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Volume Growth and Finite Topological Type

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1. Introduction

The structure of an open manifold with nonnegative Ricci curvature has received much attention recently. On one hand, Sha-Yang's examples [**SY1**, **SY2**] demonstrate that such a manifold could have infinite topological type. On the other hand, the beautiful result of U. Abresch and D. Gromoll [**AG**] shows that if the manifold M^n is not too "large" at infinity and the sectional curvature is bounded from below, then it must be of finite topological type. Here the "largeness" at infinity is measured by the notion of diameter growth introduced in [**AG**]. Precisely they require that the diameter growth $\mathscr{D}(p, r) = o(r^{1/n})$ for some point p. Following this line, the first author also proved variants of this result [**S**].

A natural generalization of Abresch-Gromoll's result would be to place conditions on the volume growth instead of diameter growth, as the fact that the diameter growth is not more than r^{α} implies that the volume growth is not more than $r^{1+(n-1)\alpha}$. In fact it is generally believed that Abresch-Gromoll's result would continue to hold for a nonnegatively Ricci curved manifold whose volume growth is not more than r^2. The purpose of this paper is to present the following result.

We assume that Riemmannian manifolds under consideration will always have "weak bounded geometry." We say a complete manifold M has weak bounded geometry, if it satisfies the bounds

$$K = \inf K_M > -\infty, \qquad v = \inf \operatorname{vol}(B(x, 1)) > 0. \tag{1}$$

THEOREM 1.1. *Let M^n be complete with weak bounded geometry* (1) *and* $\operatorname{Ric}_{(k)} \geq 0$ *outside a geodesic ball $B(p, D)$ for some $2 \leq k \leq n-1$. There*

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is a constant $c = c(n, k, K, v, D)$ *such that if*

$$\overline{\lim}_{r\to+\infty} \frac{\mathrm{vol}(B(p, r))}{r^{1+1/(k+1)}} < c,$$

then there exists a compact set, C, *such that* $M^n \backslash C$ *contains no critical points of* p. *In particular,* M^n *has finite topological type.*

The proof of Theorem 1.1 will be given in §3. Here for some $1 \le k \le n-1$, we say that the kth Ricci curvature is nonnegative, $\mathrm{Ric}_{(k)} \ge 0$, at some point x, if for every $(k+1)$-dimensional subspace $V \subset T_x M$,

$$\sum_{i=1}^{k+1} \langle R(e_i, v)v, e_i \rangle \ge 0 \quad \text{for all } v \in V,$$

where $\{e_1, \dots, e_{k+1}\}$ is any orthonormal basis for V. Thus for example $\mathrm{Ric}_{(1)} \ge 0$ means exactly the same as $K_M \ge 0$ while $\mathrm{Ric}_{(n-1)} \ge 0$ is the same as $\mathrm{Ric} \ge 0$.

As a special case of Theorem 1.1, we have

COROLLARY 1.2. *Let* M^n *be a complete manifold with weak bounded geometry,* $\mathrm{Ric} \ge 0$ *outside a compact set and* $\mathrm{vol}(B(p, r)) = o(r^{1+1/n})$ *for some* $p \in M^n$. *Then* M^n *has finite topological type.*

REMARK. By the Bishop-Gromov comparison theorem, one can show that for any complete open manifold M^n with $\mathrm{Ric}(M) \ge 0$,

$$\mathrm{vol}(B(p, r)) \ge c(n)\mathrm{vol}(B(p, 1)) \cdot r, \qquad p \in M, r \ge 1;$$

see [CGT], Calabi and Yau [Y]. Thus our condition on the volume growth requires it to be close to the minimum growth. On the other hand there are manifolds with nonnegative Ricci curvature that do not have weak bounded geometry; see [CK].

2. Critical points of distance functions

The fundamental notion involved in such finite topological type result is that of the critical point of a distance function, first introduced by Grove and Shiohama. For this and the following fundamental lemma, the reader is referred to [G, C].

ISOTOPY LEMMA. *If* $r_1 \le r_2 \le +\infty$ *and if a connected component* C *of* $\overline{B(p, r_2)} \backslash B(p, r_1)$ *is free of critical points of* p, *then* C *is homeomorphic to* $C_1 \times [r_1, r_2]$, *where* C_1 *is a topological submanifold without boundary.*

We now look for conditions that will enable us to tell whether a point is a critical point of p or not. For $r > 0$ and a point p on a complete manifold

M, let

$$R(p, r) = \{\gamma(r), \gamma \text{ a ray from } p\}.$$

$R(p, r)$ consists of points of intersections of the geodesic sphere of radius r with all the rays emanating from p. Let $R_p(x) = d(x, R(p, r))$, where $r = d(p, x)$. The excess function is defined as

$$e_p(x) = \lim_{t\to\infty}(d(p, x) + d(x, S(p, t)) - t),$$

where $S(p, t) := \{x \in M;\ d(p, x) = t\}$. Clearly,

$$e_p(x) \le \lim_{t\to+\infty}(d(p, x) + d(x, \gamma_p(t)) - t) \tag{2}$$

for any ray γ_p from p. The following inequality is then trivial.

$$e_p(x) \le R_p(x). \tag{3}$$

The significance of the excess function lies in the following

LEMMA 2.1 [S, Lemma 10]. *Suppose that M has sectional curvature $K_M \ge -K$ $(K > 0)$. Then for any critical point x of p,*

$$e_p(x) \ge \frac{1}{\sqrt{K}} \log \frac{e^{\sqrt{K}\, d(p, x)}}{\cosh \sqrt{K}\, d(p, x)}.$$

For our purpose, we need to obtain a better estimate for $e_p(x)$ than (3) in the case that $\mathrm{Ric}_{(k)} \ge 0$.

PROPOSITION 2.2. *Suppose $\mathrm{Ric}_{(k)} \ge 0$ on $M\backslash B(p, D)$ for some $2 \le k \le n-1$. Then for all $x \in M\backslash B(p, 2D)$*

$$e_p(x) \le 8R_p(x)\left(\frac{R_p(x)}{d(p, x)}\right)^{1/k}. \tag{4}$$

PROOF. If $\mathrm{Ric}_{(k)} \ge 0$ on all of M, then the estimate (4) is known; see [S, Lemma 12], also compare [AG, C] in the case $k = n-1$. Now if $\mathrm{Ric}_{(k)} \ge 0$ only on $M\backslash B(p, D)$, the estimate (4) can be obtained by a modification of the argument in [S]. We need the following

LEMMA 2.3 [S, Lemma 11]. *Let M^n be complete and $p, q \in M^n$. Suppose that q is not on the cut-locus of p (hence $d_p(x) := d(p, x)$ is smooth near q) and $\mathrm{Ric}_{(k)} \ge 0$ along the minimal geodesic σ from q to p. Then for any orthonormal set $\{e_1, \dots, e_{k+1}\}$ in T_qM with $\dot\sigma(0) \in \mathrm{span}\{e_i\}$,*

$$\sum_{i=1}^{k+1} D^2 d_p(e_i, e_i) \le \frac{k}{d(p, q)}.$$

Fix any point $x \in M \setminus B(p, 2D)$. Let γ_p be the ray emanating from p such that for $z = \gamma_p(r) \in R(p, r)$, where $r = d(p, x)$,

$$R_p(x) = d(x, z). \tag{5}$$

We may assume that $R_p(x) > 0$, otherwise it is done. Since $d(x, \gamma_p(t)) - t$ is nonincreasing in t, it follows from (2) that for all $0 \le t_1 < t_2$,

$$e_p(x) \le d(\gamma_p(t_1), x) + d(\gamma_p(t_2), x) - (t_2 - t_1). \tag{6}$$

Let $\eta > 0$ be a small number $(\eta \le 2R_p(x))$ and $R = R_p(x) + \eta$. It is easy to see that if a minimal geodesic σ from $\gamma_p(t_o)$ to a point y in $B(p, R)$ intersects with $B(p, D)$, then $t_o < D + 2R_p(x) + \eta$. Take $t_1 = D + 4R_p(x)$ and $t_2 = 2d(p, x) - t_1$. Let

$$e_{p_1p_2}(y) = d(p_1, y) + d(p_2, y) - d(p_1, p_2),$$

where $p_i = \gamma_p(t_i)$. Then it follows from (6) that

$$e_p(x) \le e_{p_1p_2}(x). \tag{7}$$

Following the line in [**AG**], we will show that

$$e_{p_1p_2}(x) \le \frac{2k}{k-1}\left(\frac{C}{2(k+1)}R_p(x)^{k+1}\right)^{1/k}, \tag{8}$$

where $C = \frac{k}{d(p_1, x) - R_p(x)} + \frac{k}{d(p_2, x) - R_p(x)}$.

Now if $R_p(x) > \frac{1}{20}d(p, x)$, then (4) follows from (3). Thus from now on we will always assume that $R_p(x) \le \frac{1}{20}d(p, x)$. In this case

$$C \le \frac{8k}{d(p, x)}. \tag{9}$$

Assuming (8) and making use of (7) and (9), one can easily obtain (4).

The outline of the proof of (8) is given as follows. Let

$$\phi_R(t) = \frac{1}{(k-1)(k+1)}(t^{k-1} - R^{1-k})R^{k+1} + \frac{1}{2(k+1)}(t^2 - R^2).$$

Let

$$f(y) = C'\phi_R(d(x, y)) - e_{p_1p_2}(y), \qquad y \in B(p, R),$$

where $C' > \frac{k}{d(p_1, x) - R} + \frac{k}{d(p_2, x) - R}$. By the choices of t_i, one see that for any $y \in B(p, R_p + \eta)$, the minimal geodesics from y to p_i and x, respectively, do not intersect with the bad set $B(p, D)$. Following the proof of Lemma 12 in [**S**] and using Lemma 2.3, one can show that for any $y \in B(p, R) \setminus \{x\}$, there is an orthonormal set $\{e_1, \ldots, e_{k+1}\}$ in T_yM such that the following

inequality holds in a generalized sense (see [S]):

$$\sum_{i=1}^{k+1} D^2 f(e_i, e_i) \geq C' - \frac{k}{d(p_1, y)} - \frac{k}{d(p_2, y)}$$
$$\geq C' - \frac{k}{d(p_1, x) - R} - \frac{k}{d(p_2, x) - R} > 0.$$

Thus f has no locally maximal point in $B(x, R)$. Notice that $f|_{S(x,R)} \leq 0$ and $f(z) > 0$, where z is defined in (5). One has that for any $0 < \rho < R$,

$$0 < f(z) \leq C'\phi_R(\rho) - \min_{y \in S(x,\rho)} e_{p_1 p_2}(y),$$

which implies

$$e_{p_1 p_2}(x) \leq \min_{y \in S(x,\rho)} e_{p_1 p_2}(y) + 2\rho \leq C'\phi_R(\rho) + 2\rho,$$

where we have made use of $|e_{p_1 p_2}(x) - e_{p_1 p_2}(y)| \leq 2\,d(x, y)$. One obtains

$$e_{p_1 p_2}(x) \leq \min_{0<\rho<R} \{C'\phi_R(\rho) + 2\rho\}$$
$$\leq \frac{2k}{k-1} \left(\frac{C'}{2(k+1)} R^{k+1} \right)^{1/k}.$$

Letting $\eta \to 0$ (hence $R \to R_p(x)$) and $C' \to C$, one obtains (8). □

Note that $\frac{1}{2} \leq \log(e^r/\cosh r)$ for $r \geq 10$. Hence combining Proposition 2.2 and Lemma 2.1, we have

LEMMA 2.4. *Let M^n be complete with $\mathrm{Ric}_{(k)} \geq 0$ on $M \backslash B(p, D)$ and $K_M \geq -K$ $(K > 0)$. If $d(p, x) \geq \max\{10/\sqrt{K}, 2D\}$ and*

$$R_p(x) \leq \frac{1}{16} K^{-k/(2(k+1))} d(p, x)^{1/(k+1)},$$

then x is not a critical point of p.

Now we shall introduce a notion of essential diameter of ends (compare [C]). Let M be a complete manifold and $p \in M$. For any $r > 0$, the essential diameter of ends at distance r from p is defined by

$$\mathscr{D}(p, r) = \sup_{\Sigma} \mathrm{diam}(\Sigma),$$

where the supremum is taken over all boundary components Σ of $M \setminus \overline{B(p, r)}$, with $\Sigma \cap R(p, r) \neq \emptyset$. Let Σ_r be a boundary component of $M \setminus \overline{B(p, r)}$ with $\Sigma_r \cap R(p, r) \neq \emptyset$. By the definition of $R_p(x)$, one has that for any $x \in \Sigma_r$, $R_p(x) \leq \mathscr{D}(p, r)$. Thus one has the following theorem which is proved in [S] in the case $\mathrm{Ric}_{(k)} \geq 0$ on all of M^n.

THEOREM 2.5. *Let* M^n *be complete with* $\mathrm{Ric}_{(k)} \geq 0$ *on* $M \setminus B(p, D)$ *and* $K_M \geq -K$ $(K > 0)$. *Suppose that*

$$\overline{\lim}_{r\to+\infty} \frac{\mathscr{D}(p, r)}{r^{1/(k+1)}} < \frac{1}{16} K^{-k/(2(k+1))}.$$

Then there is a compact subset C *in* M^n *such that* $M \setminus C$ *contains no critical point of* p.

PROOF. Applying Lemma 2.4, one concludes that there is a large number r_0 such that for any $r \geq r_0$, if $\sum_r$ is a boundary component of $M \setminus \overline{B(p, r)}$ with $\sum_r \cap R(p, r) \neq \emptyset$, then $\sum_r$ is free of critical points. Now fixing $\sum_{r_0}$, let γ_p be a ray from p with $\gamma(r_0) \in \sum_{r_0}$. Denote by $\sum_t$ the boundary component of $M \setminus \overline{B(p, t)}$ with $\gamma(t) \in \sum_t$. Thus the set

$$U = \bigcup_{t \geq r_0} \sum_t$$

is free of critical point of p. By using the Isotopy Lemma one can see that U coincides with an unbounded connected component of $M \setminus B(p, r_0)$, and U is homeomorphic to $\sum_{r_0} \times [0, +\infty)$. It is clear that there are only finitely many (bounded and unbounded) connected components of $M \setminus \overline{B(p, r_0)}$, which contain points with distance from p at least $r_0 + 1$. Thus there is $r_1 \geq r_0$ such that $M \setminus \overline{B(p, r_1)}$ is contained in the union of the unbounded components of $M \setminus \overline{B(p, r_0)}$. Therefore $M \setminus \overline{B(p, r_1)}$ is free of critical points. □

3. Volume growth and diameter growth

The purpose of this section is to provide some relations between the volume growth and diameter growth for complete manifolds with nonnegative Ricci curvature.

LEMMA 3.1. *Let* M^n *be complete with* $v = \inf \mathrm{vol}(B(x, 1)) > 0$. *Then for any pont* $p \in M^n$ *and* $r > 2$,

$$\mathscr{D}(p, r) \leq \frac{8}{v} \mathrm{vol}(B(p, r+2) \setminus B(p, r-2)).$$

PROOF. The observation here is that under our assumption the volume of annuli can be used to estimate the essential diameter of ends. Let $\sum_r$ be a boundary component of $M \setminus \overline{B(p, r)}$ such that $\sum_r \cap R(p, r) \neq \emptyset$. Then there is a ray γ_p such that $\gamma_p(r) \in \sum_r$.

Let $\{B(p_j, 1)\}_{j=1}^{N}$ be a maximal set of disjoint balls with radius 1 and center $p_j \in \sum_r$. Then

$$\bigcup_{j=1}^{N} B(p_j, 2) \supset \sum_r,$$

and

$$N \leq \frac{1}{v}\mathrm{vol}(B(p, r+2)\backslash B(p, r-2)).$$

By the connectedness of $\sum_r$, one can show that for any point $x \in \sum_r$, there is a subset of $p_i's$, say, $q_1, \ldots, q_k$, $k \leq N$, such that $x \in B(q_1, 2)$, $\gamma_p(r) \in B(q_k, 2)$, and

$$B(q_t, 2) \cap B(q_{t+1}, 2) \neq \emptyset, \qquad 1 \leq t \leq k-1.$$

Now one can easily construct a piecewise smooth geodesic c joining x and $\gamma_p(r)$ through q_i's. Thus

$$d(x, \gamma_p(r)) \leq \mathrm{length}(c) \leq 4N \leq \frac{4}{v}\mathrm{vol}(B(p, r+2)\backslash B(p, r-2)).$$

Therefore

$$\mathscr{D}(p, r) \leq \frac{8}{v}\mathrm{vol}(B(p, r+2)\backslash B(p, r-2)). \quad \square$$

PROPOSITION 3.2. *Let M^n be complete with* $\mathrm{Ric} \geq 0$ *on* $M\backslash B(p, D)$ *and* $\mathrm{Ric} \geq -(n-1)H^2$ *on* $B(p, D)$. *Then there is a constant* $c = c(n, H, D)$ *such that for all* $R \geq r \geq 2D$,

$$\mathrm{vol}(B(p, R)\backslash B(p, r)) \leq c\int_r^R \frac{1}{t}\mathrm{vol}(B(p, t))\, dt,$$

where $c = c(n, H, D)$.

PROOF. Let U_pM be the unit sphere in T_pM. For each $v \in U_pM$, let t_v denote the cut value of v. Thus if $U_p = \{tv \in T_pM | t < t_v\}$, then $\exp_p|_{U_p}$ is an imbedding onto $M\backslash C_p$, where C_p denotes the cut-locus, which is of measure zero.

For $0 < r < R$,

$$\begin{aligned}\mathrm{vol}(B(p, R)\backslash B(p, r)) &= \int_{U_p \cap \{tv | r<t<R\}} \det(\exp_{p*})\, d\lambda \\ &= \int_r^R \left(\int_{U_p \cap S_t^{n-1}} \det(\exp_{p*})\, dv\right) dt.\end{aligned}$$

Now fix a vector $v \in U_pM$. Let $\gamma_v(t)$ be the geodesic from p with $\dot{\gamma}(0) = v$. Denote $f_v(t) = \det(\exp_{p*}|_{tv})$. By the Basic Index Comparison [CE], we have

for $r \le t_v$,

$$\frac{f_v'(r)}{f_v(r)} \le \int_0^r (n-1)g'(t)^2 - \mathrm{Ric}(\dot\gamma_v, \dot\gamma_v)g(t)^2\,dt - (n-1)r^{-1}$$

for all smooth functions $g(t)$ with $g(0)=0$, $g(r)=1$. By taking $g(t) = tr^{-1}(r \ge D)$, we have

$$\frac{f_v'(r)}{f_v(r)} \le \int_0^D (n-1)H^2 r^{-2}t^2\,dt.$$

Thus

$$\begin{aligned}(\log f_v(r))|_r^R &\le \frac{1}{3}(n-1)D^3H^2r^{-1} - \frac{1}{3}(n-1)D^3H^2R^{-1}\\ &\le \frac{1}{3}(n-1)D^2H^2 \qquad (t_v \ge R \ge r \ge D).\end{aligned}$$

Therefore $f_v(R) \le cf_v(r)$, where we have put $c = \exp(\frac{1}{3}(n-1)D^2H^2)$. Let $\bar f_v(t) = 0$ for $t \ge t_v$ and $\bar f_v(t) = f_v(t)$ for $t < t_v$. Then $\bar f_v(R) \le c\bar f_v(r)$ $(R \ge r \ge D)$. Now

$$\frac{r^{n-1}}{R^{n-1}}\int_{S_R^{n-1}} \bar f_v(R)\,dv = \int_{S_r^{n-1}} \bar f_v(R)\,dv \le c\int_{S_r^{n-1}} \bar f_v(r)\,dv.$$

Assuming $R \ge 2D$ and integrating with respect to r from D to R, we have

$$\int_{S_R^{n-1}} \bar f_v(R)\,dv \le \frac{nR^{n-1}}{R^n - D^n}c\int_D^R \int_{S_r^{n-1}} \bar f_v(r)\,dv\,dr \le \frac{2n}{R}c\mathrm{vol}B(p, R).$$

Hence for $R \ge r \ge 2D$,

$$\begin{aligned}\mathrm{vol}(B(p, R)\backslash B(p, r)) &= \int_r^R \left(\int_{S_t^{n-1}} \bar f_v(t)\,dv\right) dt\\ &\le 2nc\int_r^R \frac{1}{t}\mathrm{vol}(B(p, t))\,dt. \quad \square\end{aligned}$$

COROLLARY 3.3. *Let M^n be complete with $\mathrm{Ric} \ge 0$ on $M\backslash B(p, D)$ and $\mathrm{Ric} \ge -(n-1)H^2$ on $B(p, D)$. Suppose $v = \inf \mathrm{vol}(B(x, 1)) > 0$. Then there is a constant $c = c(n, H, D)$ such that for $r \ge 2D + 10$,*

$$\mathscr{D}(p, r) \le c(n, H, D)v^{-1}\frac{\mathrm{vol}(B(p, r+2))}{r+2}.$$

Now Theorem 1.1 follows from Theorem 2.5 and Corollary 3.3.

4. Sha-Yang's examples

Sha-Yang's examples are the first kind of the nonnegatively Ricci curved complete manifolds having infinite topological type. It is thus of much interest to compute its various geometric quantities such as the diameter growth

and the volume growth. In fact in [SY1] it is pointed out that for the 7-dimensional example constructed there, the degree of diameter growth is $\frac{2}{3}$. What we presented here is a detailed computation for this and other examples in [SY2]. We acknowledge gratefully here the helpful conversations with D. Yang.

Sha-Yang's examples are built upon manifolds obtained by rotating a curve. We first establish a simple lemma about such manifolds.

LEMMA 4.1. *Let* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be a smooth function. The equation* $\rho = f(x)$ *defines a surface of revolution in* $\mathbb{R}^{n+1}$, *where* $\rho = \sqrt{y_1^2 + \cdots + y_n^2}$ *and* $(x, y_1, \ldots, y_n)$ *are the coordinates of* $\mathbb{R}^{n+1}$. *Assume for* x *large,* $f(x) = O(x^\alpha)$, $\alpha \leq 1$. *Then the diameter growth of such a surface is at most* r^α *and hence the volume growth is at most* $r^{1+(n-1)\alpha}$.

PROOF. Let $r = d(0, (x, y))$. Then $r = \int_0^x \sqrt{1 + (f'(x))^2}\, dx$. Our assumption implies that $f'(x) = O(x^{\alpha-1}) \leq O(1)$. Hence

$$x \leq r \leq Cx, \tag{10}$$

for some positive constant C.

Now clearly the essential diameter $\mathscr{D}(0, r) \leq 2\pi f(x) = O(x^\alpha)$. Hence by (10) $\mathscr{D}(0, r) \leq O(r^\alpha)$. □

Recall that Sha-Yang's examples $\mathbb{R}^{n,m}$ $(m, n \geq 2)$ are obtained by performing surgery infinitely many times on $\mathbb{R}^{m+1} \times S^{n-1}$,

$$\mathbb{R}^{n,m} = S^{n-1} \times \left(\mathbb{R}^{m+1} \setminus \coprod_{k=0}^{\infty} D_k^{m+1}\right) \cup_{\mathrm{Id}} D^n \times \coprod_{k=0}^{\infty} S_k^m.$$

Here the metric on $\mathbb{R}^{m+1}$ is obtained by realizing it as a surface of revolution $\rho = f(x)$ of parabolic type in $\mathbb{R}^{m+2}$, and D_k^{m+1}'s are a sequence of disjoint geodesic balls having constant sectional curvature. In order for the surgery to preserve nonnegative Ricci curvature the geodesic balls D_k^{m+1} must be chosen to have larger and larger radius; see [SY2, (16)] and (11) below. This can be achieved by constructing f so that the surface $\mathbb{R}^{m+1}$ contains larger and larger spherical shells. One first chooses an infinite sequence $a_k > 0$ such that $\sum_{k=0}^{\infty} a_k = \frac{\pi}{2}$. See Figure 1 on next page.

The first step of construction is to enlarge the circular arc with opening angle a_2 by going out radially so that the arclength is a prescribed number R_2. Clearly the radius N_2 is related to a_2 and R_2 by the relation $N_2 = a_2^{-1} R_2$. Now one slides the circular sector thus obtained parallel down the x-axis a distance s_2 so that the circular arc with opening angle a_0 can be connected with this slide-down circular arc by a smooth curve of parabolic

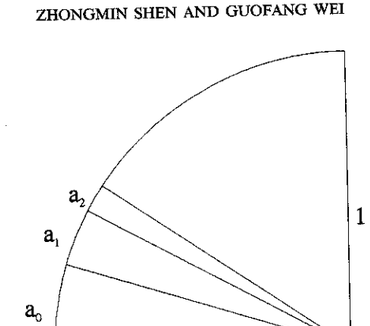

FIGURE 1

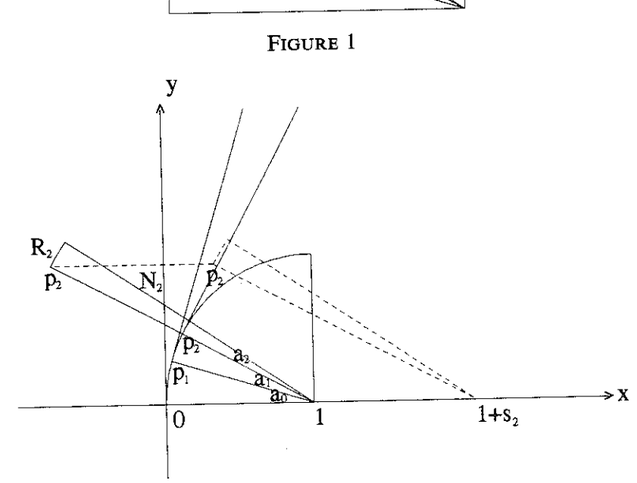

FIGURE 2

type. Elementary trigonometry shows that this is possible when

$$\frac{N_2 \cos a_1 - N_0}{\cos a_0} < s_2 < \frac{(N_2/N_0 - 1)N_2}{\cos(a_0 + a_1)},$$

i.e. p_2'' should lie in the region enclosed by the tangent lines at p_1 and p_2; see Figure 2.

Inductively, at the kth step, one enlarges the circular arc with the opening angle a_{2k} by going out radially so that its arclength is a prescribed number R_{2k}. Thus the radius $N_{2k} = a_{2k}^{-1} R_{2k}$. Now in order that the enlarged circular arc can be connected with the previous one already in position one has to slide it down a distance $s_0 + s_1 + \cdots + s_k$ where

$$\frac{N_{2k} \cos a_{2k-1} - N_{2k-2}}{\cos \sum_{j=0}^{2k-2} a_j} < s_k < \frac{(N_{2k}/N_{2k-2} - 1)N_{2k}}{\cos \sum_{j=0}^{2k-1} a_j}.$$

Now as is shown in [SY2], $\mathbb{R}^{n,m}$ will have nonnegative Ricci curvature if

$$R_k \sim a_k^{-2/\alpha}, \tag{11}$$

where $\alpha = 2(m-1)/n$.

Since the surgery reduces diameter, the diameter growth is controlled by the diameter growth of the hypersurface $\mathbb{R}^{m+1}$. By Lemma 4.1 we only need to find out the growth of the function that generates $\mathbb{R}^{m+1}$. From the above discussion, at $x = s_0 + s_1 + \cdots + s_k$, $f(x) = N_{2k} \sin \sum_{j=0}^{2k-1} a_j$. By taking $a_k = \frac{\pi}{4} 2^{-k}$, one easily finds $f(x) = O(x^\beta)$, where $\beta = \frac{\alpha+2}{2\alpha+2}$.

Thus for example, $\mathbb{R}^{2,2}$ $(m = 2,\ n = 2)$ will have diameter growth of degree $\frac{3}{4}$, hence volume growth of degree $\frac{5}{2}$. We remark that as $m \to \infty$, $\beta \to \frac{1}{2}$. Therefore the degree of diameter growth of such manifold could be arbitrarily close to $\frac{1}{2}$.

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