ZHONG-YANG TYPE EIGENVALUE ESTIMATE WITH INTEGRAL CURVATURE CONDITION

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Abstract. We prove a sharp Zhong-Yang type eigenvalue lower bound for closed Riemannian manifolds with control on integral Ricci curvature.

1. Introduction

One trend in Riemannian geometry since the 1950's has been the study of how curvature affects global quantities like the eigenvalues of the Laplacian. On a closed Riemannian manifold (M^n, g), assuming that $\mathrm{Ric}_M \geq (n-1)H$ $(H > 0)$, Lichnerowicz [Lic58] proved the lower bound $\lambda_1(M) \geq Hn$, where λ_1 is the first nonzero eigenvalue of the Laplace-Beltrami operator in (M^n, g),

$$\mathrm{div}(\nabla u) := \Delta u = -\lambda_1 u.$$

Obata [Oba62] proved the rigidity result that equality holds if and only if M^n is isometric to $\mathbb{S}^n_H$. In the case $\mathrm{Ric}_M \geq 0$, one can not prove a positive lower bound without a constraint on the diameter of M. Note that the diameter constraint is automatic when $H > 0$ by Myers' theorem. If the diameter of M is D, Li and Yau [LY80, Li12] proved a gradient estimate for the first nontrivial eigenfunction and showed that

$$\lambda_1(M) \geq \frac{\pi^2}{2D^2}.$$

Zhong and Yang [ZY84] improved this result, and obtained the optimal estimate

$$\lambda_1(M) \geq \frac{\pi^2}{D^2}.$$

Hang and Wang [HW07] proved the rigidity result that equality holds if and only if M is isometric to $\mathbb{S}^1$ with radius $\frac{D}{\pi}$. When $H < 0$, improving Li-Yau's estimate [LY80], Yang showed the following explicit estimate in [Yan90],

$$\lambda_1(M) \geq \frac{\pi^2}{D^2} e^{-C_n\sqrt{(n-1)|H|}D}, \tag{1.1}$$

where $C_n = \max\{\sqrt{n-1}, \sqrt{2}\}$. The general optimal lower bound estimate for $\lambda_1(M)$ for all H is proved in [Krö92, CW97, AC13, ZW17] using gradient estimate, probabilistic 'coupling method' and modulus of continuity, respectively, see also [BQ00, AN12]. The general lower bound gives a comparison of the first eigenvalue to a one-dimensional model and an explicit lower bound was given by Shi and Zhang [SZ07] which takes the form

$$\lambda_1(M) \geq 4(s - s^2)\frac{\pi^2}{D^2} + s(n-1)H \quad \text{for all } s \in (0,1). \tag{1.2}$$

In recent years, there has been an increasing interest in relaxing the pointwise curvature assumption, by assuming a bound in an L^p sense as in [Gal88]. In the fundamental work [PW97] the basic Laplacian comparison and Bishop-Gromov volume comparison have been extended to

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integral Ricci curvature. Many topological invariants can be expressed in terms of L^p norms of the curvature, and these bounds are also more suitable than pointwise bounds in the study of Ricci flow. To be more precise, let $\rho(x)$ be the smallest eigenvalue for the Ricci tensor. For a constant $H \in \mathbb{R}$, let ρ_H be the amount of Ricci curvature lying below $(n-1)H$, i.e.

$$\rho_H = \max\{-\rho(x) + (n-1)H, 0\}. \tag{1.3}$$

We will be concerned mainly with ρ_0, the negative part of Ric. The following quantity measures the amount of Ricci curvature lying below $(n-1)H$ in an L^p sense

$$\bar{k}(p,H) = \left(\frac{1}{\mathrm{vol}(M)}\int_M \rho_H^p dv\right)^{\frac{1}{p}} = \left(⨍_M \rho_H^p dv\right)^{\frac{1}{p}}.$$

Clearly $\bar{k}(p,H) = 0$ iff $\mathrm{Ric}_M \geq (n-1)H$.

In [Gal88], Gallot obtained a lower bound for $\lambda_1(M)$ for closed manifolds with diameter bounded from above and $\bar{k}(p,H)$ small by a heat kernel estimate, see Theorem 2.1 below. The estimate is not optimal though. When $H > 0$, Aubry [Aub07] obtained an optimal lower bound estimate for the first nonzero eigenvalue, which recovers the Lichnerowicz estimate. Recently the second and third authors [SW17] extended this to the p-Laplacian.

In this paper we obtain an optimal estimate for $H = 0$, recovering the Zhong-Yang estimate. Namely,

Theorem 1.1. Let (M^n, g) be a closed Riemannian manifold with diameter $\leq D$ and $\lambda_1(M)$ be the first nonzero eigenvalue. For any $\alpha \in (0,1)$, $p > \frac{n}{2}$, $n \geq 2$, there exists $\epsilon(n,p,\alpha,D) > 0$ such that if $\bar{k}(p,0) \leq \epsilon$, then

$$\lambda_1(M) \geq \alpha \frac{\pi^2}{D^2}.$$

The constant ϵ can be explicitly computed, see §5.

The proofs for $\lambda_1(M)$ we mentioned before for pointwise lower Ricci curvature bound do not work well with integral curvature condition. Here we use a gradient estimate similar to the one of Li and Yau. Their technique can not be applied directly in the integral curvature case, as it relies on a pointwise lower bound to control the term $\mathrm{Ric}(\nabla u, \nabla u)$ coming from the Bochner formula. To overcome this difficulty, we use the technique developed by Zhang and Zhu in [ZZ17] and [ZZ18]. See [Car, Ros19, RO19] for other applications of this technique. The strategy consists in introducing an auxiliary function J via a PDE that absorbs the curvature terms appearing in the Bochner formula, and to find appropriate bounds for J (see §2). We also follow the approach of [Li12], that uses an ODE comparison technique instead of the original barrier functions of [ZY84] (see §3).

Remark 1.1. In general, $p > \frac{n}{2}$ and the smallness of $\bar{k}(p,0)$ are both necessary conditions, see e.g. [DWZ18]. In particular, for our estimate, the example of the dumbbells of Calabi shows that only assuming that $\bar{k}(p,0)$ is bounded is not enough: consider a dumbbell $D_\epsilon \subseteq \mathbb{R}^3$, consisting of two equal spheres joined by a thin cylinder of length l and radius ϵ, with smooth necks. Assume without loss of generality that $\mathrm{vol}(D_\epsilon) \leq 1$. Then, as explained in [Che70], $\lambda_1(D_\epsilon) \leq C\epsilon^2$, so no positive lower bound is possible, as we can consider a sequence of dumbbells with $\epsilon \to 0$. Notice that in this example, $\bar{k}(p,0)$ can not be made small. This follows from Gauss-Bonnet: since D_ϵ is homeomorphic to S^2, the integral of the sectional curvature over D_ϵ is 4π. However, over the two spheres it is close to 8π, and over the cylinder it's 0. Hence, the two necks contain negative curvature, that amounts to almost -4π. This implies that $\mathrm{vol}(D_\epsilon)\bar{k}(1,0) \approx 4\pi$. We can make the construction so that $\bar{k}(1,0) > 2\pi$, thus for $p > \frac{n}{2} = 1$ we have $\bar{k}(p,0) \geq \bar{k}(1,0) > 2\pi$, so the integral curvature can not be made small.

Remark 1.2. Since in the case $\mathrm{Ric}_M \geq 0$ one has rigidity, a natural question would be to ask if one could get almost rigidity in the Gromov-Hausdorff sense (see [Sak05]), i.e. if $\lambda_1(M)$ is close to $\lambda_1(S^1) = \frac{\pi^2}{D^2}$, can we conclude that M is close to S^1 in the Gromov-Hausdorff topology? As was explained in [HW07] page 8, this is not true, as one could consider the shrinking sequence of boundaries of ϵ-neighborhoods of a line segment with length π, whose eigenvalues converge to 1, but that converges in the Gromov-Hausdorff sense to the line segment.

Remark 1.3. As the proof given in [Yan90] for the case $H < 0$ is similar to the case $H = 0$, adjusting our proof accordingly should yield an integral curvature version of the estimate (1.1). We conjecture that integral curvature versions of the estimate (1.2) and the optimal lower bound estimate when $H < 0$ should also hold.

The paper is organized as follows: in §2 we prove estimates on the auxiliary function J mentioned above, in §3 we prove the sharp gradient estimate needed to derive our main theorem, and we prove Theorem 1.1 in §4. Finally, in §5 we have an appendix with explicit estimates on ϵ (the upper bound of $\bar{k}(p,0)$) depending on the Sobolev and Poincaré constants, p, n and D.

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2. Estimates on the auxiliary function

In what follows, for $f \in L^p(M)$, we use the notation

$$\|f\|_p^* := \left(\fint_M |f|^p dv \right)^{\frac{1}{p}}.$$

First we recall an earlier eigenvalue and Sobolev constant estimate for closed manifolds with integral Ricci curvature bounds which we will use.

Theorem 2.1. [Gal88, Theorem 3,6] Given M^n closed Riemannian manifold with diameter D, for $p > n/2$, $H \in \mathbb{R}$, there exist $\varepsilon(n,p,H,D) > 0$, $C(n,p,H,D) > 0$ such that if $\bar{k}(p,H) \leq \varepsilon$, then $\mathrm{Is}(M) \leq C(n,p,H,D)/\mathrm{vol}(M)^{\frac{1}{n}}$, where $\mathrm{Is}(M) = \sup\{\frac{\mathrm{vol}(\Omega)^{1-\frac{1}{n}}}{\mathrm{vol}(\partial\Omega)} : \Omega \subset M, \mathrm{vol}(\Omega) \leq \frac{1}{2}\mathrm{vol}(M)\}$. In particular,

$$\lambda_1(M) \geq \Lambda_{rough}(n,p,H,D) > 0, \tag{2.1}$$

and for any $u \in W^{1,2}(M)$,

$$\|u - \bar{u}\|^*_{\frac{2p}{p-1}} \leq C_s \|\nabla u\|_2^* \tag{2.2}$$

$$\|u\|^*_{\frac{2p}{p-1}} \leq C_s \|\nabla u\|_2^* + \|u\|_2^* \tag{2.3}$$

where $\bar{u} = \fint_M u$, the average of u, and $C_s = \left(\frac{p}{p-1}\right)^{\frac{1}{2}} C(n,p,H,D)$.

In [Gal88] the weaker isoperimetric constant was obtained, it was improved to the optimal power above in [PS98].

Remark 2.1. From the estimate on λ_1 we can derive a Poincaré inequality. Notice that

$$\Lambda_{rough} \leq \lambda_1 = \inf_{\fint_M f dv = 0} \frac{\fint_M |\nabla f|^2 dv}{\fint_M f^2 dv} \leq \frac{\fint_M |\nabla w| dv}{\fint_M |w - \overline{w}|^2 dv},$$

where $w \in H^1(M)$. Hence we have

$$(\|w-\overline{w}\|_2^*)^2 \le \Lambda_{rough}^{-1}(\|\nabla w\|_2^*)^2. \tag{2.4}$$

As mentioned in the introduction, in the proof of the gradient estimate in Proposition 3.5 we introduce an auxiliary function J that absorbs the curvature terms. To be able to derive a sharp lower bound for $\lambda_1(M)$, we need to construct and estimate J from a PDE as follows. For $\tau > 1$ and $\sigma \ge 0$, consider

$$\Delta J - \tau\frac{|\nabla J|^2}{J} - 2J\rho_0 = -\sigma J. \tag{2.5}$$

Here ρ_0 is defined as in (1.3). Using the transformation $J = w^{-\frac{1}{\tau-1}}$, we see that this equation is equivalent to

$$\Delta w + Vw = \tilde{\sigma} w, \tag{2.6}$$

where $V = 2(\tau-1)\rho_0$ and $\tilde{\sigma} = (\tau-1)\sigma$. We choose $-\tilde{\sigma}$ be the first eigenvalue of the operator $\Delta + V$; in particular, if $\mathrm{Ric}_M \ge 0$ we have $\sigma = \tilde{\sigma} = 0$, and $J \equiv 1$ is a solution to (2.5). The main goal of this section is to prove the following propositions.

Proposition 2.2. There exists $\epsilon(n,p,D,\tau) > 0$ such that if $\bar{k}(p,0) \le \epsilon$, then there is a number σ and a corresponding function J solving (2.5) such that

$$0 \le \sigma \le 4\epsilon. \tag{2.7}$$

Proof. Since $-\tilde{\sigma}$ is the first eigenvalue, w doesn't change sign. In particular, by possibly scaling w, we can assume that $w \ge 0$, $\|w\|_2^* = 1$. By integrating equation (2.6) over M we get

$$\int_M Vw dv = \tilde{\sigma}\int_M w dv.$$

Since $w \ge 0$ and $V \ge 0$, we conclude that $\tilde{\sigma} \ge 0$, so $\sigma \ge 0$.

To obtain the upper bound, multiply equation (2.6) by w, integrate over M, and divide by $\mathrm{vol}(M)$. This way, we obtain:

$$\tilde{\sigma} = \tilde{\sigma}\fint_M w^2 dv = \fint_M w\Delta w dv + \fint_M Vw^2 dv = -\fint_M |\nabla w|^2 dv + \fint_M Vw^2 dv.$$

Define the average of w

$$\overline{w} := \fint_M w dv \le \|w\|_2^* = 1.$$

Then, using the Sobolev inequality (2.2), for $p > n/2$

$$\begin{aligned}
\tilde{\sigma} &= -\fint_M |\nabla w|^2 dv + \fint_M V(w+\overline{w}-\overline{w})^2 dv \\
&\le -\fint_M |\nabla w|^2 dv + 2\fint_M V(w-\overline{w})^2 dv + 2\fint_M V(\overline{w})^2 dv \\
&\le -\fint_M |\nabla w|^2 dv + 2\|V\|_p^*\left(\fint_M (w-\overline{w})^{\frac{2p}{p-1}} dv\right)^{\frac{p-1}{p}} + 2\|V\|_1^* \\
&\le -\fint_M |\nabla w|^2 dv + 2C_s^2\|V\|_p^*\fint_M |\nabla w|^2 dv + 2\|V\|_1^*.
\end{aligned}$$

where C_s is the Sobolev constant. Since $2C_s^2\|V\|_p^* \le 4C_s^2(\tau-1)\epsilon$, choosing $\epsilon > 0$ small enough so that $4C_s^2(\tau-1)\epsilon \le 1$, we deduce

$$\tilde{\sigma} \le 2\|V\|_1^* \le 2\|V\|_p^* \le 4(\tau-1)\epsilon.$$

Hence

$$\sigma \leq 4\epsilon.$$

□

Proposition 2.3. For any $\delta > 0$, there exists $\epsilon(n, p, D, \tau) > 0$ and a solution J to (2.5) such that if $\bar{k}(p, 0) \leq \epsilon$ then

$$|J - 1| \leq \delta. \tag{2.8}$$

Proof.
Claim 1: $\|w - \overline{w}\|_2^* \leq K_1\sqrt{\epsilon}$, where $K_1 = K_1(p, n, D, \tau)$.

Going back to equation (2.6), we have that, since $\overline{w} \leq 1$,

$$\begin{aligned}
-\Delta w(w - \overline{w}) &= (V - \tilde{\sigma})w(w - \overline{w}) \\
&= (V - \tilde{\sigma})(w - \overline{w})^2 + \overline{w}(w - \overline{w})(V - \tilde{\sigma}) \\
&\leq |V - \tilde{\sigma}|(w - \overline{w})^2 + \overline{w}((w - \overline{w})^2|V - \tilde{\sigma}| + |V - \tilde{\sigma}|) \\
&\leq 2(w - \overline{w})^2|V - \tilde{\sigma}| + |V - \tilde{\sigma}|.
\end{aligned}$$

After integration by parts and dividing by $\mathrm{vol}(M)$:

$$\begin{aligned}
\fint_M |\nabla w|^2 dv &\leq 2\fint_M |V - \tilde{\sigma}|(w - \overline{w})^2 dv + \fint_M |V - \tilde{\sigma}| dv \\
&\leq 2\left(\fint_M |V - \tilde{\sigma}|^p dv\right)^{1/p}\left(\fint_M (w - \overline{w})^{\frac{2p}{p-1}} dv\right)^{\frac{p-1}{p}} + \|V - \tilde{\sigma}\|_1^* \\
&\leq 2C_s^2\left(\|V\|_p^* + \|\tilde{\sigma}\|_p^*\right)\fint_M |\nabla w|^2 dv + \|V\|_1^* + \|\tilde{\sigma}\|_1^* \\
&\leq 12C_s^2(\tau - 1)\epsilon \fint_M |\nabla w|^2 dv + 6(\tau - 1)\epsilon.
\end{aligned}$$

Then choosing ϵ small enough so that $12C_s^2(\tau - 1)\epsilon \leq \frac{1}{2}$, we deduce

$$\fint_M |\nabla w|^2 dv \leq 12(\tau - 1)\epsilon.$$

Finally, using Poincaré inequality (2.4),

$$(\|w - \overline{w}\|_2^*)^2 = \fint_M |w - \overline{w}|^2 dv \leq \Lambda_{rough}^{-1}\fint_M |\nabla w|^2 dv \leq 12\Lambda_{rough}^{-1}(\tau - 1)\epsilon \equiv K_1^2\epsilon.$$

Claim 2: $\|w - \overline{w}\|_\infty \leq K_2(\sqrt{\epsilon} + \epsilon)$.

Denote $h := w - \overline{w}$. To derive the L^∞ bound for h, we will use Moser's iteration on a closed manifold. The technique written below is a slight modification of the one used in [WY09], introducing a potential term, (c.f. [HL11]). Notice that h satisfies

$$-\Delta h = (V - \tilde{\sigma})h + \overline{w}(V - \tilde{\sigma}).$$

Let $g = V - \tilde{\sigma}$ and $f = \overline{w}(V - \tilde{\sigma})$, then h satisfies

$$-\Delta h = gh + f. \tag{2.9}$$

Thus, h is a weak solution, in the sense that

$$\int_M \nabla h \nabla \phi dv = \int_M gh\phi dv + \int_M f\phi dv$$

for all nonnegative $\phi \in W^{1,2}(M)$.

Define $\overline{h} = h^+ + k$, where $k = \|f\|_p^*$. Notice that $\nabla\overline{h} = 0$ if $h \le 0$, and $\nabla h = \nabla\overline{h}$ if $h > 0$. Consider $\phi = \frac{\overline{h}^l}{\mathrm{vol}(M)} \in W^{1,2}(M)$. Then for some $l \ge 1$ we have

$$l \fint_M |\nabla\overline{h}|^2 \overline{h}^{l-1} dv \le \fint_M gh\overline{h}^l dv + \fint_M f\overline{h}^l dv.$$

Hence

$$\begin{aligned} l \fint_M |\nabla\overline{h}|^2 \overline{h}^{l-1} dv &\le \fint_M |g|\overline{h}^{l+1} dv + \fint_M |f|\frac{\overline{h}^{l+1}}{k} dv \\ &\le \fint_M \psi\overline{h}^{l+1} dv \\ &\le \|\psi\|_p^* \left(\|\overline{h}\|_{(l+1)\frac{p}{p-1}}^*\right)^{l+1}, \end{aligned} \tag{2.10}$$

where $\psi = |g| + \frac{|f|}{k}$ (in particular $1 \le \|\psi\|_p^* \le 1 + \|g\|_p^*$). Note that

$$\left|\nabla\left(\overline{h}^{\frac{l+1}{2}}\right)\right|^2 = \left|\frac{l+1}{2}\overline{h}^{\frac{l-1}{2}}\nabla\overline{h}\right|^2 = \frac{(l+1)^2}{4}\overline{h}^{l-1}|\nabla\overline{h}|^2.$$

So

$$\left(\|\nabla(\overline{h}^{\frac{l+1}{2}})\|_2^*\right)^2 \le \frac{(l+1)^2}{4l}\|\psi\|_p^* \left(\|\overline{h}\|_{(l+1)\frac{p}{p-1}}^*\right)^{l+1}.$$

Let $s = \frac{2p}{n} > 1$ and $r = \frac{s+1}{2} > 1$. Note that $\frac{p}{p-1} = \frac{sn}{sn-2}$. Using the Sobolev inequality (2.3) for $u \in W^{1,2}(M)$ and $q > \frac{n}{2}$,

$$\|u\|_{\frac{2q}{q-1}}^* \le C_s\|\nabla u\|_2^* + \|u\|_2^*,$$

and let $q = \frac{rn}{2}$. We make this choice, $q > n/2$, so that we can treat the cases $n = 2$ and $n > 2$ together. Note that if $q = n/2$, then $\frac{2q}{(q-1)} = \frac{2n}{(n-2)}$ is the usual Sobolev exponent. Then we have

$$\begin{aligned} \left(\|\overline{h}\|_{(l+1)\frac{rn}{rn-2}}^*\right)^{\frac{l+1}{2}} = \|\overline{h}^{\frac{l+1}{2}}\|_{\frac{2rn}{rn-2}}^* &\le C_s\|\nabla(\overline{h}^{\frac{l+1}{2}})\|_2^* + \|\overline{h}^{\frac{l+1}{2}}\|_2^* \\ &\le \mathcal{A}_l(\|\overline{h}\|_{(l+1)\frac{sn}{sn-2}}^*)^{\frac{l+1}{2}} + (\|\overline{h}\|_{l+1}^*)^{\frac{l+1}{2}}, \end{aligned} \tag{2.11}$$

where $\mathcal{A}_l = C_s\frac{l+1}{2\sqrt{l}}\sqrt{\|\psi\|_p^*}$ from (2.10).

Let $a := \frac{n(s-r)}{sn-2} > 0$ so that it satisfies

$$a + (1-a)\left(\frac{rn}{rn-2}\right) = \frac{sn}{sn-2}.$$

Then by Hölder and Young's inequality

$$xy \le \xi x^\gamma + \xi^{-\frac{\gamma^*}{\gamma}} y^{\gamma^*},$$

with $\gamma = \left((1-a)\left(\frac{rn}{rn-2}\right)\left(\frac{sn-2}{sn}\right)\right)^{-1}$ and $\gamma^* = \left(\frac{sn-2}{sn}a\right)^{-1}$,

$$\begin{aligned}(\|\overline{h}\|^*_{(l+1)\frac{sn}{sn-2}})^{\frac{l+1}{2}} &= \left(\fint_M \overline{h}^{(l+1)\frac{sn}{sn-2}}dv\right)^{\frac{sn-2}{2sn}}\\ &\le \left(\fint_M \overline{h}^{(l+1)\frac{rn}{rn-2}}dv\right)^{(1-a)\frac{sn-2}{2sn}}\left(\fint_M \overline{h}^{l+1}dv\right)^{\frac{sn-2}{2sn}a}\\ &\le \xi\left(\fint_M \overline{h}^{(l+1)\frac{rn}{rn-2}}dv\right)^{\frac{rn-2}{2rn}} + \xi^{-\frac{(1-a)}{a}\frac{rn}{rn-2}}\left(\fint_M \overline{h}^{l+1}dv\right)^{\frac{1}{2}}.\end{aligned}$$

Inserting this into inequality (2.11) and setting $\xi = \frac{1}{2}\mathcal{A}_l^{-1}$, we have

$$\begin{aligned}\left(\|\overline{h}\|^*_{(l+1)\frac{rn}{rn-2}}\right)^{\frac{l+1}{2}} &\le \mathcal{A}_l\left(\xi\left(\fint_M \overline{h}^{(l+1)\frac{rn}{rn-2}}dv\right)^{\frac{rn-2}{2rn}} + \xi^{-\frac{(1-a)}{a}\frac{rn}{rn-2}}\left(\fint_M \overline{h}^{l+1}dv\right)^{\frac{1}{2}}\right) + \left(\|\overline{h}\|^*_{l+1}\right)^{\frac{l+1}{2}}\\ &= \frac{1}{2}\left(\|\overline{h}\|^*_{(l+1)\frac{rn}{rn-2}}\right)^{\frac{l+1}{2}} + \left(2^{\frac{1-a}{a}\frac{rn}{rn-2}}\mathcal{A}_l^{\frac{1-a}{a}\frac{rn}{rn-2}+1}+1\right)\left(\|\overline{h}\|^*_{l+1}\right)^{\frac{l+1}{2}}.\end{aligned}$$

Renaming $l+1$ by $l>1$, and $\mu = \frac{rn}{rn-2}$, we obtain

$$\|\overline{h}\|^*_{l\mu} \le \left((2\mathcal{A}_{l-1})^{\frac{s}{s-r}}+2\right)^{\frac{2}{l}}\|\overline{h}\|^*_l.$$

Let $l_k = l\mu^k$. Then

$$\|\overline{h}\|^*_{l_k} \le \prod_{j=1}^{k}\left((2\mathcal{A}_{l_{j-1}-1})^{\frac{s}{s-r}}+2\right)^{\frac{2}{l_j}}\|\overline{h}\|^*_l.$$

Note that $\mathcal{A}_l = O(\sqrt{l})$ so that $\mathcal{A} = \lim_{k\to\infty}\prod_{j=1}^{k}\left((2\mathcal{A}_{l_{j-1}-1})^{\frac{s}{s-r}}+2\right)^{\frac{2}{l_j}} < \infty$ hence

$$\|\overline{h}\|_\infty \le \mathcal{A}\|\overline{h}\|^*_l \tag{2.12}$$

for $l>1$. In particular, let $l=2$ so that

$$\begin{aligned}\sup h^+ \le \|\overline{h}\|_\infty &\le \mathcal{A}\|\overline{h}\|^*_2\\ &= \mathcal{A}\|h^+ + \|f\|^*_p\|^*_2\\ &\le \mathcal{A}(\|h\|^*_2 + \|\overline{w}(V-\tilde{\sigma})\|^*_p)\\ &\le \mathcal{A}(K_1\sqrt{\epsilon}+6(\tau-1)\epsilon)\\ &\le \mathcal{A}(K_1+6(\tau-1))(\sqrt{\epsilon}+\epsilon) \equiv K_2(\sqrt{\epsilon}+\epsilon).\end{aligned}$$

Here we have used Claim 1. Since $-h$ satisfies the same equation (except for a sign in f), we conclude that

$$|w-\overline{w}| = |h| \le K_2(\sqrt{\epsilon}+\epsilon).$$

Claim 3: For ϵ small enough, $\overline{w} > \frac{1}{2}$.

From the previous claim, we know that $w \le \overline{w} + K_2(\sqrt{\epsilon}+\epsilon)$. Since $\|w\|^*_2 = 1$ and $\overline{w} \le 1$,

$$1 = \fint_M w^2 dv \le \fint_M (\overline{w}+K_2(\sqrt{\epsilon}+\epsilon))^2 dv = (\overline{w}+K_2(\sqrt{\epsilon}+\epsilon))^2 \le \overline{w}^2 + 2K_2(\sqrt{\epsilon}+\epsilon) + K_2^2(\sqrt{\epsilon}+\epsilon)^2.$$

Hence, choosing $\epsilon>0$ small enough

$$\frac{1}{4} < 1 - 2K_2(\sqrt{\epsilon}+\epsilon) - K_2^2(\sqrt{\epsilon}+\epsilon)^2 \le \overline{w}^2,$$

so that $\overline{w} > \frac{1}{2}$. This allows us to finish the proof of the lemma. Consider the function $w_2 := w/\overline{w}$. w_2 satisfies the same equation as w, and we know by claims 2 and 3 that

$$1 - \tilde{\delta} \leq w_2 \leq 1 + \tilde{\delta},$$

where $\tilde{\delta} := 2K_2(\sqrt{\epsilon} + \epsilon)$. Define $J := w_2^{-\frac{1}{\tau-1}}$. We can establish the bounds for J using the 1st order Taylor polynomial of $f(x) = x^{-\frac{1}{\tau-1}}$ near $x = 1$, on the domain $(1-\tilde{\delta}, 1+\tilde{\delta})$. We know that $f(x) = 1 + R_1(x)$, where the remainder can be estimated by

$$|R_1| = |f'(x^*)(x-1)| \leq \frac{2\tilde{\delta}}{(\tau-1)(1-\tilde{\delta})^{\frac{\tau}{\tau-1}}},$$

where $x^* \in (1-\tilde{\delta}, 1+\tilde{\delta})$. Choosing $\epsilon > 0$ small enough so that $\frac{2\tilde{\delta}}{(\tau-1)(1-\tilde{\delta})^{\frac{\tau}{\tau-1}}} \leq \delta$, we get the estimate

$$|J - 1| \leq \delta,$$

concluding the proof of the proposition. □

3. Sharp gradient estimate

As in the $\mathrm{Ric}_M \geq 0$ case, to obtain a sharp estimate we need to consider the following ODE with an additional parameter $\eta \in \mathbb{R}$,

$$\begin{cases} (1-u^2)Z''(u) + uZ'(u) = -\eta u & \text{on } [-1,1] \\ Z(0) = 0 \\ Z(\pm 1) = 0, \end{cases} \tag{3.1}$$

which has the explicit solution

$$Z(u) = \frac{2\eta}{\pi}\left(\arcsin(u) + u\sqrt{1-u^2}\right) - \eta u.$$

Furthermore, the function $Z(u)$ satisfies the following inequalities.

Proposition 3.1. For numbers $\eta = 1 + \delta$ and $J \leq \eta$, we have

$$\eta^{-1}(Z')^2 - 2J^{-1}Z''Z + Z' \geq 0, \tag{3.2}$$

$$2Z - uZ' + 1 \geq 1 - \eta, \quad u \in [-1,1], \tag{3.3}$$

$$\eta(1-u^2) \geq 2|Z|. \tag{3.4}$$

Proof. By direct computation,

$$Z'(u) = -\eta + \frac{4\eta}{\pi}\sqrt{1-u^2},$$
$$Z''(u) = -\frac{4\eta}{\pi}\frac{u}{\sqrt{1-u^2}}.$$

We first show the inequality (3.2). It was shown in [Li12] that the solution to (3.1) when $\eta = 1$, namely, $z(u) = \frac{2}{\pi}(\arcsin(u) + u\sqrt{1-u^2}) - u$ satisfies

$$(z')^2 - 2z''z + z' \geq 0.$$

For $\eta > 0$, the solution is simply given by rescaling $Z = \eta z$. Note that z, u, and $-z''$ shares sign, so that $-zz'' \geq 0$ on $[-1,1]$. Since $J \leq \eta$,

$$\eta^{-1}(Z')^2 - 2J^{-1}Z''Z + Z' = \eta(z')^2 - 2J^{-1}\eta^2 z''z + \eta z' \geq \eta((z')^2 - 2z''z + z') \geq 0.$$

Similarly, for (3.4), the case $\eta = 1$ was shown in [Li12] and we obtain the result by rescaling. Direct computation yields (3.3). $\square$

We mention that in Proposition 3.1, u, J are generic parameters, and u is a variable. In order to prove the main result, we need to make the following choice.

Let ϕ be a nontrivial eigenfunction corresponding to λ_1 and normalized so that for $0 \le a < 1$, $a+1 = \sup \phi$ and $a-1 = \inf \phi$. Set $u := \phi - a$, so that $\Delta u = -\lambda_1(u+a)$.

Proposition 3.2. Let $\delta > 0$ be fixed. Let J be the solution (2.5) with $\tau = \frac{3+4\delta}{2\delta}$. Suppose M is a closed Riemannian manifold with $\bar{k}(p,0) \le \epsilon$ so that Proposition 2.2 and 2.3 holds for this choice of δ. Then u defined as above satisfies the gradient estimate

$$J|\nabla u|^2 \le \tilde{\lambda}(1-u^2) + 2a\lambda_1 Z(u), \tag{3.5}$$

where $\tilde{\lambda} := C_1\lambda_1 + C_2$ with

$$C_1 := \frac{1+\delta+\sqrt{A}}{1-\sqrt{B}} \tag{3.6}$$

and

$$C_2 := \frac{\sigma}{2(1-\sqrt{B})}\left(\frac{\tilde{Z}}{\sqrt{A}} + \frac{1}{2\sqrt{B}}\right), \tag{3.7}$$

where $A = A(\delta) := 2\delta(1+\delta)$, $B = B(\delta) := \dfrac{\delta(5+\delta)}{(1-\delta)}$, and $Z(u)$ is defined as in (3.1) for $\eta = 1+\delta$ with $\sup\limits_{[-1,1]} Z = \tilde{Z} \le (0.116)\eta$, and σ is given in Proposition 2.2.

Remark 3.1. Note that when $\delta \to 0$, then $C_1 \to 1$. Also, when $\sigma/\sqrt{\delta} \to 0$, then $C_2 \to 0$.

Proof. Consider $\xi < 1$, and denote, for simplicity, $u := \xi(\phi - a)$. Note that comparing with the previously defined u, there is an extra factor ξ so that $-\xi \le u \le \xi$. Then this u satisfies

$$\Delta u = -\lambda_1(u + \xi a). \tag{3.8}$$

Consider for a constant c,

$$Q := J|\nabla u|^2 - c(1-u^2) - 2a\lambda_1 Z(u).$$

To obtain (3.5), we want to show that $Q \le 0$. By (3.4) and compactness of M, we can choose a suitable c so that $Q = 0$ at the maximum point of Q. For the rest of the proof, we fix such a constant c so that $\max Q = 0$.

Our goal is to show that $c \le C_1\lambda_1 + C_2$ for some constants C_1, C_2 such that $(C_1, C_2) \to (1, 0)$ as $\delta \to 0$, $\sigma/\sqrt{\delta} \to 0$. Taking $\xi \to 1$ will give us the gradient estimate.

If $c \le (1+\delta)\lambda_1$, then we are done so we can assume that

$$c > (1+\delta)\lambda_1, \tag{3.9}$$

and in particular, $c > J\lambda_1$. Let the maximum point of Q be x_0. Then $|\nabla u(x_0)| > 0$, since if $\nabla u(x_0) = 0$, then

$$\begin{aligned} 0 &= -c(1-u^2(x_0)) - 2a\lambda_1 Z(u(x_0)) \\ &\le -c(1-\xi^2) + a\lambda_1\eta(1-\xi^2) \\ &= (a\lambda_1\eta - c)(1-\xi^2) < 0, \end{aligned}$$

a contradiction.

For convenience, we write $Z = Z(u)$. By direct computation,

$$\nabla Q = (\nabla J)|\nabla u|^2 + J\nabla(|\nabla u|^2) + 2cu\nabla u - 2a\lambda_1 Z'\nabla u,$$

which at the maximum point is

$$\nabla(|\nabla u|^2) = -\frac{\nabla J}{J}|\nabla u|^2 - 2J^{-1}cu\nabla u + 2J^{-1}a\lambda_1 Z'\nabla u. \tag{3.10}$$

Let $e_1 = \frac{\nabla u}{|\nabla u|}$ and complete to an orthonormal basis $\{e_j\}_{j=1}^n$. Then at the maximal point we have

$$\operatorname{Hess} u(\nabla u, \nabla u) = -\frac{1}{2J}\langle \nabla J, \nabla u\rangle|\nabla u|^2 - c\frac{u}{J}|\nabla u|^2 + \frac{a\lambda_1 Z'}{J}|\nabla u|^2$$

so that

$$\operatorname{Hess} u(e_1, e_1) = -\frac{1}{2J}\langle \nabla J, \nabla u\rangle - c\frac{u}{J} + \frac{a\lambda_1 Z'}{J}, \tag{3.11}$$

and for $j \neq 1$,

$$\operatorname{Hess} u(e_1, e_j) = -\frac{1}{2J}\langle \nabla J, e_j\rangle|\nabla u|. \tag{3.12}$$

Again by direct computation,

$$\begin{aligned}\Delta Q = &(\Delta J)|\nabla u|^2 + 2\langle \nabla J, \nabla(|\nabla u|^2)\rangle + J\Delta|\nabla u|^2\\ &+ 2c|\nabla u|^2 + 2cu\Delta u - 2a\lambda_1 Z''|\nabla u|^2 - 2a\lambda_1 Z'\Delta u.\end{aligned}$$

By Bochner formula and (3.8),

$$\begin{aligned}\Delta Q = &(\Delta J)|\nabla u|^2 + 2\langle \nabla J, \nabla(|\nabla u|^2)\rangle\\ &+ J\left(2|\operatorname{Hess} u|^2 - 2\lambda_1|\nabla u|^2 + 2\operatorname{Ric}(\nabla u, \nabla u)\right)\\ &+ 2c|\nabla u|^2 - 2cu\lambda_1(u + \xi a) - 2a\lambda_1 Z''|\nabla u|^2 + 2a\lambda_1^2 Z'(u + \xi a).\end{aligned}$$

At the maximal point of Q, using $\operatorname{Ric} \geq -\rho_0$ and (3.10), we have

$$\begin{aligned}0 \geq &(\Delta J)|\nabla u|^2 - 2\frac{|\nabla J|^2}{J}|\nabla u|^2 - 4c\frac{u}{J}\langle \nabla J, \nabla u\rangle + 4\frac{a\lambda_1 Z'}{J}\langle \nabla J, \nabla u\rangle\\ &+ 2J|\operatorname{Hess} u|^2 - 2\lambda_1 J|\nabla u|^2 - 2J\rho_0|\nabla u|^2\\ &+ 2c|\nabla u|^2 - 2cu\lambda_1(u + \xi a) - 2a\lambda_1 Z''|\nabla u|^2 + 2a\lambda_1^2 Z'(u + \xi a).\end{aligned}$$

Using (3.11) and (3.12) we also have the lower bound of the Hessian term

$$\begin{aligned}|\operatorname{Hess} u|^2 \geq &\sum_{j=1}^n (\operatorname{Hess} u(e_1, e_j))^2\\ = &\frac{1}{4J^2}|\nabla J|^2|\nabla u|^2 + c\frac{u}{J^2}\langle \nabla J, \nabla u\rangle - \frac{a\lambda_1 Z'}{J^2}\langle \nabla J, \nabla u\rangle + c^2\frac{u^2}{J^2} - \frac{2cua\lambda_1 Z'}{J^2} + \frac{a^2\lambda_1^2(Z')^2}{J^2}.\end{aligned}$$

Inserting the Hessian lower bound we have

$$\begin{aligned}0 \geq &\left((\Delta J) - \frac{3}{2}\frac{|\nabla J|^2}{J} - 2J\rho_0\right)|\nabla u|^2 - 2c\frac{u}{J}\langle \nabla J, \nabla u\rangle + 2\frac{a\lambda_1 Z'}{J}\langle \nabla J, \nabla u\rangle\\ &+ \frac{2c^2u^2}{J} - \frac{4cua\lambda_1 Z'}{J} + \frac{2a^2\lambda_1^2(Z')^2}{J} - 2\lambda_1 J|\nabla u|^2\\ &+ 2c|\nabla u|^2 - 2cu\lambda_1(u + \xi a) - 2a\lambda_1 Z''|\nabla u|^2 + 2a\lambda_1^2 Z'(u + \xi a).\end{aligned} \tag{3.13}$$

Let $\beta = \dfrac{2\delta}{1+\delta}$. By Cauchy-Schwarz, we bound the mixed term as follows,

$$2\frac{a\lambda_1 Z'}{\sqrt{J}}\left\langle \frac{\nabla J}{\sqrt{J}}, \nabla u\right\rangle \geq -\beta\frac{a^2\lambda_1^2(Z')^2}{J} - \frac{|\nabla J|^2}{\beta J}|\nabla u|^2$$

and

$$-2c\frac{u}{\sqrt{J}}\left\langle \frac{\nabla J}{\sqrt{J}}, \nabla u\right\rangle \geq -\delta c^2\frac{u^2}{J} - \frac{|\nabla J|^2}{\delta J}|\nabla u|^2.$$

Plugging these into (3.13) we deduce

$$\begin{aligned}
0 \geq &\left((\Delta J) - \left(\frac{3+4\delta}{2\delta}\right)\frac{|\nabla J|^2}{J} - 2J\rho_0\right)|\nabla u|^2 \\
&+ (2-\delta)\frac{c^2u^2}{J} - \frac{4cua\lambda_1 Z'}{J} + (2-\beta)\frac{a^2\lambda_1^2(Z')^2}{J} - 2\lambda_1 J|\nabla u|^2 \\
&+ 2c|\nabla u|^2 - 2cu\lambda_1(u+\xi a) - 2a\lambda_1 Z''|\nabla u|^2 + 2a\lambda_1^2 Z'(u+\xi a).
\end{aligned}$$

Now using the fact that $Q = 0$ at the maximal point, written explicitly

$$|\nabla u|^2 = cJ^{-1}(1-u^2) + 2a\lambda_1 J^{-1}Z, \tag{3.14}$$

we substitute to the second and third lines of the above so that

$$\begin{aligned}
0 \geq &\left((\Delta J) - \left(\frac{3+4\delta}{2\delta}\right)\frac{|\nabla J|^2}{J} - 2J\rho_0\right)|\nabla u|^2 \\
&- \frac{4cua\lambda_1 Z'}{J} - 2\lambda_1(c(1-u^2) + 2a\lambda_1 Z) \\
&+ 2cJ^{-1}(c(1-u^2) + 2a\lambda_1 Z) - 2cu\lambda_1(u+\xi a) \\
&- 2a\lambda_1 Z'' J^{-1}(c(1-u^2) + 2a\lambda_1 Z) + 2a\lambda_1^2 Z'(u+\xi a) \\
&+ (2-\delta)\frac{c^2u^2}{J} + (2-\beta)\frac{a^2\lambda_1^2(Z')^2}{J}.
\end{aligned}$$

Using the equation (2.5) with $\tau = \frac{3+4\delta}{2\delta}$, substituting (3.14) again, and noting that $2-\beta = 2\eta^{-1}$,

$$\begin{aligned}
0 \geq &-\sigma(c(1-u^2) + 2a\lambda_1 Z) \\
&+ 2a^2\lambda_1^2(\xi Z' - 2J^{-1}Z''Z + \eta^{-1}J^{-1}(Z')^2)) \\
&- 2ac\lambda_1 J^{-1}((1-u^2)Z'' + uZ' + \xi uJ) \\
&+ 2a\lambda_1(cJ^{-1} - \lambda_1)(2Z - uZ') \\
&- 2\lambda_1 c + 2c^2J^{-1} - 2ac\lambda_1 J^{-1} + 2a\lambda_1^2 \\
&- \delta J^{-1}c^2u^2 + 2ac\lambda_1 J^{-1} - 2a\lambda_1^2.
\end{aligned}$$

After some rearranging we have

$$\begin{aligned}
0 \geq\ &2a^2\lambda_1^2(\xi Z' - 2J^{-1}Z''Z + \eta^{-1}(Z')^2) + 2a^2\lambda_1^2\eta^{-1}(J^{-1}-1)(Z')^2 \\
&- 2ac\lambda_1 J^{-1}((1-u^2)Z'' + uZ' + \xi uJ) \\
&+ 2a\lambda_1(cJ^{-1} - \lambda_1)(2Z - uZ' + 1) \\
&+ 2(cJ^{-1} - \lambda_1)(c - a\lambda_1) \\
&+ (c\sigma - c^2J^{-1}\delta)u^2 - c\sigma - 2a\sigma\lambda_1 Z.
\end{aligned}$$

The first line is grouped by terms with $a^2\lambda_1^2$, then using (3.9), we group terms as products of $cJ^{-1}-\lambda_1$ since it has a sign, the second line grouped from the remaining terms with a factor $ac\lambda_1$, and the last are remaining terms which would mostly be zero for the pointwise bounded case since $\sigma = 0$. Using the ODE (3.1), the inequalities (3.2),(3.3), and (3.9), we have

$$\begin{aligned}
0 \geq\ & 2a^2\lambda_1^2(\xi-1)Z' + 2a^2\lambda_1^2\eta^{-1}(J^{-1}-1)(Z')^2 \\
& - 2ac\lambda_1 J^{-1}(\xi J-\eta)u \\
& + 2a\lambda_1(cJ^{-1}-\lambda_1)(1-\eta) \\
& + 2(cJ^{-1}-\lambda_1)(c-a\lambda_1) \\
& - c\sigma - 2a\sigma\lambda_1 Z - c^2J^{-1}\delta.
\end{aligned}$$

Since $\eta > 1$ and $u \geq -\xi \geq -1$, $-\eta \leq Z' \leq \eta(\frac{4}{\pi}-1) \leq \eta$, and using Proposition 2.3 we replace J by either $1-\delta$ or $1+\delta$ appropriately, and noting that $2a^2\lambda_1^2\eta^{-1}(J^{-1}-1)(Z')^2 \geq -2\delta c^2$ we get

$$\begin{aligned}
0 \geq\ & -2a^2\lambda_1^2\eta(1-\xi) \\
& - 2ac\lambda_1\left(\eta-\xi+\frac{\delta\eta}{1-\delta}\right) \\
& - 2a\lambda_1\left(c-\lambda_1+\frac{c\delta}{1-\delta}\right)(\eta-1) \\
& + 2\left(c-\lambda_1-\delta c\right)\left(c-a\lambda_1\right) \\
& - c\sigma - 2a\sigma\lambda_1 Z - \frac{c^2\delta}{1-\delta} - 2\delta c^2.
\end{aligned}$$

Recall that $\tilde{Z} := \sup Z = O(\eta)$. Our goal now is to obtain a quadratic inequality in terms of $(c-\lambda_1)$. Using $\eta \geq 1$, we first rewrite the first term and split the remaining terms so that

$$\begin{aligned}
0 \geq\ & -2a^2\lambda_1^2\eta(\eta-\xi) \\
& - 2ac\lambda_1(\eta-\xi) - 2ac\lambda_1\frac{\delta\eta}{1-\delta} \\
& - 2a\lambda_1(c-\lambda_1)(\eta-1) - 2a\lambda_1\frac{c\delta}{1-\delta}(\eta-1) \\
& + 2(c-\lambda_1)(c-a\lambda_1) - 2\delta c(c-a\lambda_1) \\
& - c\sigma - 2a\sigma\lambda_1\tilde{Z} - \frac{c^2\delta}{1-\delta} - 2\delta c^2.
\end{aligned}$$

Since $0 \leq a < 1$ and noting that $\xi \leq 1$, $\eta \geq 1$, and $c \geq \lambda_1$, we replace a by 1 or 0 according to the sign so that

$$\begin{aligned}
0 \geq\ & -2\lambda_1^2\eta(\eta-\xi) \\
& - 2c\lambda_1(\eta-\xi) - 2c\lambda_1\frac{\delta\eta}{1-\delta} \\
& - 2\lambda_1(c-\lambda_1)(\eta-1) - 2\lambda_1\frac{c\delta}{1-\delta}(\eta-1) \\
& + 2(c-\lambda_1)^2 - 4\delta c^2 \\
& - c\sigma - 2\sigma\lambda_1\tilde{Z} - \frac{c^2\delta}{1-\delta}.
\end{aligned}$$

Combining the first two terms, and adding and subtracting λ_1, we get

$$\begin{aligned}0 \geq -2\lambda_1(\eta-\xi)(c-\lambda_1+\lambda_1\eta+\lambda_1) - 2c\lambda_1\frac{\delta\eta}{1-\delta} \\ - 2\lambda_1(c-\lambda_1)(\eta-1) - 2\lambda_1\frac{c\delta}{1-\delta}(\eta-1) \\ + 2(c-\lambda_1)^2 - 4\delta c^2 - c\sigma - 2\sigma\lambda_1\tilde{Z} - \frac{c^2\delta}{1-\delta}.\end{aligned}$$

After further rearranging we have,

$$\begin{aligned}0 \geq -2\lambda_1^2(\eta-\xi)(\eta+1) + 2\left(c-\lambda_1\right)^2 - 4\delta c^2 - c\sigma - 2\sigma\lambda_1\tilde{Z} - \frac{c^2\delta}{1-\delta} \\ - 2\lambda_1\left(c-\lambda_1\right)(2\eta-\xi-1) - 2\lambda_1\left(\frac{c\delta}{1-\delta}\right)(2\eta-1).\end{aligned}$$

Completing the square in terms of $(c-\lambda_1)$, we get

$$\begin{aligned}\left((c-\lambda_1) - \frac{\lambda_1}{2}(2\eta-\xi-1)\right)^2 \leq \lambda_1^2\frac{4(\eta-\xi)(\eta+1)+(2\eta-\xi-1)^2}{4} + \sigma\lambda_1\tilde{Z} \\ + c\lambda_1\left(\frac{\delta}{1-\delta}\right)(2\eta-1) + c^2\left(\frac{\delta(4-\delta)}{(1-\delta)}\right) + c\frac{\sigma}{2}.\end{aligned}$$

Using $\lambda_1 \leq c$ we get

$$\begin{aligned}\left((c-\lambda_1) - \frac{\lambda_1}{2}(2\eta-\xi-1)\right)^2 \leq \lambda_1^2\frac{4(\eta-\xi)(\eta+1)+(2\eta-\xi-1)^2}{4} + \sigma\lambda_1\tilde{Z} \\ + c^2\left(\left(\frac{\delta}{1-\delta}\right)(2\eta-1) + \frac{\delta(4-\delta)}{(1-\delta)}\right) + c\frac{\sigma}{2}.\end{aligned}$$

Let

$$\begin{aligned}A = A(\eta,\xi) &:= \frac{4(\eta-\xi)(\eta+1)+(2\eta-\xi-1)^2}{4}, \\ B = B(\delta) &:= \left(\frac{\delta(5+\delta)}{(1-\delta)}\right).\end{aligned}$$

Then

$$\begin{aligned}\left((c-\lambda_1) - \frac{\lambda_1}{2}(2\eta-\xi-1)\right)^2 &\leq A\left(\lambda_1 + \frac{\sigma\tilde{Z}}{2A}\right)^2 - \frac{\sigma^2\tilde{Z}^2}{4A} + B\left(c+\frac{\sigma}{4B}\right)^2 - \frac{\sigma^2}{16B} \\ &\leq A\left(\lambda_1 + \frac{\sigma\tilde{Z}}{2A}\right)^2 + B\left(c+\frac{\sigma}{4B}\right)^2.\end{aligned}$$

Using the inequality $\sqrt{a^2+b^2} \leq a+b$, for $a, b \geq 0$,

$$c - \lambda_1 \leq \frac{\lambda_1}{2}(2\eta-\xi-1) + \sqrt{A}\left(\lambda_1 + \frac{\sigma\tilde{Z}}{2A}\right) + \sqrt{B}\left(c+\frac{\sigma}{4B}\right).$$

Letting $\xi \to 1$, we have

$$c \leq \frac{\lambda_1}{1-\sqrt{B}}\left(\eta + \sqrt{A} + \frac{\sigma\tilde{Z}}{2\lambda_1\sqrt{A}}\right) + \left(\frac{\sigma}{4\sqrt{B}(1-\sqrt{B})}\right)$$

recalling that $\eta = 1+\delta$ so that $A := A(1+\delta, 1) = 2\delta(1+\delta)$. □

4. Proof of Theorem 1.1

We now give the sharp eigenvalue lower bound. By Proposition 3.5 we have

$$J|\nabla u|^2 \le (C_1\lambda_1 + C_2)(1-u^2) + 2a\lambda_1 Z(u).$$

By (2.1), we have a rough lower bound of the first eigenvalue, $\lambda_1 \ge \Lambda_{\text{rough}} > 0$ so that

$$\lambda_1 \ge \frac{J|\nabla u|^2}{(C_1 + C_2\Lambda_{\text{rough}}^{-1})(1-u^2) + 2aZ(u)}.$$

Let $b := C_1 + C_2\Lambda_{\text{rough}}^{-1}$ and let γ be the shortest geodesic connecting the minimum and maximum point of u with length at most D. Recalling that $\max u = 1$ and $\min u = -1$ from the construction given above Proposition 3.5, integrating the gradient estimate along the geodesic and using change of variables $x(s) = u(\gamma(s))$ and that Z is odd,

$$\begin{aligned}
D\sqrt{\lambda_1} &\ge \sqrt{\lambda_1}\int_\gamma ds \\
&\ge (1-\delta)\int_\gamma \frac{|\nabla u| ds}{\sqrt{b(1-u^2) + 2aZ(u)}} \\
&= (1-\delta)\int_{-1}^{1} \frac{dx}{\sqrt{b(1-x^2) + 2aZ(x)}} \\
&= (1-\delta)\int_0^1 \left(\frac{1}{\sqrt{b(1-x^2)+2aZ(x)}} + \frac{1}{\sqrt{b(1-x^2)-2aZ(x)}}\right) dx \\
&= \frac{1-\delta}{\sqrt{b}}\int_0^1 \frac{1}{\sqrt{1-x^2}}\left(\frac{1}{\sqrt{1+\frac{2aZ(x)}{b(1-x^2)}}} + \frac{1}{\sqrt{1-\frac{2aZ(x)}{b(1-x^2)}}}\right) dx \\
&\ge \frac{1-\delta}{\sqrt{b}}\int_0^1 \frac{1}{\sqrt{1-x^2}}\left(2 + \frac{3a^2(Z(x))^2}{b^2(1-x^2)^2}\right) dx \\
&\ge \frac{1-\delta}{\sqrt{b}}\pi,
\end{aligned}$$

so that

$$\lambda_1 \ge \frac{(1-\delta)^2}{b}\frac{\pi^2}{D^2}.$$

Recall in the limiting case (c.f. Remark 3.1) that $b \to 1$ as $\delta \to 0$ and $\sigma/\sqrt{\delta} \to 0$. Let $\alpha := \dfrac{(1-\delta)^2}{b}$ to obtain the result.

5. Appendix: estimate of ϵ

In this appendix we will give explicit bounds for ϵ depending on p, n, D, and in terms of the Sobolev C_s and Poincaré Λ_{rough}^{-1} constants, which have explicit expressions in [Gal88]. Note that the Sobolev and Poincaré constants can be estimated and do not change for ϵ smaller than some fixed number. We show that it suffices to choose

$$\epsilon < \min\left\{(n-1)\left(\frac{1}{BD}\ln\left(1+\frac{1}{2^{p+1}}\right)\right)^2, \frac{\delta}{12C_s^2(3+2\delta)}, \left(\frac{\sqrt{7}-2}{K_2}\right)^2, \frac{1}{8K_2\left(\frac{4}{3+2\delta}\right)^{\frac{9+6\delta}{3+2\delta}}}\right\} \tag{5.1}$$

where $B(p,n) = \left(\frac{2p-1}{p}\right)^{\frac{1}{2}} (n-1)^{1-\frac{1}{2p}} \left(\frac{2p-2}{2p-n}\right)^{\frac{p-1}{2p}}$, $K_2 = \mathcal{A}(K_1 + \frac{9+6\delta}{\delta})$, $K_1 = \sqrt{6\Lambda_{rough}^{-1}\left(2+\frac{3}{\delta}\right)}$ and $\mathcal{A}(n,p,D)$ is the constant that appears from Moser's iteration (2.12).

It suffices to choose ϵ smaller than the worst of the following conditions:

(1) To apply the Sobolev inequalities that follow from Theorem 2.1, ϵ needs to be small enough so that the theorem holds. Using our notation, the condition that needs to be satisfied in [Gal88, Theorem 3, 6] is

$$\fint_M \left(\frac{\rho_0}{(n-1)\tilde{\alpha}^2} - 1\right)_+^p dv \leq \frac{1}{2}\left(e^{B\tilde{\alpha}D} - 1\right)^{-1} \tag{5.2}$$

for some $\tilde{\alpha} > 0$. Multiplying (5.2) by $\tilde{\alpha}^{2p}$ and raising both sides to the power $1/p$ we get

$$\left(\fint_M \left(\frac{\rho_0}{n-1} - \tilde{\alpha}^2\right)_+^p dv\right)^{\frac{1}{p}} \leq \frac{\tilde{\alpha}^2}{2^{\frac{1}{p}}}(e^{B\tilde{\alpha}D} - 1)^{-\frac{1}{p}}.$$

Note that

$$\left(\fint_M \left(\frac{\rho_0}{n-1} - \tilde{\alpha}^2\right)_+^p dv\right)^{\frac{1}{p}} \leq \left\|\frac{\rho_0}{n-1} - \tilde{\alpha}^2\right\|_p^* \leq \frac{\bar{k}(p,0)}{n-1} + \tilde{\alpha}^2 \leq \frac{\epsilon}{n-1} + \tilde{\alpha}^2.$$

Thus, it suffices to impose that for some fixed $\tilde{\alpha} > 0$ we have

$$\frac{\epsilon}{n-1} + \tilde{\alpha}^2 \leq \frac{\tilde{\alpha}^2}{2^{\frac{1}{p}}}(e^{B\tilde{\alpha}D} - 1)^{-\frac{1}{p}},$$

or equivalently

$$\epsilon \leq (n-1)\tilde{\alpha}^2\left(\frac{1}{2^{\frac{1}{p}}(e^{B\tilde{\alpha}D} - 1)^{\frac{1}{p}}} - 1\right).$$

So choosing $\tilde{\alpha} = \frac{1}{BD}\ln\left(1 + \frac{1}{2^{p+1}}\right)$ we obtain

$$\epsilon \leq (n-1)\left(\frac{1}{BD}\ln\left(1 + \frac{1}{2^{p+1}}\right)\right)^2 \tag{5.3}$$

(2) In Proposition 2.2 and in Claim 1 of Proposition 2.3 we need ϵ to satisfy

$$\epsilon \leq \frac{1}{24C_s^2(\tau - 1)}, \tag{5.4}$$

where C_s is the Sobolev constant of (2.2). From the proof of Proposition 3.5, $\tau = \frac{3+4\delta}{2\delta}$ so we get

$$\epsilon < \frac{\delta}{12C_s^2(3+2\delta)}.$$

(3) In Claim 3 of Proposition 2.3 we need ϵ to satisfy

$$\frac{1}{4} < 1 - 2K_2(\sqrt{\epsilon} + \epsilon) - K_2^2(\sqrt{\epsilon} + \epsilon)^2.$$

This implies that

$$\epsilon < \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\sqrt{7} - 2}{2K_2}}\right)^2.$$

To get a cleaner estimate, notice that since $\mathcal{A} > 1$, then $K_2 > 6$, thus $\frac{\sqrt{7}-2}{2K_2} < \frac{\sqrt{7}-2}{12} < \frac{3}{4}$. Using that for $0 < x < 3$ we have that $-1 + \sqrt{1+x} > x$, letting $x = \frac{2(\sqrt{7}-2)}{K_2}$, we get

$$\epsilon < \left(\frac{\sqrt{7}-2}{K_2}\right)^2. \tag{5.5}$$

(4) In Claim 3 of Proposition 2.3 we also need ϵ to be small enough so that

$$\frac{2\tilde{\delta}}{(\tau-1)(1-\tilde{\delta})^{\frac{\tau}{\tau-1}}} \leq \delta,$$

where $\tilde{\delta} = 2K_2(\sqrt{\epsilon}+\epsilon)$. This condition is equivalent to

$$\left(\frac{1}{\tilde{\delta}}-1\right)\tilde{\delta}^{\frac{1}{\tau}} \geq \left(\frac{2}{\delta(\tau-1)}\right)^{\frac{\tau-1}{\tau}} = \left(\frac{4}{3+2\delta}\right)^{\frac{3+2\delta}{3+4\delta}} \equiv C_3(\delta).$$

Note that since $\delta < 1$, we have that $\tau > \frac{3}{2}$. Then we get that $\tilde{\delta}^{\frac{1}{\tau}} > \tilde{\delta}^{\frac{2}{3}}$. It suffices to choose ϵ small enough so that

$$\left(\frac{1}{\tilde{\delta}}-1\right)\tilde{\delta}^{\frac{2}{3}} \geq C_3.$$

Notice that if $\delta < \frac{1}{2}$, then $C_3 > 1$. Thus, choosing $\tilde{\delta} < \frac{1}{8C_3^3}$ we get that

$$\left(\frac{1}{\tilde{\delta}}-1\right)\tilde{\delta}^{\frac{2}{3}} > (2C_3)^3 - \frac{1}{(2C_3)^2} > C_3 + 7C_3^3 - \frac{1}{4C_3^2} > C_3 + 7 - \frac{1}{4} > C_3,$$

so the condition is satisfied. This gives us the last condition

$$\epsilon < -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{16K_2\left(\frac{4}{3+2\delta}\right)^{\frac{9+6\delta}{3+2\delta}}}}.$$

As above, to get a cleaner estimate, notice that if $\delta < \frac{1}{2}$ then

$$\frac{1}{16K_2\left(\frac{4}{3+2\delta}\right)^{\frac{9+6\delta}{3+2\delta}}} < \frac{1}{16K_2} < \frac{1}{96} < \frac{3}{4},$$

so using the same property as in (3) we get that it suffices to assume

$$\epsilon < \frac{1}{8K_2\left(\frac{4}{3+2\delta}\right)^{\frac{9+6\delta}{3+2\delta}}}. \tag{5.6}$$

From (5.3),(5.4),(5.5),and (5.6), we arrive at (5.1).

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