MONOTONICITY FORMULAS FOR BAKRY-EMERY RICCI CURVATURE

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Abstract. Motivated and inspired by the recent work of Colding [5] and Colding-Minicozzi [6] we derive several families of monotonicity formulas for manifolds with nonnegative Bakry-Emery Ricci curvature, extending the formulas in [5, 6].

1. Introduction

The Bakry-Emery Ricci tensor is a Ricci tensor for smooth metric measure spaces, which are Riemannian manifolds with measures conformal to the Riemannian measures. Formally a smooth metric measure space is a triple $(M^n, g, e^{-f}dvol_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on M. These spaces occur naturally as smooth collapsed limits of manifolds under the measured Gromov-Hausdorff convergence.

The N-Bakry-Emery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{N} df \otimes df \quad \text{for } 0 \le N \le \infty. \tag{1.1}$$

The purported dimension of the space is related to N, i.e. it is $n + N$. When $N = 0$, we assume f is constant and $\mathrm{Ric}_f^N = \mathrm{Ric}$, the usual Ricci curvature. When N is infinite, we denote $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f$. Note that if $N_1 \ge N_2$ then $\mathrm{Ric}_f^{N_1} \ge \mathrm{Ric}_f^{N_2}$ so $\mathrm{Ric}_f^N \ge \lambda g$ implies $\mathrm{Ric}_f \ge \lambda g$.

The Einstein equation $\mathrm{Ric}_f = \lambda g$ (λ a constant) is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow. On the other hand, the equation $\mathrm{Ric}_f^N = \lambda g$, for N a positive integer, corresponds to warped product Einstein metric on $M \times_{e^{-\frac{f}{N}}} F^N$, where F^N is some N dimensional Einstein manifold, see [2].

Recently Colding [5] and Colding-Minicozzi [6] introduced some new monotonicity formulas associated to positive Green's function of the Laplacian. These formulas are very useful and related to other known monotonicity formulas, see [7]. In particular, using one of the monotonicty formula Colding-Minicozzi showed that for any Ricci-flat manifold with Euclidean volume growth, tangent cones at infinity are unique as long as one tangent cone has a smooth cross-section [8]. Our paper is motivated and inspired by these work of Colding and Colding-Minicozzi.

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With respect to the measure $e^{-f}dvol$ the natural self-adjoint f-Laplacian is $\Delta_f = \Delta - \nabla f \cdot \nabla$. Consider the positive Green's function $G(x_0, \cdot)$ of the f-Laplacian of $(M^n, g, e^{-f}dvol)$ (see Definition 2.1). For any real number $k > 2$, let $b = G^{\frac{1}{2-k}}$. For $\beta, l, p \in \mathbb{R}$, when b is proper, we consider

$$A_f^\beta(r) = r^{1-l} \int_{b=r} |\nabla b|^{\beta+1} e^{-f},$$

$$V_f^{\beta,p}(r) = r^{p-l} \int_{b \le r} \frac{|\nabla b|^{2+\beta}}{b^p} e^{-f}.$$

While $A_f^\beta(r)$ is well defined for all $r > 0$, $V_f^{\beta,p}(r)$ is only well defined when

$$C(n,k,p) = (n-2)(k-p) - \beta(k-n) > 0. \tag{1.2}$$

See the proof of Lemma 4.1 for detail. When $k = l = n, \beta = 2, p = 0$, these reduce to $A(r), V(r)$ in [5]. When $k = l = n, p = 2$, these are A_β, V_β in [6].

First we obtain the following gradient estimate for b.

Proposition 1.1. *If a smooth metric measure space $(M^n, g, e^{-f}dvol)$ $(n \ge 3)$ has $\mathrm{Ric}_f^N \ge 0$, then for $k = n + N$, there exists $r_0 > 0$, such that on $M \setminus B(x_0, r_0)$,*

$$|\nabla b(y)| \le C(n, N, r_0). \tag{1.3}$$

Remark In [5, Theorem 3.1] Colding obtained the sharp estimate that if $\mathrm{Ric}_{M^n} \ge 0$ $(n \ge 3)$, then $|\nabla b| \le 1$ for $f = 0, k = n$ in above. From (2.2) this can not be true when $k > n$ as $|\nabla b(y)| \to \infty$ as $y \to x_0$. For $\mathrm{Ric}_f \ge 0$, $|\nabla b(y)|$ may not be bounded as $y \to \infty$, see Example 6.5.

We prove many families of monotonicity formulas, which, besides recovering the ones in [5, 6] when $k = l = n$ and f is constant, give some new ones even in this case. For example, when N is finite, we have

Theorem 1.2. *If $M^n (n \ge 3)$ has $\mathrm{Ric}_f^N \ge 0$, then, for $k \ge n + N$, $k \le l \le 2k - 2$, $\alpha = 3k - p - l - 2$ and $C(n,k,p) > 0$,*

$$\begin{aligned}(A_f^\beta - \alpha V_f^{\beta,p})'(r)\\ \ge r^{p-1-l} \int_{b \le r} \frac{\beta |\nabla b|^{\beta-2}}{4b^p} \left\{ \left| \mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g \right|^2 + 4(\beta-2) b^2 |\nabla |\nabla b||^2 \right\} e^{-f}.\end{aligned} \tag{1.4}$$

Hence if in addition $\beta \ge 2$, then $A_f^\beta - \alpha V_f^{\beta,p}$ is nondecreasing in r.

See discussion in Section 5 for full generality. Note that when $N = 0$ (i.e. f is constant and $\mathrm{Ric} \ge 0$), and $p = 0, \beta \ge 2$, $k = n$ we get monotonicity for all $n \le l \le 2n-2$; in the case when $\beta = 2, l = n$, this is the first monotonicity formula in [5].

Theorem 1.3. *If $M^n (n \ge 3)$ has $\mathrm{Ric}_f^N \ge 0$, then for $\beta \ge 2$, $k = l = n + N$, $(A_f^\beta)'(r) \le 0$ and $(V_f^{\beta,p})'(r) \le 0$ for $p < n + N - \frac{\beta N}{n-2}$. In fact*

$$(A_f^\beta)'(r) \le -\frac{\beta}{4} r^{k-3} \int_{b \ge r} b^{2-2k} |\nabla b|^{\beta-2} \left| \mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g \right|^2 e^{-f}. \tag{1.5}$$

Again this reduces to a formula in [5] when $k = l = n$ and f is constant, which is used in [8] to show that for any Ricci-flat manifold with Euclidean volume growth, tangent cones at infinity are unique as long as one tangent cone has a smooth cross-section.

Without any assumption on f we also establish monotonicity formulas when N is infinite. For example,

Theorem 1.4. *If $M^n (n \geq 4)$ has $\mathrm{Ric}_f \geq 0$, then for $\beta = 2, p = 0, k \geq 12, l = \frac{3}{2}k - 1$, we have $(A_f^\beta - (\frac{3}{2}k-1)V_f^{\beta,p})'(r) \geq 0$; for $\beta = 2, k \geq 12, l = \frac{3}{2}(k-1)$, $r_2 \geq r_1 > 0$, we have $(A_f^\beta)'(r_2) \geq (A_f^\beta)'(r_1)$.*

Here $l > k$. See Theorem 6.1 for more general statement.

When $k = l$, similar monotonicity is not true any more. In fact for $k = l = n$ we show while for $n = 3$ several monotonity still holds for the Bryant solitons for r large, it does not hold when $n \geq 5$. In fact it is monotone in the opposite direction for r large, see Example 6.5. With some condition on f we still get several monotonicity, see e.g. Corollary 6.4.

As in [6], one can use the term $\left|\mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g\right|^2$ to allow smaller β (one only needs $\beta \geq 1 - \frac{1}{n-1}$ instead of $\beta \geq 2$, see the last part of Section 5).

The paper is organized as follows. In the next section, we discuss the existence and the basic properties of Green's function for f-Laplacian and prove Proposition 1.1. In Section 3, using the following Bochner formula (see e.g. [17])

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{1.6}$$

we compute the f-laplacian of $(b^{2q}|\nabla b|^\beta)$, a key formula needed for deriving the monotonicity formulas in Section 4. In Section 5, 6 we apply these formulas to the case when $\mathrm{Ric}_f^N \geq 0$ for N finite and infinite respectively.

As with many other monotonicity formulas, we expect our formulas will have nice applications, especially for quasi-Einstein manifolds and steady gradient Ricci solitons.

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2. Green's function

Definition 2.1. *Given a smooth metric measure space $(M^n, g, e^{-f}dvol)$, $x_0 \in M$, $G = G(x_0, \cdot)$ is the Green's function of the f-Laplacian (with pole at x_0) if*

$$\Delta_f G = -\delta_{x_0}.$$

In [13, 10] it is shown that on any complete Riemannian manifold there exists a symmetric Green's function of the Laplacian. Same proof carries over for f-Laplacian.

M is called f-nonparabolic if it has positive Green's function. When $f = 0$ and $\mathrm{Ric}_M \geq 0$, the existence of positive Green's function is well understood [18], see also [11, 16]. Namely M is nonparabolic if and only if $\int_1^\infty \frac{r}{\mathrm{Vol} B(x_0,r)} dr < \infty$, and $G \to 0$ at infinity. Same result also holds when $\mathrm{Ric}_f^N \geq 0$ (N finite) or $\mathrm{Ric}_f \geq 0$

and f is bounded. This and other existence results will be studied in [1]. In there we observe that all nontrivial steady Ricci solitons are f-nonparabolic.

In this paper we assume M is f-nonparabolic and $G \to 0$ at infinity so b is proper. Also, near the pole (after normalization), for $n \geq 3$, by [9], we have

$$G(y) = d^{2-n}(x_0, y)(1 + o(1)), \quad |\nabla G(y)| = (n-2)d^{1-n}(x_0, y)(1 + o(1)), \tag{2.1}$$

where $o(1)$ is a function with $o(y) \to 0$ as $y \to x_0$.

Recall for $k > 2$, $b = G^{\frac{1}{2-k}}$. Then

$$b(y) = d^{\frac{n-2}{k-2}}(x_0, y)(1 + o(1)), \quad |\nabla b(y)| = \frac{n-2}{k-2} d^{\frac{n-k}{k-2}}(x_0, y)(1 + o(1)). \tag{2.2}$$

To prove Proposition 1.1, we need the following two propositions for $\mathrm{Ric}_f^N \geq 0$. Similar to the case when $\mathrm{Ric} \geq 0$, one has the following Laplacian comparison and gradient estimate for $\mathrm{Ric}_f^N \geq 0$ [14, 15], see also [17, 12]. Let $r(x) = d(x, x_0)$ be the distance function, then

$$\Delta_f r \leq \Delta_{\mathbb{R}^{n+N}} r = \frac{n+N}{r}. \tag{2.3}$$

If u is a f-harmonic function on $B(x, R)$, then on $B(x, R/2)$,

$$|\nabla \log u| \leq \frac{C(n+N)}{R}. \tag{2.4}$$

Proof of Proposition 1.1. For any $y \in M \setminus \{x_0\}$, G is a smooth harmonic function on $B(y, r)$ with $r = r(y) = d(y, x_0)$. By (2.4),

$$|\nabla \log G| \leq \frac{C(k)}{r} \quad \text{on} \quad B(y, r/2).$$

Now $|\nabla \log b| = \frac{1}{k-2}|\nabla \log G|$. Therefore

$$|\nabla b|(y) = |\nabla \log b| \cdot b \leq \frac{C(k)}{k-2} \cdot \frac{b}{r}.$$

Hence (1.3) follows if we show $b \leq C_1(n, N, r_0)r$, i.e. $G \geq C_2(n, N, r_0)r^{2-k}$ for some $r_0 > 0$ on $M \setminus B(x_0, r_0)$.

For any $\epsilon > 0$, we have on $M \setminus \{x_0\}$,

$$\Delta_f (G - \epsilon\, r^{2-k}) = -\epsilon\, \Delta_f (r^{2-k}) \leq -\epsilon\, \Delta_{\mathbb{R}^k}(r^{2-k}) = 0,$$

where the inequality follows from the Laplacian comparison (2.3). Since $\lim_{r\to\infty}(G - \epsilon\, r^{2-k}) = 0$. Also by (2.1), there exists $r_0 > 0$ small such that $G(y) \geq \frac{1}{2}r_0^{2-n}$ on $\partial B(x_0, r_0)$. Take $\epsilon = \frac{1}{2}r_0^{k-n}$, we have $(G - \frac{1}{2}r_0^{k-n}r^{2-k}) \geq 0$ on $\partial B(x_0, r_0)$. Therefore by the maximum principle we have $G(y) \geq \frac{1}{2}r_0^{k-n}r^{2-k}$ when $r(y) \geq r_0$. Namely $b \leq \left(2r_0^{n-k}\right)^{\frac{1}{k-2}} r$. □

3. The f-Laplacian of b and $|\nabla b|$

Lemma 3.1. *For any real number β,*

$$\Delta_f b = \frac{k-1}{b}|\nabla b|^2, \tag{3.5}$$

$$\Delta_f b^\beta = \beta(\beta + k - 2)b^{\beta-2}|\nabla b|^2. \tag{3.6}$$

In particular,

$$\Delta_f b^2 = 2k|\nabla b|^2. \tag{3.7}$$

Proof: For any positive function v, we have

$$\Delta_f v^\beta = \beta v^{\beta-1}\left[(\beta-1)\frac{|\nabla v|^2}{v} + \Delta_f v\right] \tag{3.8}$$

Since $\Delta_f b^{2-k} = 0$, this gives $(2-k-1)\frac{|\nabla b|^2}{b} + \Delta_f b = 0$, namely (3.5). Combining (3.8) and (3.5) gives (3.6). □

The following important formulas holds for any positive f-harmonic function G, not just Green's function.

Proposition 3.2.

$$\begin{aligned}\Delta_f|\nabla b|^\beta &= \frac{\beta}{4b^2}|\nabla b|^{\beta-2}\Big\{\big|\mathrm{Hess}\, b^2\big|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2) + 2(k-2)\langle\nabla b^2,\nabla|\nabla b|^2\rangle \\ &\quad +4(\beta-2)b^2|\nabla|\nabla b||^2 - 4k|\nabla b|^4\Big\}.\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\Delta_f(b^{2q}|\nabla b|^\beta) \\ &= \frac{\beta}{4}b^{2q-2}|\nabla b|^{\beta-2}\Bigg\{\big|\mathrm{Hess}\, b^2\big|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2) + 2(k-2+2q)\langle\nabla b^2,\nabla|\nabla b|^2\rangle \\ &+4(\beta-2)b^2|\nabla|\nabla b||^2 + \left[\frac{8q}{\beta}(k-2+2q) - 4k\right]|\nabla b|^4\Bigg\}\end{aligned} \tag{3.10}$$

Proof: Since the formulas are in terms of the function b^2, we first compute $\Delta_f|\nabla b^2|^2$. Applying the Bochner formula (1.6) to b^2 and using (3.7), we have

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla b^2|^2 &= |\mathrm{Hess}\, b^2|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2) + \langle\nabla b^2,\nabla(\Delta_f b^2)\rangle \\ &= |\mathrm{Hess}\, b^2|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2) + 2k\langle\nabla b^2,\nabla|\nabla b|^2\rangle.\end{aligned} \tag{3.11}$$

Now we compute $\Delta_f|\nabla b|^2$. Since $|\nabla b^2|^2 = 4b^2|\nabla b|^2$,

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla b^2|^2 &= 2\left(|\nabla b|^2\Delta_f b^2 + b^2\Delta_f|\nabla b|^2 + 2\langle\nabla b^2,\nabla|\nabla b|^2\rangle\right) \\ &= 4k|\nabla b|^4 + 2b^2\Delta_f|\nabla b|^2 + 4\langle\nabla b^2,\nabla(|\nabla b|^2)\rangle.\end{aligned}$$

Combine this with (3.11), we have

$$\begin{aligned}2b^2\Delta_f|\nabla b|^2 &= |\mathrm{Hess}\, b^2|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2) + (2k-4)\langle\nabla b^2,\nabla|\nabla b|^2\rangle \\ &\quad -4k|\nabla b|^4.\end{aligned} \tag{3.12}$$

Then,

$$\begin{aligned}\Delta_f|\nabla b|^\beta &= \Delta_f(|\nabla b|^2)^{\frac{\beta}{2}} \\ &= \frac{\beta}{2}|\nabla b|^{\beta-2}\Delta_f|\nabla b|^2 + \beta(\beta-2)|\nabla b|^{\beta-2}|\nabla|\nabla b||^2 \\ &= \frac{\beta}{4b^2}|\nabla b|^{\beta-2}\left\{|\mathrm{Hess}\, b^2|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2) + (2k-4)\langle\nabla b^2,\nabla|\nabla b|^2\rangle\right. \\ &\quad \left. +4(\beta-2)b^2|\nabla|\nabla b||^2 - 4k|\nabla b|^4\right\},\end{aligned}$$

which is (3.9).

For the second one, by the product formula for Laplacian and using (3.6), we get

$$\begin{aligned}\Delta_f(b^{2q}|\nabla b|^\beta) &= b^{2q}\Delta_f|\nabla b|^\beta + |\nabla b|^\beta\Delta_f(b^{2q}) + 2\langle\nabla|\nabla b|^\beta, \nabla b^{2q}\rangle \\ &= b^{2q}\Delta_f|\nabla b|^\beta + 2q(2q+k-2)b^{2q-2}|\nabla b|^{2+\beta} \\ &\quad +\beta q|\nabla b|^{\beta-2}b^{2q-2}\langle\nabla b^2, \nabla|\nabla b|^2\rangle.\end{aligned}$$

Plug in (3.9) we obtain (3.10). □

4. Monotonicity Formulas

Recall for $l, \beta, p \in \mathbb{R}$,

$$A_f^\beta(r) = r^{1-l}\int_{b=r}|\nabla b|^{\beta+1}e^{-f},$$

$$V_f^{\beta,p}(r) = r^{p-l}\int_{b\le r}\frac{|\nabla b|^{2+\beta}}{b^p}e^{-f}.$$

As $r \to 0$, we have the following information.

Lemma 4.1. *Let M^n be a smooth manifold with $n \geq 3$. Denote $C(n,k,l) = (k-l)(n-2)+(n-k)\beta$. Then*

$$\lim_{r\to 0} A_f^\beta(r) = \begin{cases} 0 & \text{if } C(n,k,l) > 0 \\ \left(\frac{n-2}{k-2}\right)^{1+\beta} Vol(\partial B_1(0))e^{-f(x_0)} & \text{if } C(n,k,l) = 0 \\ \infty & \text{if } C(n,k,l) < 0 \end{cases}, \tag{4.1}$$

$$\lim_{r\to 0} V_f^{\beta,p}(r) = \begin{cases} 0 & \text{if } C(n,k,l) > 0 \\ \left(\frac{n-2}{k-2}\right)^{1+\beta} \frac{n-2}{C(n,k,p)}\mathrm{Vol}(\partial B_1(0))e^{-f(x_0)} & \text{if } C(n,k,l) = 0 \\ \infty & \text{if } C(n,k,l) < 0 \end{cases}, \tag{4.2}$$

where $\mathrm{Vol}(\partial B_1(0))$ *is the volume of the unit sphere in* $\mathbb{R}^n$.

Proof: From (2.2),

$$\begin{aligned} A_f^\beta(r) &= r^{1-l}\int_{b=r}|\nabla b|^{1+\beta}e^{-f} \\ &= \left(\frac{n-2}{k-2}\right)^{1+\beta}\left(r^{1-l+\frac{n-k}{n-2}(1+\beta)+\frac{k-2}{n-2}(n-1)}\right)(1+o(1))e^{-f(x_0)}\mathrm{Vol}(\partial B_1(0)),\end{aligned}$$

where $o(1) \to 0$ as $r \to 0$. Note that $1-l+\frac{n-k}{n-2}(1+\beta)+\frac{k-2}{n-2}(n-1) = (k-l)(n-2)+(n-k)\beta$. This gives (4.1).

Similarly,

$$\begin{aligned} V_f^{\beta,p}(r) &= r^{p-l}\int_0^r\int_{b=s}\frac{|\nabla b|^{1+\beta}}{b^p}e^{-f} \\ &= \left(\frac{n-2}{k-2}\right)^{1+\beta} r^{p-l}\int_0^r s^{\frac{n-k}{n-2}(1+\beta)-p+\frac{k-2}{n-2}(n-1)}ds(1+o(1))\cdot Vol(\partial B_1(0))e^{-f(x_0)}\end{aligned}$$

The integral exists if the constant in (1.2), $C(n,k,p) > 0$, and

$$V_f^{\beta,p}(r) = \frac{n-2}{C(n,k,p)}\left(\frac{n-2}{k-2}\right)^{1+\beta} r^{\frac{C(n,k,l)}{n-2}}(1+o(1))Vol(\partial B_1(0))e^{-f(x_0)}$$

This gives (4.2). $\square$

Since

$$V_f^{\beta,p}(r) = r^{p-l}\int_0^r \int_{b=s} \frac{|\nabla b|^{1+\beta}}{b^p} e^{-f}.$$

we have

$$\begin{aligned}(V_f^{\beta,p})'(r) &= (p-l) r^{p-l-1} \int_0^r \int_{b=s} \frac{|\nabla b|^{1+\beta}}{b^p} e^{-f} + r^{p-l}\int_{b=r} \frac{|\nabla b|^{1+\beta}}{b^p} e^{-f} \\ &= \frac{p-l}{r} V_f^{\beta,p}(r) + \frac{1}{r} A_f^{\beta}(r). \qquad (4.3)\end{aligned}$$

To find the derivative of $A_f(r)$, we use the following formula.

Lemma 4.2. *For a smooth function $u : M \setminus x \to \mathbb{R}$, let*

$$I_u(r) = \int_{b=r} u|\nabla b|\, e^{-f}.$$

Then for any $0 < r_0 \le r$,

$$\begin{aligned} I_u'(r) &= \frac{k-1}{r} I_u(r) + \int_{b=r} \langle \nabla u, \nu\rangle e^{-f} \qquad (4.4)\\ &= \frac{k-1}{r} I_u(r) + \int_{r_0\le b\le r} (\Delta_f u)\, e^{-f} + \int_{b=r_0} \langle \nabla u, \nu\rangle e^{-f}, \qquad (4.5)\end{aligned}$$

where $\nu = \frac{\nabla b}{|\nabla b|}$ is the unit normal direction.

This formula can be derived using the diffeomorphisms generated by $\frac{\nabla b}{|\nabla b|^2}$, see [4, Appendix]. For completeness, we give a simple proof using just the divergent theorem and the co-area formula.

Proof: Note that

$$\begin{aligned} I_u(r) - I_u(r_0) &= \int_{b=r} u|\nabla b|\, e^{-f} - \int_{b=r_0} u|\nabla b|\, e^{-f} \\ &= \int_{r_0\le b\le r} \operatorname{div}(ue^{-f}\nabla b) \\ &= \int_{r_0}^r \int_{b=s} \frac{\operatorname{div}(ue^{-f}\nabla b)}{|\nabla b|}.\end{aligned}$$

Take derivative of this equation both sides with respect to r and using (3.5) gives

$$\begin{aligned} I_u' &= \int_{b=r} \frac{\operatorname{div}(ue^{-f}\nabla b)}{|\nabla b|} \\ &= \int_{b=r} \frac{\langle \nabla u, \nabla b\rangle e^{-f} + ue^{-f}\Delta_f b}{|\nabla b|} \\ &= \int_{b=r} \langle \nabla u, \frac{\nabla b}{|\nabla b|}\rangle e^{-f} + \int_{b=r} ue^{-f}\frac{k-1}{b}|\nabla b| \\ &= \int_{r_0\le b\le r} (\Delta_f u)\, e^{-f} + \int_{b=r_0} \langle \nabla u, \frac{\nabla b}{|\nabla b|}\rangle e^{-f} + \frac{k-1}{r}\int_{b=r} u|\nabla b|\, e^{-f}.\end{aligned}$$

$\square$

In order to match the derivative of A_f^{β} with $V_f^{\beta,p}$, we write $A_f^{\beta}(r) = r^{p-l-1}\int_{b=r} b^{2-p}|\nabla b|^{\beta+1} e^{-f}$.

Applying (4.5) to $u = b^{2-p}|\nabla b|^{\beta}$, if $C(n,k,p) > 0$, the integral on $b = r_0$ goes to zero as $r_0 \to 0$, and we have

Corollary 4.3.

$$(A_f^\beta)'(r) = \frac{k-l-2+p}{r} A_f^\beta(r) + r^{p-1-l}\int_{b\le r}\left(\Delta_f(b^{2-p}|\nabla b|^\beta)\right)e^{-f}. \tag{4.6}$$

As in [5] we derive a formula which will give the first monotonicity.

Theorem 4.4. *When $k>2$, $C(n,k,p)>0$, for any $\alpha\in\mathbb{R}$,*

$$\begin{aligned}(A_f^\beta - \alpha V_f^{\beta,p})'(r) = &\\ & r^{p-1-l}\int_{b\le r}\frac{\beta|\nabla b|^{\beta-2}}{4b^p}\left\{\left|\operatorname{Hess} b^2\right|^2 + \operatorname{Ric}_f(\nabla b^2,\nabla b^2) + 4(\beta-2)b^2|\nabla|\nabla b||^2\right\}e^{-f}\\ &+\frac{1}{r}\left(\lambda_1 A_f^\beta(r) + \lambda_2 V_f^{\beta,p}(r)\right),\end{aligned} \tag{4.7}$$

where

$$\lambda_1 = 3k-p-l-2-\alpha, \tag{4.8}$$

$$\lambda_2 = (p+2-2k)(k-p)-\beta k-\alpha(p-l). \tag{4.9}$$

Proof: From (4.6) we would like to compute $\int_{b\le r}\left(\Delta_f(b^{2-p}|\nabla b|^\beta)\right)e^{-f}$. By (3.10),

$$\begin{aligned}\Delta_f(b^{2-p}|\nabla b|^\beta) &= \frac{\beta|\nabla b|^{\beta-2}}{4b^p}\left\{\left|\operatorname{Hess} b^2\right|^2 + \operatorname{Ric}_f(\nabla b^2,\nabla b^2) + 2(k-p)\langle\nabla b^2,\nabla|\nabla b|^2\rangle\right.\\ &\quad\left. +4(\beta-2)b^2|\nabla|\nabla b||^2 + \left[\frac{8-4p}{\beta}(k-p)-4k\right]|\nabla b|^4\right\}.\end{aligned} \tag{4.10}$$

To compute the third term, by Stokes' theorem and (3.6),

$$\begin{aligned}&\int_{b\le r} b^{-p}\frac{\beta}{2}|\nabla b|^{\beta-2}\langle\nabla b^2,\nabla|\nabla b|^2\rangle e^{-f}\\ &= \int_{b\le r}\frac{2}{2-p}\langle\nabla b^{2-p},\nabla\left(|\nabla b|^2\right)^{\beta/2}\rangle e^{-f}\\ &= \int_{b=r} b^{-p}\langle\nabla b^2,\frac{\nabla b}{|\nabla b|}\rangle|\nabla b|^\beta e^{-f} - \frac{2}{2-p}\int_{b\le r}\Delta_f(b^{2-p})|\nabla b|^\beta e^{-f}\\ &= 2r^{1-p}\int_{b=r}|\nabla b|^{\beta+1}e^{-f} - 2(k-p)\int_{b\le r} b^{-p}|\nabla b|^{\beta+2}e^{-f}\\ &= 2r^{l-p}A_f^\beta + 2(p-k)r^{l-p}V_f^{\beta,p}.\end{aligned} \tag{4.11}$$

In the above proof we assume $p\neq 2$. By taking limit of (4.11) as $p\to 2$, we see (4.11) also holds for $p=2$. In the second equality, we use $C(n,k,p)>0$ so the integral for $r\to 0$ is zero.

Combining (4.11),(4.10), (4.6) and (4.3), we have

$$\begin{aligned}(A_f^\beta - \alpha V_f^{\beta,p})'(r) = &\\ & r^{p-1-l}\int_{b\le r}\frac{\beta|\nabla b|^{\beta-2}}{4b^p}\left\{\left|\operatorname{Hess} b^2\right|^2 + \operatorname{Ric}_f(\nabla b^2,\nabla b^2) + 4(\beta-2)b^2|\nabla|\nabla b||^2\right\}e^{-f}\\ &+\frac{3k-p-l-2-\alpha}{r}A_f^\beta(r) + \frac{1}{r}\left[(p+2-2k)(k-p)-\beta k-\alpha(p-l)\right]V_f^{\beta,p}(r).\end{aligned}$$

□

Letting $k=l, \alpha=2k-p-2$ in (4.7) gives

Corollary 4.5. *When $k>2$, $C(n,k,p)>0$, and $k=l$, we have*

$$(A_f^\beta-(2k-p-2)V_f^{\beta,p})'(r)=r^{p-1-l}\int_{b\le r}\frac{\beta|\nabla b|^{\beta-2}}{4b^p}\Big\{\left|\mathrm{Hess}\, b^2\right|^2 + \mathrm{Ric}_f(\nabla b^2,\nabla b^2)+4(\beta-2)b^2|\nabla|\nabla b||^2-4k|\nabla b|^4\Big\}e^{-f}. \tag{4.12}$$

Following is a formula which will give second monotonicity formula.

Theorem 4.6. *For $c,d\in\mathbb{R}$, let $g(r)=r^c\left(r^dA_f^\beta(r)\right)'$. Then for $0<r_1<r_2$,*

$$\begin{aligned} g(r_2)-g(r_1) &= \int_{r_1\le b\le r_2}\frac{\beta}{4}b^{c+d-l-1}|\nabla b|^{\beta-2}\Big\{|\mathrm{Hess}\, b^2|^2+\mathrm{Ric}_f(\nabla b^2,\nabla b^2)+\lambda_3|\nabla b|^4 \\ &+ 4(\beta-2)b^2|\nabla|\nabla b||^2\Big\}e^{-f}+\lambda_4\int_{r_1\le b\le r_2}b^{c+d-l}\langle\nabla b,\nabla|\nabla b|^\beta\rangle e^{-f}, \end{aligned} \tag{4.13}$$

where

$$\lambda_3 = \frac{4}{\beta}(k+d-l+c-1)(k+d-l)-4k, \tag{4.14}$$

$$\lambda_4 = 3k-2l-3+c+2d. \tag{4.15}$$

Proof: From (4.4) with $u=|\nabla b|^\beta$, we have

$$(A_f^\beta)'(r)=r^{1-l}\int_{b=r}\langle\nabla|\nabla b|^\beta,\nu\rangle e^{-f}+(k-l)r^{-l}\int_{b=r}|\nabla b|^{1+\beta}e^{-f}.$$

Hence

$$g(r)=\int_{b=r}b^{c+d+1-l}\langle\nabla|\nabla b|^\beta,\nu\rangle e^{-f}+(k+d-l)\int_{b=r}b^{c+d-l}|\nabla b|^{1+\beta}e^{-f},$$

and

$$\begin{aligned} g(r_2)-g(r_1) &= \int_{r_1\le b\le r_2}\mathrm{div}(b^{c+d+1-l}e^{-f}\nabla|\nabla b|^\beta) \\ &\quad +(k+d-l)\int_{r_1\le b\le r_2}\mathrm{div}(b^{c+d-l}|\nabla b|^\beta e^{-f}\nabla b). \end{aligned}$$

Since

$$\begin{aligned} &\mathrm{div}(b^{c+d+1-l}e^{-f}\nabla|\nabla b|^\beta) \\ &\quad = (c+d+1-l)b^{c+d-l}\langle\nabla b,\nabla|\nabla b|^\beta\rangle e^{-f}+b^{c+d+1-l}(\Delta_f|\nabla b|^\beta)e^{-f} \end{aligned}$$

and

$$\begin{aligned} \mathrm{div}(b^{c+d-l}|\nabla b|^\beta e^{-f}\nabla b) &= (c+d-l)b^{c+d-1-l}|\nabla b|^{2+\beta}e^{-f}+ \\ &\quad b^{c+d-l}\langle\nabla|\nabla b|^\beta,\nabla b\rangle e^{-f}+b^{c+d-l}|\nabla b|^\beta(\Delta_f b)e^{-f}, \end{aligned}$$

plugging $\Delta_f|\nabla b|^\beta$ and $\Delta_f b$ with (3.9) and (3.5) to the above, we get

$$\begin{aligned}
&g(r_2)-g(r_1)\\
&= \int_{r_1\le b\le r_2}\Big\{\frac{\beta}{4}b^{c+d-1-l}|\nabla b|^{\beta-2}\Big[\left|\operatorname{Hess} b^2\right|^2+\operatorname{Ric}_f(\nabla b^2,\nabla b^2)-4k|\nabla b|^4+\\
&\quad 4(\beta-2)b^2|\nabla|\nabla b||^2\Big]e^{-f}+(2k-3+c+d-l)b^{c+d-l}\langle\nabla b,\nabla|\nabla b|^\beta\rangle e^{-f}\Big\}\\
&\quad +(k+d-l)\int_{r_1\le b\le r_2} b^{c+d-l}\left[(c+d-l+k-1)b^{-1}|\nabla b|^{2+\beta}+\langle\nabla b,\nabla|\nabla b|^\beta\rangle\right]e^{-f}.
\end{aligned}$$

This is (4.13) after grouping. □

5. Monotonicity for $\operatorname{Ric}_f^N\ge 0$

From Theorem 4.4 and Theorem 4.6, if $\operatorname{Ric}_f^N\ge 0$, we get many families of monotonicity quantities.

Since $\operatorname{Ric}_f(\nabla b^2,\nabla b^2)=\operatorname{Ric}_f^N(\nabla b^2,\nabla b^2)+\frac{\langle\nabla b^2,\nabla f\rangle^2}{N}$ and

$$|\operatorname{Hess} b^2|^2=\left|\operatorname{Hess} b^2-\frac{\Delta b^2}{n}g\right|^2+\frac{(\Delta b^2)^2}{n}, \tag{5.1}$$

we have

$$\begin{aligned}
\left|\operatorname{Hess} b^2\right|^2+\operatorname{Ric}_f(\nabla b^2,\nabla b^2) &\ge \frac{(\Delta b^2)^2}{n}+\frac{\langle\nabla b^2,\nabla f\rangle^2}{N}\\
&\ge \frac{\left(\Delta_f b^2\right)^2}{n+N}=\frac{4k^2}{n+N}|\nabla b|^4.
\end{aligned}$$

Here we used the basic inequality $\frac{a^2}{p}+\frac{b^2}{q}\ge\frac{(a+b)^2}{p+q}$.

Therefore, by Theorem 4.4, $(A_f^\beta-\alpha V_f^{\beta,p})'(r)\ge 0$ if $\beta\ge 2$, $\lambda_1\ge 0$ and $\lambda_2+\frac{\beta k^2}{n+N}\ge 0$. There are many solutions to these.

For example, if we let $\lambda_1=0$, namely $\alpha=3k-p-l-2$, and $k\ge n+N$, then $\lambda_2+\frac{\beta k^2}{n+N}\ge 0$ when $k\le l\le 2k-2$,which gives Theorem 1.2.

Letting $k=l=n+N$ in Theorem 1.2, we get

Corollary 5.1. *If $M^n (n\ge 3)$ has $\operatorname{Ric}_f^N\ge 0$, then, for $k=l=n+N$, $\beta\ge 2$, $p<n+N-\frac{\beta N}{n-2}$, and $0<r_1<r_2$,*

$$(A_f^\beta-(2n+2N-p-2)V_f^{\beta,p})(r_2)\ge(A_f^\beta-(2n+2N-p-2)V_f^{\beta,p})(r_1).$$

Similarly, by Theorem 4.6, $g(r)=r^c\left(r^dA_f^\beta(r)\right)'$ is nondecreasing if $\beta\ge 2,\lambda_3+\frac{4k^2}{n+N}\ge 0$ and $\lambda_4=0$. Again there are many solutions. $\lambda_4=0$ requires that $c+2d=3-3k+2l$. When $c=k-1,d=l-2k+2$ or $c=3-3k+2l,d=0$, and $k=l=n+N$, we have

Proposition 5.2. *If $M^n (n\ge 3)$ has $\operatorname{Ric}_f^N\ge 0$, then for $0<r_1<r_2$, $k=l=n+N$,*

$$\begin{aligned}
&r_2^{k-1}(r^{2-k}A_f^\beta)'(r_2)-r_1^{k-1}(r^{2-k}A_f^\beta)'(r_1)\\
&\ge \int_{r_1\le b\le r_2}\frac{\beta}{4}b^{-k}|\nabla b|^{\beta-2}\left\{\left|\operatorname{Hess} b^2-\frac{\Delta b^2}{n}g\right|^2+4(\beta-2)b^2|\nabla|\nabla b||^2\right\}e^{-f},
\end{aligned}$$

and

$$r_2^{3-k}(A_f^\beta)'(r_2) - r_1^{3-k}(A_f^\beta)'(r_1)$$
$$\geq \int_{r_1\le b\le r_2} \frac{\beta}{4} b^{2-2k}|\nabla b|^{\beta-2}\left\{\left|\mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g\right|^2 + 4(\beta-2)b^2|\nabla|\nabla b||^2\right\} e^{-f}. \tag{5.2}$$

Again, when $N = 0$, $\beta = 2$, these are the second and third monotonicity formula in [5].

Now we prove Theorem 1.3 which we restate it here.

Theorem 5.3. *If $M^n (n \geq 3)$ has $\mathrm{Ric}_f^N \geq 0$, then for $\beta \geq 2$, $k = l = n + N$, $(A_f^\beta)'(r) \leq 0$ and $(V_f^{\beta,p})'(r) \leq 0$ for $p < n + N - \frac{\beta N}{n-2}$. In fact*

$$(A_f^\beta)'(r) \leq -\frac{\beta}{4} r^{k-3} \int_{b\geq r} b^{2-2k}|\nabla b|^{\beta-2}\left|\mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g\right|^2 e^{-f}. \tag{5.3}$$

Proof: First we show $A_f^\beta(r)$ is bounded as $r \to \infty$. By (1.3), $|\nabla b(y)| \leq C(n, N, r_0)$ on $M \setminus B(x_0, r_0)$. Hence for $r \geq r_0$,

$$\begin{aligned} A_f^\beta(r) &= r^{1-k}\int_{b=r} |\nabla b|^{1+\beta} e^{-f} \\ &\leq C(n, N, r_0, \beta) r^{1-k} \int_{b=r} |\nabla b| e^{-f}. \end{aligned}$$

Define $h(r) = r^{1-k}\int_{b=r} |\nabla b|e^{-f}$, by (4.4)

$$h'(r) = r^{1-k}\int_{b=r} \langle \nabla 1, \nu\rangle e^{-f} = 0.$$

By (4.1), $\lim_{r\to 0} h(r) = \frac{n-2}{k-2}\mathrm{Vol}(\partial B_1(0))e^{-f(x_0)}$. Therefore

$$A_f^\beta(r) \leq C(n, N, r_0, \beta)e^{-f(x_0)} \quad \text{for } r \geq r_0. \tag{5.4}$$

Next we prove that $(A_f^\beta)'(r) \leq 0$. By (5.2), for $0 < r_1 < r_2$

$$r_2^{3-k}(A_f^\beta)'(r_2) \geq r_1^{3-k}(A_f^\beta)'(r_1).$$

If there is some $\bar{r} > 0$ such that $(A_f^\beta)'(\bar{r}) > 0$, then for all $r \geq \bar{r}$,

$$(A_f^\beta)'(r) \geq \left(\frac{r}{\bar{r}}\right)^{k-3}(A_f^\beta)'(\bar{r}) \geq (A_f^\beta)'(\bar{r}) > 0.$$

Namely $A_f^\beta(r) \to \infty$ as $r \to \infty$, contradicting to (5.4).

To show $(V_f^{\beta,p})'(r) \leq 0$, note that by Corollary 4.12, $(A_f^\beta - (2n + 2N - p - 2)V_f^{\beta,p})'(r) \geq 0$. Hence

$$(2n + 2N - p - 2)(V_f^{\beta,p})'(r) \leq (A_f^\beta)'(r) \leq 0.$$

Now (5.3) follows from (5.2) since the fact that $(A_f^\beta)'(r) \leq 0$ and $A_f^\beta(r)$ is bounded as $r \to \infty$ imply there are sequence $r_j \to \infty$ such that $r_j^{3-k}(A_f^\beta)'(r_j) \to 0$. □

As in [6], one can decompose $\left|\mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g\right|^2$ further more to allow smaller β. Denote

$$B = \mathrm{Hess}\, b^2 - \frac{\Delta b^2}{n} g, \tag{5.5}$$

the trace free symmetric bilinear form, where g is the Riemannian metric. One can decompose B into the normal and tangential components. Let B_0, g_0 be the restriction of B, g to the level set of b, define $B(\nu)$ the vector such that $\langle B(\nu), v\rangle = B(\nu, v)$, and $B(\nu)^T$ its tangential component, so $|B(\nu)|^2 = (B(\nu,\nu))^2 + \left|B(\nu)^T\right|^2$. Then

$$\begin{aligned} |B|^2 &= (B(\nu,\nu))^2 + 2\left|B(\nu)^T\right|^2 + |B_0|^2 \\ &= |B(\nu)|^2 + |B(\nu)^T|^2 + \frac{(\mathrm{tr} B_0)^2}{n-1} + \left|B_0 - \frac{\mathrm{tr} B_0}{n-1} g_0\right|^2 \\ &= \frac{n}{n-1}|B(\nu)|^2 + \frac{n-2}{n-1}\left|B(\nu)^T\right|^2 + \left|B_0 - \frac{\mathrm{tr} B_0}{n-1} g_0\right|^2. \end{aligned} \tag{5.6}$$

Here we used the fact that B is trace free so $\mathrm{tr} B_0 = -B(\nu,\nu)$. Now

Lemma 5.4.

$$|B(\nu)|^2 = 4b^2|\nabla|\nabla b||^2 + \lambda^2 + 4\lambda b\langle\nabla|\nabla b|, \nu\rangle, \tag{5.7}$$

where $\lambda = 2|\nabla b|^2 - \frac{\Delta b^2}{n} = 2(1-\frac{k}{n})|\nabla b|^2 - \frac{1}{n}\langle\nabla b^2, \nabla f\rangle$.

Proof: Since

$$\mathrm{Hess}\, b^2 = 2b\mathrm{Hess}\, b + 2\nabla b \otimes \nabla b,$$

we have

$$\begin{aligned} B(\nabla b) &= 2b\mathrm{Hess}\, b(\nabla b) + 2|\nabla b|^2\nabla b - \frac{\Delta b^2}{n}\nabla b \\ &= b\nabla|\nabla b|^2 + \lambda\nabla b, \end{aligned}$$

and

$$B(\nu) = 2b\nabla|\nabla b| + \lambda\nu.$$

Hence

$$|B(\nu)|^2 = 4b^2|\nabla|\nabla b||^2 + \lambda^2 + 4\lambda b\langle\nabla|\nabla b|, \nu\rangle.$$

□

Therefore we have

Remark When $\lambda = 0$ (as in the case in [6]) or $\langle\nabla|\nabla b|, \nu\rangle = 0$, $|B|^2 \geq \frac{4n}{n-1} b^2 |\nabla|\nabla b||^2$. We also get all above monotonicity for $\beta \geq 1 - \frac{1}{n-1}$ instead of $\beta \geq 2$.

6. Monotonicity for $\mathrm{Ric}_f \geq 0$

When one only assumes $\mathrm{Ric}_f \geq 0$, one does not have the extra term $\frac{\langle\nabla b^2, \nabla f\rangle^2}{N}$ to combine with $\left|\mathrm{Hess}\, b^2\right|^2$ to get $(\Delta_f b^2)^2$. First we study the monotonicity without using $\left|\mathrm{Hess}\, b^2\right|^2$. Interestingly we still get many monotonic formulas.

Theorem 6.1. *If $M^n (n \geq 3)$ has $\mathrm{Ric}_f \geq 0$, then for $\beta \geq 2$, k, p such that $C(n,k,p) > 0, (k-2)^2 - 4\beta k \geq 0$, $l_1 \leq l \leq l_2$, where l_1 and l_2 are the solutions of $l^2 + (2-3k)l + 2k^2 - 2k + \beta k = 0$, and $\alpha = 3k - p - l - 2$, we have $(A_f^\beta - \alpha V_f^{\beta,p})'(r) \geq 0$.*

Proof: From (4.4), we only need to make sure $\lambda_1 \geq 0, \lambda_2 \geq 0$. By the choice of α we have $\lambda_1 = 0$ and $\lambda_2 \geq 0$ if and only if $l^2 + (2-3k)l + 2k^2 - 2k + \beta k \leq 0$. This has real solution when $(k-2)^2 - 4\beta k \geq 0$. □

When $\beta = 2, p = 0, n \geq 4$, then for any $k \geq 12$, $l = \frac{3}{2}k - 1$ is between the l_1, l_2 occurred above, which is the first part of Theorem 1.4.

Similarly, by Theorem 4.6, if $\beta = 2, c = d = 0, l = \frac{3}{2}(k-1), k \geq 12$, then $\lambda_4 = 0, \lambda_3 > 0$, we get the second statement in Theorem 1.4.

To get monotonicity for $\mathrm{Ric}_f \geq 0$ when $k = l$, we need to use the $\left|\mathrm{Hess}\, b^2\right|^2$ term. Here are some formulas when $k = l = n$, which recover the formulas in [6] when f is constant and $p = 2$.

Theorem 6.2. *When $k = l = n, p < n$, we have*

$$\begin{aligned}
&(A_f^\beta - (2n-p-2)V_f^{\beta,p})'(r) \\
&\quad = r^{p-1-n} \int_{b\leq r} \frac{\beta |\nabla b|^{\beta-2}}{4b^p} \Big\{ |B|^2 + \mathrm{Ric}_f(\nabla b^2, \nabla b^2) \\
&\quad +4(\beta-2)b^2 |\nabla|\nabla b||^2 + 4|\nabla b|^2 \langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n} \Big\} e^{-f},
\end{aligned} \tag{6.1}$$

where B is given in (5.5).

Proof: From (4.12)

$$\begin{aligned}
(A_f^\beta - (2n-p-2)V_f^{\beta,p})'(r) = r^{p-1-n} \int_{b\leq r} \frac{\beta |\nabla b|^{\beta-2}}{4b^p} \Big\{ \left|\mathrm{Hess}\, b^2\right|^2 \\
+\mathrm{Ric}_f(\nabla b^2, \nabla b^2) + 4(\beta-2)b^2 |\nabla|\nabla b||^2 - 4n|\nabla b|^4 \Big\} e^{-f}.
\end{aligned}$$

By (5.1) and (3.7),

$$\begin{aligned}
\left|\mathrm{Hess}\, b^2\right|^2 &= |B|^2 + \frac{(\Delta_f b^2 + \langle \nabla b^2, \nabla f\rangle)^2}{n} \\
&= |B|^2 + 4n|\nabla b|^4 + 4|\nabla b|^2 \langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n}.
\end{aligned}$$

Plug this into above gives (6.1). □

Similarly, letting $k = l = n,\ d = 0,\ c = 3-n$ in Theorem 4.6 gives

Theorem 6.3. *When $k = l = n$, for $0 < r_1 < r_2$,*

$$\begin{aligned}
&r_2^{3-n}(A_f^\beta)'(r_2) - r_1^{3-n}(A_f^\beta)'(r_1) \\
&\quad = \int_{r_1 \leq b \leq r_2} \frac{\beta}{4} b^{2-2n} |\nabla b|^{\beta-2} \Big\{ |B|^2 + \mathrm{Ric}_f(\nabla b^2, \nabla b^2) \\
&\quad +4(\beta-2)b^2 |\nabla|\nabla b||^2 + 4|\nabla b|^2 \langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n} \Big\} e^{-f}.
\end{aligned}$$

Corollary 6.4. *If $\mathrm{Ric}_f \geq 0$, $\beta \geq 2$, $k = l = n$, $p < n$, and*

$$|B|^2 + 4|\nabla b|^2 \langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n} \geq 0, \tag{6.2}$$

then

$$(A_f^\beta - (2n-p-2)V_f^{\beta,p})'(r) \geq 0$$

and for $0 < r_1 < r_2$,

$$r_2^{3-n}(A_f^\beta)'(r_2) \geq r_1^{3-n}(A_f^\beta)'(r_1).$$

Note (6.2) holds in particular if $\langle \nabla b^2, \nabla f\rangle \geq 0$ or $\langle \nabla b^2, \nabla f\rangle \leq -4n|\nabla b|^2$. In general $|B|^2 + 4|\nabla b|^2\langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n}$ could be negative and monotonicity in Corollary 6.4 could even be reversed. We illustrate this in the following example.

Example 6.5. *Bryant soliton is a rotationally symmetric steady gradient Ricci soliton. It is* $\mathbb{R}^n (n \geq 3)$ *with the metric*

$$g = dr^2 + \phi(r)^2 g_{S^{n-1}} \tag{6.3}$$

where, as $r \to \infty$, $C^{-1}r^{1/2} \leq \phi(r) \leq Cr^{1/2}$ *and* $\phi'(r) = O(r^{-1/2})$ *for some positive constant* C. *And the potential function* $f(r) = -r + O(\ln r)$ *as* $r \to \infty$. *See e.g.* [3].

Then for $r >> 0$, $|B|^2 + 4|\nabla b|^2\langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n}$ *is positive when* $n = 3$ *and negative when* $n \geq 5$, *showing that for all* $r_1 \leq r_2$ *big,*

$$\begin{aligned} r_2^{3-n}(A_f^\beta)'(r_2) &\geq r_1^{3-n}(A_f^\beta)'(r_1) \text{ if } n = 3, \\ r_2^{3-n}(A_f^\beta)'(r_2) &\leq r_1^{3-n}(A_f^\beta)'(r_1) \text{ if } n \geq 5. \end{aligned}$$

Proof: The Green's function G of the f-Laplacian with pole at the origin is $G = G(r)$, with

$$G'' + (n-1)\frac{\phi'}{\phi}G' - f'G' = 0.$$

Hence $G' = -e^f\phi^{1-n}$, and $G(r) = \int_r^\infty e^f\phi^{1-n}ds$, both go to 0 as $r \to \infty$.

With $b = G^{\frac{1}{2-n}}$, we have $\frac{b'}{b} = \frac{1}{2-n}\frac{G'}{G}$, and

$$\begin{aligned} \frac{b''}{b'} &= \frac{G''}{G'} - \frac{G'}{G} + \frac{b'}{b} \\ &= f' - (n-1)\frac{\phi'}{\phi} - \frac{(n-1)G'}{(n-2)G}. \end{aligned}$$

Combine this with (5.6) and (5.7), we have

$$\begin{aligned} |B|^2 &= \frac{n}{n-1}|B(\nu)|^2 = \frac{n}{n-1}\left|2b\nabla|\nabla b| - \frac{1}{n}\langle \nabla b^2, \nabla f\rangle \nu\right|^2 \\ &= \frac{n}{n-1}\left(2bb'' - \frac{1}{n}2bb'f'\right)^2 \\ &= (2bb')^2 n(n-1)\left(\frac{G'}{(2-n)G} + \frac{1}{n}f' - \frac{\phi'}{\phi}\right)^2. \end{aligned}$$

Now

$$4|\nabla b|^2\langle \nabla b^2, \nabla f\rangle = (2bb')^2 2f'\frac{b'}{b} = (2bb')^2\frac{2}{2-n}f'\frac{G'}{G}.$$

Hence

$$\begin{aligned} &|B|^2 + 4|\nabla b|^2\langle \nabla b^2, \nabla f\rangle + \frac{\langle \nabla b^2, \nabla f\rangle^2}{n} \\ &= (2bb')^2\left[n(n-1)\left(\frac{G'}{(2-n)G} + \frac{1}{n}f' - \frac{\phi'}{\phi}\right)^2 - \frac{2}{n-2}f'\frac{G'}{G} + \frac{(f')^2}{n}\right]. \end{aligned}$$

Since $\lim_{r\to\infty} f'(r) = -1$, $\lim_{r\to\infty} \frac{\phi'}{\phi} = 0$ and using L'Hospital's rule

$$\lim_{r\to\infty} \frac{G'}{G} = \lim_{r\to\infty} \frac{G''}{G'} = \lim_{r\to\infty} (f' + (1-n)\frac{\phi'}{\phi}) = -1,$$

we see that

$$\lim_{r\to\infty} \left[n(n-1) \left(\frac{G'}{(2-n)G} + \frac{1}{n} f' - \frac{\phi'}{\phi} \right)^2 - \frac{2}{n-2} f' \frac{G'}{G} + \frac{(f')^2}{n} \right] = \frac{-n^2+4n}{n(n-2)^2},$$

which is positive when $n = 3$ and negative when $n \geq 5$.

Now the last claim follows from Theorem 6.3. $\square$

References

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