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COMPARISON GEOMETRY WITH INTEGRAL CURVATURE BOUNDS

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Abstract

In this paper we shall generalize a formula of Heintze and Karcher for the volume of normal tubes around geodesics to a situation where one has integral bounds for the sectional curvature. This formula leads to a generalization of Cheeger's lemma for the length of the shortest closed geodesic and to a generalization of the Grove-Petersen finiteness result to a situation where one has integral curvature bounds.

1 Introduction

In this paper we shall be concerned with generalizing certain finiteness and compactness theorems. The three classical results are the Cheeger, Gromov, and Grove-Petersen Finiteness Theorems (see [C1], [GrWu], [Gro1], [GrovPe2], [GrovPeWu], [Pet1], [Pet2]). For these one considers classes of Riemannian manifolds with some uniform curvature bounds, upper diameter bounds, and lower volume bounds (except for the Betti number estimate). There have been many efforts to generalize especially Cheeger's results to situations with weaker curvature bounds. One direction has been to assume Ricci curvature bounds, however, in all but one case (see [AC]) one needs stronger side conditions than mere lower volume bounds (see e.g. [GrovPe1] for a survey containing most of the known results). Another direction has been to assume L^p curvature bounds, but also here stronger side conditions seem to be necessary. In addition one also runs into some very fundamental problems. Namely, much of Comparison Geometry relies heavily on having uniform curvature bounds. So for these extensions one cannot even be sure that the classes one studies are precompact in the Gromov-Hausdorff topology. In this paper we will show how some parts of

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comparison geometry can be recovered in the situation where one has L^p curvature bounds.

In order to explain these generalizations we need some notation. On a Riemannian manifold M define the two functions $f, g : M \to [0, \infty)$ as $f(x) =$ smallest sectional curvature of a plane in T_xM and $g(x) =$ the smallest eigenvalue for $\mathrm{Ric} : T_xM \to T_xM$. Now consider

$$K(\lambda, p) = \int_M \big(\max\{-f(x) + \lambda, 0\} \big)^p d\,\mathrm{vol}\,,$$
$$k(\lambda, p) = \int_M \big(\max\{-g(x) + (n-1)\lambda, 0\} \big)^p d\,\mathrm{vol}\,,$$

and in addition define $K(p) = K(0,p)$ and $k(p) = k(0,p)$. Apparently these quantities measure how much of the sectional curvature or Ricci curvature lies below λ in the L^p norm. In particular, we have that $\sec(M) \ge \lambda$ iff $K(\lambda, p) = 0$, and similarly $\mathrm{Ric}(M) \ge (n-1)\lambda$ iff $k(\lambda, p) = 0$.

It is classical that one obtains bounds on the volume of tubes around hypersurfaces and metric balls given uniform lower bounds on the Ricci curvature. In the two papers [G] and [Y] the authors obtain volume bounds in those same situations in terms of $k(\lambda, p)$, when $\lambda \le 0$. One can get many nice results out of these generalized volume estimates. In [G] the author shows how one can bound rational Betti numbers in terms of diameter, $\int_M \|R\|^p \, d\,\mathrm{vol}$ and λ, when $k(\lambda, p)$ is small, thus partially generalizing [Gro1]. It is also explained why it is necessary that $k(\lambda, p)$ has to be small. In other words bounds on diameter and $K(p)$ do not imply Betti number bounds. There are numerous other interesting results in [G] which have had applications in other areas of geometry. The 3-dimensional isospectral compactness theorem in [BPerPe] for instance relies heavily on the results in both [G] and [Y]. Another interesting compactness result using [Y] was also obtained in [Hi]. This compactness result is probably the most general such result given that one has a lower bound on the injectivity radius. In [Y] the author obtains a compactness result with L^p curvature bounds (no smallness is necessary), but it is also assumed that for some c, r one has $\mathrm{vol}\, B(p,t) \ge ct^n$ for $t \le r$. Thus instead of just a lower volume bound it is necessary to have a local volume growth condition. In [A] this volume growth condition was replaced by a lower bound on the shortest closed geodesic (see also [GrovPe1] for an account of how all these theorems and conditions are connected).

To generalize the Cheeger and Grove-Petersen finiteness results it is necessary that one knows how to bound the volume of normal tubes around geodesics or simply how to bound the volume of the whole manifold using a closed geodesic. It was Cheeger who first realized the importance of this in [C1], where he used it to prove his finiteness theorem and also in [C2] for some very general pinching results. Later in [HK] a new method was found for estimating the volume of tubes around closed geodesics. In analogy with this result of Heintze and Karcher we shall show

Theorem 1.1. *Let $N \subset M$ be a geodesic and $\lambda \leq 0$, then the volume of the normal tube around N satisfies*

$$\operatorname{vol} T\left(N, r\right) \leq F\left(n, p, a, b, c, r\right) ,$$

where

$$\begin{aligned} a &= \ell\left(N\right) = \text{length of } N\,, \\ b &= |\lambda|^p\ , \\ c &= K\left(\lambda, p\right)\ , \qquad p > n-1\,. \end{aligned}$$

Furthermore, as $a, c \to 0$ we have that $F\left(n, p, a, b, c, r\right) \to 0$.

It would be more natural if we only needed to assume $p > n/2$ as is the case in other results of this nature. The more restrictive assumption $p > n-1$ comes as a further technical condition from some new estimates. These estimates, which appear in section 3, are of a very technical nature and cannot be done if $p \leq n-1$. We believe that these estimates are completely new to Riemannian geometry and hope they might prove very useful in situations where one has integral curvature bounds.

We can also extend the work of [HK] to the situation where $N \subset M$ is a submanifold. But, as we do not use this for our applications we decided to discuss this slight extension in another paper.

Our proof of this volume estimate is not just a mere extension of Gallot and Yang's work, although we have been much inspired by the beautiful new ideas presented in [G]. In fact we have some new comparison estimates in section 3 which are of a completely different nature to what one is used to in comparison geometry.

Using this volume estimate we shall present generalizations of 1) Cheeger's estimate for the shortest closed geodesic and 2) the Grove-Petersen Finiteness Theorem. The volume estimate will enable us to obtain compactness and pinching results where in addition to assuming lower volume bounds and upper diameter bounds one has some sort of L^p curvature bounds. However, the techniques developed in [PeW] give better results so we shall defer the discussion of these results to that paper.

Theorem 1.2. *Given an integer $n \geq 2$, and numbers $p > n-1$, $\lambda \leq 0$, $v > 0$, $D < \infty$, one can find $\varepsilon = \varepsilon(n, p, \lambda, v, D) > 0$ and $\delta = \delta\,(n, p, \lambda, v, D) > 0$ such that a closed Riemannian n-manifold M with*

$$\operatorname{vol}(M) \geq v\,,$$
$$\operatorname{diam}(M) \leq D\,,$$
$$K\,(\lambda, p) \leq \varepsilon\,(n, p, \lambda, v, D)$$

has the property that the length of the shortest closed geodesic $\geq \delta(n, p, \lambda, v, D)$.

For the second type of finiteness results we have

Theorem 1.3. *Given an integer $n \geq 2$, and numbers $p > n-1$, $\lambda \leq 0$, $v > 0$, and $D < \infty$, one can find an $\varepsilon = \varepsilon\,(n, p, \lambda, v, D)$ such that the class of closed Riemannian n-manifolds M satisfying*

$$\operatorname{vol}(M) \geq v\,,$$
$$\operatorname{diam}(M) \leq D\,,$$
$$K\,(\lambda, p) \leq \varepsilon\,(n, p, \lambda, v, D)$$

contains finitely many simple homotopy types, moreover, when $n \neq 3$ it contains only finitely many homeomorphism types.

With the techniques developed in this paper it is not clear that this class is precompact in the Gromov-Hausdorff topology. However, this will be taken care of in [PeW]. Nevertheless, one can use [GrPe] to establish this and the results there in turn depend heavily on the deep work on the filling radius from [Gro2].

We would like to point out that the original Homotopy Finiteness Theorem was improved in [P]. In there it was shown that the same class of Alexandrov spaces in fact only contains finitely many homeomorphism types (in all dimensions) with a more direct approach than that used in [GrovPeWu]. We do not know as yet whether a more direct approach along these lines is possible in our case, as we do not have any of the standard distance comparison techniques available for the classes of manifolds we consider.

The paper is briefly organized as follows. In section 2 we shall prove our main result on the volume estimate for tubes around geodesics. In this section we shall defer the proof of a crucial estimate to section 3. In section 4 we present the generalization of Cheeger's lemma and also the extension of the results in [GrovPe2] and [GrovPeWu]. It is somewhat less simple to carry the proof through here. We must in fact rely on some of the lesser known results on homotopy finiteness of classes of Riemannian manifolds.

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2 The Volume Estimate

We suppose that we are given a complete Riemannian n-manifold M and a k-dimensional submanifold $N \subset M$. The submanifold does not have to be closed or complete. The *normal tube of radius* r around N is defined as $T(N,r) = \{x \in M \mid x = \exp_N(tv) \text{ where } |t| \le r \text{ and } v \in \nu(N)\}$. Here $\nu(N)$ is the normal bundle of N in M consisting of vectors perpendicular to N and $\exp_N : \nu(N) \to M$ is the normal exponential map. Therefore, if N is not closed in M, then there is a difference between the normal tube and the set of points with distance r of N. For the applications we have in mind it suffices to consider the case where N is a geodesic $\gamma : (a,b) \to M$. More general formulas for volumes of tubes around submanifolds will be considered in a different paper. Using that N is a geodesic we can use a parallel framing along N to coordinatize the normal tubes $T(N,r)$ as (t,s,θ) where t measures the distance to N, s is the arclength parameter on N, and $\theta_{n-2} \in S^{n-2}$ is the angular parameter from the unit normal bundle. We know that these coordinates are always coordinates on a set of full measure on $T(N,r)$. Now write the Riemannian volume element as $d\,\mathrm{vol} = \omega dt \wedge ds \wedge d\theta_{n-2}$. As t increases ω becomes undefined but we can just define it to be zero for those t.

In [G] and [Y] the cases where N is a hypersurface or point respectively are studied. There the volume element is written in a similar way $d\,\mathrm{vol} = \omega dt \wedge ds$ or $d\,\mathrm{vol} = \omega dt \wedge d\theta_{n-1}$. They then consider the function J defined by $J^{n-1} = \omega$. The key point is that this function in both cases satisfies $J'' \le -(n-1)^{-1}\,\mathrm{Ric}(\partial_t, \partial_t)\, J$. In the first case one has the initial conditions $J(0) = 1$ and $J'(0) = h/(n-1)$ (mean curvature of S), while in the second one has $J(0) = 0$ and $J'(0) = 1$. If in our case we use $J^{n-1} = \omega$ we will of course get the same differential inequality for J. The initial conditions, however, will be $J(0) = 0$ and $J'(0) = \infty$ which is quite useless. Instead we define $J^{n-2}=\omega$, then the initial conditions become $J(0)=0$ (since N is totally geodesic) and $J'(0)=1$. Note that at this point we must restrict attention to $n>2$. However when $n=2$ a geodesic is a hypersurface which is the situation covered in [G]. To find the correct differential inequality we need a little more notation. We have the radial field ∂_t which is the unit gradient field for the distance function to N. Now define the *shape operator* $S(X)=\nabla_X\partial_t$ and the curvature $R_{\partial_t}(X)=R(X,\partial_t)\partial_t$, then we have the formula

$$\nabla_{\partial_t}S+S^2=S'+S^2=-R_{\partial_t}\,.$$

By taking traces and defining $h=\operatorname{tr}S$ we obtain

$$h'+\frac{h^2}{n-1}\le h'+\operatorname{tr}S^2=-\operatorname{Ric}(\partial_t,\partial_t)\,.$$

We know that $\omega'=h\omega$, therefore, if we define J by $J^{n-1}=\omega$, then

$$J''=\frac{1}{n-1}\left(h'+\frac{h^2}{n-1}\right)J\,.$$

Going to our case where $J^{n-2}=\omega$, we get instead

$$J''=\frac{1}{n-2}\left(h'+\frac{h^2}{n-2}\right)J\,.$$

Unfortunately the quantity $\left(h'+\frac{h^2}{n-2}\right)$ cannot be estimated in terms of the curvature as in the other situation. In order to handle this quantity more easily let $\varphi_1\le\cdots\le\varphi_{n-1}$ be the nontrivial eigenvalues of S (0 is always an eigenvalue with eigenvector ∂_t). From our setup we know that $\varphi_1(0)=0$, $\varphi_2(t),\dots,\varphi_{n-1}(t)\approx 1/t$ as $t\to 0$, and that the products $\varphi_1(t)\varphi_2(t),\dots,\varphi_1(t)\,\varphi_{n-1}(t)$ are bounded as $t\to 0$. We can now prove

Lemma 2.1. *Suppose $n>2$, then*

$$\left(h'+\frac{h^2}{n-2}\right)\le\left(-\operatorname{Ric}(\partial_t,\partial_t)+\frac{2}{n-2}\varphi_1(t)\left(\varphi_2(t)+\cdots+\varphi_{n-1}(t)\right)\right)\,.$$

Proof. First use that $h'=-\operatorname{tr}S^2-\operatorname{Ric}(\partial_t,\partial_t)$ to get

$$\left(h'+\frac{h^2}{n-2}\right)=\left(-\operatorname{Ric}(\partial_t,\partial_t)-\operatorname{tr}S^2+\frac{h^2}{n-2}\right)\,.$$

Then we only need to establish

$$\left(-\operatorname{tr}S^2+\frac{h^2}{n-2}\right)\le\frac{2}{n-2}\varphi_1(t)\left(\varphi_2(t)+\cdots+\varphi_{n-1}(t)\right)\,.$$

A direct computation shows

$$-\operatorname{tr}S^2+\frac{h^2}{n-2}$$

$$= -(\varphi_1^2 + \cdots + \varphi_{n-1}^2) + \frac{(\varphi_1 + \cdots + \varphi_{n-1})^2}{n-2}$$
$$= -(\varphi_1^2 + \cdots + \varphi_{n-1}^2) + \frac{\varphi_1^2}{n-2} + \frac{(\varphi_2 + \cdots + \varphi_{n-1})^2}{n-2} + \frac{2\varphi_1 (\varphi_2 + \cdots + \varphi_{n-1})}{n-2}$$
$$\leq \frac{2}{n-2} \varphi_1 (\varphi_2 + \cdots + \varphi_{n-1}) . \qquad \square$$

Now we have to find an appropriate estimate for $\varphi_1 \varphi_{n-1}$ in terms of curvature. One can clearly not find an upper bound for this quantity in terms of lower curvature bounds. Just imagine that one has positive curvature, then one does get upper bounds for φ_1 and φ_{n-1} separately, however both quantities tend to be negative after a while and so one can certainly not expect to bound their product from above.

Let us define $\varphi = \max\{\varphi_1, 0\}$ and $\psi = \max\{\varphi_{n-1}, 0\}$, i.e. we take the nonnegative parts of the smallest and largest eigenvalues. Furthermore, define $\rho = \max\{-\operatorname{Ric}(\partial_t, \partial_t), 0\}$, then we can prove

LEMMA 2.2. *Suppose $n > 2$ and $p > \frac{n-1}{2}$, then*

$$\frac{J' - 1}{J^\delta} \leq \left(C_1 (n,p) \int_0^t (2\varphi\psi + \rho)^p \, \omega dt \right)^{\frac{1}{2p-1}} ,$$

where

$$\delta = \frac{2p - n + 1}{2p - 1} ,$$
$$C_1 (n,p) = \frac{(2p-1)^p (p-1)^p}{(n-2)^p p^p (2p - n + 1)^{p-1}} .$$

Proof. In order to prove this we have to consider the two different cases where $h \geq 0$ and $h < 0$. Better yet, observe that as long as $J' < 1$ there is nothing to prove and that this will certainly be the case when $h < 0$. So suppose we are on an interval $[a,b]$ where $J'(a) = 1$ and $J' \geq 1$. In this case we have

$$\varphi_1 (\varphi_2 + \cdots + \varphi_{n-1}) \leq (n-2) \varphi\psi .$$

Whence $J'' \leq \frac{1}{n-2}(2\varphi\psi - \operatorname{Ric}(\partial_t, \partial_t))J$ on $[a,b]$. A straightforward calculation as in [Y, Lemma 7.3] shows that

$$\left(\frac{J' - 1}{J^\delta} \right)' + \delta \frac{(J'-1) J'}{J^{\delta+1}} \leq \frac{1}{n-2} (2\varphi\psi + \rho) J^{1-\delta} .$$

Since $J' \geq 1$ we have that

$$(J' - 1) J' = (J'-1)^2 + J' - 1 \geq (J'-1)^2 .$$

Therefore

$$\left(\frac{J'-1}{J^\delta}\right)' + \delta J^{\delta-1}\left(\frac{J'-1}{J^\delta}\right)^2 \le \frac{1}{n-2}(2\varphi\psi+\rho)J^{1-\delta}\,.$$

Now applying the inequality $\alpha x \le (1+x)_+^p$, where $\alpha = p^p\,(p-1)^{1-p}$, we obtain

$$\alpha\delta^{p-1}J^{-(p-1)(1-\delta)}\left(\frac{J'-1}{J^\delta}\right)^{2(p-1)}\left(\frac{J'-1}{J^\delta}\right)' \le \frac{1}{(n-2)^p}\,(2\varphi\psi+\rho)^p\,J^{p(1-\delta)}\,.$$

This implies that

$$\frac{d}{dt}\left(\frac{J'-1}{J^\delta}\right)^{2p-1} \le (2p-1)\,\alpha^{-1}\delta^{1-p}\frac{1}{(n-2)^p}\,(2\varphi\psi+\rho)^p\,J^{(2p-1)(1-\delta)}\,.$$

Now set $\delta = (2p-n+1)(2p-1)^{-1}$, i.e. $J^{(2p-1)(1-\delta)} = J^{n-2} = \omega$, and integrate on both sides of the inequality to obtain

$$\frac{J'-1}{J^\delta} \le \left(C_1\,(n,p)\int_a^t (2\varphi\psi+\rho)^p\,\omega dt\right)^{\frac{1}{2p-1}}, \qquad t \le b\,.$$

But then we can simply use

$$\int_a^t (2\varphi\psi+\rho)^p\,\omega dt \le \int_0^t (2\varphi\psi+\rho)^p\,\omega dt$$

to arrive at the desired inequality. □

We now have to estimate $\varphi\psi$. Even though we have gotten rid of the negative part of these eigenvalues it is still not true that this quantity can be estimated as it stands in terms of curvature. To see this, consider an example where the curvature is -1 for $t \le 1$ but 0 when $t \ge 1$. Then both functions will remain positive, however, when one enters the region where the curvature is 0 it will certainly not happen that $\varphi\psi$ becomes nonpositive, in fact it will look like $1/t^2$.

Denote by $\sigma = \max\{0, \min(\mathrm{spec}(-R_{\partial_t}))\}$. Here $\mathrm{spec}(-R_{\partial_t})$ is the set of eigenvalues of the operator $v \to -R_{\partial_t}(v)$. In the next section we shall prove

$$\int_0^t (\varphi\psi)^p\omega dt \le C_2(n,p)\int_0^t \sigma^p\omega dt\,.$$

Using $\sigma \ge \rho$ we can then conclude that

$$\frac{J'-1}{J^\delta} \le \left(C_3(n,p)\int_0^t \sigma^p\omega dt\right)^{\frac{1}{2p-1}}\,.$$

Define $(v(r))^{n-2} = \int_{S^{n-2}}\int_N \omega ds d\theta =$ area of level set $t = r$. Then we have

LEMMA 2.3. *Suppose $n > 2$ and $p > n-1$, then*

$$v' \leq a + bv^{\delta},$$
$$v(0) = 0,$$
$$\delta = \frac{2p-n+1}{2p-1},$$

where

$$a = \left(\ell(N)\operatorname{vol}(S^{n-2})\right)^{\frac{1}{n-2}},$$
$$b = \left(C_3(n,p)K(p)\right)^{\frac{1}{2p-1}}.$$

Proof. First we observe that

$$v'(r) \leq \left(\int_{S^{n-2}}\int_N \omega ds d\theta\right)^{\frac{1}{n-2}-1}\left(\int_{S^{n-2}}\int_N J^{n-3}J' ds d\theta\right).$$

Using

$$\frac{J'-1}{J^{\delta}} \leq \left(C_3(n,p)\int_0^t \sigma^p \omega dt\right)^{\frac{1}{2p-1}},$$

we get that

$$\begin{aligned}
v'(r) &\leq \left(\int_{S^{n-2}}\int_N \omega ds d\theta\right)^{\frac{1}{n-2}-1}\Bigg\{\left(\int_{S^{n-2}}\int_N J^{n-3} ds d\theta\right) \\
&\quad + \int_{S^{n-2}}\int_N J^{n-3}J^{\delta}\left(C_3(n,p)\int_0^r \sigma^p\omega dt\right)^{\frac{1}{2p-1}} ds d\theta\Bigg\} \\
&\leq \left(\int_{S^{n-2}}\int_N \omega ds d\theta\right)^{\frac{3-n}{n-2}}\Bigg\{\left(\int_{S^{n-2}}\int_N J^{n-2} ds d\theta\right)^{\frac{n-3}{n-2}} \\
&\quad \cdot\left(\int_{S^{n-2}}\int_N 1 ds d\theta\right)^{\frac{1}{n-2}} + \left(\int_{S^{n-2}}\int_N J^{(n-2)} ds d\theta\right)^{\frac{n-3+\delta}{n-2}} \\
&\quad \cdot\left(C_3(n,p)\int_{S^{n-2}}\int_N\left(\int_0^r \sigma^p\omega dt\right) ds d\theta\right)^{\frac{1}{2p-1}}\Bigg\} \\
&= \left(\int_{S^{n-2}}\int_N 1\, ds d\theta\right)^{\frac{1}{n-2}} + v^{\delta}(r)\left(C_3(n,p)\int_{S^{n-2}}\int_N\int_0^r \sigma^p\omega dt ds d\theta\right)^{\frac{1}{2p-1}},
\end{aligned}$$

which yields

$$v' \leq \left(\ell(N)\operatorname{vol}(S^{n-2})\right)^{\frac{1}{n-2}} + \left(C_3(n,p)K(p)\right)^{\frac{1}{2p-1}}v^{\delta}. \qquad \square$$

We can now prove our main estimate on the volume of tubes

Theorem 2.4. *If $N \subset M$ is a geodesic, then the volume of the normal tubes can be estimated as follows:*

$$\operatorname{vol} T(N,r) \le (a+b)^{n-2} \int_0^r \left(\frac{e^{bt}-1}{b} \right)^{n-2} dt\,,$$

where

$$a = \left(\ell(N) \operatorname{vol}(S^{n-2})\right)^{\frac{1}{n-2}},$$
$$b = \left(C_3(n,p)\, K(p)\right)^{\frac{1}{2p-1}}.$$

Proof. We have that $v(t)$ satisfies

$$v' \le a + bv^{\delta},$$
$$v(0) = 0,$$
$$\delta = \frac{2p-n+1}{2p-1} \in (0,1)\ .$$

From this inequality we get that

$$v' \le (a+b) + bv\,,$$
$$v(0) = 0\,,$$

which immediately implies

$$v(t) \le (a+b)\left(\frac{e^{bt}-1}{b} \right).$$

Now we have from the definition of v that

$$\operatorname{vol} T(N,r) = \int_0^r v^{n-2}(t)\,dt\,.$$

This gives the desired inequality. □

If we want an estimate in terms of $K(\lambda,p)$, where $\lambda < 0$, then we need to use that $\sigma \le \max\{\sigma+\lambda, 0\} - \lambda$. Using this we see

$$\sigma^p \le 2^{p-1}\left((\max\{\sigma+\lambda,0\})^p + |\lambda|^p\right).$$

From this inequality we get

$$\begin{aligned} v' &\le \left(\ell(N) \operatorname{vol}(S^{n-2})\right)^{\frac{1}{n-2}} \\ &\quad + \left(C_4(n,p) \left(\int_{S^{n-2}} \int_N \int_0^r |\lambda|^p \,\omega dt ds d\theta + K(\lambda,p) \right)\right)^{\frac{1}{2p-1}} v^{\delta} \\ &= \left(\ell(N) \operatorname{vol}(S^{n-2})\right)^{\frac{1}{n-2}} \\ &\quad + \left(C_4(n,p) \left(|\lambda|^p \int_0^r v^{n-2} dt + K(\lambda,p) \right)\right)^{\frac{1}{2p-1}} v^{\delta}, \end{aligned}$$

which implies

$$v' \le a + \left(b \int_0^r v^{n-2} dt + c\right)^{\frac{1}{2p-1}} v^\delta ,$$
$$v(0) = 0 ,$$
$$a = \left(\ell(N)\,\mathrm{vol}(S^{n-2})\right)^{\frac{1}{n-2}} ,$$
$$b = C_4(n,p)\,|\lambda|^p ,$$
$$c = C_4(n,p)\,K(\lambda,p) .$$

We can rewrite this by defining

$$\begin{aligned} V &= \int_0^r v^{n-2} dt , && V(0) = 0 , \\ V' &= v^{n-2} , && V'(0) = 0 , \\ V'' &= (n-2)v^{n-3}v' = (n-2)\left(V'\right)^{\frac{n-3}{n-2}} v' . \end{aligned}$$

Then substitution yields

$$\begin{aligned} V'' &\le (n-2)\big(a + (bV+c)^{\frac{1}{2p-1}} (V')^{\frac{\delta}{n-2}}\big)(V')^{\frac{n-3}{n-2}} \\ &= (n-2)\big((V')^{\frac{n-3}{n-2}} a + (V')^{\frac{n-3+\delta}{n-2}} (bV+c)^{\frac{1}{2p-1}}\,\big) , \end{aligned}$$

which implies

$$\begin{aligned} V''(V')^{\frac{1-\delta}{n-2}} &\le (n-2)\left((V')^{\frac{n-2-\delta}{n-2}} a + (V')(bV+c)^{\frac{1}{2p-1}}\,\right) \\ &\le (n-2)\left(a + aV' + (V')(bV+c)^{\frac{1}{2p-1}}\,\right) . \end{aligned}$$

After integrating this (recall that $b \neq 0$), we obtain

$$(V')^{\frac{n-1-\delta}{n-2}} \le (n-1-\delta)\left(ar + aV + \frac{2p-1}{2pb}(bV+c)^{\frac{2p}{2p-1}}\right) ,$$

or

$$\begin{aligned} V' &\le (n-1-\delta)^{\frac{n-2}{n-1-\delta}} \left(ar + aV + \frac{2p-1}{2pb}(bV+c)^{\frac{2p}{2p-1}}\right)^{\frac{n-2}{n-1-\delta}} \\ &\le (n-1-\delta)^{\frac{n-2}{n-1-\delta}} \left((ar+aV)^{\frac{n-2}{n-1-\delta}} + \left(\frac{2p-1}{2pb}\right)^{\frac{n-2}{n-1-\delta}} (bV+c)\right) \\ &\le (n-1-\delta)^{\frac{n-2}{n-1-\delta}} \left((ar+aV)^{\frac{n-2}{n-1-\delta}} + \left(\frac{2p-1}{2pb}\right)^{\frac{n-2}{n-1-\delta}} (bV+c)\right) \\ &\le (n-1-\delta)^{\frac{n-2}{n-1-\delta}} \left(a^{\frac{n-2}{n-1-\delta}}(1+r+V) + \left(\frac{2p-1}{2pb}\right)^{\frac{n-2}{n-1-\delta}} (bV+c)\right) \\ &= \alpha + \beta r + \gamma V , \end{aligned}$$

where

$$\alpha = a^{\frac{n-2}{n-1-\delta}} + \left(\frac{2p-1}{2pb}\right)^{\frac{n-2}{n-1-\delta}} c\,,$$
$$\beta = a^{\frac{n-2}{n-1-\delta}}\,,$$
$$\gamma = a^{\frac{n-2}{n-1-\delta}} + \left(\frac{2p-1}{2p}\right)^{\frac{n-2}{n-1-\delta}} b^{\frac{1-\delta}{n-1-\delta}}\,.$$

Using that $V(0) = 0$ we see

$$V(r) \le \frac{\alpha\gamma+\beta}{\gamma^2}\big(\exp(\gamma r) - 1\big) - \frac{1}{\gamma}\beta r\,.$$

Combining this with the previous theorem then implies

Theorem 2.5. *Let $N \subset M$ be a geodesic and $\lambda \le 0$, then the volume of the normal tube around N satisfies*

$$\operatorname{vol} T(N,r) \le F(n,p,a,b,c,r)\ ,$$

where

$$a = \ell(N) = \text{ length of } N\,,$$
$$b = \lambda\,,$$
$$c = K(\lambda,p)\ , \quad p > n-1\,.$$

Furthermore as $a,c \to 0$ we have that $F(n,p,a,b,c,r) \to 0$.

We shall also need a modified version of [Y, Theorem 7.1]. The purpose is to measure the volume of cones from a point rather than just metric balls.

Theorem 2.6. *Suppose $p > n/2$. Let $N = \{x\}$ and $\widehat{S}$ a subset of $S^{n-1} =$ unit sphere in T_xM. If we define $B(\widehat{S},r) = \{y = \exp_x t\theta \mid t \le r, \theta \in \widehat{S}\}$, then*

$$\operatorname{vol} B(\widehat{S},r) \le G(n,p,a,b,c,r)\ ,$$

where

$$a = \operatorname{vol}\widehat{S}\,,$$
$$b = \lambda\,,$$
$$c = k(\lambda,p)\ ,$$

and $G(n,p,a,b,c,r) \to 0$ as $a,c \to 0$.

Proof. We proceed as in the proof of the above theorem using the following inequality from [Y, Lemma 7.3]

$$\begin{aligned} v' &\leq \alpha + \beta v^{\delta} , \\ v(0) &= 0 , \\ \delta &= \frac{2p-n}{2p-1} , \end{aligned}$$

where $(v(r))^{n-1} = \int_{\widehat{S}} \omega(r, \theta_{n-1}) d\theta_{n-1}$, and $d\,\mathrm{vol} = \omega dt \wedge d\theta_{n-1}$ defines the density of the volume element in normal polar coordinates around x, and

$$\begin{aligned} \alpha &= (\mathrm{vol}\,\widehat{S})^{\frac{1}{n-1}} , \\ \beta &= \left(\left(\frac{2p-1}{p} \right)^{p} \left(\frac{p-1}{2p-n} \right)^{p-1} k(p) \right)^{\frac{1}{2p-1}} . \end{aligned}$$ □

3 Eigenvalue Comparison

We shall not need to use any geometry in this section. All the results are estimates that hold regardless of their geometric content.

We have the shape operator $S(t)$ which we shall think of as a function of t. It is a symmetric matrix which satisfies

$$S' + S^2 = -R_{\partial_t} .$$

Since S is actually symmetric we know that the unordered set of eigenvalues $\{\varphi_1, \ldots, \varphi_{n-1}\}$ from the previous section varies smoothly (see [Ka, II Theorem 6.8]). We are going to consider the two functions $\varphi = \max\{\min\{\varphi_1, \ldots, \varphi_{n-1}\}, 0\}$ and $\psi = \max\{\max\{\varphi_1, \ldots, \varphi_{n-1}\}, 0\}$ as before. These functions are obviously absolutely continuous and since $\sigma \geq 0$ they also satisfy

$$\begin{aligned} \varphi' + \varphi^2 &\leq \sigma , \\ \psi' + \psi^2 &\leq \sigma . \end{aligned}$$

The initial conditions for these two functions are $\varphi(0) = 0$, $\psi(t) \approx t^{-1}$, and $\varphi(t)\psi(t)$ is bounded as $t \to 0$. In addition we shall use the density of the volume form ω which satisfies $\omega' = h\omega \leq (\varphi + (n-2)\psi)\omega$. Our main result is

LEMMA 3.1. *There is a constant $C_2(n,p)$ such that when $p > n-1$ we have*

$$\int_0^r (\varphi\psi)^p \omega dt \leq C_2(n,p) \int_0^r \sigma^p \omega dt .$$

Proof. Consider the differential inequality
$$\varphi' + \varphi^2 \leq \sigma \,.$$
Multiply through by $(\varphi\psi)^{p-1}\,\omega$ and integrate to get
$$\int_0^r \varphi'\,(\varphi\psi)^{p-1}\,\omega dt + \int_0^r \varphi^2(\varphi\psi)^{p-1}\omega dt \leq \int_0^r \sigma\,(\varphi\psi)^{p-1}\,\omega dt \,.$$
Integration by parts yields
$$\begin{aligned}
&\int_0^r \varphi'(\varphi\psi)^{p-1}\omega dt \\
&= \frac{1}{p}\varphi^p\psi^{p-1}\omega\Big|_0^r - \frac{p-1}{p}\int_0^r \varphi^p\psi^{p-2}\psi'\omega dt - \frac{1}{p}\int_0^r \varphi^p\psi^{p-1}h\omega dt \\
&\geq 0 + \frac{p-1}{p}\int_0^r \varphi^p\psi^{p-2}\left(\psi^2-\sigma\right)\omega dt - \frac{1}{p}\int_0^r \varphi^p\psi^{p-1}\left(\varphi + (n-2)\,\psi\right)\omega dt \\
&= \frac{p+1-n}{p}\int_0^r \varphi^p\psi^p\omega dt - \frac{p-1}{p}\int_0^r \varphi^p\psi^{p-2}\sigma\omega dt - \frac{1}{p}\int_0^r \varphi^{p+1}\psi^{p-1}\omega dt \,.
\end{aligned}$$
Inserting this in the above inequality we obtain
$$\begin{aligned}
\frac{p+1-n}{p}\int_0^r \varphi^p\psi^p\omega dt + \left(1-\frac{1}{p}\right)\int_0^r \varphi^{p+1}\psi^{p-1}\omega dt - \frac{p-1}{p}\int_0^r \varphi^p\psi^{p-2}\sigma\omega dt \\
\leq \int_0^r \sigma(\varphi\psi)^{p-1}\omega dt \,.
\end{aligned}$$
Since $0 \leq \varphi \leq \psi$ we then obtain
$$\begin{aligned}
\frac{p+1-n}{p}\int_0^r \varphi^p\psi^p\omega dt &\leq \int_0^r \sigma(\varphi\psi)^{p-1}\omega dt + \frac{p-1}{p}\int_0^r \varphi^p\psi^{p-2}\sigma\omega dt \\
&\leq \int_0^r (\varphi\psi)^{p-1}\sigma\omega dt + \frac{p-1}{p}\int_0^r (\varphi\psi)^{p-1}\sigma\omega dt \\
&\leq \frac{2p-1}{p}\left(\int_0^r \sigma^p\omega dt\right)^{\frac{1}{p}}\left(\int_0^r (\varphi\psi)^p\omega dt\right)^{1-\frac{1}{p}} .
\end{aligned}$$
This is the crucial place where we must have that $p > n-1$, for otherwise the left hand side will become negative. After dividing this inequality by $\left(\int_0^r (\varphi\psi)^p\,\omega dt\right)^{1-\frac{1}{p}}$ we get
$$\frac{p+1-n}{p}\left(\int_0^r \varphi^p\psi^p\omega dt\right)^{1/p} \leq \frac{2p-1}{p}\left(\int_0^r \sigma^p\omega dt\right)^{1/p} .$$
This finishes the proof. □

There are other inequalities of the same type which can be proved in a similar manner. One can for example show that
$$\int_0^r (\varphi t^{-1})^p\omega dt \leq C_7\,(n,p)\int_0^r (\sigma)^p\,\omega dt \,, \qquad p > n-1$$

$$\int_0^r \widetilde{\psi}^{2p}\omega dt \le C_8(n,p)\int_0^r (\sigma)^p\omega dt\,, \qquad \widetilde{\psi} = \max\{\psi - t^{-1}, 0\} \text{ and } p > \tfrac{n}{2}$$

It is clearly significant that the last inequality holds when $p > n/2$, while the others do not. Unfortunately $\widetilde{\psi}$ does not dominate φ so we cannot take advantage of this.

We should also point out that after having proved all of the above inequalities it becomes possible to find pointwise bounds for quantities like $\left(h - \frac{n-2}{t}\right)_+^q\omega$, $\varphi^q\omega$, etc. in terms of $\int_0^r \sigma^p\omega dt$. The way to obtain such inequalities is simply to note that when we used integration by parts we threw away these terms since they were non-negative. Instead of discarding them we can leave them there and then use our integral estimates to obtain pointwise estimates in terms of the integral of the curvature.

This obviously leads to the possibility of some sort of distance comparison. We will use and investigate these matters in a separate paper.

4 Applications

We first establish Cheeger's lemma.

Theorem 4.1. *Given an integer $n \ge 2$, and numbers $p > n-1$, $\lambda \le 0$, $v > 0$, $D < \infty$ one can find $\varepsilon = \varepsilon(n,p,\lambda,v,D) > 0$ and $\delta = \delta\,(n,p,\lambda,v,D) > 0$ such that a closed Riemannian n-manifold M with*

$$\begin{aligned} \operatorname{vol}(M) &\ge v\,,\\ \operatorname{diam}(M) &\le D\,,\\ K(\lambda,p) &\le \varepsilon\,(n,p,\lambda,v,D) \end{aligned}$$

has the property that the length of the shortest closed geodesic $\ge \delta\,(n,p,\lambda,v,D)$.

Proof. Let $C \subset M$ be a closed geodesic. Parametrize it by arclength $\gamma : [0,l] \to C$ and consider $N = \gamma(0,l)$, i.e. one point has been deleted. If we let $r = D$ then we have that the normal tube $T(N,D)$ contains all of M except for a set of measure zero. We can therefore conclude that

$$v \le \operatorname{vol} M = \operatorname{vol} T\,(N,D) \le F\,(n,p,a,b,c,D)\ ,$$

where

$$\begin{aligned} a &= l\,,\\ b &= |\lambda|^p\ , \qquad p > n-1\,,\\ c &= K\,(\lambda,p)\ . \end{aligned}$$

Now notice that the quantity $F(n,p,a,b,c,D) \to 0$ as $a,c \to 0$. Next observe that $a \to 0$ if $l \to 0$, and $c \to 0$ if $K(\lambda,p) \to 0$. It is therefore clear that if we are to maintain the inequality

$$v \le F(n,p,a,b,c,D)$$

for some fixed v and D the quantities l or $K(\lambda,p)$ must be bounded from below. Therefore, if we suppose from the begining that we have ε,δ chosen so small $F(n,p,a,b,c,D) \le v/2$ whenever $l \le \delta$ and $K(\lambda,p) \le \varepsilon$, then we clearly must have that $l > \delta$ as long as $K(\lambda,p) \le \varepsilon$. □

In a separate paper we will show how one can estimate the length of the shortest closed geodesic loop at a point $p \in M$ if one assumes for some small r that $\operatorname{vol} B(p,r) \ge \left(\frac{1}{2}+\varepsilon\right)\omega_{n-1}r^n$, where $\omega_{n-1} = \operatorname{vol}(S^{n-1})$. This requires some modifications of many of the existing volume estimates.

We are now ready to prove the homotopy finiteness result. The proof will in outline follow what was done in [GrovPe2], with the exception that precompactness of the class under consideration only becomes apparent after we have proved the analogue of [GrovPe2, Main Lemma 1.3]. We shall also need to use some of the ideas presented in [Gr] to establish this main lemma.

If we have points $p,q \in M$ then we denote by $\Gamma_{pq} \subset S_p \subset T_pM$ the set of unit vectors tangent to a segment from p to q. Furthermore define

$$\begin{aligned}\Gamma_{pq}(\theta) &= \left\{u \in S_p \mid \measuredangle(u,\Gamma_{pq}) \le \theta\right\},\\ \Gamma'_{pq}(\theta) &= \left\{u \in S_p \mid \measuredangle(u,\Gamma_{pq}) > \theta\right\}.\end{aligned}$$

With this notation we can now establish

Lemma 4.2. *Given an integer $n \ge 2$, and numbers $p > n-1$, $\lambda \le 0$, $v > 0$, $D < \infty$ one can find $\varepsilon = \varepsilon(n,p,\lambda,v,D) > 0$, $\alpha = \alpha(n,p,\lambda,v,D) > 0$, and $\delta = \delta(n,p,\lambda,v,D) > 0$ such that a closed Riemannian n-manifold M with*

$$\operatorname{vol}(M) \ge v\,,$$
$$\operatorname{diam}(M) \le D\,,$$
$$K(\lambda,p) \le \varepsilon(n,p,\lambda,v,D)$$

has the property that any two points $p,q \in M$ with $\Gamma_{pq}\left(\frac{\pi}{2}+\alpha\right) = S_p$ and $\Gamma_{qp}\left(\frac{\pi}{2}+\alpha\right) = S_q$ satisfy $d(p,q) \ge \delta(n,p,\lambda,v,D)$.

Proof. Suppose $\Gamma_{pq}\left(\frac{\pi}{2}+\alpha\right) = S_p$ and $\Gamma_{qp}\left(\frac{\pi}{2}+\alpha\right) = S_q$ for some $\alpha \in \left(0,\frac{\pi}{2}\right)$. Using Helly's theorem as in [Gr] we can extract $2(n+1)$ segments N_i, $i = 1,\dots,2(n+1)$ from p to q such that if $\Gamma_p \subset \Gamma_{pq}\left(\frac{\pi}{2}+\alpha\right)$ and $\Gamma_q \subset \Gamma_{qp}\left(\frac{\pi}{2}+\alpha\right)$ denote the corresponding tangent vectors then we still have $\Gamma_p\left(\frac{\pi}{2}+\alpha\right) = S_p$ and $\Gamma_q\left(\frac{\pi}{2}+\alpha\right) = S_q$.

For any $x \in M$ we can always find the shortest connection to the union of our closed segments N_i, $i = 1, \dots, 2(n+1)$. This geodesic first of all has length $\leq D$ and is in addition either perpendicular to one of these segments or connects x to p (or q) in such a way that it is not perpendicular to any N_i at p (or q). In other words we have that either x lies in one of the tubes $T(N_i, D)$, $i = 1, \dots, 2(n+1)$, or in one of the two "cones" $B\left(\Gamma_p'\left(\frac{\pi}{2}\right), D\right)$ or $B\left(\Gamma_q'\left(\frac{\pi}{2}\right), D\right)$. With this we can now make our choices for $\varepsilon, \delta, \alpha$.

First choose δ and ε such that if $d(p,q) \leq \delta$, then

$$\begin{aligned}\operatorname{vol} T\left(N_i, D\right) d &\leq F\left(n, p, a, b, c, D\right)\\ &\leq \frac{v}{4\left(n+1\right)}\,, \qquad \text{for all } i = 1, \dots, 2\left(n+1\right)\,.\end{aligned}$$

This is possible since

$$\begin{aligned}a &= d\left(p, q\right) \leq \delta\,,\\ b &= |\lambda|^p\,,\\ c &= K\left(\lambda, p\right) \leq \varepsilon\,.\end{aligned}$$

Next observe that $\operatorname{vol}\Gamma_p'\left(\frac{\pi}{2}\right) \leq \operatorname{vol}\Gamma_\alpha'\left(\frac{\pi}{2}\right)$, where Γ_α is a set consisting of two unit vectors forming an angle $\pi - 2\alpha$ and volumes are measured as subsets of the unit sphere (see [GrovPe2, Appendix]). As $\alpha \to 0$ we obviously have that $\operatorname{vol}\Gamma_\alpha'\left(\frac{\pi}{2}\right) \to 0$ as well, we can therefore choose ε and α using Theorem 2.6 such that

$$\begin{aligned}&\operatorname{vol}\left(B\left(\Gamma_p'\left(\frac{\pi}{2}\right), D\right)\right) \text{ and } \operatorname{vol}\left(B\left(\Gamma_q'\left(\frac{\pi}{2}\right), D\right)\right)\\ &\quad\leq G(n, p, a, b, c, D)\\ &\quad< \frac{v}{4}\,,\end{aligned}$$

since we have

$$\begin{aligned}a^{n-1} &= \operatorname{vol}\Gamma_p'\left(\frac{\pi}{2}\right) \leq \operatorname{vol}\Gamma_\alpha'\left(\frac{\pi}{2}\right)\\ b &\leq 2^{p-1}\left(\frac{2p-1}{p}\right)^p\left(\frac{p-1}{2p-n}\right)^{p-1}|\lambda|^p\\ c &\leq 2^{p-1}\left(\frac{2p-1}{p}\right)^p\left(\frac{p-1}{2p-n}\right)^{p-1}\varepsilon\,.\end{aligned}$$

The observation from above then implies

$$\begin{aligned} v &\leq \operatorname{vol} M \\ &\leq \operatorname{vol}\left(B\left(\Gamma_p'\left(\frac{\pi}{2}\right), D\right)\right) + \operatorname{vol}\left(B\left(\Gamma_q'\left(\frac{\pi}{2}\right), D\right)\right) + \sum_{i=1}^{2(n+1)} \operatorname{vol} T\left(N_i, D\right) \\ &< \frac{v}{4} + \frac{v}{4} + 2\left(n+1\right)\frac{v}{4\left(n+1\right)} = v\,, \end{aligned}$$

provided $d(p,q) \leq \delta$. As this is a contradiction we must conclude $d(p,q) > \delta$. □

With this result behind us we can now argue as in [GrovPe2, Section 2] that there are numbers $R > 0$ and $L > 1$ such that if M is a manifold satisfying the hypotheses of the above lemma, then M has the property that any metric ball $B(x,r)$ of radius $r \leq R$ is contractible in the larger concentric ball $B(x, Lr)$. In particular M is $LGC(\rho)$, where $\rho(r) = Lr$ for $r \leq R$ (see [Pe]).

In addition we must observe that Theorem 2.6 yields an upper bound for the volume of M,

$$\operatorname{vol} M = \operatorname{vol} B\left(S_p, D\right) \leq G\left(n, p, a, b, c, D\right)$$

where all the quantities a, b, c are bounded in terms of n, p, D, λ and ε.

Theorem 1.3 from the introduction is now a consequence of [GrPe, Theorem 2] (see also [F1] and [F2] for some more general results of this type).

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