Eigenvalue estimates for Kato-type Ricci curvature conditions

Christian Rose[*]   Guofang Wei[†]

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Abstract

We prove that optimal lower eigenvalue estimates of Zhong-Yang type as well as a Cheng-type upper bound for the first eigenvalue hold on closed manifolds assuming only a Kato condition on the negative part of the Ricci curvature. This generalizes all earlier results on L^p-curvature assumptions. Moreover, we introduce the Kato condition on compact manifolds with boundary with respect to the Neumann Laplacian, leading to Harnack estimates for the Neumann heat kernel and lower bounds for all Neumann eigenvalues, what provides a first insight in handling variable Ricci curvature assumptions in this case.

1 Introduction

A classical field in Riemannian geometry is the investigation of eigenvalue estimates of the Laplace-Beltrami operator depending on geometric assumptions. More precisely, given a compact Riemannian manifold $M = (M^n, g)$ of dimension $n \in \mathbb{N}$, $n \geq 2$, with possibly non-empty smooth boundary ∂M, the aim is to bound the eigenvalues λ, η of the problems

$$\Delta u = -\lambda u, \quad \text{if} \quad \partial M = \emptyset, \tag{1.1}$$

and

$$\Delta u = -\eta u, \quad \partial_\nu u = 0 \quad \text{if} \quad \partial M \neq \emptyset, \tag{1.2}$$

from above and below in terms of curvature restrictions, diameter, and volume. Here, ∂_ν denotes differentiation w.r.t. the inward normal of ∂M, and $-\Delta \geq 0$ the Laplace-Beltrami operator acting in $W^{1,2}(M)$, of course w.r.t. Neumann boundary conditions if $\partial M \neq \emptyset$. Problems (1.1) and (1.2) are called the closed and Neumann eigenvalue problem, respectively. Since in both situations the Laplace-Beltrami operator has compact resolvent, the spectra $\sigma_D(-\Delta)$ and $\sigma_N(-\Delta)$ in the cases $\partial M = \emptyset$ and $\partial M \neq \emptyset$, respectively, are purely discrete and consist of unbounded non-decreasing sequences of eigenvalues with finite multiplicties,

$$\sigma_D(-\Delta) = \{\lambda_k \colon k \in \mathbb{N}_0\} \quad \text{and} \quad \sigma_N(-\Delta) = \{\eta_k \colon k \in \mathbb{N}_0\}.$$

Trivially, $\lambda_0 = \eta_0 = 0$, and both eigenvalues are simple. For $k \geq 1$, it is known that geometric quantities naturally come into play.

Problem (1.1) is well investigated under pointwise uniform lower Ricci curvature bounds $\mathrm{Ric} \geq K$, $K \in \mathbb{R}$. The first result was obtained by Lichnerowicz, who proved that if $K > 0$, then $\lambda_1(M) \geq nK$ [Lic58]. In the case $\mathrm{Ric} \geq 0$, imposing diameter bounds is necessary, while $\mathrm{Ric} \geq K > 0$ automatically

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[*]University of Potsdam, Germany, christian.rose@uni-potsdam.de
[†]University of California Santa Barbara, USA, wei@math.ucsb.edu

implies a diameter bound by Myers' theorem. If Ric ≥ 0 and diam $M \leq D$, Zhong and Yang proved a sharp estimate for λ_1 in [ZY84], improving earlier estimate of Li and Yau [LY80]. Many extensions of these estimates can be found e.g. in [Yan90, Krö92, CW97, HW07, SZ07, AC13, ZW17]. Bounds for higher eigenvalues have been proven in [LY86, Gro80] for closed manifolds satisfying Ric $\geq -K$, $K \geq 0$, and diam $M \leq D$. Upper bounds for λ_1 under those assumptions have been provided by Cheng [Che75] by comparison principles.

In the last decades there was an increasing interest in relaxing the uniform pointwise Ricci curvature assumption to integral Ricci curvature bounds. Those provide estimates that are more stable under perturbations of the metric. Let

$$\rho \colon M^n \to \mathbb{R}, \quad x \mapsto \inf(\sigma(\mathrm{Ric}_x)),$$

where the Ricci tensor Ric is viewed as a pointwise endomorphism on the cotangent bundle, and $\sigma(A)$ denotes the spectrum of an operator A. For $x \in \mathbb{R}$, denote $x_- = \max(0, -x)$. The fundamental Laplacian and volume comparison for uniform Ricci curvature lower bounds have been generalized to smallness assumptions on L^p- means of ρ_- in [PW97] for $p > n/2$. This enabled several authors to extend many results for uniform pointwise Ricci curvature lower bound to the case when the L^p-mean of ρ_- is small for some $p > n/2$. For the first eigenvalue λ_1, in the seminal work [Gal88], Gallot obtained some lower bound. Later, in [PS98] a Cheng-type comparison estimate for λ_1 was proven. Moreover, Aubry showed an optimal Lichnerowicz-type estimate [Aub07]. Recently, in [ROSWZ20] an optimal Zhong-Yang type estimate was also obtained.

In the case of compact manifolds with boundary, i.e., considering Problem (1.2), the fundamental lower bound is given by Cheeger's inequality [Che70] as in the closed case. Namely $\eta_1 \geq \frac{h^2}{4}$, where h is the Neumann isoperimetric constant. In [Bus82], Buser obtained a lower bound for h, therefore η_1, for starlike domain in terms of inner, outer radius and Ricci curvature lower bound, see also [CL97]. In general one needs additional control on the geometry of the boundary. Li and Yau showed lower bounds for η_k if ∂M is convex and Ricci curvature is bounded from below [LY86]. Later, Chen generalized those in [Che90] in terms of a uniform lower bound on the Ricci curvature and the second fundamental form plus the so-called interior rolling-R-ball condition, see Definition 1.5 (the choice of R depends on the upper bound of the sectional curvature near the boundary). In [Wan97], higher eigenvalue estimates are derived for this set-up.

In the L^p-Ricci curvature case, nothing is known so far about estimates for η_k, $k \geq 1$.

Another generalization of pointwise Ricci curvature lower bound is the Kato condition for ρ_-. A measuable function $V \colon M \to [0, \infty]$ satisfies the Kato condition if there is a $T > 0$ such that

$$\kappa_T(V) := \sup_{n \in \mathbb{N}} \| \int_0^T P_t(V \wedge n)\mathrm{d}t \|_\infty < 1,$$

where $x \wedge y := \min(x, y)$, and $(P_t)_{t \geq 0}$ denotes the heat semigroup, i.e.,

$$P_t f(x) := \mathrm{e}^{t\Delta} f(x) = \int_M p_t(x, y) f(y) \mathrm{dvol} y,$$

for $t > 0$, where p_t is the heat kernel of M. (Note that the truncation procedure ensures that $\kappa_T(V)$ is well-defined, since P_t maps L^∞ to itself.)

The Kato condition is a tool from perturbation theory that is used to show semiboundedness of functions V w.r.t. a generator of a Dirichlet form, e.g., $-\Delta$. Moreover, mapping properties of the

heat semigroup carry over to the semigroup which is generated by $-\Delta + V$ when V_- satisfies the Kato condition. For an overview we refer to [RS20]. See Section 2 for some background results.

The Kato condition generalizes L^p-conditions for $p > n/2$. Namely for closed manifolds the smallness of the L^p-mean of ρ_- implies the Kato condition [Gün14], see also Lemma 2.1. Another advantage of the Kato condition is that for the Kähler-Ricci flow, it is known that the L^4-norm of the negative part of the Ricci curvature is preserved [TZ16], but this is not known for $p > 4$. On the other hand a version of the Kato condition on ρ_- is preserved [TZ15]. Apart from the L^p-curvature assumptions, in [CR20, Theorem 4.3], another geometric condition is given which implies the Kato condition. These make the Kato condition a very interesting assumption to study. Note that generalizations of the Laplacian and volume comparison of [PW97] for $L^p(p > n/2)$ Ricci curvature are unavailable for Kato condition.

In [RS17], the authors utilized smallness of $\kappa_T(\rho_-)$ to bound the first Betti number in terms of $\|\rho_-\|_p$, $p > n/2$. The first author showed in [Ros19b] a Li-Yau gradient estimate on $(0, T)$ and an estimate on the first Betti number assuming $\kappa_T(\rho_-)$ is small enough via a technique adapted from [ZZ18]. Carron explored the implications of smallness of $\kappa_T(\rho_-)$ further in [Car19]. Moreover, in [CR20], Carron and the first author showed an optimal Lichnerowicz estimate. Compactness results and diameter bounds under a smallness condition on $\kappa_T((\rho - K)_-)$ for $K > 0$ have been investigated in [CR20, Ros19a]. Note that the Kato condition has also been applied on graphs equipped with Bakry-Émery Ricci curvature in [MR19].

In this paper, we prove all the eigenvalue estimates mentioned above assuming only a Kato-type condition on the negative part of Ricci curvature. First, we generalize the Zhong-Yang type eigenvalue estimate obtained in [ROSWZ20] to the Kato condition.

Theorem 1.1. *Let $D > 0$ and $n \geq 2$. For any $\alpha \in (0, 1)$ there is a $\delta = \delta(n, D, \alpha) > 0$ such that for all closed manifolds (M^n, g) with $\mathrm{diam}(M) \leq D$ satisfying*

$$\kappa_{D^2}(\rho_-) < \delta,$$

we have

$$\lambda_1(M) \geq \alpha \frac{\pi^2}{D^2}$$

with $\alpha \to 1$ if $\delta \to 0$.

Under the same assumptions a lower bound on λ_1 had been obtained by Carron [Car19] (see (2.8)), but it is not optimal.

Remark 1.2. It is known that a heat kernel upper bound for all times implies lower estimates for all λ_k, $k \geq 2$. For the Kato condition, small time estimates for the heat kernel had been obtained first in [Ros19b] and sharpened and extended to all times in [Car19]. Combining the short time bounds, cf. Ineq. (2.7), and our sharp quantitative lower bounds for λ_1, we can improve the constants which would follow from Carron's estimates.

We also obtain a Cheng-type upper estimate for λ_1:

Theorem 1.3. *For all closed manifolds (M^n, g), there is a constant $C(n) > 0$ such that if*

$$\kappa_{\mathrm{diam}(M)^2}(\rho_-) \leq \frac{1}{16n},$$

then

$$\lambda_1(M) \leq C(n) \, \mathrm{diam}(M)^{-2}. \tag{1.3}$$

An eigenvalue comparison theorem assuming smallness of the normalized L^p-condition, $p > n/2$, for the negative part of the Ricci curvature results from [PS98, Theorem 5.1], yielding a similar bound as in Theorem 1.3. The proof there uses Laplacian comparison for the distance function which is not available for the Kato condition. Theorem 1.3 follows from the fact that the Kato condition implies volume doubling. It is well-kown, see, e.g., [HSC01], that the latter property yields the upper estimate on the eigenvalue. For completeness, we will give a proof.

Using the sufficient geometric condition for the Kato condition obtained in [CR20, Theorem 4.3], we have the following corollary.

Corollary 1.4. *Let $D > 0$ and $n \geq 2$, (M^n, g) be a closed manifold with $\mathrm{diam}(M) \leq D$.*

(i) For any $\alpha \in (0, 1)$ there is a $\delta = \delta(n, D, \alpha) > 0$ such that if

$$\int_0^D \frac{r}{\mathrm{Vol}(B(x,r))} \int_{B(x,r)} \rho_- \mathrm{dvol} dr \leq \delta,$$

then

$$\lambda_1 \geq \alpha \frac{\pi^2}{D^2}.$$

(ii) There are $c(n), \varepsilon(n) > 0$ such that

$$\int_0^{\mathrm{diam}(M)} \frac{r}{\mathrm{Vol}(B(x,r))} \int_{B(x,r)} \rho_- \mathrm{dvol} dr \leq \varepsilon(n)$$

implies

$$\lambda_1 \leq c(n) \mathrm{diam}(M)^{-2}.$$

Now we turn to Problem (1.2). To generalize eigenvalue lower bounds on compact manifolds with smooth boundary from the convex to the non-convex case, the following condition was studied by Chen [Che90].

Definition 1.5. The boundary ∂M of a Riemannian manifold M satisfies the interior R-rolling ball condition if for any $y \in \partial M$ there exists a ball $B(q, R) \Subset M$ such that $\overline{B(q, R)} \cap \partial M = \{y\}$.

Consider the heat equation for $-\Delta \geq 0$ with Neumann boundary conditions on ∂M, i.e.,

$$\partial_t u = \Delta u, \quad \text{in } M, \quad \partial_\nu u = 0 \quad \text{on } \partial M. \tag{1.4}$$

Similar to the closed case, there is the Neumann heat semigroup $(H_t)_{t \geq 0}$ given by

$$H_t f(x) = \int_M h_t(x, y) f(y) \mathrm{dvol} y, \quad x \in M, f \in L^2(M), t \geq 0,$$

where h_t denotes the Neumann heat kernel. Since $-\Delta \geq 0$ with Neumann boundary condition on ∂M generates a Dirichlet form, we introduce an analogous Kato condition for manifolds with boundary:

$$\mu_T(\rho_-) := \| \int_0^T H_t \rho_- \mathrm{d}t \|_\infty < 1. \tag{1.5}$$

By proving a Li-Yau gradient estimate for h_t (which clearly also imply upper upper bounds for h_t) assuming smallness of $\mu_T(\rho_-)$, we get the following lower bounds for η_1. Those are the first results for the Neumann boundary problem for the Kato condition. In [RO19], the Neumann condition for L^p-Ricci assumptions is studied, but only for compact domains lying in an ambient manifold. In contrast to our results, the estimates obtained there depend highly on the ambient space.

Theorem 1.6. *Let $n \in \mathbb{N}$, $D, H, T, R > 0$. Suppose M is a compact Riemannian manifold with boundary of dimension n, such that the second fundamental form of ∂M is bounded below by $-H$, and that satisfies the interior R-rolling ball condition for R small enough. Assume that $\operatorname{diam} M \leq D$ and*

$$\mu_T(\rho_-) < 7 \left[16 \left((3 + 2(1+H)^2)(4 + 8n^2(1+H)^2) - 1 \right) \right]^{-1}.$$

Then, the first non-zero Neumann eigenvalue satisfies

$$\eta_1 \geq \frac{1}{2D^2} 2^{-C_2 \mathrm{e}^{16\mu_T(\rho_-)}} \exp\left(-C_1/2D^2 - \mathrm{e}^{16\mu_T(\rho_-)}(1+H)^2 D^2 \right),$$

where $C_i = C_i(n, H, R)$, $i \in \{1, 2\}$.

Remark 1.7. As in the closed case, utilizing a Neumann heat kernel upper obtained by a Harnack inequality for the Neumann heat kernel, lower bounds for η_k, $k \geq 2$ can be derived.

Remark 1.8. As in [Che90, Wan97] the assumption that the interior R-rolling ball condition holds for $R > 0$ small enough depends implicitly on the upper bound K_R of the sectional curvature of the R-tubular neighborhood of ∂M in M. More precisely, $R \in (0, 1)$ has to be chosen such that

$$\sqrt{K_R} \tan\left(R\sqrt{K_R} \right) \leq \frac{1}{2}(1+H) \quad \text{and} \quad \frac{H}{\sqrt{K_R}} \tan\left(R\sqrt{K_R} \right) \leq \frac{1}{2}.$$

The plan of this paper is as follows: in Section 2 we focus on the closed eigenvalue Problem (1.1). We review known results from [Ros19b, Car19] that will be used to derive Theorems 1.1. Then, we adapt techniques from [RS17] to show certain estimates for perturbed semigroups generated by the Schrödinger operator $-\Delta - \rho_-$ assuming smallness of $\kappa_T(\rho_-)$. We utilize them to show estimates for the smallest eigenfunction and the associated eigenvalue of this operator. Those estimates also hold for general continuous potentials V instead of ρ_-. Then, we adapt the proof of [ROSWZ20, Theorem 1.1] to derive Theorem 1.1. After that, we prove upper bounds on the first Dirchlet eigenvalue in balls to derive Theorem 1.3. Although we derive an estimate of $\lambda_1(M)$ similar to Cheng's explicit bounds, we are neither able to prove a sharp bound nor sharp comparison results for Dirichlet eigenvalues in balls. This might be a good direction for further investigations.

In Section 3, we focus on Problem (1.2), i.e., manifolds with non-empty boundary $\partial M \neq \emptyset$. Transferring the technique from [Wan97, RO19] to our setting, we derive a Harnack inequality for positive solutions of the heat equation with Neumann boundary conditions under the interior R-rolling ball condition and lower-bounded second fundamental form. In contrast to the main results in [RO19], smallness of $\mu_T(\rho_-)$ surprisingly allows to get rid of all extra assumptions made in the latter article to get an a priori Neumann heat kernel bound. The proof is almost the same and we explain the necessary changes. To obtain a lower bound for η_1, we adapt the technique from [CL97] via a much shorter proof than in the latter article. This is possible by utilizing the global Harnack inequality obtained by us, which was not known in their setting. Finally, we use the Harnack inequality to provide upper bounds for the Neumann heat kernel h_t.

We want to emphasize that the Kato condition on the Neumann heat semigroup could be generalized to Robin boundary conditions. We conjecture that similar estimates as for the Neumann eigenvalues should hold under sufficiently regular assumptions on the boundary functions.

2 Eigenvalue bounds for closed manifolds

2.1 Preliminaries

The spectrum of a selfadjoint operator A in a Hilbert space X will always be denoted by $\sigma(A)$. $\|A\|_{p,q}$ will be the operator norm of a bounded operator $A\colon L^p(M) \to L^q(M)$. For a closed Riemannian manifold M let dvol be its volume measure, and for $x \in M$ and $r > 0$, let $B(x,r)$ the open ball with center x and radius r. It is convenient to abbreviate

$$V(x,r) := \mathrm{Vol}(B(x,r)).$$

Denote by $((-\Delta + \alpha)^{-1})_{\alpha>0}$ the resolvents of Δ. The heat semigroup and the resolvents can be extended to $L^p(M)$ for $p \in [1,\infty]$.

Note that the functions ρ and ρ_- introduced above are continuous, such that we can restrict our considerations involving the Kato constant to continuous functions. Thus, for some continuous function $V\colon M \to [0,\infty]$, $\alpha, T > 0$, we simplify

$$\kappa_T(V) := \left\| \int_0^T P_t V \, \mathrm{d}t \right\|_\infty \tag{2.1}$$

and

$$c_\alpha(V) := \|(-\Delta + \alpha)^{-1}V\|_\infty. \tag{2.2}$$

Note that those two quantities are closely related [RS17, Gün14]:

$$\left(1 - \mathrm{e}^{-\alpha\beta}\right) c_\alpha(V) \leq \kappa_\beta(V) \leq \mathrm{e}^{\alpha\beta} c_\alpha(V). \tag{2.3}$$

Sometimes it is more convenient to work with κ_T instead of c_α and vice versa. We say that $V \geq 0$ continuous satisfies the Kato condition if

$$\kappa_T(V) < 1 \quad \text{or} \quad c_\alpha(V) < 1$$

for some $T > 0$ or $\alpha > 0$ respectively. Of particular interest for us is the Kato condition for ρ_-:

$$\kappa_T(\rho_-) < 1 \quad \text{or} \quad c_\alpha(\rho_-) < 1$$

or variants of those with functions depending on ρ.

Denote

$$\|f\|_p^* = \left(\frac{1}{\mathrm{Vol}(M)} \int_M |f|^p \, \mathrm{dvol} \right)^{\frac{1}{p}}.$$

The next small lemma shall illustrate how smallness of $\|\rho_-\|_p^*$ implies smallness of the Kato constant. It can be found, e.g., in [RS17], and is a special case of [Gün14].

Lemma 2.1. *Let $n \geq 3$, $p > n/2$, $D > 0$. There exist $\varepsilon = \varepsilon(n,p,D) > 0$ and $C = C(n,p,D) > 0$ such that for all manifolds $M = (M^n, g)$ satisfying $\|\rho_-\|_p^* \leq \varepsilon$, we have*

$$\kappa_{D^2}(\rho_-) \leq C\|\rho_-\|_p^*.$$

Proof. The assumption $\|\rho_-\|_p^* \leq \varepsilon$ implies a heat kernel estimate [Gal88, PS98] of the form

$$p_t(x,y) \leq \frac{C(n,p,D)}{\mathrm{Vol}(M)} t^{-n/2}, \quad x,y \in M, t \in (0, D^2].$$

Moreover, we have by Dunford-Pettis' theorem

$$\|P_t\|_{1,\infty} = \sup_{x,y \in M} p_t(x,y),$$

provided the right-hand side is bounded. The Riesz-Thorin interpolation theorem implies

$$\|P_t\|_{p,\infty} = \sup_{x,y \in M} p_t(x,y)^{1/p}.$$

Thus, considering ρ_- as an operator from $L^\infty(M)$ to $L^p(M)$ for $p > n/2$ and factorizing through L^p via the Grothendieck theorem, we get

$$\|P_t \rho_-\|_\infty = \|P_t \rho_-\|_{\infty,\infty} \leq \|P_t\|_{p,\infty} \|\rho_-\|_{\infty,p} = \|P_t\|_{p,\infty} \|\rho_-\|_p.$$

Thus, for $p > n/2$,

$$\kappa_{D^2}(\rho_-) \leq \|\rho_-\|_p \int_0^{D^2} \|P_t\|_{p,\infty} \mathrm{d}t \leq C \|\rho_-\|_p^* \int_0^{D^2} t^{-n/2p} \mathrm{d}t.$$

$\square$

We recall the following estimates for the heat kernel, the volume doubling condition, and the smallest positive eigenvalue of the Laplacian proved in [Car19, Ros19b]: Let $\nu := e^2 n$ and T be the largest possible time such that $\kappa_T(V) \leq \frac{1}{16n}$.

Theorem 2.2 ([Car19, Ros19b])**.** *Let $T > 0$, $n \geq 2$. For all n-dimensional compact M satisfying*

$$\kappa_T(\rho_-) \leq \frac{1}{16n},$$

we have for any positive solution u satisfying

$$\partial_t u = \Delta u$$

and all $t \in (0, T]$

$$e^{-2} \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{\nu}{2t}. \tag{2.4}$$

In particular, we have the Harnack inequality for such u: for all $x, y \in M$, $0 < s \leq t \leq T$,

$$u(s,x) \leq \left(\frac{t}{s}\right)^{\frac{\nu}{2}} e^{2 \frac{d(x,y)^2}{t-s}} u(t,y). \tag{2.5}$$

Note that the statement above indeed holds without any restrictions on the diameter. In the articles cited in theorem, the authors also obtained heat kernel estimates. Carron explored in [Car19] the implications of the Harnack inequality further, and obtained also the volume doubling property as well as an eigenvalue and a heat kernel lower bound. By choosing $T = D^2$, the result below follows immediately from this article.

Theorem 2.3 ([Car19]). *Let $D > 0$, $n \geq 2$ For all M satisfying $\mathrm{diam}(M) \leq D$ and*

$$\kappa_{D^2}(\rho_-) \leq \frac{1}{16n},$$

then

(i) *there is a doubling constant $c_d(n, D) > 0$ such that for all $x \in M$, $0 < r \leq R \leq D$,*

$$\frac{V(x, R)}{V(x, r)} \leq c_d \left(\frac{R}{r}\right)^\nu. \tag{2.6}$$

(ii) *there is a constant $c(n) > 0$ such that for all $t \in (0, D^2/2]$, $x, y \in M$,*

$$p_t(x, y) \leq \frac{c(n) D^\nu}{\mathrm{Vol}(M)} t^{-\nu/2}. \tag{2.7}$$

(iii) *there is a constant $c = c(n) > 0$ such that*

$$\lambda_1(M) \geq \Lambda := \frac{c}{D^2}. \tag{2.8}$$

In the following, we let

$$\bar{w} = \frac{1}{\mathrm{Vol}(M)} \int w \, \mathrm{dvol}$$

.

Note that (2.8) implies a Poincaré inequality:

$$\Lambda \leq \lambda_1(M) = \inf_{f \perp 1} \frac{\int |\nabla f|^2 \mathrm{dvol}}{\int f^2 \mathrm{dvol}} = \inf_{w \neq constant} \frac{\int |\nabla w|^2 \mathrm{dvol}}{\int |w - \bar{w}|^2 \mathrm{dvol}}, \tag{2.9}$$

such that for all $w \in W^{1,2}(M)$, we have

$$\|w - \bar{w}\|_2^2 \leq \Lambda^{-1} \|\nabla w\|_2^2. \tag{2.10}$$

2.2 Proofs of Theorems 1.1 and 1.3

The proof of Theorem 1.1 is adapted from [ROSWZ20]. In [ROSWZ20] the key estimate is to control the auxilary function J which is the solution of the following problem,

$$\Delta J - \tau \frac{|\nabla J|^2}{J} - 2J\rho_- = -\sigma J \tag{2.11}$$

for $\tau > 1$ and $\sigma \geq 0$. Using $J = w^{-\frac{1}{\tau-1}}$, this equation is equivalent to

$$\Delta w + Vw = \tilde{\sigma} w, \tag{2.12}$$

where $V := 2(\tau - 1)\rho_-$ and $\tilde{\sigma} := (\tau - 1)\sigma$. The proof of those estimates is based on bounds of the Sobolev constant and smallness of $\|\rho_-\|_p^*$ using a Moser-iteration. This does not carry over to our setting, and we use semigroup techniques instead.

We will study the lowest eigenfunction of the general Schrödinger operator $-\Delta - V$ for $V : M \to [0, \infty)$ being continuous and satisfing the Kato condition since this might be of independent interest. We show the necessary comparison estimate for the eigenfunction w and the function J depending on the Kato condition. Then we apply to $V = 2(\tau - 1)\rho_-$ and proving Proposition 2.5 and 2.9, replacing Propositions 2.2 and 2.3 in [ROSWZ20],

From the heat kernel upper bound (2.7), we infer the following mapping properties for perturbed heat semigroups.

Proposition 2.4. *Let $T, D > 0$, $n \geq 2$. For all M^n with $\mathrm{diam}(M) \leq D$ and $\kappa_{D^2}(\rho_-) < \frac{1}{16n}$, and all continuous functions $V \colon M \to [0, \infty)$ satisfying*

$$\kappa = \kappa_T(V) < 1,$$

we have for all $t \in (0, D^2/2]$,

$$\|\mathrm{e}^{-t(-\Delta - V)}\|_{2,\infty} \leq \left(\frac{1}{1-\kappa}\right)^{\frac{1}{2}\left(1+\frac{t}{T}\right)} \left(\left(\frac{2}{1-\kappa}\right)^{\frac{1+\kappa}{1-\kappa}\left(1+\frac{t}{T}\right)+\nu/2} \frac{c_n D^{\nu/2}}{\mathrm{Vol}(M)} t^{-\nu/2}\right)^{\frac{1}{2}}. \qquad (2.13)$$

Proof. By (2.7) for all $t \in (0, D^2/2]$

$$\|\mathrm{e}^{t\Delta}\|_{1,\infty} = \sup_{x,y\in M} p_t(x,y) \leq \frac{c(n)D^{\nu/2}}{\mathrm{Vol}(M)} t^{-\nu/2}. \qquad (2.14)$$

Moreover, we have for all $t > 0$ as in [RS17] and [Voi77]

$$\|\mathrm{e}^{t(\Delta+V)}\|_{1,1} \leq \left(\frac{1}{1-\kappa}\right)^{1+\frac{t}{T}}. \qquad (2.15)$$

Following the proof of [RS17, Corollary 5.4], we get for all $t \in (0, D^2/2]$,

$$\|\mathrm{e}^{t(\Delta+V)}\|_{2,\infty} \leq \left(\frac{1}{1-\kappa}\right)^{\frac{1}{2}\left(1+\frac{t}{T}\right)} \left(\left(\frac{2}{1-\kappa}\right)^{\frac{1+\kappa}{1-\kappa}\left(1+\frac{t}{T}\right)+\nu/2} \frac{c_n D^{\nu/2}}{\mathrm{Vol}(M)} t^{-\nu/2}\right)^{\frac{1}{2}}. \qquad (2.16)$$

$\square$

Let $-\tilde{\sigma}$ be the the smallest eigenvalue of $-\Delta - V$:

$$\Delta w + V w = \tilde{\sigma} w. \qquad (2.17)$$

We give an upper and lower bound on $\tilde{\sigma}$ in terms of $c_\alpha(V) < 1$ (see (2.2) for the definition of $c_\alpha(V)$).

Proposition 2.5. *Assume that $V \geq 0$ and $c_\alpha(V) < 1$ for some $\alpha > 0$. then we have*

$$0 \leq \tilde{\sigma} \leq \alpha c_\alpha.$$

Proof. Since $\tilde{\sigma}$ is the biggest eigenvalue of $\Delta + V$, we can choose $w \geq 0$ such that $\|w\|_2^* = 1$. Integrating (2.17) over M, we get

$$\int V w \, \mathrm{dvol} = \tilde{\sigma} \int w \, \mathrm{dvol}.$$

Since V and w are non-negative, $\tilde{\sigma} \geq 0$.

For the upper bound, recall general perturbation theory implies [RS20, SV96, Sim82]

$$\int V \varphi^2 \leq c_\alpha \int |\nabla \varphi|^2 + \alpha c_\alpha \int \varphi^2, \quad \varphi \in W^{1,2}(M). \qquad (2.18)$$

Now multiply (2.17) by w, integrate over M, and use (2.18) to obtain

$$\mathrm{Vol}(M)\tilde{\sigma} = \tilde{\sigma} \int w^2 \mathrm{dvol} = -\int |\nabla w|^2 \mathrm{dvol} + \int V w^2 \mathrm{dvol}$$

$$\leq -\int |\nabla w|^2 + c_\alpha \int |\nabla w|^2 \, \mathrm{dvol} + \alpha c_\alpha \int w^2 \, \mathrm{dvol}$$

$$\leq \alpha c_\alpha \, \mathrm{Vol}(M),$$

due to $c_\alpha(V) < 1$. Thus,

$$\tilde{\sigma} \leq \alpha c_\alpha.$$

$\square$

Now we estimate the first eigenfunction w.

Lemma 2.6. *Let w be a first eigenfunction such that $\|w\|_2^* = 1$ and assume $c_\alpha(V) < 1$. Then,*

$$\int |\nabla w|^2 \leq \frac{\alpha c_\alpha}{1 - c_\alpha} \operatorname{Vol}(M) \tag{2.19}$$

and

$$\|w - \bar{w}\|_2^2 \leq \Lambda^{-1} \frac{\alpha c_\alpha}{1 - c_\alpha} \operatorname{Vol}(M). \tag{2.20}$$

Proof. From (2.17) we infer

$$\int |\nabla w|^2 \mathrm{dvol} = - \int w \Delta w \mathrm{dvol} = \int w(V - \tilde{\sigma})w \mathrm{dvol} = \int V w^2 \mathrm{dvol} - \tilde{\sigma} \int w^2 \mathrm{dvol}$$

$$\leq c_\alpha \int |\nabla w|^2 \mathrm{dvol} + (\alpha c_\alpha - \tilde{\sigma}) \int w^2 \mathrm{dvol},$$

where we use (2.18) in the last inequality. Thus,

$$\int |\nabla w|^2 \leq \frac{\alpha c_\alpha - \tilde{\sigma}}{1 - c_\alpha} \operatorname{Vol}(M). \tag{2.21}$$

Inequality (2.10) implies

$$\|w - \bar{w}\|_2^2 \leq \Lambda^{-1} \|\nabla w\|_2^2 \leq \Lambda^{-1} \frac{\alpha c_\alpha - \tilde{\sigma}}{1 - c_\alpha} \operatorname{Vol}(M). \tag{2.22}$$

$\square$

Lemma 2.7. *Suppose $c_\alpha(V) < 1$, $\kappa_T(V) < \frac{1}{2}$ and $\kappa_{D^2}(\rho_-) \leq \frac{1}{16n}$. We have*

$$\|w - \bar{w}\|_\infty \leq 2^{\frac{3D^2}{2T}} c_{n,\nu} D^{-\nu/2} \frac{1 - \kappa_T(V)}{1 - 2\kappa_T(V)} \Lambda^{-1/2} \sqrt{\frac{\alpha c_\alpha(V)}{1 - c_\alpha(V)}}. \tag{2.23}$$

Proof. Define $h := w - \bar{w}$ and consider the heat equation

$$\begin{cases} \Delta f + V f = \partial_t f & \text{on } M \times (0, T], \\ f = |h| & \text{on } M \times \{0\}. \end{cases} \tag{2.24}$$

The general solution of this problem is given by $f = \mathrm{e}^{-t(-\Delta - V)}|h|$. To derive an upper bound on $\|h\|_\infty$, we transform (2.24) into an integral equation via Duhamel's formula:

$$f(x, t) = |h|(x) + \int_0^t \int_M p_{t-s}(x, y) V(y) f(x, s) \mathrm{dvol}(y) \mathrm{d}s, \tag{2.25}$$

where p_t denotes the heat kernel. Define $g(t) := \max_{x \in M, s \in [0,t]} f(x, s)$. Note that g is non-decreasing in t. Thus, for $t \in (0, T]$,

$$g(t) \leq \|h\|_\infty + \sup_{x \in M, s \in [0,t]} \int_0^s \int_M p_{s-\tau}(x, y) V(y) f(x, \tau) \mathrm{dvol}(y) \mathrm{d}\tau$$

$$\leq \|h\|_\infty + \sup_{s \in [0,t]} g(s) \sup_{x \in M, s \in [0,t]} \int_0^s \int_M p_{s-\tau}(x, y) V(y) \mathrm{dvol}(y) \mathrm{d}\tau$$

$$\leq \|h\|_\infty + g(t) \sup_{x \in M, s \in [0,t]} \int_0^s \int_M p_\tau(x, y) V(y) \mathrm{dvol}(y) \mathrm{d}\tau$$

$$\leq \|h\|_\infty + g(t) \kappa_T(V).$$

Hence,
$$g(t) \leq \frac{\|h\|_\infty}{1 - \kappa_T(V)}.$$

Thus, we infer from (2.25)

$$
\begin{aligned}
|h|(x) &\leq |f(x,t)| + g(t)\|\int_0^t \int_M p_{t-s}(x,y)V(y)\mathrm{dvol}(y)\mathrm{d}s\|_\infty \\
&\leq |f(x,t)| + \frac{\|h\|_\infty}{1 - \kappa_T(V)}\kappa_T(V) \\
&= |\mathrm{e}^{-t(-\Delta-V)}|h|(x)| + \frac{\|h\|_\infty}{1 - \kappa_T(V)}\kappa_T(V) \\
&\leq \|\mathrm{e}^{-t(-\Delta-V)}|h|\|_\infty + \frac{\|h\|_\infty}{1 - \kappa_T(V)}\kappa_T(V) \\
&\leq \|\mathrm{e}^{-t(-\Delta-V)}\|_{2,\infty}\|h\|_2 + \frac{\|h\|_\infty}{1 - \kappa_T(V)}\kappa_T(V).
\end{aligned}
$$

Hence,

$$\|h\|_\infty \leq \frac{1 - \kappa_T(V)}{1 - 2\kappa_T(V)}\|\mathrm{e}^{-t(-\Delta-V)}\|_{2,\infty}\|h\|_2, \tag{2.26}$$

which is finite if $\kappa_T(V) < \frac{1}{2}$. From Proposition 2.4, we get with $t = D^2/2$

$$\|\mathrm{e}^{t(\Delta+V)}\|_{2,\infty} \leq 2^{\frac{3D^2}{2T}} c_{n,\nu} \frac{D^{-\nu/2}}{\mathrm{Vol}(M)^{\frac{1}{2}}}, \tag{2.27}$$

where $c_{n,\nu} = 8^{\frac{3}{2}+\frac{\nu}{4}} c(n)^{\frac{1}{2}}$. Thus,

$$
\begin{aligned}
\|h\|_\infty &\leq 2^{\frac{3D^2}{2T}} c_{n,\nu} \frac{D^{-\nu/2}}{\mathrm{Vol}(M)^{\frac{1}{2}}} \frac{1 - \kappa_T(V)}{1 - 2\kappa_T(V)}\|h\|_2 \\
&\leq 2^{\frac{3D^2}{2T}} c_{n,\nu} \frac{D^{-\nu/2}}{\mathrm{Vol}(M)^{\frac{1}{2}}} \frac{1 - \kappa_T(V)}{1 - 2\kappa_T(V)} \Lambda^{-1/2}\sqrt{\frac{\alpha c_\alpha(V)}{1 - c_\alpha(V)} \mathrm{Vol}(M)}.
\end{aligned}
$$

$\square$

Lemma 2.8. *In the situation above, if*

$$2^{\frac{3D^2}{2T}} c_{n,\nu} D^{-\nu/2} \frac{1 - \kappa_T(V)}{1 - 2\kappa_T(V)} \Lambda^{-1/2}\sqrt{\frac{\alpha c_\alpha(V)}{1 - c_\alpha(V)}} < \frac{1}{4},$$

we have $\bar{w} > \frac{1}{2}$.

Proof. The lemma above tells us that
$$w \leq \bar{w} + K,$$

where we set
$$K := 2^{\frac{3D^2}{2T}} c_{n,\nu} D^{-\nu/2} \frac{1 - \kappa_T(V)}{1 - 2\kappa_T(V)} \Lambda^{-1/2}\sqrt{\frac{2\alpha c_\alpha}{1 - c_\alpha}}.$$

Since $\|w\|_2^* = 1$ and $\bar{w} \leq 1$, we have

$$\mathrm{Vol}(M) = \int w^2 \mathrm{dvol} \leq \int (\bar{w} + K)^2 \mathrm{dvol} \leq \mathrm{Vol}(M)\left(\bar{w}^2 + 2K + K^2\right).$$

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Hence, if $K < \frac{1}{4}$, we get

$$\frac{1}{4} < 1 - 2K - K^2 \le \bar{w}^2.$$

$\square$

We are now in the position to prove estimates on the auxiliary function J defined in (2.11), which is the key to derive Theorem 1.1 from the proof of [ROSWZ20, Theorem 1.1].

Proposition 2.9. *Suppose $D > 0$, $n \ge 2$. For any $\delta \in [0,1)$ there exists a $\tau_0 > 1$ such that for all $\tau \ge \tau_0$ and all manifolds satisfying $\mathrm{diam}(M) \le D$ and*

$$\kappa_{D^2}(\rho_-) < \frac{1}{2}(\tau - 1)^{-3}, \tag{2.28}$$

we have

$$|J - 1| \le \delta.$$

Proof. The choice $\tau \ge \sqrt{8n} + 1$ implies $k_{D^2}(\rho_-) \le \frac{1}{16n}$. Since $V = 2(\tau - 1)\rho_-$ the condition (2.28) implies $\kappa_{D^2}(V) < \frac{1}{2}$. Note that $(\tau - 1)^2 - 2 \ge 7$, such that (2.3) gives for all $\alpha \ge D^{-2}\ln\left(\frac{1}{2}(\tau-1)^2\right)$

$$c_\alpha(V) \le 2(\tau - 1)\kappa_{D^2}(\rho_-)\left(1 - \mathrm{e}^{-\alpha D^2}\right)^{-1} \le (\tau - 1)^{-2}\left(1 - \mathrm{e}^{-\alpha D^2}\right)^{-1} < \frac{1}{2}.$$

Hence,

$$\|w - \bar{w}\|_\infty \le C_{D,\Lambda}\sqrt{\alpha c_\alpha(V)} \tag{2.29}$$

with $C_{D,\Lambda} := 4c_{n,\nu}\frac{\sqrt{8n}}{\sqrt{8n-2}}D^{-\nu/2}\Lambda^{-1/2}$. If we choose $\alpha := D^{-2}(\tau - 1)$, then $1 - \mathrm{e}^{-\alpha D^2} \ge \frac{1}{2}$ and

$$\|w - \bar{w}\|_\infty \le \sqrt{2}C_{D,\Lambda}D^{-1}(\tau - 1)^{-1/2}. \tag{2.30}$$

To get $\bar{w} > \frac{1}{2}$, choose τ such that

$$\sqrt{2}C_{D,\Lambda}D^{-1}(\tau - 1)^{-1/2} < \frac{1}{4}. \tag{2.31}$$

Thus, it suffices to take

$$\tau \ge \tau_1 := \max\left(\sqrt{8n} + 1, 2\left(1 + 32\frac{C_{D,\Lambda}^2}{D^2}\right)\right)$$

such that the above estimates are valid.

With those choices, we can finish the proof of the proposition as in [ROSWZ20]. Consider the function $w_2 := w\bar{w}^{-1}$. Then, by (2.23) above,

$$1 - \tilde{\delta} \le w_2 \le 1 + \tilde{\delta},$$

where $\tilde{\delta} := 2\sqrt{2}C_{D,\Lambda}D^{-1}(\tau - 1)^{-1/2}$. We define $J := w_2^{-\frac{1}{\tau-1}}$. Use the first order Taylor expansion of $f(x) = x^{-\frac{1}{\tau-1}}$ near one on the domain $(1 - \tilde{\delta}, 1 + \tilde{\delta})$, i.e., $f(x) = 1 + R_1(x)$ with

$$|R_1| \le |f'(x^*)(x - 1)| \le \frac{2\tilde{\delta}}{(\tau - 1)(1 - \tilde{\delta})^{\frac{\tau}{\tau-1}}},$$

where $x^* \in (1 - \tilde{\delta}, 1 + \tilde{\delta})$. From above, we know that we have to choose $\tau \ge \tau_1$. Moreover, τ must be so large such that

$$\frac{2\tilde{\delta}}{(\tau - 1)(1 - \tilde{\delta})^{\frac{\tau}{\tau-1}}} \le \delta$$

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is satisfied. Hence, we can choose

$$\tau_2 := 1 + \max\left(8\frac{C_{D,\Lambda}^2}{D^2}, \left(\frac{162 C_{D,\Lambda}^2}{D^2 \delta^2}\right)^{\frac{1}{3}}\right)$$

and $\tau_0 := \max\{\tau_1, \tau_2\}$. □

Now, we turn to the proof of Theorem 1.3. The statement follows from a simple test-function argument using the volume doubling condition and Courant's principle on the first eigenvalue.

Lemma 2.10. *Assume*

$$\kappa_{\operatorname{diam} M^2}(\rho_-) \leq \frac{1}{16n}.$$

Then there is a $D = D(n, \nu) > 0$ such that for any $R < \operatorname{diam} M$, we have

$$\lambda_1(B(x, R)) \leq \frac{D}{R^2}.$$

Proof. Fix $x_0 \in M$ and $R < \operatorname{diam} M$. Define $f(x) := \varphi(d(x, x_0))$ with

$$\varphi(r) = \begin{cases} 1 & : r \in [0, R/2] \\ -\frac{2}{R}r + 2 & r \in (R/2, R]. \end{cases}$$

Then $f \in W_0^1(M)$ with $\nabla f(x) = \varphi'(d(x_0, x)) \nabla d(x_0, x)$ a.e. and $|\nabla f|^2 = \frac{4}{R^2} \mathbf{1}_{B(x_0, R) \setminus B(x_0, R/2)}$. Thus,

$$\|\nabla f\|_{B(x_0, R)}^2 = \frac{4}{R^2} V(x_0, R) \quad \text{and} \quad \|f\|_{2, B(x_0, R)}^2 \geq V(x_0, R/2).$$

Hence, using f as test function for the Rayleigh quotient of $\lambda_1(B(x_0, R))$ and the volume doubling property, see Theorem 2.3(i),

$$\lambda_1(B(x_0, R)) \leq \frac{\|\nabla f\|_{2, B(x_0, R)}^2}{\|f\|_{2, B(x_0, R)}^2} \leq \frac{4}{R^2} \frac{V(x_0, R)}{V(x_0, R/2)} \leq \frac{4 c_d 2^\nu}{R^2}.$$

□

Proof of Theorem 1.3. Lemma 2.10 tells us that for all balls of radius $R < \operatorname{diam} M$, we have

$$\lambda_1(B(x, R)) \leq \frac{D}{R^2}.$$

Take $R = \operatorname{diam} M/2$ and note that there are points $x_0, x_1 \in M$ such that $B(x_0, R) \cap B(x_1, R) = \emptyset$. By Courant's principle,

$$\lambda_1(M) \leq \max(\lambda_1(B(x_0, R)), \lambda_1(B(x_1, R))) \leq \frac{D}{R^2}.$$

□

3 Eigenvalue bounds for compact manifolds with boundary

3.1 Gradient estimates and Harnack inequality for the Neumann heat kernel

We emphasized in the introduction that to obtain a lower bound on η_1, we will prove a Li-Yau gradient estimate for positive solutions of the Neumann heat equation (1.4) under the Neumann Kato condition and suitable regularity of the boundary. Then, we adapt the technique from [CL97] to our setting, giving a much shorter proof than in the latter article. The proof of the Li-Yau inequality is an adaption from [RO19]. In this article, the author proved a Li-Yau gradient estimate for positive solutions of the Neumann heat equation in relatively compact domains $M \subset N$ in a globally doubling Riemannian manifold N assuming the interior R-rolling ball condition in ∂M, second fundamental form bounded below, and uniform smallness of $\|\rho_-\|_p^*$ in M. Those conditions are made to ensure an a priori upper bound on the Neumann heat kernel, which is used to prove the Li-Yau gradient estimate. In contrast, using the Kato condition for the Neumann heat semigroup, we neither need the existence of an ambient manifold N nor the doubling condition to derive a Li-Yau gradient estimate. In partiular, we do not need an a priori Neumann heat kernel bound to derive the Li-Yau inequality, and the heat kernel bound follows a posteriori.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $H, T, R > 0$. Suppose M is a compact Riemannian manifold with boundary of dimension n, such that the second fundamental form of ∂M is bounded below by $-H$, and that satisfies the interior R-rolling ball condition for R small enough (see Remark 1.8). Assume*

$$\mu_T(\rho_-) < 7 \left[16 \left((3 + 2(1+H)^2)(4 + 8n^2(1+H)^2) - 1 \right) \right]^{-1}.$$

Then, we have for any positive solution of the Neumann heat equation on M:

$$\frac{1}{2(1+H)^2} J \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq C_1 + \frac{C_2}{Jt}, \quad t \in (0, T], \tag{3.1}$$

where

$$\mathrm{e}^{-16\mu_T(\rho_-)} \leq J \leq 1$$

and

$$C_1 = 4n^2(1+H)^2 \sqrt{\frac{4(1 + 2n^2(1+H)^2)}{4(1 + 2n^2(1+H)^2) - 1}} c_1,$$

$$c_1 = \frac{128n^2 H^2}{R^2} + \frac{H}{2R(1+H)(1 + 2n^2(1+H)^2)} + \frac{16H(1+H)}{R},$$

$$C_2 = \frac{1}{2n^2(1+H^2)} \left(1 - \frac{1}{4 + 8n^2(1+H)^2} \right).$$

To compare this statement with the result in [LY86], we give the following corollary where the boundary is assumend to be convex.

Corollary 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$, $T, R > 0$. Suppose M is a compact Riemannian manifold with boundary of dimension n, such that the second fundamental form of ∂M is non-negative. Assume*

$$\mu_T(\rho_-) < 7 \left[16(19 + 40n^2) \right]^{-1}.$$

Then, we have for any positive solution of the Neumann heat equation on M:

$$\frac{1}{2} J \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq C_1 + \frac{C_2}{Jt}, \quad t \in (0, T], \tag{3.2}$$

where

$$\mathrm{e}^{-16\kappa_T(\rho_-)} \le J \le 1$$

and

$$C_1 = 4n^2\sqrt{\frac{4(1+2n^2)}{4(1+2n^2)-1}}, \quad C_2 = \frac{1}{2n^2}\left(1 - \frac{1}{4+8n^2}\right)$$

The main ingredient of the above theorem are bounds of the solution of the following problem. In [RO19], it was solved using that global volume doubling of an ambient space N, an upper bound for the heat kernel on N, and the induced doubling condition in $M \subset N$ imply an a priori Neumann heat kernel bound. We do not need anything of this and use the Kato condition instead.

Lemma 3.3. *Assume $c > 1$ and*

$$\mu_T(\rho_-) \le \frac{7}{16(c-1)}.$$

There exists a unique solution for

$$
\begin{cases}
\Delta J - \partial_t J - c\frac{|\nabla J|^2}{J} - 2J\rho_- = 0 & \text{in } M \times (0,T], \\
\partial_\nu J = 0 & \text{on } \partial M \times (0,T], \\
J = 1 & \text{on } M \times \{0\},
\end{cases}
\tag{3.3}
$$

satisfying

$$\mathrm{e}^{-16\mu_T(\rho)} \le J \le 1.$$

Proof. Using the transformation $w = J^{-(c-1)}$, the above problem transforms into

$$
\begin{cases}
\partial_t w = \Delta w + Vw, & \text{in } M \times (0,T], \\
\partial_\nu w = 0 & \text{on } \partial M \times (0,T] \\
w = 1 & \text{on } M \times \{0\},
\end{cases}
\tag{3.4}
$$

where $V = 2(c-1)\rho_-$. Due to our assumption $\mu_T(V) \le 7/8$. By stochastic completeness of M, we have $\|P_t\|_{\infty,\infty} = 1$. Furthermore, $\mathcal{C}^1(M)$ is dense in $H^1(M) \cap \mathrm{dom}(V)$. [Voi77, Theorem 1.1] implies that $-\Delta - V$ generates a strongly continuous semigroup in $L^1(M)$, and we have $w = \mathrm{e}^{-t(-\Delta-V)}\mathbf{1}$. Trotter-Kato gives $w \ge 1$, implying $J \le 1$. For the upper bound on w, we use [Voi77, Theorem 1.1] again:

$$\|w\|_\infty \le \|\mathrm{e}^{-t(-\Delta-V)}\|_{\infty,\infty} \le \left(\frac{1}{1-\mu_T(V)}\right)^{1+\frac{t}{T}} \le \mathrm{e}^{16(c-1)\mu_T(\rho_-)}, \quad t \in (0,T],$$

where we used the upper bound on $\mu_T(V)$. This implies the lower bound on J. $\qquad\square$

Proof of Theorem 3.1. To obtain our result, we need to follow the proof of [RO19, Theorem 1.1]. We repeat the necessary steps and explain where the Kato condition enters. Denote by $\psi\colon [0,\infty) \to [0,\infty)$ a C^2-function such that

$$
\begin{cases}
\psi(r) \le H: & r \in [0,1/2) \\
\psi(r) = H: r \in [1,\infty)
\end{cases},
$$

$\psi(0) = 0$, $0 \le \psi' \le 2H$, $\psi'(0) = H$, and $\psi'' \ge -H$. Let $r\colon M \to [0,\infty)$ the distance function to the boundary, $\psi(x) := \psi(r(x)/R)$, $\varphi := (1+\psi)^2$, and $\tilde\varphi := \alpha\varphi$ for $\alpha > 0$. Then, we have

$$\alpha \le \tilde\varphi \le \alpha(1+H)^2, |\nabla\tilde\varphi| \le \frac{4}{R}\alpha H(1+H), \Delta\tilde\varphi \ge -2\alpha(1+H)\left(\frac{H}{R^2} + \frac{2}{R}(n-1)H(3H+1)\right). \tag{3.5}$$

Let u be a positive solution of the Neumann heat equation and set $f := \ln u$, $\beta, \varepsilon > 0$, J the solution of (3.3), $c = (3 + \alpha^{-1})\beta^{-1}$ (α, β are chosen later so that $c > 1$), and

$$G(x,t) := t\left(\tilde{\varphi}J(|\nabla f|^2 + \varepsilon) - \partial_t f\right).$$

Let (p, t_0) is a maximum of G in $M \times (0, T]$, for $T > 0$. Then w.l.o.g. $t_0 > 0$. Assuming $p \in \partial M$ leads to a contradiction due to the interior R-rolling ball condition and second fundamental form bounded below, cf. [Wan97]. Thus, $p \in M \setminus \partial M$ and (p, t_0) is a local maximum. Hence, $\nabla G = 0$, $\partial_t G \geq 0$, $\Delta G \leq 0$, and $\Delta G - \partial_t G \leq 0$ in (p, t_0). W.l.o.g., we can assume $G(p, t_0) > 0$. Putting $Q = G/t$ and $g = |\nabla f|^2 + \varepsilon$, a lengthy calculation using Cauchy-Schwarz and the binomial formula several times gives

$$(\Delta - \partial_t)Q + 2\nabla f\nabla Q \geq Jg\Delta\tilde{\varphi} + 2J\nabla\tilde{\varphi}\nabla(|\nabla f|^2) + 2gJ\nabla\tilde{\varphi}\nabla f - \beta|\nabla\tilde{\varphi}|^2 gJ \tag{3.6}$$

$$+ 2(1-\beta)\tilde{\varphi}J|\partial_i\partial_j f|^2 - \beta Jg\tilde{\varphi}|\nabla f|^2 + \tilde{\varphi}g\left(\Delta J - \partial_t J - \left(3 + \frac{1}{\alpha}\right)\frac{1}{\beta}\frac{|\nabla J|^2}{J} - 2J\rho_-\right). \tag{3.7}$$

Now we use Lemma 3.3 with our choice for c, which is the step that is solved in [RO19] by the a priori bound of the Neumann heat kernel. This gives

$$(\Delta - \partial_t)Q + 2\nabla f\nabla Q \geq Jg\Delta\tilde{\varphi} + 2J\nabla\tilde{\varphi}\nabla(|\nabla f|^2) + 2gJ\nabla\tilde{\varphi}\nabla f - \beta|\nabla\tilde{\varphi}|^2 gJ \tag{3.8}$$

$$+ 2(1-\beta)\tilde{\varphi}J|\partial_i\partial_j f|^2 - \beta Jg\tilde{\varphi}|\nabla f|^2. \tag{3.9}$$

The relation of G and Q, the fact that (p, t_0) is a local maximum, the binomial formula, and

$$\sum_{i,j=1}^{n} |\partial_i\partial_j f|^2 \geq \frac{1}{n^2}\left(|\nabla f|^2 - \partial_t f\right)^2$$

imply

$$Q \geq t_0 J\left(g\Delta\tilde{\varphi} + 2g\nabla\tilde{\varphi}\nabla f - \beta|\nabla\tilde{\varphi}|^2 g\right. \tag{3.10}$$

$$\left. + \frac{2(1-\beta)\tilde{\varphi} - \alpha}{n^2}\left(|\nabla f|^2 - \partial_t f\right)^2 - \frac{4}{\alpha}|\nabla\tilde{\varphi}|^2|\nabla f|^2 - 2\beta g\tilde{\varphi}|\nabla f|^2\right). \tag{3.11}$$

This implies

$$0 \geq t_0 J\left(|\nabla f|^2(\Delta\tilde{\varphi} - (\beta + \frac{4}{\alpha})|\nabla\tilde{\varphi}|^2 + O(\varepsilon)) - 2|\nabla\tilde{\varphi}||\nabla f|^3 + \frac{2(1-\beta)\tilde{\varphi} - \alpha}{n^2}Q^2\right. \tag{3.12}$$

$$\left. + |\nabla f|^4\left((1 - \tilde{\varphi}J)^2\frac{2(1-\beta)\tilde{\varphi} - \alpha}{n^2} - \beta\tilde{\varphi}\right)\right) - (1 + O(\varepsilon))Q + O(\varepsilon) \tag{3.13}$$

Choosing

$$\alpha = \frac{1}{2(1+H)^2}, \quad \beta = (4 + 8n^2(1+H)^2)^{-1}, \quad A = (16n^2(1+H)^2)^{-1},$$

$$C = \frac{2H}{R(1+H)}\left((4 + 8n^2(1+H)^2)^{-1} + 8(1+H)^2\right), \quad B = \frac{4H}{R(1+H)},$$

$$E = \frac{1}{2n^2(1+H)^2}\left(1 - \frac{1}{4 + 8n^2(1+H)^2}\right),$$

we get

$$(1 - \tilde{\varphi}J)^2\frac{2(1-\beta)\tilde{\varphi} - \alpha}{n^2} - \beta\tilde{\varphi} \geq A, \quad \Delta\tilde{\varphi} - (\beta + \frac{4}{\alpha} \geq -C + O(\varepsilon) =: -C_\varepsilon,$$

$$-2|\nabla\tilde{\varphi}| \geq -B, \quad \frac{2(1-\beta)\tilde{\varphi} - \alpha}{n^2} \geq E.$$

Hence,

$$0 \geq t_0 J \left(-C_\varepsilon |\nabla f|^2 - B|\nabla f|^3 + A|\nabla f|^4 + EQ^2 \right) - (1 + O(\varepsilon))Q + O(\varepsilon). \tag{3.14}$$

Denoting $y = |\nabla f|^2$ and

$$\tilde{D} = 8n^2(1+H)^2 \left(\frac{128n^2H^2}{R^2} + \frac{2H}{4R(1+H)(1+2n^2(1+H)^2} + \frac{16H(1+H)}{R} \right),$$

we have

$$Ay^2 - By^{3/2} - C_\varepsilon y \geq -\tilde{D} + O(\varepsilon).$$

Thus,

$$0 \geq EJG^2 - (1 + O(\varepsilon))G - \tilde{D}t_0^2 J + O(\varepsilon).$$

Solving for G and keeping in mind that G is positive in (p, t_0) leads to the claim.

$\square$

From Theorem 3.1, we infer by standard techniques the following.

Proposition 3.4. *Let $n \in \mathbb{N}$, $n \geq 2$, $H, T, R > 0$. Suppose M is a compact Riemannian manifold with boundary of dimension n, such that the second fundamental form of ∂M is bounded below by $-H$, and that satisfies the interior R-rolling ball condition for R small enough as in Remark 1.8. Assume*

$$\mu_T(\rho_-) < 7 \left[16 \left((3 + 2(1+H)^2)(4 + 8n^2(1+H)^2) - 1 \right) \right]^{-1}.$$

Then, we have for any positive solution u of the Neumann heat equation on M and $0 < t_1 \leq t_2 \leq T$, $x, y \in M$,

$$u(t_1, x) \leq \left(\frac{t_2}{t_1} \right)^{C_1 T + C_2} u(t_2, x) \tag{3.15}$$

and

$$u(x, t_1) \leq u(y, t_2) \left(\frac{t_2}{t_1} \right)^{C_2 e^{16\mu_T(\rho_-)}} \exp \left(C_1(t_2 - t_1) + \frac{e^{16\mu_T(\rho_-)}(1+H)^2 d(x,y)^2}{2(t_2 - t_1)} \right), \tag{3.16}$$

with C_1, C_2 given in Theorem 3.1.

From this, it is easy to get the following Neumann heat kernel bound.

Theorem 3.5. *Let $n \in \mathbb{N}$, $T, R > 0$. Suppose M is a compact Riemannian manifold with boundary of dimension n, such that the second fundamental form of ∂M is bounded below by $-H$, and that satisfies the interior R-rolling ball condition for R small enough as in Remark 1.8. Assume*

$$\mu_T(\rho_-) < 7 \left[16 \left((3 + 2(1+H)^2)(4 + 8n^2(1+H)^2) - 1 \right) \right]^{-1}.$$

Then, the Neumann heat kernel satisfies for all $x \in M$ and $t \in (0, T/2]$

$$h_t(x, x) \leq \frac{C_{T,\mu,H,R}}{\mathrm{Vol}(B(x, \sqrt{t}))}, \tag{3.17}$$

where

$$C_{T,\mu,H,R} = 2^{16C_2\mu_T(\rho_-)} \exp \left(C_1 T + \frac{e^{16\mu_T(\rho_-)}}{2}(1+H)^2 \right)$$

and C_1 and C_2 are given in Theorem 3.1.

Proof. According to Proposition 3.4 we have for all $t \in (0, T/2]$ all $x \in M$ and $y \in M$ with $d(x, y) \leq \sqrt{t}$

$$h_t(x, x) \leq h_{2t}(x, y) C_{T,\mu,H,R}.$$

Hence, we get

$$V(x, \sqrt{t}) h_t(x, x) = \int_{B(x,\sqrt{t})} h_t(x, x) \mathrm{dvol} y \leq C_{T,\mu,H,R} \int_{B(x,\sqrt{t})} h_{2t}(x, y) \mathrm{dvol} y \leq C_{T,\mu,H,R},$$

where in the last inequality we used non-negativity of h_t and stochastic completeness. $\qquad\square$

3.2   Proof of Theorem 1.6

Proof of Theorem 1.6. The first steps of the proof are the same as in the proof of [CL97, Theorem 1]. The proof differs after (3.24).

The function

$$F(x, t) := \int_\Omega h(x, y, t) f(y) dy \tag{3.18}$$

solves the heat equation on M with Neumann boundary conditions on ∂M and initial condition $F(x, 0) = f(x)$. Consider the function

$$g(x, t) := \int_\Omega h(x, y, t) \left(f(y) - F(x, t)\right)^2 dy. \tag{3.19}$$

By definition, we have

$$
\begin{aligned}
\int_M g(x, t) dx &= \int_M \int_M h(x, y, t) f^2(y) dy dx - \int_M F^2(x, t) dx \\
&= \int_M f^2(y) dy - \int_M F^2(x, t) dx \\
&= -\int_0^t \frac{\partial}{\partial s} \left(\int_M F^2(x, s) dx \right) ds \\
&= -2 \int_0^t \int_M F(x, s) \Delta F(x, s) dx ds \\
&= 2 \int_0^t \int_M |\nabla F|^2(x, s) dx ds.
\end{aligned}
\tag{3.20}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} \int_M |\nabla F|^2(x, t) dx &= 2 \int_M \nabla F \cdot \nabla F_t(x, t) dx \\
&= -2 \int_M \Delta F F_t(x, t) dx \\
&= -2 \int_M F_t^2(x, t) dx \leq 0,
\end{aligned}
$$

hence we conclude that for any $t > 0$

$$\int_M |\nabla F|^2(x, t) dx \leq \int_M |\nabla f|^2(x) dx. \tag{3.21}$$

Thus, from (3.20) we have the estimate

$$\int_M g(x, t) dx \leq 2t \int_M |\nabla f|^2(x) dx. \tag{3.22}$$

At this point, the proof differs from the one in [CL97]. We have

$$\int_M g(x,t)dx = \int_M \int_M h(x,y,t)\left(f(y)-F(x,t)\right)^2 dydx$$

$$\geq \int_M \inf_{y\in M} h(x,y,t) \int_M \left(f(y)-F(x,t)\right)^2 dydx$$

$$\geq \inf_{a\in\mathbb{R}} \int_M \left(f(y)-a\right)^2 dy \int_M \inf_{y\in M} h(x,y,t)dx. \tag{3.23}$$

Using the variational principle, we have from (3.22) and (3.23) that

$$\eta_1 \geq \frac{1}{2t} \int_M \inf_{y\in M} h(x,y,t)dx. \tag{3.24}$$

Note that, since M is compact and h is smooth, the infimum is attained at some point in M. Since M is stochastically complete, for a fixed $\tilde{t} > 0$, for all $x \in M$ there exists $y^* \in M$ such that $h(x,y^*,\tilde{t}) \geq \frac{1}{2\operatorname{Vol}(M)}$ (since otherwise $\int_M h(x,y,\tilde{t})dx \leq \frac{1}{2}$).

Fix $x \in M$. For $y,z \in M$, we can use (3.16) on $h(x,y,t)$ for $t_1 = \frac{t}{2}$ and $t_2 = t$ $t \leq T$, where $t > 0$ will be determined later, to obtain

$$h_{t/2}(x,y) \leq h_t(x,z)2^{C_2 e^{16\mu_T(\rho_-)}} \exp\left(C_1 t/2 + e^{16\mu_T(\rho_-)}(1+H)^2 D^2\right). \tag{3.25}$$

Let $y^* \in M$ be such that $h_{t/2}(x,y^*) \geq \frac{1}{2\operatorname{Vol}(M)}$. Then choosing $y = y^*$ and letting z be the point where h is minimum in (3.25), we have that

$$\inf_{y\in\Omega} h_t(x,y) = h_t(x,z)$$

$$\geq h_{t/2}(x,y^*)2^{-C_2 e^{16\mu_T(\rho_-)}} \exp\left(-C_1 t/2 - e^{16\mu_T(\rho_-)}(1+H)^2 D^2\right)$$

$$\geq \frac{1}{2\operatorname{Vol}(M)}2^{-C_2 e^{16\mu_T(\rho_-)}} \exp\left(-C_1 t/2 - e^{16\mu_T(\rho_-)}(1+H)^2 D^2\right).$$

Thus, since the choice of $x \in M$ was arbitrary, using (3.24), we get

$$\eta_1 \geq \frac{1}{4t}2^{-C_2 e^{16\mu_T(\rho_-)}} \exp\left(-C_1 t/2 - e^{16\mu_T(\rho_-)}(1+H)^2 D^2\right). \tag{3.26}$$

Choosing $t = \min(T, D^2/2)$ yields the claim. $\qquad\square$

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