Spaces with Ricci Curvature Lower Bounds

Jiayin Pan

Department of Mathematics, University of California, Santa Cruz, USA

E-mail: jpan53@ucsc.edu

Guofang Wei

Department of Mathematics, University of California, Santa Barbara, USA

E-mail: wei@ucsb.edu

Abstract We give a survey on authors' resent works on the geometry and topology of manifolds with Ricci curvature lower bounds or Ricci limit spaces. The topics include the Busemann functions and fundamental groups on complete manifolds with nonnegative Ricci curvature, as well as the Hausdorff dimension and local topology of Ricci limit spaces.

Key words Ricci curvature, Busemann function, fundamental group, Hausdorff dimension.

1. Busemann function of manifolds with nonnegative Ricci curvature

1.1 Busemann function

Let M^n be a complete noncompact manifold. For any $p \in M$, there exists a unit speed ray $\gamma(t) : [0, +\infty) \to M$, i.e. $\mathrm{d}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in [0, \infty)$. The Busemann function associated to γ is a renormalized distance function to infinity along γ, which plays an important in the study of noncompact manifolds.

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Definition 1. The Busemann function associated to a ray γ is a function $b_\gamma : M \to \mathbb{R}$ defined by

$$b_\gamma(x) = \lim_{t\to\infty} \Big(t - \mathrm{d}(x, \gamma(t))\Big).$$

Note that the sequence is monotone and bounded so the limit exists.

Remark 2.

- *b_γ is Lipschitz with Lipschitz constant* 1.
- *Along γ, $b_\gamma(\gamma(t)) = t$ is linear in t.*

Example 3. Let $M = \mathbb{R}^n$ with the usual Euclidean metric. Then all rays are of the form $\gamma(t) = \gamma(0) + t\gamma'(t)$, and $b_\gamma(x) = \langle x - \gamma(0), \gamma'(0)\rangle$.

Theorem 4 (Cheeger-Gromoll, 71', 72')**.**

- *[8] If the sectional curvature $K_M \geq 0$, then* $\mathrm{Hess}\, b_\gamma \geq 0$.
- *[7] If the Ricci curvature* $\mathrm{Ric}_M \geq 0$*, then* $\Delta b_\gamma \geq 0$.

  (both in barrier sense)

Remark 5.

1. *The first result plays an important role in the proof of Soul's theorem, while the second one leads to the splitting theorem.*
2. *By Laplacian comparison we have $\Delta r \leq \Delta_{\mathbb{R}^n} \bar{r} = \frac{n-1}{r}$. So intuitively, $\Delta\Big(t - \mathrm{d}(x, \gamma(t))\Big) \geq -\frac{n-1}{\mathrm{d}(x,\gamma(t))} \to 0$ as $t \to \infty$.*

Definition 6 (Busemann function of a point)**.** $b_p(x) := \sup_\gamma b_\gamma(x)$, where the supremum is taken among all rays γ starting from p.

When M^n is polar with pole at p, then $b_p(x) = \mathrm{d}(p, x)$. Still $b_p(x)$ is convex when $K_M \geq 0$, and subharmonic when $\mathrm{Ric}_M \geq 0$ in barrier sense.

The convexity of b_p implies $b_p(x)$ is proper. In fact, it implies that the sublevel set $b_p^{-1}(-\infty, a] = \cap_\gamma b_\gamma^{-1}(-\infty, a]$ is compact.

Question 7 (Open problem since 70's)**.** *Is b_p proper when* $\mathrm{Ric}_M \geq 0$*?*

It has been shown that the answer is yes in many special cases:

- When M is polar with pole at p, we have $b_p(x) = \mathrm{d}(p, x)$, proper.
- If $\lim_{r\to\infty}\sup \frac{\mathrm{diam}\left(\partial B(p,r)\right)}{r} = \epsilon < 1$, then $\lim_{x\to\infty}\inf \frac{b_p(x)}{\mathrm{d}(p,x)} \geq 1 - \epsilon > 0$, which implies b_p is proper.

- **Shen, 1996** [25]: When M^n has Euclidean volume growth, b_p is proper.
- **Sormani 1998** [26]: When M^n has linear volume growth, i.e., $Cr \le \mathrm{vol}B(x,r) \le C'r$, then b_p is proper.

In Section 1.3, we will see that the answer in general is negative.

1.2 Nabonnand's example of manifolds with positive Ricci curvature

We first recall Nabonnand's example [15], which is the first example of a manifold with positive Ricci curvature and infinite fundamental group π_1.

Example 8. Let $M = \mathbb{R}^k \times \mathbb{S}^1$ equipped with the doubly warped metric

$$[0,\infty) \times_f S^{k-1} \times_h S^1, \quad g = \mathrm{d}r^2 + f^2(r)\,\mathrm{d}s_{k-1}^2 + h^2(r)\,\mathrm{d}s_1^2$$

with $f(0) = 0, f'(0) = 1, f''(0) = 0, h(0) > 0, h'(0) = 0$. Denote $H = \frac{\partial}{\partial r}$, u a unit vector tangent to $\mathbb{S}^{k-1}$, and v a unit vector tangent to $\mathbb{S}^1$. Then one can compute

$$\mathrm{Ric}(H,H) = -(k-1)\frac{f''}{f} - \frac{h''}{h} \tag{1.1}$$

$$\mathrm{Ric}(u,u) = -\frac{f''}{f} - \frac{k-2}{f^2}\left(1-(f')^2\right) - \frac{f'h'}{fh} \tag{1.2}$$

$$\mathrm{Ric}(v,v) = -\frac{h''}{h} - (k-1)\frac{f'h'}{fh}. \tag{1.3}$$

When $0 < f' < 1, f'' < 0$, $h' < 0$, and $k \ge 2$, it is easy to see that $\mathrm{Ric}(u,u) > 0$.

Choose $h = f'$, then $\mathrm{Ric}(v,v) = \mathrm{Ric}(H,H)$. Let f be the solution of the ODE:

$$\begin{cases} f' = (1-\varphi(f))^{\frac{1}{2}} \\ f(0) = 0, \end{cases}$$

where $\varphi(x) = \frac{\sqrt{3}}{\pi}\int_0^x \frac{\arctan u^3}{u^2}\,\mathrm{d}u$. Here we choose an explicit φ, there are many other choices of φ for the construction. As $\int_0^\infty \frac{\arctan u^3}{u^2}\,\mathrm{d}u = \frac{\pi}{\sqrt{3}}$, we have $0 < \varphi(x) < 1$ for $x \in (0,\infty)$. Note that $h \to 0$ and $h \sim r^{-1/3},\ f \sim r^{2/3}$ as $r \to +\infty$.

Then one computes that $\mathrm{Ric}(H,H) > 0$ when $k \ge 3$.

Same construction works for $\mathbb{R}^k \times M^q$, where M has nonnegative Ricci curvature by modifying with $h = (f')^{1/q}$ [1].

Example 9. In [33] the second-named author constructed a metric with positive Ricci curvature on $\mathbb{R}^k \times N$, where N is a nilmanifold and k is large, as warped products

$$[0,\infty) \times_f S^{k-1} \times (N, g_r), \quad g = \mathrm{d}r^2 + f^2(r)\,\mathrm{d}s_{k-1}^2 + g_r.$$

This is the first example of manifolds with positive Ricci curvature with nilpotent fundamental group. See [2, 3] for more constructions along the line.

1.3 Example of manifolds with positive Ricci curvature and non-proper Busemann function

Theorem 10. *[22][Pan-Wei 2022] Given any integer $n \geq 4$, there is an open n-manifold with positive Ricci curvature and a non-proper Busemann function.*

Proof. First we study the geodesics in Nabonnand's example. Note that for each $x \in S^{k-1}$, the subset

$$C(x) = \{(r, \pm x, v) | r \geq 0, v \in S^1\}$$

is a totally geodesic and geodesically complete submanifold in M. Given any fixed point p at $r = 0$ in M, there are three types of geodesics starting at p:

(i) moving purely in $\mathbb{R}^k$, that is, $\gamma(t) = (t, x, y)$, where $x \in S^{k-1}$ and $y \in S^1$ are independent of t.

(ii) moving purely in the S^1 direction, that is, $\gamma(t) = (0, x, c(t))$, where $x \in S^{k-1}$ and $c(t)$ goes around the circle.

(iii) a mixture of both, that is, $\gamma(t) = (r(t), x, y(t))$, where $x \in S^{k-1}$.

In case (i), the geodesic is a ray and in case (ii), the geodesic is a closed circle. In case (iii), by Clairaut's relation $h(r(t)) \cdot (\cos\theta(t)) = \text{const} = \cos\theta(0)$, where $\theta(t)$ is the angle between γ and the parallel circle at $\gamma(t)$. Since h goes to zero as r increases and $\cos\theta$ is bounded, the geodesic stays in a bounded region and crosses $\{r = 0\}$ transversely infinite many times.

On the universal cover $\tilde{M}$ of M, a geodesic $\tilde{\gamma}$ starting at $\tilde{p}$ is a ray precisely when the projection $\pi(\tilde{\gamma})$ is case (i). This is because in cases (ii) and (iii), the geodesics are bounded in M, then they cannot lift to rays. Otherwise, the π_1-action would give a line in $\tilde{M}$, contradicting $\mathrm{Ric} > 0$.

Let g be a generator of $\pi_1(M, p) = \mathbb{Z}$. We claim $b_{\tilde{p}}(g^l \tilde{p}) = 0$ for all $l \in \mathbb{Z}$. Since the orbit at $\tilde{p}$ is noncompact as $\mathbb{Z} = \langle g \rangle$ is an infinite group, $b_{\tilde{p}}$ is not proper.

To show the claim, we prove that $b_{\tilde{\gamma}}(g^l\tilde{p}) = 0$ for any ray $\tilde{\gamma}$ at $\tilde{p}$. Note that we have shown that $\tilde{\gamma}$ must be the the lift of a geodesic described in case (i). Now

$$b_{\tilde{\gamma}}(g^l\tilde{p}) = \lim_{t\to\infty}(t - d(\tilde{\gamma}(t), g^l\tilde{p})).$$

Let $\tilde{\alpha}$ be a minimal geodesic connecting $\tilde{\gamma}(t)$ and $g^l\tilde{p}$, then

$$\begin{aligned} d(\tilde{\gamma}(t),\ g^l\tilde{p}) =& l(\tilde{\alpha}) = \text{length of the shortest curve in the homotopy class} \\ \leq& t + l\cdot 2\pi h(t). \end{aligned}$$

Covering map is distance non-increasing, hence

$$d(\tilde{\gamma}(t),\ g^l\tilde{p}) \geq d(\gamma(t), p) = t.$$

Therefore

$$-2\pi l\cdot h(t) \leq t - d(\tilde{\gamma}(t), g^l\tilde{p}) \leq 0.$$

Since $h(t)\to 0$ as $t\to\infty$, we have $b_{\tilde{\gamma}}(g^l\tilde{p}) = 0$ and proved the claim. $\square$

Question 11. *What about $n = 3$?*

Bruè-Naber-Semola [4] recently gave examples of manifolds M^n with nonnegative Ricci curvature and π_1 not finitely generated for $n \geq 7$.

Question 12. *Is the properness of the Busemann function related to the finitely generation of the π_1?*

2. Hausdorff dimension of Ricci limit spaces

2.1 Hausdorff dimension

Recall that the Hausdorff measure is an outer measure on subsets of a general metric space (X, d).

Definition 13 (Hausdorff Measure)**.** Let $0 \leq \alpha < \infty$ and $0 < \delta < \infty$. Let $A \subset X$. Define an outer measure

$$\mathcal{H}^\alpha_\delta(A) := \omega_\alpha \cdot \inf\left\{ \sum_{i=1}^\infty \left(\tfrac{\operatorname{diam} C_i}{2}\right)^\alpha \,\middle|\, A\subset \bigcup_{i=1}^\infty C_i,\ \text{with}\ \operatorname{diam} C_i \leq \delta \text{ for all } i\in\mathbb{N} \right\},$$

the infimum is taken over all countable covers of A.

The α-dimensional *Hausdorff measure* of $A \subset X$ is the outer measure

$$\mathcal{H}^\alpha(A) := \lim_{\delta\to 0}\mathcal{H}^\alpha_\delta(A).$$

Definition 14. The quantity

$$\dim_{\mathcal{H}}(A) := \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) = 0\} = \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) \neq \infty\}$$

is called the *Hausdorff dimension* of A.

Example 15. Let $\alpha \ge 1$ and let $(\mathbb{R}, d_\alpha)$ be defined by

$$d_\alpha(t_1, t_2) = |t_1 - t_2|^{1/\alpha}.$$

One can see that $\dim_{\mathcal{H}}(\mathbb{R}, d_\alpha) = \alpha$.

If $(M_i^n, p_i, \mu_i) \overset{mGH}{\to} (Y, p, \mu)$ and each M_i has $\mathrm{Ric} \ge -(n-1)H$, where $\mu_i = \mathrm{vol}(\cdot)/\mathrm{vol}(B(p_i, 1))$ is the renormalized measure on M_i, then under the limit renormalized limit measure μ we have $r_1 \ge r_2$ implying

$$\frac{\mu(B(y, r_1))}{\mu(B(y, r_2))} \ge \frac{v(n, H, r_1)}{v(n, H, r_2)}, \tag{1.4}$$

where $v(n, H, r)$ means the volume of an r-ball in the n-dimensional space form with constant curvature H. This relative volume comparison implies the following.

Proposition 16. *[6] Any such space Y has $\dim_{\mathcal{H}}(Y) \le n$.*

Let (M_i^n, g_i, p_i) be a sequence of Riemannian manifolds with $\mathrm{Ric}_{M_i} \ge (n-1)H$ and $M_i^n \overset{GH}{\to} Y$. Recall that Y is called a non-collapsed limit if

$$\mathrm{vol}(B(p_i, 1)) \ge v > 0,$$

and collapsed if

$$\mathrm{vol}(B(p_i, 1)) \to 0.$$

Let μ be a renormalized measure on Y. We have the following result

Proposition 17. *[6] If Y is a non-collapsed limit, then $\mu = c\mathcal{H}^n$ for some constant $c > 0$ and $\dim_{\mathcal{H}}(Y) = n$.*

If Y is a collapsed limit, then $\dim_{\mathcal{H}}(Y) \le n - 1$. In particular, the Hausdorff dimension of Y cannot be between n and $n - 1$.

2.2 Rectifiable dimension of Ricci limit spaces

Definition 18 (tangent cone at a point)**.** Let (X, d) be a metric space. Let $p \in X$. We take the pGH limit of $(X, p, \lambda_n d)$ where $\lambda_n \to \infty$. If such a limit exists, then it is called a tangent cone of X at p.

Remark 19. *Note that the limit may depend on the choice of sequence* λ_n*, i.e. tangent cones may not be unique. If it is unique, we denote the tangent cone at* p *as* $C_p(X)$*. The intuition is that we are zooming in the space at* p*.*

Definition 20 (asymptotic cone)**.** Let (X, d) be a (non-compact) metric space. Let $p \in X$. We take the pGH limit of $(X, p, \lambda_n d)$ where $\lambda_n \to 0$. If such a limit exists, then it is called a asymptotic cone of X.

Remark 21. *The asymptotic cone does not depend on* $p \in X$*, but may depend on* λ_n*.The intuition is that we are zooming out the space, looking from somewhere far away. Asymptotic cone is especially useful when study spaces with* $\mathrm{Ric} \geq 0$*.*

Definition 22 (Regular and singular points)**.** A point $p \in X$ is called a regular point if the tangent cone at p exists, and is unique and isometric to $\mathbb{R}^k$ for some integer k. We denote $\mathcal{R} = \{p \in X | p \text{ is regular}\}$, the collection of all regular points in X, $\mathcal{S} = X \setminus \mathcal{R}$, the set of singular points.

Remark 23. *In general, different regular points in* X *may have different* k*. For example, consider a suitable CW complex.*

Denote the k-regular set,

$$\mathcal{R}_k = \{y \in Y \mid C_y(Y) \text{ is unique and isometric to } \mathbb{R}^k\}.$$

We have the following general result for Ricci limit space Y by Colding-Naber.

Theorem 24. *[9] There exists a unique integer* k*,* $0 \leq k \leq n$ *such that* $\mathcal{R}_k$ *has full* μ*-measure for any limit renormalized measure* μ*, i.e.,* $\mu(Y \setminus \mathcal{R}_k) = 0$*.*

This k is called the rectifiable dimension or essential dimension. In general, k is not equal to the Hausdorff dimension of Y, see Section 2.3 below. However, in the non-collapsed case, we indeed have [6]

$$k = \dim_{\mathcal{H}}(Y) = \dim(M_i).$$

In general $\dim_{\mathcal{H}} \mathcal{R}_k = k$.

2.3 Examples of Ricci limit space with Hausdorff dimension different from the rectifiable dimension

Consider the doubly warped product

$$M = [0, \infty) \times_f S^{k-1} \times_h S^1, \quad g = \mathrm{d}r^2 + f^2(r)\, \mathrm{d}s^2_{k-1} + h^2(r)\, \mathrm{d}s^2_1$$

again as in Section 1.2, but with warping functions as in [33]. Namely, let $f(r) = r(1+r^2)^{-\frac{1}{4}} \sim \sqrt{r}$ and $h(r) = (1+r^2)^{-\alpha} \sim r^{-2\alpha}$, where $\alpha > 0$.

From the curvature formulas (1.1)-(1.3) one can check that Ric > 0 if $k \geq \max\{4\alpha+3, 16\alpha^2+8\alpha+1\}$.

Let $\tilde{M}$ be its universal cover, which is diffeomorphic to $\mathbb{R}^{k+1}$. We denote the asymptotic cone, singular set, regular set of $\tilde{M}$ by $Y, \mathcal{S}, \mathcal{R}_2$, respectively.

Theorem 25. *[21] [Pan-Wei, 2022]*

1. $Y = [0,+\infty) \times \mathbb{R},\ \mathcal{S} = \{0\} \times \mathbb{R},\ \mathcal{R}_2 = (0,+\infty) \times \mathbb{R}$
2. $\dim_{\mathcal{H}}(\mathcal{S}) = 1 + 2\alpha,\ \dim_{\mathcal{H}}(\mathcal{R}_2) = 2$

Theorem 26 ([10])**.** *Y is the metric completion of an incomplete Riemannian metric $g_Y = \mathrm{d}r^2 + r^{-4\alpha}\,\mathrm{d}v^2$ on $(0,+\infty)\times\mathbb{R}$.*

Remark 27. *Y with this metric is called 2α-Grushin halfplane, an almost Riemannian and RCD space at the same time. We remark that the whole Grushin plane is not an RCD space as the singular set cuts the plane into two disjoint parts but the regular set of RCD space should be path connected.*

Theorem 26 has some immediate interesting consequences.

For $\lambda > 0$, consider $F_\lambda(r,v) = (\lambda r, \lambda^{1+2\alpha} v)$. Then we have $F_* g_Y = \lambda^2 g_Y$, therefore $\mathrm{d}\Big(F_\lambda(y_1), F_\lambda(y_2)\Big) = \lambda\,\mathrm{d}(y_1,y_2)$. In other words, $\{F_\lambda\}$ are metric dilations of Y. Apply above with $\lambda = v^{\frac{1}{1+2\alpha}}$. Then

$$\mathrm{d}\Big((0,v),(0,0)\Big) = \mathrm{d}\Big(F_\lambda(0,1), F_\lambda(0,0)\Big) = v^{\frac{1}{1+2\alpha}}\,\mathrm{d}\Big((0,1),(0,0)\Big).$$

This implies $\dim_{\mathcal{H}}(\mathcal{S}) = 1 + 2\alpha$ from Example 15.

Proof of Theorem 26. We have $\tilde{M} = \mathbb{R}^k \times \mathbb{R}^1$, with doubly warped metric

$$g = dr^2 + r^2(1+r^2)^{-\frac{1}{2}} ds_{k-1}^2 + (1+r^2)^{-2\alpha} dv^2.$$

Given any $\lambda > 1$, let $s = \lambda^{-1} r$, $w = \lambda^{-2\alpha} v$. Then we get

$$\begin{aligned}\lambda^{-2} g_{\tilde{M}} &= \lambda^{-2}\left[dr^2 + r^2(1+r^2)^{-\frac{1}{2}} ds_{k-1}^2 + (1+r^2)^{-2\alpha} dv^2\right]\\ &= ds^2 + \frac{s^2}{1+\lambda^2 s^2} ds_{k-1}^2 + (1+\lambda^2 s^2)^{-2\alpha}\lambda^{4\alpha} dw^2\end{aligned}$$

As we take the limit $\lambda \to \infty$, this metric approaches $ds^2 + s^{-4\alpha} dw^2$. □

Question 28. *Given $\alpha > 1$, what is the smallest dimension n such that a Ricci limit space $X \in \mathcal{M}(n, -1)$ admits an isometric $\mathbb{R}$-orbit with Hausdorff dimension α?*

3. Characterizing virtual abelianness / nilpotency of $\pi_1(M)$

3.1 Small escape rate and virtual abelianness

We have mentioned in Section 1.2 that $\mathbb{R}^k \times N$, where N is a nilmanifold, admits a metric with positive Ricci curvature when k is large. This is distinct from open manifolds with nonnegative sectional curvature, whose fundamental groups are virtually abelian (i.e., contains an abelian subgroup of finite index). In fact, if M has $\sec_M \geq 0$, then it follows from Cheeger-Gromoll soul theorem that M is homotopic to a closed submanifold S with $\sec_S \geq 0$ in M, thus $\pi_1(M) = \pi_1(S)$ is virtually abelian.

Therefore, it is natural to investigate on what additional conditions $\pi_1(M)$ is virtually abelian for nonnegative Ricci curvature; or equivalently, we can ask how virtual abelianness or nilpotency of $\pi_1(M)$ is related to the geometry of M.

It follows from Cheeger-Gromoll splitting theorem that we have the following result on virtual abelianness.

Proposition 29. *Let (M, x) be an open manifold of $\mathrm{Ric} \geq 0$. If $\sup_{\gamma \in \Gamma} d_H(x, c_\gamma) < \infty$, where c_γ is a minimal representing loop of $\gamma \in \pi_1(M, x)$ at x, then $\pi_1(M, x)$ is virtually abelian.*

The assumption in Proposition 29 always holds when $\sec \geq 0$. This follows from Cheeger-Gromoll soul theorem and Sharafutdinov retraction [8, 24]. On the other hand, the assumption in Proposition 29 is quite restrictive for manifolds with nonnegative Ricci curvature. In fact, if M has positive Ricci curvature and an infinite fundamental group, then it follows from the Cheeger-Gromoll splitting theorem that the representing geodesic loops c_γ will always escape from any bounded sets as γ exhausts $\pi_1(M, x)$.

To study this escape phenomenon and its relation to $\pi_1(M)$, the first-named author introduced the notion of escape rate in [16].

Definition 30. Let (M, x) be an open manifold with an infinite fundamental group. We define the *escape rate* of (M,x), a scaling invariant, as

$$E(M,x) = \limsup_{|\gamma|\to\infty} \frac{d_H(x,c_\gamma)}{|\gamma|},$$

where $\gamma \in \pi_1(M,x)$, $|\gamma| = \text{length}(c_\gamma)$, and d_H is the Hausdorff distance. If $\pi_1(M)$ is finite, then we set $E(M,x)=0$ as a convention.

By definition, the escape rate takes value within $[0,1/2]$. It is known that $E(M,p) < 1/2$ implies the finite generation of $\pi_1(M)$ by Sormani's halfway lemma [27]. It is unclear whether its converse holds.

Question 31. *Let (M,p) be an open n-manifold with* $\text{Ric} \geq 0$ *and a finitely generated fundamental group. Is it true that $E(M,p) < 1/2$?*

Regarding examples as doubly warped products $[0,\infty) \times_f S^{p-1} \times_h S^1$, their escape rates are determined by the warping function h. As $h(r)$ decreases to 0 as $r \to \infty$, the representing geodesic loop will take advantage of the thin end to shorten its length, which will in turn enlarge its size. In other words, one should expect that the faster h decays, the larger the escape rate is. In fact, if $h(r)$ has polynomial decay, then $E(M,x)$ is positive; if $h(r)$ has logarithm decay or converges to a positive constant, then $E(M,x)=0$.

As the main result in [17], small escape rate implies virtual abelianness. Note that this greatly generalized Proposition 29.

Theorem 32. *[Pan 2022] Given n, there exists a constant $\epsilon(n) > 0$ such that if (M,x) is an open n-manifold with* $\text{Ric} \geq 0$ *and $E(M,x) \leq \epsilon(n)$, then $\pi_1(M,x)$ is virtually abelian.*

In other words, if $\pi_1(M,x)$ contains a torsion-free nilpotent subgroup of nilpotency step ≥ 2, then its minimal representing loops must escape any bounded subsets at a relatively fast rate compared to their lengths.

The proof of Theorem 32 depends on the equivariant asymptotic geometry of $(\widetilde{M}, \pi_1(M,x))$. The key is to show that any asymptotic π_1-orbit at $\tilde{x}$ is Gromov-Hausdorff close to a Euclidean factor. See [16, 17] for details.

3.2 Nilpotency step and Hausdorff dimension

Given Theorem 32, it is naturally to further study the equivariant asymptotic geometry without the smallness of escape rate.

To better motivate the next result, we recall the structure of Carnot groups [23]. Let Γ be a finitely generated virtually nilpotent group with nilpotency step l. Any finite generating set S of Γ defines a word length metric d_S on Γ. The asymptotic structure of (Γ, d_S) was studied by Gromov [13] and Pansu [23]. For any sequence $r_i \to \infty$, Gromov-Hausdorff convergence holds:

$$(r_i^{-1}\Gamma, e, d_S) \xrightarrow{GH} (G, e, d).$$

The unique limit space (G, d) is a Carnot group, that is, a simply connected stratified nilpotent Lie group G with nilpotency step l and a distance d induced by a left-invariant subFinsler metric. Moreover, $\dim_{\mathcal{H}}(Ly) = l$ for any one-parameter subgroup L in $\zeta_{l-1}(G)$, the last nontrivial subgroup in the lower central series. This structure also applies to closed manifolds. For a closed Riemannian manifold (M, g) with a virtually nilpotent fundamental group Γ, although g cannot have nonnegative Ricci curvature when Γ has nilpotency step ≥ 2, the blow-down sequence of the universal cover $(r_i^{-1}\widetilde{M}, \tilde{p}, \tilde{g})$ actually converges in the Gromov-Hausdorff topology to a limit space (G, e, d) as described above.

When it comes to open manifolds with $\mathrm{Ric} > 0$ and torsion-free nilpotent fundamental groups mentioned in Example 9, the asymptotic cone of the universal cover always has an isometric $\mathbb{R}$-orbit with Hausdorff dimension ≥ 2; see [19, Section 3] for details.

There is indeed a relation between the nilpotency step of $\pi_1(M)$ and the Hausdorff dimension of $\mathbb{R}$-orbits in asymptotic cones [19]:

Theorem 33. *[Pan 2023] Let (M, p) be an open n-manifold with* $\mathrm{Ric} \geq 0$ *and $E(M, p) \neq 1/2$. Let $\mathcal{N}$ be a torsion-free nilpotent subgroup of $\pi_1(M, p)$ with nilpotency step l and finite index. Then there exists an asymptotic cone (Y, y) of $\widetilde{M}$, the universal cover of M, and a closed $\mathbb{R}$-subgroup L of* $\mathrm{Isom}(Y)$ *such that $\dim_{\mathcal{H}}(Ly) \geq l$.*

We say that $\widetilde{M}$ is conic at infinity, if any asymptotic cone (Y, y) of $\widetilde{M}$ is a metric cone with vertex y. For a metric cone (Y, y), any isometric $\mathbb{R}$-orbit at y must have Hausdorff dimension exactly 1. Thus Theorem 33 implies the virtual abelianness of π_1 in this case. When $\widetilde{M}$ has Euclidean volume growth, by a result of Cheeger-Colding, it is conic at infinity. We can also control the index of the abelian subgroup by the dimension and volume growth rate [18].

Theorem 34. *[Pan 2022] Let (M,p) be an open n-manifold with* $\mathrm{Ric} \geq 0$ *and* $E(M,p) \neq 1/2$.
(1) If its Riemannian universal cover is conic at infinity, then $\pi_1(M)$ is virtually abelian.
(2) If its Riemannian universal cover has Euclidean volume growth of constant at least L, then $\pi_1(M)$ has an abelian subgroup of index at most $C(n,L)$, a constant only depending on n and L.

We close this section with a question below, which may be related to Question 28 in the light of Theorem 33.

Question 35. *What is the smallest dimension n such that there is an open n-manifold with* $\mathrm{Ric} \geq 0$ *and a torsion-free nilpotent fundamental group?*

4. Topology of Ricci limit/RCD spaces

This chapter concerns the universal covers and semi-local simple connectedness of Ricci limit spaces or RCD spaces. In general, a Ricci limit space, even non-collapsing, may have infinite second or higher Betti number locally [12]. Recently, Hupp, Naber, and Wang have constructed a collapsing Ricci limit space such that any open set U has infinitely generated $H^2(U)$ [11]. Thus it is natural to ask whether a Ricci limit space is semi-locally simply connected, or equivalently, whether it has a simply connected universal cover.

4.1 Relative δ-covers

The universal cover is often defined as the simply connected cover. Here we do not assume it is simply connected, instead as the cover of all covers.

Definition 36. [30, Page 82] We say $\widetilde{X}$ is a universal cover of a path-connected space X if $\widetilde{X}$ is a cover of X such that for any other cover $\bar{X}$ of X, there is a commutative triangle formed by a covering map $f : \widetilde{X} \to \bar{X}$ and the two covering projections as below:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & \bar{X} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Let $\mathcal{U}$ be any open covering of X. For any $x \in X$, by [30, Page 81], there is a covering space $\widetilde{X}_{\mathcal{U}}$ of X with covering group $\pi_1(X,\mathcal{U},p)$, where $\pi_1(X,\mathcal{U},x)$ is a normal subgroup of $\pi_1(X,p)$ generated by homotopy classes of closed paths having a representative of the form $\alpha^{-1}\circ\beta\circ\alpha$, where β is a closed path lying in some element of $\mathcal{U}$ and α is a path from x to $\beta(0)$.

Now we recall the notion of δ-covers introduced in [28] which plays an important role in studying the existence of the universal cover.

Definition 37. Given $\delta > 0$, the δ-cover, denoted $\widetilde{X}^{\delta}$, of a length space X is defined to be $\widetilde{X}_{\mathcal{U}_\delta}$, where $\mathcal{U}_\delta$ is the open covering of X consisting of all balls of radius δ.

Intuitively, a δ-cover is the result of unwrapping all but the loops generated by small loops in X. Clearly $\widetilde{X}^{\delta_1}$ covers $\widetilde{X}^{\delta_2}$ when $\delta_1 \le \delta_2$.

Definition 38 (Relative δ-cover)**.** Suppose X is a length space, $x \in X$ and $0 < r < R$. Let

$$\pi^\delta : \widetilde{B}_R(x)^\delta \to B_R(x)$$

be the δ-cover of the open ball $B_R(x)$. A connected component of

$$(\pi^\delta)^{-1}(B(x,r)),$$

where $B(x,r)$ is a closed ball, is called a relative δ-cover of $B(x,r)$ and is denoted $\widetilde{B}(x,r,R)^\delta$.

4.2 Universal cover of Ricci limit/RCD space exists

In [29, Lemma 2.4, Theorem 2.5] it is shown that if the relative δ-cover stabilizes, then universal cover exists. This is the key tool for showing the existence of the universal cover in the works of Sormani and the second-named author.

Theorem 39. *[28,29] Let (X,d) be a length space and assume that there is $x\in X$ with the following property: for all $r>0$, there exists $R\ge r$, such that $\tilde{B}(x,r,R)^\delta$ stabilizes for all δ sufficiently small. Then (X,d) admits a universal cover $\tilde{X}$. More precisely $\tilde{X}$ is obtained as covering space $\tilde{X}_{\mathcal{U}}$ associated to a suitable open cover $\mathcal{U}$ of X satisfying the following property: for every $x\in X$ there exists $U_x\in\mathcal{U}$ such that U_x is lifted homeomorphically by any covering space of (X,d).*

Theorem 40. *[28,29] If X is the Gromov-Hausdorff limit of a sequence of complete Riemannian manifolds M_i^n with Ricci curvature $\ge K$, then X has a universal cover.*

These results on Ricci limit spaces were later generalized to RCD spaces by Mondino and the second-named author.

Theorem 41 ([14])**.** *Any* $\mathsf{RCD}^*(K,N)$ *space* $(X,\mathsf{d},\mathfrak{m})$ *admits a universal cover* $(\tilde{X},\tilde{\mathsf{d}},\tilde{\mathfrak{m}})$*, which is itself* $\mathsf{RCD}^*(K,N)$*, where* $K\in\mathbb{R}$*,* $N\in(1,+\infty)$*.*

By Theorem 39, in order to prove that the universal exists, it is enough to show the relative covers stabilize. See [14, Theorem 4.5].

Theorem 42. *[14] Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}^*(K,N)$ *space for some* $K\in\mathbb{R},N\in(1,\infty)$*. For all* $R>0$ *and* $x\in X$*, there exists* $\delta_{x,R}$ *depending on* X,x,R *such that*

$$\tilde{B}(x,\tfrac{R}{10},R)^{\delta_{x,R}}=\tilde{B}(x,\tfrac{R}{10},R)^{\delta},\qquad \forall\delta<\delta_{x,R}.$$

Theorem 42 plays an important role in the work of Wang [32] showing that RCD spaces are semi-locally simply connected.

4.3 Ricci limit/RCD spaces are semi-locally simply connected

Recall that the universal cover $\tilde{X}$ is simply connected iff X is semi-locally simply connected, which means that there exists a neighbourhood such that every loop is contractible in X.

Definition 43 (1-contractibility radius)**.**

$$\rho(t,x)=\inf\{\infty,\rho\ge t|\text{ any loop in } B_t(x) \text{ is contractible in } B_\rho(x)\}.$$

X is semi-locally simply connected if for any $x\in X$, there is $T>0$ such that $\rho(T,x)<\infty$.

In [20], for noncollapsing Ricci limit spaces, we show they are essentially locally simply connected.

Theorem 44. *[20] Any non-collapsing Ricci limit space is semi-locally simply connected. Therefore the universal cover is simply connected. In fact*

$$\lim_{t\to 0}\frac{\rho(t,x)}{t}=1.$$

In the paper, we illustrate several ways of constructing homotopy. One way is to construct a homotopy by defining it on finer and finer skeletons of closed unit disk, see [20, Lemma 4.1].

The case for general Ricci limit spaces and RCD spaces are resolved by Wang [31,32]. A key step by Wang shows that the stablity of local δ-covers indeed implies semi-local simple connectedness.

Theorem 45. *[32] For a locally compact length metric space X, if any local relative δ-cover is stable, then X is semi-locally simply connected.*

Combining this with Theorem 42 it shows that the universal cover of $\mathsf{RCD}(k, N)$, where $N < \infty$, is simply connected.

To prove Theorem 45, the key is to use stability of the local relative δ-cover to show any loop in a small neighborhood of an $\mathsf{RCD}(K, N)$ space is homotopic to some loops in very small balls by a controlled homotopy image.

Lemma 46. *[32] For any $x \in (X, \mathsf{d}, \mathfrak{m})$, an $\mathsf{RCD}(K, N)$, any $l < 1/2$, and small $\delta > 0$, there exists $\rho < l$ and $k \in \mathbb{N}$ so that any loop $\gamma \subset B_\rho(x)$ is homotopic to the union of some loops γ_i $(1 \leq i \leq k)$ in δ-balls and the homotopy image is in $B_{4l}(x)$.*

Applying the above lemma iteratively, one can construct the needed homotopy by using a similar method in [20, Lemma 4.1]. Namely, first shrink γ to smaller loops in δ_1-balls, then the second step is to shrink each new loops to even smaller loops in δ_2-balls, etc. Since the homotopy to shrink each loop is contained in a l_i-ball in the i-th step, this process converges to a homotopy map which contracts γ while the image is contained in a ball with radius $\sum_{i=1}^{\infty} l_i \leq R$.

In terms of the 1-contractibility radius, the above argument indeed proves that $\rho(t, x) \to 0$ as $t \to 0$. It is unclear whether a stronger estimate $\rho(t, x)/t \to 1$ in the non-collapsing case holds in general.

Question 47. $\lim_{t \to 0} \frac{\rho(t,x)}{t} = 1$*?*

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