# *On Riemannian Manifolds of Almost Nonnegative Curvature*

ZHONGMIN SHEN & GUOFANG WEI

**1. Introduction.** A closed riemannian manifold $M$ has almost nonnegative Ricci curvature if the diameter diam $M$ and the Ricci curvature $\mathrm{Ric}(M)$ satisfy $\mathrm{diam}(M)^2\mathrm{Ric}(M) \geq -\varepsilon(n)$ for a sufficiently small number $\varepsilon(n) > 0$ depending only on $n = \dim M$. It was proved in [CG1], [CG2] that any closed $n$-manifold $M$ with nonnegative Ricci curvature is up to finite cover diffeomorhpic to a direct product $V \times T^k$, where $V$ is simply connected manifold and $T^k$ is a $k$–torus. This theorem is not valid for manifolds of almost nonnegative Ricci curvature (see Example 1 below). T. Yamaguchi [Y1] proved that there is a constant $\varepsilon(n,D) > 0$, such that if an $n$–manifold $M$ has Ricci curvature $\mathrm{Ric}(M) \geq -\varepsilon(n,D)$, and sectional curvature $|K_M| \leq 1$, diameter $\mathrm{diam}(M) \leq D$, then $M$ fibers over a $b_1(M)$–torus; here $b_1(M)$ denotes the first Betti number of $M$. In [Y2] he proved the same result, up to finite cover, for almost nonnegatively curved manifolds $M$, i.e. $\mathrm{diam}(M)^2 K_M \geq -\varepsilon(n)$. With an additional assumption on the injectivity radius, we can prove the following:

**Theorem 1.** *Given $n$, $i_0 > 0$, there is a positive number $\varepsilon(n, i_0)$ depending only on $n$ and $i_0$ such that if the diameter, the injectivity radius and the curvature of a compact riemannian manifold $M$ satisfy $\mathrm{diam}(M) = 1$, $\mathrm{inj}(M) \geq i_0$ and $K_M \geq -\varepsilon(n, i_0)$, respectively, then $M$ is up to finite cover $C^1$–diffeomorphic to a direct product $V \times T^k$, where $V$ is a simply connected $(n-k)$–manifold of class $C^1$.*

***Example 1*** (Almost flat manifolds [G2]). A closed almost flat manifold, in general, is not a direct product of a simply connected manifold with a torus, e.g. almost flat $n$–manifolds with fundamental groups of polynomial growth with degree $\geq n+1$. Hence the above results are not true without assumption on the injectivity radius.

***Example 2*** (Almost flat manifolds [G2][BK]). Given $n > 0$, and $0 < i_0 \leq 1$, there exists a positive number $c(n)$ depending only on $n$, such that if the



Indiana University Mathematics Journal ©, Vol. 40, No. 2 (1991)

diameter, the injectivity radius and the curvature of a closed riemannain $n$–manifold $M$ satisfy the bounds

$$\operatorname{diam}(M) = 1, \quad \operatorname{inj}(M) \geq i_0, \quad |K_M| \leq c(n) i_0^2,$$

then a finite covering of $M$ is diffeomorphic to $T^n$.

*Proof.* Let $M$ be a closed $n$–manifold with diameter $\operatorname{diam}(M) = 1$. It follows from Corollary 1.5.1 in [BK] that there is a constant $\varepsilon(n)$ depending only on $n$ such that if $M$ satisfies

(1) $|K_M| \leq \varepsilon \leq \varepsilon(n)$,
(2) $\operatorname{inj}(M) > 2^{-n^3} \left(\frac{\varepsilon}{\varepsilon(n)}\right)^{1/2}$,

then $M$ is covered by a torus. □

On the other hand, the fundamental group of a closed manifold $M$ with nonnegative Ricci is almost abelian, hence has polynomial growth with degree $\leq n = \dim M$. It is generally believed that the fundamental groups of almost nonnegatively curved manifolds have polynomial growth. The sectional curvature case is conjectured by M. Gromov. The authors have been informed recently that K. Fukaya and T. Yamaguchi had an affirmative answer in this case. In Section 5, with additional assumptions on the volume or the 1–systole $\operatorname{sys}_1(M)$ of $M$, we will study the fundamental groups of manifolds $M$ with almost nonnegative Ricci curvature.

The proof of Theorem 1 goes roughly as follows. By passing to the limit for the manifolds satisfying the bounds in Theorem 1 with the lower bound of sectional curvature converging to zero, one obtains a metric space $M_\infty$ with nonnegative sectional curvature in a generalized sense. For the universal cover of such space a splitting theorem has been shown to hold in the sense of distance, cf. [GP], [Y2]. To show that $M_\infty$ is then, up to finite cover, diffeomorphic to the product of a simply-connected manifold and a torus, however, is more involved. One needs certain regularity to conclude that the isometry group is a Lie group (Section 2) and that such regularity persists after the splitting (Section 3). Anderson-Cheeger's convergence theorem provides the desired regularity to work with.

Anderson-Cheeger [AC] proved the following convergence theorem: given any $p > n$, $\alpha = 1 - \frac{n}{p}$, suppose a sequence of pointed complete $n$-manifold $(M_j, h_j, 0_j)$ satisfying the bounds

$$\operatorname{Ric}(M_j) \geq -(n-1)\lambda^2, \quad \operatorname{inj}(M_j) \geq i_0. \tag{1}$$

Then there is a subsequence of $(M_j, h_j, o_j)$, which have pointed $L^{1,p}$ bound, and converge, in the pointed $C^{\alpha'}$ topology for $\alpha' < \alpha$, to a pointed complete

$L^{1,p}$ riemannian $n$–manifolds $(M_\infty, g_\infty, o)$ with respect to the fixed $L^{2,p}$ atlas of harmonic coordinate charts for $M_\infty$. This is done by proving a $L^p$ version of Theorem 0.1 in [AC] and using the method in [C].

By definition, an $n$–dimensional manifold $M$ is of class $L^{2,p}$, for $p > n$, if it admits an atlas of coordinate charts $F_\nu : U_\nu \to B(r_\nu) \subset R^n$ for $M$, such that the overlaps $F_{\mu\nu} = F_\mu \circ F_\nu^{-1}$ are controlled in $L^{2,p}$, i.e. $\|F_{\mu\nu}\|_{L^{2,p}} < C_\nu$. A riemannian metric $g$ on $M$ is of class $L^{1,p}$, if in each coordinate chart $F_\nu = (x^a)$, the riemannian metric coefficients $g_{ab} := g\big((\partial/\partial x^a),\ (\partial/\partial x^b)\big)$ satisfy

$$Q_\nu^{-1}\delta_{ab} \le g_{ab} \le Q_\nu \delta_{ab}, \tag{2}$$

$$\|g_{ab}\|_{L^{1,p}(B(r_\nu))} \le Q_\nu, \tag{3}$$

for some constant $Q_\nu > 1$. In this case $(M, g)$ is called an $L^{1,p}$ riemannian $n$–manifold. Here $\|\cdot\|_{L^{k,p}(\Omega)}$, $k \ge 1$ denotes the $L^{k,p}$–norm for a bounded open subset $\Omega \subset R^n$. Notice that the Sobolev embedding theorem says that for any bounded domain $\Omega$ of class $C^{0,1}$ in $R^n$, $L^{k,p}(\Omega) \subset C^{k-1,1-(n/p)}(\bar\Omega)$. In particular, an $L^{1,p}$ riemannian $n$-manifold must be a $C^\alpha$ riemannian manifold for $\alpha = 1 - \frac{n}{p}$.

We say $(M_j, g_j, o_j)$ have the pointed $L^{1,p}$ bound, and converge, in the pointed $C^{\alpha'}$ topology for $\alpha' < \alpha := 1 - \frac{n}{p}$, to a pointed $L^{1,p}$ riemannian $n$–manifold $(M_\infty, g_\infty; o)$ if, with respect to the fixed $L^{2,p}$ atlas of coordinate charts for $M_\infty$, for any $r > 0$, there are $C^{1,\alpha} \cap L^{2,p}$ diffeomorphisms $f_j^r$ from the geodesic ball $B_r(o, M_\infty)$ for $M_\infty$ to a pointed open subset $(U_j, o_j)$ of $M_j$, such that in each coordinate chart $F_\nu = (x^\alpha)$ for which $U_\nu \subset B_r(o, M_\infty)$, the riemannian metric coefficients $(g'_j)_{ab}$ of $g'_j := (f_j^r)^* g_j$ satisfy (2) and (3) above for some constant $Q'_\nu > 1$ independent of $j$, and

$$\lim_{j\to\infty} \big\|(g'_j)_{ab} - (g_\infty)_{ab}\big\|_{C^{\alpha'}} = 0. \tag{4}$$

Stronger convergence theorems can be obtained if instead of $\mathrm{Ric}(M_j) \ge -(n-1)\lambda^2$, in (1), one assumes $|K_{M_j}| \le \lambda^2$. In this case, one can prove that given any $0 < \alpha < 1$, $(M_j, g_j, o_j)$ sub-converge, in the pointed $C^{1,\alpha'}$ topology for $\alpha' < \alpha$, to a pointed complete $C^{1,\alpha}$ riemannian manifold. We refer the reader to [C], [G1], [P], [GW], [K]. This so-called $C^{1,\alpha}$ convergence theorem is actually valid under the weaker hypothesis $|\mathrm{Ric}(M_j)| \le (n-1)\lambda^2$ [A1].

***Acknowledgement.*** The authors would like to thank D. Gromoll for his advice and encouragement. They also would like to thank M. Anderson, J. Cheeger, K. Grove and P. Petersen V. for many discussions and valuable suggestions.

**2. The Isometry Groups.** We begin with some basic facts for metric spaces. Let $(X,\rho)$ be a connected locally compact, complete inner metric space. Here a metric space is called an inner metric space if the distance between any two points is the infimum of the metric lengths of rectifiable curves joining them (where metric length is defined as usual). A rectifiable curve can be parameterized by arclength. A rectifiable curve $\gamma : [a,b] \to X$, parameterized proportionally to arclength, is called a minimal geodesic, if it realizes the distance, which is equivalent to that for some $c > 0$, $\rho(\gamma(s_1), \gamma(s_2)) = c|s_1 - s_2|$, $s_1$, $s_2 \in [a,b]$. A geodesic $\gamma$ will always be a locally minimal geodesic parameterized proportionally to arclength. In this case, the Hopf-Rinow theorem holds in $X$, i.e. for any points $x$, $y \in X$, there is a minimal geodesic joining them. A normal minimal geodesic $\gamma : [0,+\infty) \to X$ is called a ray. The Busemann function $b_\gamma$ associated with $\gamma$ is defined as $b_\gamma(x) = \lim_{t\to+\infty} t - \rho(x,\gamma(t))$, $x \in X$. A normal minimal geodesic $\ell : R \to X$ is called a line through $x = \ell(0)$. $\ell^\pm$ will always denote the rays $\ell^\pm(t) := \ell(\pm t)$, $t \geq 0$. The following lemma is basic, and it is well-known in the case of smooth riemannian manifolds.

**Lemma 1.** *Let $(X_i, \rho_i)$, $i = 1,2$, be locally compact, complete inner metric spaces. Let $X = X_1 \times X_2$ be the metric product with the natural induced metric $\rho = \rho_1 \oplus \rho_2$. Then for any geodesic $\gamma = (\gamma_1, \gamma_2) : [a,b] \to X$, each $\gamma_i$ is a geodesic in $X_i$, $i = 1,2$.*

*Proof.* Without loss of generality, we assume $\gamma$ is a normal minimal geodesic. We will prove each $\gamma_i$ is a minimal geodesic, i.e. for some $c_i \geq 0$, $\rho_i(\gamma_i(t), \gamma_i(t')) = c_i|t - t'|$, $t$, $t' \in [a,b]$. For any subdivision $T$ of $[a,b] : a = t_0 < t_1 < \cdots < t_N = b$,

$$\rho(\gamma(a),\gamma(b)) = \sum_{k=1}^{N-1} \rho(\gamma(t_{k+1}),\gamma(t_k))$$

$$= \sum_{k=1}^{N-1} \sqrt{\rho_1(\gamma_1(t_{k+1}),\gamma_1(t_k))^2 + \rho_2(\gamma_2(t_{k+1}),\gamma_2(t_k))^2}$$

$$\geq \sqrt{\left[\sum_{k=1}^{N-1} \rho_1(\gamma_1(t_{k+1}),\gamma_1(t)k))\right]^2 + \left[\sum_{k=1}^{N-1} \rho_2(\gamma_2(t_{k+1}),\gamma_2(t_k))\right]^2} \tag{5}$$

$$\geq \sqrt{\rho_1(\gamma_1(a),\gamma_1(b))^2 + \rho_2(\gamma_2(a),\gamma_2(b))^2} \tag{6}$$

$$= \rho(\gamma(a),\gamma(b)).$$

Hence the equality in (6) holds, which implies

$$\rho_i(\gamma_i(a),\gamma_i(b)) = \sum_{k=1}^{N-1} \rho_i(\gamma_i(t_{k+1}),\gamma_i(t_k)), \qquad i = 1,2. \tag{7}$$

Suppose $\rho_1(\gamma_1(a),\gamma_1(b)) \neq 0$. It is clear that the equality in (5) holds iff there is $\mu_T \geq 0$, such that

$$\rho_2(\gamma_2(t_{k+1}),\gamma_2(t_k)) = \mu_T\rho_1(\gamma_1(t_{k+1}),\gamma_1(t_k)), \qquad k = 1,\cdots,N-1. \tag{8}$$

Clearly, from (7) and the sum of (8) from $k = 1$ to $k = N-1$,

$$\mu_T = \rho_2(\gamma_2(a),\gamma_2(b))/\rho_1(\gamma_1(a),\gamma_1(b)) := \mu \geq 0,$$

which is independent of subdivisions $T$. Now from (8) one obtains

$$\begin{aligned} |t_{k+1} - t_k| &= \rho(\gamma(t_{k+1}),\gamma(t_k)) \\ &= \sqrt{\rho_1(\gamma_1(t_{k+1}),\gamma_1(t_k))^2 + \rho_2(\gamma_2(t_{k+1}),\gamma_2(t_k))^2} \\ &= \sqrt{1+\mu^2}\rho(\gamma_1(t_{k+1}),\gamma_1(t_k)). \end{aligned}$$

Consequently one obtains that $\rho_2(\gamma_2(t_{k+1}),\gamma_2(t_k)) = \mu/\sqrt{1+\mu^2}|t_{k+1}-t_k|$. Since the subdivision is arbitrary, this proves Lemma 1. □

An isometry of a metric space $(X,\rho)$, by definition, is a map $\varphi : X \to X$, which preserves the metric $\rho$, i.e. $\rho(\varphi(x),\varphi(y)) = \rho(x,y)$, $x$, $y \in X$. A theorem due to van Dantzig and van der Waerden [KN; Theorem 4.7] says that the isometry group of a connected, locally compact metric space is locally compact with respect to the compact-open topology. In particular if the metric space is connected and compact, then the isometry group is also compact. For the proof of Theorem 1, one needs the following:

**Proposition 1.** *Let $(X_1,\rho_1)$ be a connected, locally compact, complete inner metric space which contains no line. Let $R^k$ be the Euclidean space. Let $X = X_1 \times R^k$ with metric $\rho = \rho_1 \oplus \|\cdot\|$, where $\|\cdot\|$ is the standard flat metric of $R^k$. Then any isometry $\varphi$ of $(X,\rho)$ can be written as $(\varphi_1,\varphi_2)$, where $\varphi_1$ is an isometry of $(X_1,\rho_1)$ and $\varphi_2$ is an isometry of $(R^k,\|\cdot\|)$.*

*Proof.* This proposition was proved by Cheeger-Gromoll [CG1] in the case of smooth riemannian manifolds. Write $\varphi(x,\xi) = (\varphi_1(x,\xi),\varphi_2(x,\xi))$, $(x,\xi) \in X_1 \times R^k$. Given any $(x,\xi) \in X_1 \times R^k$, $\xi \neq 0$, consider the line $\ell(t) := (x,\frac{\xi}{|\xi|}t)$, $t \in R$. It is clear that $\varphi \circ \ell$ is a line again. From Lemma 1 it follows that $(\varphi_2 \circ \ell)(t)$ is a geodesic in $R^k$ parameterized proportionally to arclength, thus $(\varphi \circ \ell)(t) = \varphi_2(x,0) + \xi_1 t$, $t \in R$. Since $X_1$ contains no lines, $(\varphi_1 \circ \ell)(t)$ is a point, which implies $\varphi_1(x,\xi) = \varphi_1(x,\xi) = \varphi_1(x,0)$, $x$, $\xi \in X_1 \times R^k$. Now fix $x_0 \in X_1$. For any $x \in X_1$, there is a normal minimal geodesic $\sigma$ in $X_1$ joining $x_0$ and $x$.

Let $\sigma_\xi = (\sigma(t),\xi)$, $\xi \in R^k$. Consider $\varphi\circ\sigma_\xi = \big(\varphi_1(\sigma,\xi),\ \varphi_2(\sigma,\xi)\big)$. It follows from Lemma 1 that $\varphi_2(\sigma(t),\xi)$ is a geodesic through $\varphi_2(x_0,\xi)$ in $R^k$ parameterized proportionally to arclength, i.e. $\varphi_2\big(\sigma(t),\xi\big) = \varphi_2(x_0,\xi)+\xi' t$, $\xi' \in R^k$. We claim $\xi' = 0$. If $\xi' \neq 0$, then $\ell(t) := \big(\varphi_1(x_0,\xi), \varphi_2(x_0,\xi) + \frac{\xi'}{|\xi'|}t\big)$ is a line $X$. From the same argument as above it follows that $(\varphi^{-1}\circ\ell')(t) = (\xi_0,\ \xi+\xi'' t)$, $t \in R$, for some $|\xi''| = 1$. By the definition of the Busemann functions, one obtains

$$0 = b_{(\varphi^{-1}\circ\ell')^+}\big(\sigma_\xi(t)\big) = b_{(\ell')^+}\big((\varphi\circ\sigma_\xi)(t)\big) = |\xi'|.$$

It is a contradiction. Thus $\ell'$ is a point. Therefore $\varphi_2\big(\sigma(t),\xi\big)$ is independent of $t$, which implies that $\varphi_2(x,\xi) = \varphi_2(x_0,\xi)$. We complete the proof. □

Now we consider a connected $n$-dimensional $C^\alpha$ riemannian manifold $(M,g)$, $0<\alpha\leq 1$, i.e. an $n$-manifold of class $C^{1,\alpha}$ with $C^\alpha$ riemannian metric $g$. For any absolutely continuous curve in $M$, its length is defined by $g$ as usual. By definition, the distance $\rho_g$ between two points is defined as the infimum of the integral length of absolutely continuous curve joining them. It is clear that the topology induced by $\rho_g$ is equivalent to the original manifold topology. A curve in $M$ is rectifiable with respect to $\rho_g$ iff it is of bounded variation in local coordinates (cf. [My]). In particular, minimal geodesics (with respect to $\rho_g$) are Lipschitz continuous in local coordinates. Furthermore M. Morse proved that the integral length of an absolutely continuous curve is equal to its metric length defined by $\rho_g$. Thus minimal geodesics (with respect to $\rho_g$) coincide with the extremal in the variational problem associated with the integral length function. We say $(M,g)$ is complete if $(M,\rho_g)$ is complete. Hence a complete $C^\alpha$ riemannian manifold is a locally compact, complete inner metric space. Complete $C^\alpha$ riemannian manifolds $(M,g)$ have nice properties as we expect. For example, any geodesic in $(M,g)$ is of class $C^{1,\alpha}$ (cf. [CH]). However, since a geodesic may have branched points and may not be extended to $R$, the exponential map need not be defined. Thus the normal coordinates need not exist.

***Theorem*** (Calabi-Hartman [CH]). *Let $(M,g)$ be a connected $n$-dimensional $C^\alpha$ riemannian manifold, $0<\alpha\leq 1$. Then any isometry $\varphi$ of $(M,\rho_g)$ is of class $C^{1,\alpha}$. Furtermore $\varphi$ satisfies*

$$\varphi^* g = g.$$

***Remark 1.*** For complete $C^0$ riemannian metrics on $M$, the isometry need not be of class $C^1$ (cf. [CH]).

***Proposition 2.*** *Let $(M,g)$ be an $n$–dimensional connected complete $C^\alpha$ riemannian manifold, $0\leq\alpha\leq 1$. Then the isometyry group* Isom$(M)$ *of $(M,\rho_g)$ is a Lie group, which acts effectively on $M$ as a transformation group, and the mapping* Isom$(M)\times M \to M$, *$(\varphi,x)\to\varphi(x)$, is of class $C^1$.*

*Proof.* First notice that the isometry group $\mathrm{Isom}(M)$ is locally compact with respect to the compact-open topology (cf. [KN; Theorem 4.7]). Then from the Calabi-Hartman's theorem above it follows that each isometry as a homeomorphism of $M$ is of class $C^{1,\alpha}$. Thus $\mathrm{Isom}(M)$ is a Lie group and the mapping $\mathrm{Isom}(M) \times M \to M$ is of class $C^1$ as follows from Montgomery-Zippin [MZ; p. 208]. □

**3. The Splitting Theorems.** It is well known that any complete open riemannian manifold $M$ with nonnegative sectional curvature can be decomposed into a riemannian metric product $V \times R^k$ where $V$ is a riemannian manifold containing no lines. This splitting theorem is due to Cohn-Vossen in the 2–dimensional case, to Toponogov [T] in the higher dimensional case, and it is actually valid under the weaker hypothesis $\mathrm{Ric}(M) \geq 0$ [CG2].

In this section, we will prove the following:

***Theorem 2.*** *Let $(M_j, g_j; o_j)$ be a sequence of pointed complete n-manifolds satisfying the bounds*

$$K_{M_j} \geq -\lambda_j^2, \quad \mathrm{inj}(M_j) \geq i_0$$

*with $\lim_{j\to\infty} \lambda_j = 0$. Suppose that $(M_j, g_j; o_j)$ have the pointed $L^{1,p}$ bound, and converge in the pointed $C^{\alpha'}$ topology for $\alpha' < \alpha := 1 - \frac{n}{p}$, $p > n$, to a pointed complete $L^{1,p}$ riemannian manifold $(M_\infty, g_\infty; o)$. Then $(M_\infty, g_\infty)$ is isometric to a riemannian metric product $(V \times R^k,\ g_1 \oplus \delta)$, where $(V, g_1)$ is a $C^\alpha$ riemannian submanifold containing no lines, with the induced riemannian metric $g_1$ of $g_\infty$, and the isometry between $M_\infty$ and $V \times R^k$ is a $C^{1,\alpha}$ diffeomorphism preserving the riemannian metrics.*

***Remark 2.*** We conjecture that Theorem 2 is valid under the weaker hypothesis $\mathrm{Ric}(M_j) \geq -(n-1)\lambda_j^2$, instead of $K_{M_j} \geq -\lambda_j^2$. Then the proof implies that Theorem 1 is also valid under the weaker hypothesis $\mathrm{Ric}(M) \geq -\varepsilon(n, i_0)$. Partial results have been obtained by the authors. For example, if $(M_\infty, g_\infty)$ contains a line $\ell$, then the Busemann functions $b^\pm$, associated with $\ell^\pm$, satisfy $b^+ + b^- = 0$, and $b^+$ (hence $b^-$) is of class $C^{1,\alpha}$ for $0 < \alpha < 1$. The sectional curvature condition in this paper is used only in proving the convexity of $b^\pm$.

For the proof of Theorem 2, we will prove the following lemmas.

***Lemma 2.*** *For each $(M_j, g_j; o_j)$ in Theorem 2 and any point $x_j \in M_j$, the distance function $\rho_j := \rho_j(\cdot, x_j)$ satisfies*

$$\int_{M_j} \rho_j \Delta_j \varphi \, d\mu_j \leq (n-1) \int_{M_j} \varphi \lambda_j \coth \lambda_j \rho_j \, d\mu_j$$

*for all nonnegative functions $\varphi \in C_0^\infty(M_j)$, where $\rho_j(\cdot,\cdot)$, $\Delta_j$ and $d\mu_j$, respectively denote the distance function, the Laplace operator and the volume element defined by $g_j$.*

*Proof.* Let $C_{x_j}$ denote the cut-locus at $x_j$, and let $\Omega_{x_j} = M_j \setminus C_{x_j}$, which is a star-like region. It is well-known that $\rho_j$ is smooth in $\Omega_{x_j}$ and satisfies

$$\Delta_j \rho_j \leq (n-1)\lambda_j \coth \lambda_j \rho_j.$$

One can choose a sequence of star-like regions $\Omega_\varepsilon \Subset \Omega_{x_j}$ such that

$$\lim_{\varepsilon \to 0} \Omega_\varepsilon = \Omega_{x_j}$$

and $\partial \rho_j / \partial \nu > 0$ on $\partial \Omega_\varepsilon$, where $\nu$ denotes the outward-pointing unit normal to $\partial \Omega_\varepsilon$. Clearly,

$$\int_{M_j} \rho_j \Delta_j \varphi \, d\mu_j = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \rho_j \Delta_j \varphi \, d\mu_j$$

for all nonnegative functions $\varphi \in C_0^\infty(M_j)$. By Green's formula (which is valid for Lipschitz continuous functions), one concludes

$$\begin{aligned}
\int_{\Omega_\varepsilon} \rho_j \Delta_j \varphi \, d\mu_j &= \int_{\Omega_\varepsilon} (\Delta_j \rho_j) \varphi \, d\mu_{g_j} - \int_{\partial \Omega_\varepsilon} \frac{\partial \rho_j}{\partial \nu} \varphi \, ds_j \\
&\leq \int_{\Omega_\varepsilon} (\Delta_j \rho_j) \varphi \, d\mu_j \\
&\leq (n-1) \int_{\Omega_\varepsilon} \varphi \lambda_j \coth \lambda_j \rho_j \, d\mu_j.
\end{aligned}$$ □

The proof of Lemma 3 below follows from (4).

**Lemma 3.** *For any fixed point $x_0 \in M_\varepsilon$, let $\rho_\infty(\cdot, x_0)$ denote the distance function defined by $g_\infty$ in $M_\infty$ from $x_0$. Then for any compact subset $K$, $x_0 \in K$, there is a sufficiently large $r > 0$ such that $x_0 \in K \subset B_r(o; M_\infty)$, and the induced distance function $\rho_j' := (f_j^r)^* \rho_j$ converges to $\rho_\infty$, uniformly on $K$.*

**Lemma 4.** *Let $\ell_{x_0}$ be a line in $(M_\infty, g_\infty; o)$ through $x_0 = \ell_{x_0}(0)$. Let $b_{x_0}^\pm$ denote the Busemann functions associated with $\ell_{x_0}^\pm$. Then $b_{x_0}^\pm$ satisfy*

$$\int_{M_\infty} b_{x_0}^\pm \Delta_\infty \varphi \mu_\infty \geq 0$$

*for all nonnegative functions $\varphi \in C_0^\infty(M_\infty)$, where $\Delta_\infty$ and $d\mu_\infty$, respectively, denote the Laplace operator and the volume element defined by $g_\infty$.*

*Proof.* For any nonnegative function $\varphi \in C_0^\infty(M)$, and any $t > 0$, by Lemma 3 we can choose a sufficiently large $r > 0$, such that $\ell_{x_0}^{\pm}(t)$, supp $\varphi \subset B_r(o; M_\infty)$, and $\rho_j'$ converge to $\rho_\infty^t := \rho_\infty(\cdot, \ell_{x_0}^{\pm}(t))$ uniformly on supp $\varphi$. Let $\Delta_j'$ and $d\mu_j'$ denote the Laplace operator and the volume element defined by $g_j'$. From Lemma 2 it follows that

$$\int_{M_j} \rho_j' \Delta_j' \varphi \mu_j' \leq (n-1) \int_{M_\infty} \varphi \lambda_j \coth_j \rho_j' d\mu_j'.$$

Since in each coordinate $F_\nu = (x^a)$ for $M_\infty$, the riemannian metric coefficients $(g_j')_{ab}$ of $g_j'$ satisfy the bounds (2) and (3), together with Lemma 3, one obtains

$$\lim_{j\to\infty} \left| \int_{M_\infty} \rho_j' \Delta_j' \varphi \, d\mu_j' - \int_{M_\infty} \rho_\infty^t \Delta_j' \varphi \, d\mu_j' \right| = 0.$$

Further, the uniform bounds (2) and (3) for $(g_j')_{ab}$ imply that $(g_j')_{ab}$ and $(g_j')^{ap}$ converge, in the weak $L^{1,p}$ topology, to $(g_\infty)_{ab}$ and $(g_\infty)^{ab}$, respectively. Hence

$$\lim_{j\to\infty} \int_{M_\infty} \rho_\infty^t \Delta_j' \varphi \, d\mu_j' = \int_{M_\infty} \rho_\infty^t \Delta_\infty \varphi \mu_\infty.$$

Letting $j \to +\infty$, one concludes

$$\int_{M_\infty} \rho_\infty^t \Delta_\infty \varphi \, d\mu_\infty \leq \int_{M_\infty} \frac{n-1}{\rho_\infty^t} \varphi \, d\mu_\infty.$$

Clearly,

$$\int_{M_j} (t - \rho_\infty^t) \Delta_\infty \varphi \, d\mu_\infty \geq - \int_{M_\infty} \frac{n-1}{\rho_\infty^t} \varphi \, d\mu_\infty.$$

Letting $t \to +\infty$, since $t - \rho_\infty^t \to b_{x_o}^{\pm}$ uniformly on compact subsets, one obtains

$$\int_{M_\infty} b_{x_o}^{\pm} \Delta_\infty \varphi d\mu_\infty \geq 0.$$

The following splitting property is proved in [GP2], [Y2].

***Proposition 3.*** *Let $(M_j, g_j; 0_j)$ be a sequence of pointed complete $n$–manifolds whose sectional curvature satisfies the bound*

$$K_{M-j} \geq -\lambda_j^2$$

*with $\lim_{j\to\infty} \lambda_j = 0$. Let $\rho_j$ denote the metric induced by $g_j$. Suppose $(M_j, \rho_j; o_j)$ converge, in the pointed Hausdorff metric, to a pointed complete metric space $(X_\infty, \rho_j; o_j)$. Then $(X_\infty, \rho_\infty)$ is isometric to a metric product*

$$(V \times R^k, \rho_1 \oplus \| \cdot \|)$$

*for some totally convex subset $(V, \rho_1)$ containing no lines, where $\rho_1$ is the restriction of $\rho_\infty$ to $V$.*

***Sketch of proof.*** Clearly, $(X_\infty, \rho_\infty)$ is a locally compact, complete inner metric space. Further, the Hopf-Rinow theorem holds in $X_\infty$ (cf. e.g. [G1]). By Lemma 2.2 in [GP1] it is not difficult to prove that $X_\infty$ satisfies the following:

***Property T.*** *For any normal minimal geodesics $\sigma : [-a,b] \mapsto X_\infty$ and $\gamma : [0,c] \mapsto X_\infty$, with $\sigma(0) = \gamma(0)$, there is a constant $\eta \in [-1,1]$ such that for $s \in [-a,b]$, $t \in [0,c]$,*

$$\rho_\infty\big(\sigma(s),\gamma(t)\big)^2 \leq s^2 + t^2 - \eta st.$$

*where a, b, c are positive numbers.*

Property T above shows that geodesics in $X_\infty$ have no branched points. Since any totally convex subset in $X_\infty$ still statisfies Property T above it suffices to prove that if $X_\infty$ contains a line $\ell_{x_o}$ through $x_o = \ell_{x_o}(0)$, then $(X_\infty, \rho_\infty)$ is *isometric* to $(V \times R, \rho_1 \oplus \|\cdot\|)$ for some totally convex subset $(V, \rho_1)$ in $(X_\infty, \rho_\infty)$. Let $b^\pm_{x_o}$ denote the Busemann functions associated with $\ell^\pm_{x_o}$. First by Property T above, one obtains

$$b^+_{x_o} + b^-_{x_o} = 0. \tag{9}$$

Then by using Property T above and (9), one can conclude that the Buseman functions $b^\pm_{x_o}$ are linear along geodesics. Thus all level sets $V_t := (b^\pm_{x_o})^{-1}(t)$, $t \in R$, are totally convex. Furthermore, for any point $x \in X_\infty$, there is a unique line $\ell_x$ through $x \in X_\infty$ satisfying

$$b^\pm_{x_o}\big(\ell_x(t)\big) = t + b^\pm_{x_o}(x), \qquad t \in R. \tag{10}$$

Let $b^\pm_x$ denote the Busemann functions associated with $\ell^\pm_x$. Then one can easily check the following equality holds:

$$b^\pm_{x_0}(y) = b^\pm_{x_o}(x) + b^\pm_x(y), \qquad x,\ y \in X_\infty. \tag{11}$$

Now we construct a map $\Phi : (V_0 \times T, \rho_1 \oplus \delta) \mapsto (X_\infty, \rho_\infty)$ defined as $\Phi(x,t) = \ell_x(t)$, where $V_0 = (b^+_{x_o})^{-1}(0)$. We claim that $\Phi$ is a (distance-preserving) isometry. The following argument is different from that given by Grove-Petersen[GP2]. It suffices to prove that for any points $x_1$, $x_2 \in V^t$ and $s$, $t \in R$,

(i) $\rho_\infty(\ell_{x_1}(s), x_2) = \sqrt{\rho_\infty(x_1,x_2)^2 + s^2}$;
(ii) $\rho\big(\ell_{x_1}(s), \ell_{x_2}(s)\big) = \rho_\infty(x_1,x_2)$.

To prove (i), without loss of generality, we assume $s > 0$. Applying Property T above to $\ell_{x_1}|[-r,s]$ for large $r$ and a normal minimal geodesic $\gamma$ issuing from $x_1$ to $x_2$, together with (11) one concludes that

$$\rho_\infty\big(\ell_{x_1}(s), x_2\big)^2 \leq \rho_\infty(x_1,x_2)^2 + s^2. \tag{12}$$

Similarily by (11) and applying Property T above to $\ell_{y_1}|[-s,r]$, $y_1 = \ell_{x_1}(s)$, for large $r$ and a normal minimal geodesic $\gamma$ issuing from $\ell_{x_1}(s)$ to $x_2$, one obtains

$$\rho_\infty(x_1,x_2)^2 \le \rho_\infty\big(\ell_{x_1}(s),x_2\big)^2 - s^2. \tag{13}$$

Then (i) above follows from (12) and (13). To prove (ii), let $y_i = \ell_i(s)$, $i = 1,2$. Then $x_i = \ell_{y_i}(-s)$. It follows from (i) that

$$\begin{aligned}\sqrt{\rho_\infty(x_1,x_2)^2 + s^2} = \rho_\infty\big(\ell_{x_1}(s),x_2\big) &= \rho_\infty\big(y_1,\ell_{y_2}(-s)\big)\\ &= \sqrt{\rho_\infty(y_1,y_2)^2 + (-s)^2}\\ &= \sqrt{\rho_\infty\big(\ell_{x_1}(s),\ell_{x_2}(s)\big)^2 + s^2},\end{aligned}$$

from which (ii) follows. □

***Proof of Theorem 2.*** It follows from Lemma 4 and (9) that the Busemann functions satisfy

$$\int_{M_\infty} b^\pm_{x_o}\Delta_\infty\varphi\, d\mu_\infty = 0.$$

A standard result from PDE [LU; Theorem 15.1] shows that $b^\pm_{x_o}$ are of $C^{1,\alpha}\cap L^{2,p}$. Thus the level set $V_t = (b^\pm_{x_o})^{-1}(t)$ is a $C^{1,\alpha}$ submanifold of $M_\infty$. It follows from Proposition 3 that there is a (distance-preserving) isometry

$$\Phi : (V\times R^k, \rho_1\oplus\delta) \mapsto (M_\infty,\rho_\infty),$$

where $V$ is a totally convex $C^{1,\alpha}$ submanifold of $M_\infty$ with resriction $\rho_1$ of $\rho_\infty$. Let $g_1$ be the restriction of $g_\infty$ to $V$. We claim that $\rho_1$ is induced from $g_1$, i.e. for any $x$, $y\in V$, $\rho_1(x,y) = \inf_{c\subset V}\text{length}_{g_1}(c)$, where the infimum is taken over all absolutely continuous curves $c$ in $V$ joining $x$ and $y$. On one hand, for any absolutely continuous curve $c$ in $V$ joining $x$ and $y$, $\rho_1(x,y) := \rho_\infty(x,y) \le \text{length}_{g_\infty}(c) = \text{length}_{g_1}(c)$. On the other hand, there is a minimal geodesic $\gamma$ in $(M_\infty,\rho_\infty)$ joining $x$ and $y$. Since minimal geodesics must be the extremals in the calculus of variation problem associated with integral length, one has that $\rho_\infty(x,y) = \text{length}_{g_\infty}(\gamma) = \text{length}_{g_1}(\gamma)$. The convexity of $V$ implies that $\gamma\subset V$. Thus $\rho_\infty(x,y) = \text{length}_{g_\infty}(\gamma) = \text{length}_{g_1}(\gamma)$. This proves the above claim. Thus the metric $\rho_1\oplus\|\cdot\|$ on $V\times R^k$ is one induced by $g_1\oplus\delta$, where $\delta$ is the flat riemannian metric on $R^k$. From the Calabi-Hartman theorem the isometry $\Phi$ is at least of $C^{1,\alpha}$ and preserves the riemannian metrics, i.e., $\Phi^*(g_1\oplus\delta) = g_\infty$. This proves Theorem 2. □

**4. Proof of Theorem 1.**

We prove it by contradiction. Suppose Theorem 1 is false. Then there is a sequence of riemannian manifolds $(M_j, g_j)$ satisfying

- $K_{M_j} \geq -\lambda_j^2$,
- $\text{inj}(M_j, g_j) \geq i_o$,
- $\text{diam}(M_j) \leq 1$,

with $\lim_j \lambda_j = 0$, such that no finite cover of $M_j$ is diffeormorphic to $V \times T^k$ for some closed simply connected $(n-k)$–dimensional manifold $V$. From the Anderson-Cheeger's $C^\alpha$–precompactness theorem [AC] one concludes that given $p > n$ and $\alpha = 1 - \frac{n}{p}$, passing to a subsequence if necessary, there are $C^{1,\alpha} \cap L^{2,p}$ diffeomorphisms $f_j : M := M_1 \to M_j$ such that with respect to the fixed $L^{2,p}$ atlas of harmonic coordinate charts for $M$, $g_j' = f_j^* g_j$ have the $L^{1,p}$ bound, and converge, in the $C^{\alpha'}$ topology for $\alpha' < \alpha$, to an $L^{1,p}$ riemannian manifold $(M, g)$ (for definition see Section 1 above). Let $\pi : (\tilde{M}; \tilde{p}) \to (M; p)$ be a universal covering map. Let $\tilde{g}_j' := \pi^* g_j'$ and $\tilde{g} = \pi^* g$. Then, with respect to the induced $L^{2,p}$ atlas of harmonic coordinate charts for $(\tilde{M}, \tilde{g})$, $\tilde{g}_j'$ have the $L^{1,p}$ bound (independent of $j$), and coverge, in the $C^{\alpha'}$ topology, to the $L^{1,p}$ riemannnian manifold $(\tilde{M}, \tilde{g})$. Since the sectional curvature of $\tilde{g}_j'$ satisfies $K_{(\tilde{M}, \tilde{g}_j')} \geq -\lambda_j^2$, $\lim_{j \to +\infty} \lambda_j = 0$, it follows from Theorem 2 above that $(\tilde{M}, \tilde{g})$ can be decomposed into a riemannian metric product $(V \times R^k, g_1 \oplus \delta)$ for some connected totally convex $C^\alpha$ riemannian submanifold $(V, g_1)$ in $(\tilde{M}, \tilde{g})$. As proved in [CG2], one knows $V$ must be compact. From Proposition 1 one concludes that $\text{Isom}(\tilde{M})$ can be written as $\text{Isom}(V) \times \text{Isom}(R^k)$. It follows from Proposition 2 that $\text{Isom}(V)$ is a *Lie group* and the mapping $\text{Isom}(V) \times V \to V$ is of class $C^1$. Therefore the argument in [CG1] carries over to the rest of the proof. Let $\pi_1 := \pi_1(M; p) \subset \text{Isom}(\tilde{M}) \cong \text{Isom}(V) \times \text{Isom}(R^k)$ be the fundamental group. Following [CG1], let $p_1 : \pi_1 \to \text{Isom}(V)$ and $p_2 : \pi_1 \to \text{Isom}(R^k)$ be the natural projections, respectively. It is clear that $\ker(p_2)$ is finite. For the sake of simplicity we assume that $\ker(p_2) = 0$, i.e. $p_2$ is injective. The natural projection $\tilde{M} = V \times R^k \to R^k$ induces a continuous map $M = \tilde{M}/\pi_1 \to R^k/p_2(\pi_1)$. Thus $p_2(\pi_1)$ is a cocompact discrete subgroup of $\text{Isom}(R^k)$. Thus there is a abelian subgroup $\Gamma^* \subset \pi_1$ of finite index such that $p_2(\Gamma^*)$ is a lattice in the subgroup $L$ of translations in $\text{Isom}(R^k)$. Let $t : a \in R^k \to t(a) \in L$ be the natural isomorphism, where $t(a)(\xi) = \xi + a$, $\xi \in R^k$. Let $H = \overline{p_1(\Gamma^*)} \subset \text{Isom}(V)$ and let $H_e$ be the connected component of the identity in $H$. Then $H_e$ is a *connected* abelian Lie subgroup of $\text{Isom}(V)$, thus it turns out to be a torus. Let $\Gamma = \Gamma^* \cap p_1^{-1}(H_e)$ which is of finite index in $\pi_1$. Define a homomorphism $\Psi$ :

$p_2(\Gamma) \subset L \to H_e$ by $p_2(g) \to p_1(g)$, $g \in \Gamma$. Since $H_e$ is a torus, $\Psi$ can be extended to a homomorphism $\bar{\Psi} : L \to H_e$. $\Gamma$ acts on $V \times L$ by $g(x,t(a)) = (x,p_2(g)t(a))$ and on $V \times R^k$ by $g(x,a) = (p_1(g)x,p_2(g)a) = (p_1(g)x,t^{-1}(p_2(g)) + a)$, $g \in \Gamma$, respectively. We construct a $C^1$–diffeomorphism $f : V \times L \to V \times R^k = \tilde{M}$ by $f(x,t(a)) = (\bar{\Psi}(t(a))x,a)$. Clearly, $f \circ g = g \circ f$, $g \in \Gamma$. Therefore $f$ induces a $C^1$–diffeomorphism $\bar{f} : V \times L/\Gamma = V \times T^k \to (V \times \mathbb{R}^k)/\Gamma = M^*$, where $M^*$ is a finite cover of $M$. It is a contradiction. □

**5. Fundamental Groups.** It is well known that if a closed riemannian $n$–mani-
fold $M$ has nonnegative Ricci curvature, then the fundamental group, $\pi_1(M)$, has polynomial growth with degree $\leq n$ (see [M]).

Let $M$ be a closed riemannian manifold. Let $\text{sys}_1(M,p)$ denote the lower bound of the lengths of noncontractible curves at $x \in M$, and we put $\text{sys}_1(M) = \inf_{x \in M} \text{sys}_1(M,x)$, which is called the first systole of $M$. We have the old theorem due to Hilbert and Birkhoff, that if $M$ is a closed riemannian manifold, then two (not necessary distinct) points in $M$ can be connected by a geodesic curve in a prescribed homotopy class. These geodesics are obtained by minimizing the energy among all maps in the given homotopy class. We can prove the following:

***Theorem 3.*** *Given $n$, $L > 0$, there exists a constant $\varepsilon = \varepsilon(n,L) > 0$ depending only on $n$ and $L$ such that if a closed $n$-manifold $M$ admits a metric satisfying the conditions* $\text{diam}(M) = 1$, $\text{sys}_1(M) \geq L$ *and* $\text{Ric}(M) \geq -\varepsilon$, *then the fundamental group of $M$ is of polynomial growth with degree $\leq n$.*

The argument is similar to that given in [We] where the second author proved the following:

***Theorem 4*** (Wei). *Given $n$, $v > 0$, there exists a constant $\varepsilon = \varepsilon(n,v) > 0$ depending only on $n$ and $v$ such that if a closed $n$-manifold $M$ admits a metric satifying the conditions* $\text{diam}(M) = 1$, $\text{Vol}(M) \geq \nu$ *and* $\text{Ric}(M) \geq -\varepsilon$, *then the fundamental group of $M$ is of polynomial growth with degree $\leq n$.*

***Proof of Theorem 3.*** First let us assume that $M$ is in the following class:

$$\text{Ric}(M) \geq -(n-1)\lambda^2, \qquad \text{sys}_1(M) \geq L, \qquad \text{diam}(M) = 1 \tag{14}$$

for some positive constants $L, \lambda$. It was proved by M. Gromov ([G1], Proposition 5.28) that for any point $\tilde{p}$ in the universal covering $\tilde{M}$ of $M$, the fundamental group $\pi_1(M,p)$ admits a finite system of generators $\{g_\alpha\}_{\alpha=1}^N$ such that $\|g_\alpha\| :=$

$d(g_\alpha\tilde{p},\tilde{p}) \le 2$, $\operatorname{diam}(M) = 2$, and every relation is of form $g_\alpha g_\beta = g_\gamma$. Now it is clear that $B_{L/2}(g_\alpha\tilde{p},\tilde{M}) \cap B_{L/2}(g_\beta\tilde{p},\tilde{M}) = \emptyset$, for $\alpha \neq \beta$. Thus

$$\sum_{\alpha=1}^{N} \operatorname{vol}\big(B_{L/2}(g_\alpha\tilde{p},\tilde{M})\big) \le \operatorname{vol}\big(B_{2+L/2}(\tilde{p},\tilde{M})\big).$$

By the Bishop-Gromov volume comparison theorem one obtains

$$\begin{aligned}\frac{\operatorname{vol}\big(B_{L/2}(g_\alpha\tilde{p},\tilde{M})\big)}{\int_0^{L/2}\sinh^{n-1}\lambda t dt} &\ge \frac{\operatorname{vol}\big(B_{4+L/2}(g_\alpha\tilde{p},\tilde{M})\big)}{\int_0^{4+L/2}\sinh^{n-1}\lambda t dt}\\ &\ge \frac{\operatorname{vol}\big(B_{2+L/2}(\tilde{p},\tilde{M})\big)}{\int_0^{4+L/2}\sinh^{n-1}\lambda t dt}.\end{aligned}$$

Thus

$$N \le \frac{\int_0^{4+L/2}\sinh^{n-1}\lambda t dt}{\int_0^{L/2}\sinh^{n-1}\lambda t dt} := N(n,L,\lambda).$$

Therefore compare with [A1] we prove the following

***Theorem 5.*** *There are only finitely many possibilities for $\pi_1(M)$ among compact n-manifolds satisfying* (14).

Now the argument in [We] carries over to the rest part of the proof for Theorem 3. □

## REFERENCES


[A1] M. ANDERSON, *Short geodesics and gravitational instantons*, J. Diff. Geom. **31** (1990), 265-275.

[A2] ———, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), 429–445.

[AC] M. ANDERSON & J. CHEEGER, *$C^\alpha$-compactness for manifolds with Ricci curvature and injectivity radius bounded below*, (Preprint).

[BK] P. BUSER & H. KARCHER, *Gromov's almost flat manifolds, Asterisque n° 81*, Soc. Math. France (1981).

[C] J. CHEEGER, *Finiteness theorems for riemannian manifolds*, Amer. J. Math. **92** (1970), 61-74.

[CG1] J. CHEEGER & D. GROMOLL, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1974), 413-443.

[CG2] ———, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom **6** (1971), 119-128.

[CH] E. CALABI & P. HARTMAN, *On the smoothness of isometries*, Duke Math. J. **37** (1970), 741-750.

[G1] M. GROMOV, *Structures metriques pour les varietes riemanniennes*, redige par J. Lafontaine et P. Pansu, Textes Math. $n°$ 1, Cedic-Nathan, Paris, 1981.

[G2] ———, *Almost flat manifolds*, J. Diff. Geom **13** (1978), 235-241.

[GP1] K. GROVE & P. PETERSON V., *Manifolds near the boundary of existence*, J. Diff. Geom. (to appear).

[GP2] ———, *On the excess of metric spaces and manifolds*, (Preprint).

[GW] R. GREENE & H. WU, *Lipschitz convergence of riemannian manifolds*, Pac. J. Math. **131** (1988), 119-141.

[K] A. KASUE, *A convergence theorem for riemannian manifolds and some applications*, Nagoya Math J. **114** (1989), 21-51.

[KN] S. KOBAYASHI & K. NOMIZU, *Foundations of Differential Geometry I*, New York, 1963.

[LU] O.A. LADYSZHENSKAYA & N.N. URAL'TSEVA, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968 (English Translation).

[My] S.B. MYERS, *Arc length in metric and Finsler manifolds*, Ann. Math. **39** (1938), 463-471.

[M] J. MILNOR, *A note on curvature and fundamental group*, J. Diff. Geom **2** (1968), 1-7.

[MZ] D. MONTGOMERY & L. ZIPPIN, *Transformation groups*, Interscience Tracts No. 1, Interscience, New York, 1955.

[P] S. PETERS, *Convergence of riemannian manifolds*, Comp. Math. **62** (1987), 3-16.

[T] V. TOPONOGOV, *Spaces with straight lines*, Amer. Math. Soc. Translations **37** (1964), 287-290.

[Y1] T. YAMAGUCHI, *Manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **28** (1988), 157-167.

[Y2] ———, *Collapsing and pinching in lower curvature bound*, Ann. of Math. (to appear).

[We] G. WEI, *On the fundamental groups of manifolds with almost nonnegative Ricci curvature*, Proc. Amer. Math. Soc. **110** (1990), 197–199.



ZHONGMIN SHEN
Department of Mathematics
SUNY at Stony Brook
Stony Brook, New York 11794

GUOFANG WEI
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139


*Received: September 24th, 1990.*