilde{z}, \tilde{z})} \le t \cdot \limsup_{i\to\infty} \frac{\mathrm{size}(c_{l_i})}{\mathrm{length}(c_{l_i})}$$

Now we change ϵ. To emphasize the dependence on ϵ, we will write τ_ϵ and δ_ϵ instead of τ and δ, respectively. By Lemma 1.2(2), the minimal geodesics τ_ϵ satisfy
(i) $\mathrm{length}(\tau_\epsilon) \in [C\epsilon t, \epsilon t]$;
(ii) midpoint of τ_ϵ has distance δ_ϵ to $\tilde{z}$, where $\delta_\epsilon/\epsilon \to 0$ as $\epsilon \to 0$.
Together with Cheeger-Colding splitting theorem [2], we see that any tangent cone at $\tilde{z}$ is isometric to $\mathbb{R}^2$, with a direction coming from the lifting of $\mathbb{R}$ as the tangent cone of X at z, and the other direction coming from the limit of τ_{ϵ_j} for some $\epsilon_j \to 0$.

(2) This follows directly from the distance estimate in Lemma 1.3. In fact, we consider the maps

$$\phi : (Gy, d) \to (\mathbb{R}, d_0), \quad (b, 0) \mapsto b,$$
$$\psi : (\mathbb{R}, d_0) \to (Gy, d), \quad b \mapsto (b, 0),$$

where d_0 is the Euclidean distance on $\mathbb{R}$. By Lemma 1.3, the maps ϕ and ψ are Hölder continuous with exponent $1+2\alpha$ and $1/(1+2\alpha)$, respectively. Thus

$$\dim_H(\mathbb{R}, d_0) \le \frac{1}{1+2\alpha} \dim_H(Gy, d),$$
$$\dim_H(Gy, d) \le (1+2\alpha)\dim_H(\mathbb{R}, d_0).$$

We conclude that Gy has Hausdorff dimension $1+2\alpha$. □

We end our paper with some remarks related to our examples.

Remark 1.4. One can also derive that every point $z \notin Gy$ is regular from the regularity theory. If z is singular, then any point in Gz is also singular; this would contradict the path-connectedness of the regular set [6].

Remark 1.5. For any asymptotic cone (Y, y) of $(N, \tilde{p})$, Y is non-polar at y. In fact, we consider a minimal geodesic σ from y to a point $gy \neq y$. We claim that σ cannot be extended further across gy as a minimal geodesic. Suppose that we can extend σ to $\overline{\sigma}$. First note that the extended part is not contained in Gy; otherwise, it would have infinite length by Lemma 1.3. Thus the minimal geodesic $\overline{\sigma}$ contains two regular points with a singular point gy in between; this would contradict the Hölder continuity of tangent cones [6, 7].

Remark 1.6. If the warping function h has logarithm decay $\ln^{-\alpha}(r)$ or it decays to a positive number $c > 0$, then (Y, y), the corresponding asymptotic cone of the universal cover, splits isometrically as $\mathbb{R} \times [0, \infty)$; see [13].

Remark 1.7. We suspect that the structure of fundamental group $\pi_1(M)$ is related to the structure of singular set of Y: if $\pi_1(M)$ contains a torsion-free nilpotent subgroup of class ≥ 2, is it true that for some asymptotic cone (Y, y, G) of the universal cover $(N, \tilde{p}, \pi_1(M, p))$, its orbit Gy must be non-rectifiable? The first author proved that for such a fundamental group, in some (Y, y, G) there is a geodesic connecting two points in Gy but definite away from Gy; see [13, 14].

Remark 1.8. We can construct a compact Ricci limit space with same feature as in Theorem A as follows. Let $\beta = 2\alpha$, (M, g_α), and $(N, \tilde{g_\alpha})$ as constructed before. For a sequence $r_i \to \infty$, we choose $\gamma_i = \gamma^{l_i} \in \Gamma$, where $l_i \to \infty$, so that

$$(r_i^{-1} N, \tilde{p}, \gamma^{l_i}, \Gamma) \xrightarrow{GH} (Y, y, \gamma_\infty, G),$$

where $\gamma_\infty \in G$ with $d(\gamma_\infty y, y) = 1$. Let $(\overline{M_i}, \bar{p}_i) = (N, \tilde{p})/\langle \gamma_i \rangle$. Then

$$(r_i^{-1} \overline{M_i}, \bar{p}_i) \xrightarrow{GH} (Y/\mathbb{Z}, \bar{y}),$$

where $\mathbb{Z} = \langle \gamma_\infty \rangle$ is a discrete subgroup of G and the limit space $Y/\mathbb{Z}$ has Hausdorff dimension $1 + \beta$ when $\beta > 1$.

Next, we glue two copies of a bounded region in $r_i^{-1}\overline{M_i}$ along the boundary to obtain a compact Ricci limit space. Let $K_i = \pi_i^{-1}(B_{r_i}(p)) \subseteq \overline{M_i}$, where $\pi_i : \overline{M_i} \to M$ is the covering map. The principal eigenvalues of the second fundamental form of $\partial B_r(p)$ are $\frac{f'(r)}{f(r)}, \frac{h'(r)}{h(r)}$, which decay like r^{-1} as $r \to \infty$. Thus the second fundamental forms of $r_i^{-1}(\partial K_i)$ are uniformly bounded. Now one can extending the boundary of K_i with definite length to have totally geodesic boundary, then we can double it to have closed manifolds with Ricci curvature uniformly bounded from below and diameter uniformly bounded from above. The Hausdorff dimension of its Gromov-Hausdorff limit is also $1 + \beta$ when $\beta > 1$.

References

[1] Lionel Bérard-Bergery. Quelques exemples de variétés riemanniennes complètes non compactes à courbure de Ricci positive. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(4):159–161, 1986.

[2] Jeff Cheeger and Tobias H. Colding. Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)*, 144(1):189–237, 1996.

[3] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. i. *J. Differential Geom.*, 46(3):406–480, 1997.

[4] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. ii. *J. Differential Geom.*, 54(1):13–35, 2000.

[5] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. iii. *J. Differential Geom.*, 54(1):37–74, 2000.

[6] Tobias H. Colding and Aaron Naber. Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. *Annals of Mathematics*, 176(2):1173–1229, 2012.
[7] Qin Deng. Hölder continuity of tangent cones in RCD(K,N) spaces and applications to non-branching. *arXiv:2009.07956*, 2020.
[8] Kenji Fukaya. Theory of convergence for Riemannian orbifolds, *Japan. J. Math. (N.S.)*, 12 (1):121–160, 1986.
[9] Kenji Fukaya and Takao Yamaguchi. The fundamental groups of almost nonnegatively curved manifolds. *Annals of Mathematics*, 136(2):253–333, 1992.
[10] Xavier Menguy. Examples of nonpolar limit spaces. *Amer. J. Math.*, 122(5):927–937, 2000.
[11] Aaron Naber. Conjectures and open questions on the structure and regularity of spaces with lower Ricci curvature bounds. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 104, 8, 2020.
[12] Philippe Nabonnand. Sur les variétés riemanniennes complètes à courbure de Ricci positive. *C. R. Acad. Sci. Paris Sér. A-B*, 291(10):A591–A593, 1980.
[13] Jiayin Pan. On the escape rate of geodesic loops in an open manifold with nonnegative Ricci curvature. *Geom. & Topol.*, 25(2):1059–1085, 2021.
[14] Jiayin Pan. Nonnegative Ricci curvature and escape rate gap. *J. Reine Angew. Math.*, DOI:10.1515/crelle-2021-0065, 2021.
[15] Guofang Wei. Examples of complete manifolds of positive Ricci curvature with nilpotent isometry groups. *Bull. Amer. Math. Soc. (N.S.)*, 19(1):311–313, 1988.

(Jiayin Pan) Department of Mathematics, University of California, Santa Barbara, CA, USA.

Current address: The Fields Institute for Research in Mathematical Sciences, Toronto, Canada.
Email address: `jypan10@gmail.com`

(Guofang Wei) Department of Mathematics, University of California, Santa Barbara, CA, USA.

Email address: `wei@math.ucsb.edu`