VIASM SUMMER SCHOOL IN DIFFERENTIAL GEOMETRY 2023 LECTURE NOTES

GUOFANG WEI

TITLE: SOME RECENT RESULTS FOR SPACES WITH RICCI CURVATURE LOWER BOUNDS

Abstract. Recently there are tremendous developments in the study of manifolds with Ricci curvature lower bounds, their Gromov-Hausdorff limits (Ricci limit spaces), and/or RCD spaces. We will first recall the basic tools like Bochner formula, Bishop-Gromov volume comparison and some generalizations, and Cheng-Yau's gradient estimate, then go on to the almost volume rigidity. Then we will present examples showing the Busemann function of manifolds with nonnegative Ricci curvature may not be proper (which was open since the seventies), the Hausdorff dimension of singular set of Ricci limit spaces may be bigger than the Hausdorff dimension of the regular set, which answer a question of Cheeger-Colding more than twenty years ago. In the end we will present the topological result that Ricci limit space/RCD spaces are semi-locally simply connected.

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For general references about manifolds with Ricci curvature bounds, see [5, 25].

1. Lecture 1: Basic Tools for Ricci Curvature

1.1. **Bochner formula.** For a smooth function u on a Riemannian manifold (M^n, g), the gradient of u is the vector field ∇u such that $\langle \nabla u, X\rangle = X(u)$ for all vector fields X on M. The Hessian of u is the symmetric bilinear form

$$\mathrm{Hess}\,(u)(X,Y) = XY(u) - \nabla_X Y(u) = \langle \nabla_X \nabla u, Y\rangle,$$

and the Laplacian is the trace $\Delta u = \mathrm{tr}(\mathrm{Hess}\ u)$. For a bilinear form A, we denote $|A|^2 = \mathrm{tr}(AA^t)$.

The Bochner formula for functions is

Theorem 1.1 (Bochner's Formula)**.** For a smooth function u on a Riemannian manifold (M^n, g),

$$\frac{1}{2}\Delta|\nabla u|^2 = |\,\mathrm{Hess}\ u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{1.1}$$

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The Bochner formula simplifies whenever $|\nabla u|$ or Δu are simply. Hence it is natural to apply it to the distance functions, harmonic functions, and the eigenfunctions among others, getting many applications.

Proof. We can derive the formula by using local geodesic frame and commuting the derivatives. Fix $x \in M$, let $\{e_i\}$ be an orthonormal frame in a neighborhood of x such that, at x, $\nabla_{e_i} e_j(x) = 0$ for all i, j. At x,

$$\begin{aligned} \frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\sum_i e_i e_i \langle \nabla u, \nabla u\rangle \\ &= \sum_i e_i \langle \nabla_{e_i} \nabla u, \nabla u\rangle = \sum_i e_i \,\text{Hess } u(e_i, \nabla u) \\ &= \sum_i e_i \,\text{Hess } u(\nabla u, e_i) = \sum_i e_i \langle \nabla_{\nabla u} \nabla u, e_i\rangle \\ &= \sum_i \langle \nabla_{e_i} \nabla_{\nabla u} \nabla u, e_i\rangle \\ &= \sum_i \left[\langle \nabla_{\nabla u} \nabla_{e_i} \nabla u, e_i\rangle + \langle \nabla_{[e_i, \nabla u]} \nabla u, e_i\rangle + \langle R(e_i, \nabla u)\nabla u, e_i\rangle\right]. \end{aligned} \tag{1.2}$$

Now at x,

$$\begin{aligned} \sum_i \langle \nabla_{\nabla u} \nabla_{e_i} \nabla u, e_i\rangle &= \sum_i \left[\nabla u \langle \nabla_{e_i} \nabla u, e_i\rangle - \langle \nabla_{e_i} \nabla u, \nabla_{\nabla u} e_i\rangle\right] \\ &= \nabla u(\Delta u) = \langle \nabla u, \nabla(\Delta u)\rangle, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \sum_i \langle \nabla_{[e_i, \nabla u]} \nabla u, e_i\rangle &= \sum_i \text{Hess } u([e_i, \nabla u], e_i) \\ &= \sum_i \text{Hess } u(e_i, \nabla_{e_i} \nabla u) \\ &= \sum_i \langle \nabla_{e_i} \nabla u, \nabla_{e_i} \nabla u\rangle = |\,\text{Hess } u|^2. \end{aligned} \tag{1.4}$$

Combining (1.2), (1.3) and (1.4) gives (1.1). ∎

Applying the Cauchy-Schwarz inequality $|\,\text{Hess } u|^2 \geq \frac{(\Delta u)^2}{n}$ to (1.1) we obtain the following inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + \text{Ric}(\nabla u, \nabla u), \tag{1.5}$$

with equality if and only if Hess $u = hI_n$ for some $h \in C^\infty(M)$. If Ric $\geq (n-1)H$, if and only if the following holds for all $u \in \mathcal{C}^3(M)$.

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + (n-1)H|\nabla u|^2. \tag{1.6}$$

1.2. **Mean Curvature/Laplacian Comparison.** Here we apply the Bochner formula to distance functions. We call $\rho : U \to \mathbb{R}$, where $U \subset M^n$ is open, is a distance function if $|\nabla \rho| \equiv 1$ on U.

Example 1.1. Let $A \subset M$ be a submanifold, then $\rho(x) = d(x, A) = \inf\{d(x, y)|y \in A\}$ is a distance function on some open set $U \subset M$. When $A = q$ is a point, the distance function $r(x) = d(q, x)$ is smooth on $M \setminus \{q, C_q\}$, where C_q is the cut locus of q. When A is a hypersurface, $\rho(x)$ is smooth outside the focal points of A.

For a smooth distance function $\rho(x)$, Hess ρ is the covariant derivative of the normal direction $\partial_r = \nabla \rho$. Hence Hess $\rho = II$, the second fundamental form of the level sets $\rho^{-1}(r)$, and $\Delta \rho = m$, the mean curvature. For $r(x) = d(q, x)$, $m(r, \theta) \sim \frac{n-1}{r}$ as $r \to 0$; for $\rho(x) = d(x, A)$, where A is a hypersurface, $m(y, 0) = m_A$, the mean curvature of A, for $y \in A$.

Putting $u(x) = \rho(x)$ in (1.1), we obtain the Riccati equation along a radial geodesic,

$$0 = |II|^2 + m' + \mathrm{Ric}(\partial_r, \partial_r). \tag{1.7}$$

By the Cauchy-Schwarz inequality,

$$|II|^2 \geq \frac{m^2}{n-1}.$$

Thus we have the Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \mathrm{Ric}(\partial_r, \partial_r). \tag{1.8}$$

If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then

$$m' \leq -\frac{m^2}{n-1} - (n-1)H. \tag{1.9}$$

From now on, unless specified otherwise, we assume $m = \Delta r$, the mean curvature of geodesic spheres. Let M^n_H denote the complete simply connected space of constant curvature H and m_H (or m^n_H when dimension is needed) the mean curvature of its geodesics sphere, then

$$m'_H = -\frac{m_H^2}{n-1} - (n-1)H. \tag{1.10}$$

Let $\mathrm{sn}_K(r)$ be the solution to

$$\mathrm{sn}''_K + K\,\mathrm{sn}_K = 0$$

such that $\mathrm{sn}_K(0) = 0$ and $\mathrm{sn}'_K(0) = 1$, i.e. sn_K are the coefficients of the Jacobi fields of the model spaces $\mathbb{M}^n_K$. Let $\mathrm{cs}_K(s) = \mathrm{sn}'_K(s)$. Explicitly we have

$$\mathrm{sn}_K(s) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}s), & K > 0 \\ s, & K = 0 \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}s) & K < 0, \end{cases} \quad \text{and} \quad \mathrm{cs}_K(s) = \begin{cases} \cos(\sqrt{K}s), & K > 0 \\ 1, & K = 0 \\ \cosh(\sqrt{-K}s), & K < 0. \end{cases}$$

Then

$$m_H = (n-1)\frac{\mathrm{sn}'_H}{\mathrm{sn}_H}. \tag{1.11}$$

As $r \to 0$, $m_H \sim \frac{n-1}{r}$. The mean curvature comparison is

Theorem 1.2 (Mean Curvature/Laplace Comparison)**.** If $\mathrm{Ric}_{M^n} \geq (n-1)H$, then along any minimal geodesic segment from q,

$$\Delta r = m(r) \leq m_H(r) = \Delta_H(r). \tag{1.12}$$

Moreover, equality holds if and only if all radial sectional curvatures are equal to H.

This follows from Ricatti equation comparison. Here we give a simple proof.

Proof. From (1.9) and (1.10) we have

$$(m - m_H)' \leq -\frac{1}{n-1}(m^2 - m_H^2). \tag{1.13}$$

Let $m_+^H = (m - m_H)_+ = \max\{m - m_H, 0\}$, amount of mean curvature comparison failed.

We only need to work on the interval where $m - m_H \geq 0$. On this interval $-(m^2 - m_H^2) = -m_+^H(m - m_H + 2m_H) = -m_+^H(m_+^H + 2m_H)$. Thus (1.13) gives

$$(m_+^H)' \leq -\frac{(m_+^H)^2}{n-1} - 2\frac{m_+^H \cdot m_H}{n-1} \leq -2\frac{m_+^H \cdot m_H}{n-1} = -2\frac{sn_H'}{sn_H} m_+^H.$$

Hence $(sn_H^2 m_+^H)' \leq 0$. Since $sn_H^2(0)m_+^H(0) = 0$, we have $sn_H^2 m_+^H \leq 0$ and $m_+^H \leq 0$. Namely $m \leq m_H$. □

The local Laplacian comparison immediately gives us Myers' theorem, a diameter comparison. Let S_H^n be the sphere with radius $1/\sqrt{H}$.

Theorem 1.3 (Myers, 1941)**.** If $\mathrm{Ric}_M \geq (n-1)H > 0$, then $\mathrm{diam}(M) \leq \mathrm{diam}(S_H^n) = \pi/\sqrt{H}$. In particular, $\pi_1(M)$ is finite.

Proof. If $\mathrm{diam}(M) > \pi/\sqrt{H}$, let $q, q' \in M$ such that $d(q,q') = \pi/\sqrt{H} + \epsilon$ for some $\epsilon > 0$, and γ be a minimal geodesic connecting q, q' with $\gamma(0) = q, \gamma(\pi/\sqrt{H} + \epsilon) = q'$. Then $\gamma(t) \notin C_q$ for all $0 < t \leq \pi/\sqrt{H}$. Let $r(x) = d(q,x)$, then r is smooth at $\gamma(\pi/\sqrt{H})$, therefore Δr is well defined at $\gamma(\pi/\sqrt{H})$. By (1.12) $\Delta r \leq \Delta_H r$ at all $\gamma(t)$ with $0 < t < \pi/\sqrt{H}$. Now $\lim_{r \to \pi/\sqrt{H}} \Delta_H r = -\infty$ so Δr is not defined at $\gamma(\pi/\sqrt{H})$. This is a contradiction. □

The Laplacian comparison also works for radial functions (functions composed with the distance function). In geodesic polar coordinate, we have

$$\Delta f = \tilde{\Delta} f + m(r,\theta)\frac{\partial}{\partial r} f + \frac{\partial^2 f}{\partial r^2}, \tag{1.14}$$

where $\tilde{\Delta}$ is the induced Laplacian on the sphere and $m(r,\theta)$ is the mean curvature of the geodesic sphere in the inner normal direction. Therefore

Theorem 1.4 (Global Laplacian Comparison)**.** If $\mathrm{Ric}_{M^n} \geq (n-1)H$, in all the weak senses above, we have

$$\Delta f(r) \leq \Delta_H f(r) \quad (\text{if } f' \geq 0), \tag{1.15}$$

$$\Delta f(r) \geq \Delta_H f(r) \quad (\text{if } f' \leq 0). \tag{1.16}$$

1.3. **Volume Comparison.** Let $dvol = \mathcal{A}(r,\theta)\,drd\theta_{n-1}$ be the volume element of M in geodesic polar coordinate at q. Then we have the following lemma.

Lemma 1.1. The relative rate of change of the volume element is given by the mean curvature,

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = m(r,\theta). \tag{1.17}$$

This combines with the mean curvature comparison gives the volume element comparison.

Theorem 1.5. Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Let $dvol = \mathcal{A}(r,\theta)\,drd\theta_{n-1}$ be the volume element of M in geodesic polar coordinate at q and let $dvol_H = \mathcal{A}_H(r,\theta)\,drd\theta_{n-1}$ be the volume element of the model space M^n_H. Then

$$\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)} \text{ is nonincreasing along any minimal geodesic segment from } q. \tag{1.18}$$

Integrate this gives

Theorem 1.6 (Bishop-Gromov's Relative Volume Comparison)**.** Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Then

$$\frac{\mathrm{vol}\partial B(x,r))}{\mathrm{vol}\partial B_H(r))} \text{ and } \frac{\mathrm{vol}\,(B(x,r))}{\mathrm{vol}(B_H(r))} \text{ are nonincreasing in } r. \tag{1.19}$$

In particular,

$$\mathrm{vol}\,(B(x,r)) \leq \mathrm{vol}_H(B(r)) \quad \text{for all} \;\; r>0, \tag{1.20}$$

$$\frac{\mathrm{vol}\,(B(x,r))}{\mathrm{vol}\,(B(x,R))} \geq \frac{\mathrm{vol}_H(B(r))}{\mathrm{vol}_H(B(R))} \quad \text{for all} \;\; 0<r\leq R, \tag{1.21}$$

and equality holds if and only if $B(x,r)$ is isometric to $B_H(r)$.

Actually from Theorem 1.5 we can also get the annulus volume comparison. Denote the annulus by $A(p,r,R) = \{x \mid r < \mathrm{d}(r,x) \leq R\}$. Then for $0 \leq r_1 \leq R_1 \leq R_2$, $0 \leq r_1 \leq r_2 \leq R_2$,

$$\frac{\mathrm{vol}A(p,r_1,R_1)}{\mathrm{vol}A_H(r_1,R_1)} \geq \frac{\mathrm{vol}A(p,r_2,R_2)}{\mathrm{vol}A_H(r_2,R_2)}.$$

- If $r_1 = r_2 = 0$, this is just the relative volume comparison for balls.
- If we let $r_1 = 0$, $R_1 = R_2 = R$, and $r_2 = R-\epsilon$, then by dividing by $R_2 - r_2 = \epsilon$ we have
$$\frac{\mathrm{vol}A(p,R-\epsilon,R)}{\epsilon} = \frac{\int_{R-\epsilon}^{R} \mathrm{vol}\partial B(p,s)\,\mathrm{d}s}{\epsilon} \to \mathrm{vol}\partial B(p,R), \quad \text{as } \epsilon \to 0.$$
Therefore
$$\frac{\mathrm{vol}B(p,R)}{\mathrm{vol}B_H(R)} \geq \frac{\mathrm{vol}A(p,R-\epsilon,R)}{\mathrm{vol}A_H(R-\epsilon,R)} \to \frac{\mathrm{vol}\partial B(p,R)}{\mathrm{vol}\partial B_H(R)}.$$
- One can also just integrate along a sector of S^{n-1}. So the volume comparison holds for any star-shaped domain.

These basic Laplacian and volume comparsions have been generalized to integral Ricci curvature [19] and Bakry-Emery Ricci curvature [27].

2. Lecture 2: Rigidity and Almost volume cone rigidity

2.1. **Some Rigidity Results.** Some well known rigidity results are two " maximal diameter rigidity".

Theorem 2.1 (Cheng's Maximal Diameter Theorem 1975 [10])**.** Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H > 0$. If $\mathrm{diam}_M = \pi/\sqrt{H}$, then M is isometric to the sphere $\mathbb{S}^n_H$ with radius $1/\sqrt{H}$.

One way to prove this is to show there exists u such that $|\nabla u| \neq 0$ and $\mathrm{Hess}\ u = -ug$ (Given by first eigenfunction)

Theorem 2.2 (Splitting Theorem, Cheeger-Gromoll 1971 [9])**.** Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_M \geq 0$. If M has a line, then M is isometric to the product $\mathbb{R} \times N^{n-1}$, where N is an $n-1$ dimensional manifold with $\mathrm{Ric}_N \geq 0$.

Key to the proof is to find a function u such that $|\nabla u| = 1$, $\mathrm{Hess}\ u = 0$. (Given by Busemann function, see Section 3 for definition)

Theorem 2.3. A Riemannian manifold (M, g) is a warped product $((a,b) \times_f N,\ dr^2 + f^2(r) g_N)$ if and only if there is a nontrivial smooth function u on M such that

$$\nabla u \neq 0, \quad \mathrm{Hess}\ u = hg$$

for some function $h: M \longrightarrow \mathbb{R}$. ($f = u'$ up to a multiplicative constant)

Proof. If $g = dr^2 + f^2(r) g_0$ simply take $u(r) = \int f(t)dt$. Then $u'(r) = \frac{\partial}{\partial r} u = f(r) > 0$, $\mathrm{Hess}\ u = u''(r) g = f'(r) g$.

Conversely, if u satisfies $\mathrm{Hess}\ u = hg$, then for any vector field X,

$$X\left(\frac{1}{2}|\nabla u|^2\right) = \mathrm{Hess}\ u(X, \nabla u) = h \cdot g(X, \nabla u)$$

which shows that $|\nabla u|$ is constant on level sets of u. Let $N = u^{-1}(c)$, a level set of u, g_N the metric restricted to this level set, and r the signed distance to N defined by requiring that ∇r and ∇u point in the same direction. Then it is easy to see that u is a function of r: $u = u(r)$. Hence,

$$\nabla u = u' \nabla r, \quad \mathrm{Hess}\ u = u'' dr^2 + u' \,\mathrm{Hess}\ r.$$

Comparing with the equation $\mathrm{Hess}\ u = hg$ shows that $h = u''$ and that

$$\mathrm{Hess}\ r = \frac{u''}{u'} g$$

on the orthogonal complement of ∇r. On the other hand, $g = dr^2 + g_r$ with g_r the restriction of g on the level set of r. Since

$$L_{\nabla r} g_r = 2 \,\mathrm{Hess}\ r = 2\frac{u''}{u'} g_r.$$

Again the Hessian here is restricted to the orthogonal complement of ∇r. Thus, $g = dr^2 + (ku')^2 g_N$ where $ku'(0) = 1$. □

For warped product metrics, the curvatures are easy to compute. Double warped product metrics are used in many constructions.

2.2. **Metric cone and Volume cone.**

Definition 2.1 (Euclidean metric cone)**.** Let Z be a metric space. We define the Euclidean metric cone of Z by

$$C(Z) := \text{metric completion of } (0,\infty)\times Z$$

w.r.t. the metric

$$d((r_1,z_1),(r_2,z_2)) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\max\{d(z_1,z_2),\pi\}}$$

Note we have used Euclidean cosine law to define the metric, hence a Euclidean metric cone.

Example 2.1. If Z^{n-1} is a Riemannian manifold, then $C(Z)$ is exactly the cone manifold (except the vertex) with the Riemannian metric

$$g = dr^2 + r^2 g_Z.$$

Let $h = \frac{r^2}{2n}$, then $\Delta h = 1, \operatorname{Hess} h = \frac{1}{n}g$. Also, note that

$$\frac{\mathrm{vol}\partial B(z^*,r)}{\mathrm{vol}\partial B(z^*,R)} = \left(\frac{r}{R}\right)^{n-1}, \tag{2.1}$$

where z^* is the vertex.

It turns out that metric cone $\Longleftrightarrow$ there exists h so that $\operatorname{Hess} h = \frac{\Delta h}{n} g$.

Definition 2.2 (Volume cone)**.** A volume cone of Z is a conical space such that (2.1) holds.

By volume rigidity, if $\mathrm{Ric} \geq 0$, then volume cone $\Longleftrightarrow$ metric cone.

Cheeger-Colding [8] proved the following almost rigidity.

Theorem 2.4 (Almost rigidity)**.** If a Riemannian manifold M^n satisfies $\mathrm{Ric} \geq -(n-1)\delta$, and

$$\frac{\mathrm{vol}B(p,r)}{\mathrm{vol}\partial B(p,r)}(1-\delta) \leq \frac{\mathrm{vol}B_{-\delta}(r)}{\mathrm{vol}\partial B_{-\delta}(r)}, \tag{2.2}$$

then

$$d_{GH}(B(p,r), B(z^*,r)) \leq \psi(\delta|n,R)$$

for some space Z. The ball $B(z^*,r)$ is the r-ball centered at the vertex of $C(Z)$.

In proving this, Bochner's formula, Cheng-Yau's gradient estimate, Poincare inequality, Segment inequality are the basic tools.

2.3. **Cheng-Yau's gradient estimate.**

Cheng-Yau's gradient estimate [11] is an important tool in geometric analysis. The following combines the versions in [6] and [13]. The proof uses Bochner's formula, maximal principle, cut-off function and Laplacian comparison.

Theorem 2.5. Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_{M^n} \geq -(n-1)H^2 (H \geq 0)$. If u is a positive function defined on the closed ball $B(q,2R) \subset M$ satisfying $\Delta u = K(u)$. Then

$$|\nabla(\log u)(x)| \leq (n-1)H + c(n,H)R^{-1} + \left[(n-1)\sup_{B(q,2R)}\left(|K'| + \frac{|K|}{u}\right)\right]^{1/2} \quad \text{on } B(q,R). \tag{2.3}$$

In particular, for positive harmonic function u defined on M,

$$|\nabla(\log u)(x)| \leq (n-1)H. \tag{2.4}$$

Estimate (2.4) is optimal. When equality occurs, it implies strong rigidity: the manifold is a warped product, see [13]. When $H = 0$, (2.4) gives the following Loiuville type result.

Corollary 2.1. Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_{M^n} \geq 0$, then all positive harmonic functions are constant. In particular, all bounded harmonic functions are constant.

Proof of Theorem 2.5. : Let $h = \log u$. Then $\nabla h = \frac{\nabla u}{u}, \Delta h = \frac{\Delta u}{u} - \left|\frac{\nabla u}{u}\right|^2 = \frac{K(u)}{u} - |\nabla h|^2$. Apply the Bochner formula (1.1) to the function h we have

$$\begin{aligned} \frac{1}{2}\Delta|\nabla h|^2 &= |\operatorname{Hess} h|^2 + \langle \nabla h, \nabla(\Delta h)\rangle + \mathrm{Ric}(\nabla h, \nabla h) \\ &\geq |\operatorname{Hess} h|^2 + \langle \nabla h, \nabla \tfrac{K(u)}{u}\rangle - \langle \nabla h, \nabla(|\nabla h|^2)\rangle - (n-1)H^2|\nabla h|^2. \end{aligned} \tag{2.5}$$

For the Hessian term one could use the Schwarz inequality

$$|\operatorname{Hess} h|^2 \geq \frac{(\Delta h)^2}{n}. \tag{2.6}$$

Indeed, when $H = 0$, this estimate is enough. For $H > 0$, using this estimate one would get $(n(n-1))^{1/2}$ instead of $n-1$ for the coefficient of H in (2.4). To get the best constant for $H > 0$, note that (2.6) is only optimal when Hessian at all directions are same. Harmonic functions in the model spaces are radial functions, so their Hessian along the radial direction would be different from the spherical directions. Therefore one computes the norm by separating the radial direction and uses Schwarz inequality in the spherical directions. Let $\{e_i\}$ be an orthonormal basis with $e_1 = \frac{\nabla h}{|\nabla h|}$, the potential radial direction, denote $h_{ij} = \operatorname{Hess} h(e_i, e_j)$. We compute

$$\begin{aligned} |\operatorname{Hess} h|^2 &= h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \sum_{i,j\geq 2} h_{ij}^2 \\ &\geq h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{(\Delta h - h_{11})^2}{n-1} \\ &= h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{(|\nabla h|^2 + h_{11})^2}{n-1} \\ &\geq \frac{n}{n-1}\sum_{j=1}^n h_{1j}^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}|\nabla h|^2 h_{11}. \end{aligned}$$

Now $h_{1j} = \frac{1}{|\nabla h|}\langle \nabla_{e_j}\nabla h, \nabla h\rangle = \frac{1}{2|\nabla h|} e_j(|\nabla h|^2)$ and $h_{11} = \frac{1}{2|\nabla h|^2}\langle \nabla|\nabla h|^2, \nabla h\rangle$. Therefore

$$|\operatorname{Hess} h|^2 \geq \frac{n}{4(n-1)}\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{|\nabla h|^4 + \langle \nabla|\nabla h|^2, \nabla h\rangle}{n-1}, \tag{2.7}$$

and equality holds if and only if Hess h are same on the level set of h, i.e.

$$\operatorname{Hess} h = -\frac{|\nabla h|^2}{n-1}\left(g - \frac{1}{|\nabla h|^2} dh \otimes dh\right).$$

Plug (2.7) into (2.5) we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 \quad &\geq \quad \frac{n}{4(n-1)}\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{|\nabla h|^4}{n-1} - \frac{n-2}{n-1}\langle\nabla|\nabla h|^2, \nabla h\rangle \\ &\quad +(K' - \tfrac{K}{u})|\nabla h|^2 - (n-1)H^2|\nabla h|^2. \qquad (2.8)\end{aligned}$$

If $|\nabla h|^2$ achieves a local maximum inside $B(q, 2R)$ then we are done. Assume $|\nabla h|(q_0)$ is the maximum for some $q_0 \in B(q, 2R)$, then $\nabla|\nabla h|^2(q_0) = 0, \Delta|\nabla h|^2(q_0) \leq 0$. Plug these into (2.8) gives

$$0 \geq \frac{|\nabla h|^4}{n-1} + (K' - \tfrac{K}{u})|\nabla h|^2 - (n-1)H^2|\nabla h|^2.$$

Hence $|\nabla h|^2 \leq (n-1)^2H^2 + (n-1)\sup_{B(q,2R)}\left(|K'| + \frac{|K|}{u}\right)$.

In general the maximum could occur at the boundary and one has to use a cut-off function to force the maximum is achieved in the interior. Let $f : [0, 2R] \to [0, 1]$ be a smooth function with

$$f|_{[0,R]} \equiv 1, \ \mathrm{supp} f \subset [0, 2R), \qquad (2.9)$$

$$-cR^{-1}f^{1/2} \leq f' \leq 0, \qquad (2.10)$$

$$|f''| \leq cR^{-2}, \qquad (2.11)$$

where $c > 0$ is a universal constant. Let $\phi : \overline{B(q, 2R)} \to [0, 1]$ with $\phi(x) = f(r(x))$, where $r(x) = d(x, q)$ is the distance function. Set $G = \phi|\nabla h|^2$. Then G is nonnegative on M and has compact support in $B(q, 2R)$. Therefore it achieves its maximum at some point $q_0 \in B(q, 2R)$. We can assume $r(q_0) \in [R, 2R)$ since if $q_0 \in B(q, R)$, then $|\nabla h|^2$ achieves maximal at q_0 on $B(q, R)$ and previous argument applies.

If q_0 is not a cut point of q then ϕ is smooth at q_0 and we have

$$\Delta G(q_0) \leq 0, \quad \nabla G(q_0) = 0. \qquad (2.12)$$

At the smooth point of r, $\nabla G = \nabla\phi|\nabla h|^2 + \phi\nabla|\nabla h|^2$,

$$\Delta G = \Delta\phi|\nabla h|^2 + 2\langle\nabla\phi, \nabla|\nabla h|^2\rangle + \phi\Delta|\nabla h|^2.$$

Using (2.8), (2.12) and express $|\nabla h|^2, \nabla|\nabla h|^2$ in terms of $G, \nabla G$ we get, At the maximal point q_0 of G,

$$\begin{aligned}0 \geq \Delta G(q_0) \quad &\geq \quad \frac{\Delta\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G + \frac{n}{2(n-1)}\frac{|\nabla\phi|^2}{\phi^2}G + \frac{2}{n-1}\frac{G^2}{\phi} \\ &\quad +\frac{2(n-2)}{n-1}\langle\nabla h, \nabla\phi\rangle\frac{G}{\phi} + 2(K' - \tfrac{K}{u})G - 2(n-1)H^2G. \qquad (2.13)\end{aligned}$$

Since $\phi(x) = f(r(x))$ is a radial function, $|\nabla\phi| = |f'|$, by (2.10),

$$\langle\nabla h, \nabla\phi\rangle \geq -|\nabla h||\nabla\phi| = -G^{\frac{1}{2}}\frac{|\nabla\phi|}{\phi^{1/2}} \geq -G^{\frac{1}{2}}\frac{c}{R}.$$

Also $\Delta\phi = f'\Delta r + f''$. Since $f' \leq 0$, by the Laplacian comparison (1.16) and (2.11)

$$\Delta\phi \geq f'\Delta_H r - cR^{-2},$$

where $\Delta_H r = (n-1)H\coth(Hr)$, which is $\le (n-1)H\coth(HR)$ on $[R,2R]$. Hence

$$\begin{aligned}\Delta\phi &\ge -cR^{-1}\left((n-1)H\coth(HR)+R^{-1}\right)\\ &\ge -cR^{-1}\left[(n-1)(2R^{-1}+4H)+R^{-1}\right]\end{aligned}$$

Multiply (2.13) by $\frac{(n-1)\phi}{G}$ and plug these in, we get

$$\begin{aligned}0 \ \ge\ & 2G-2(n-2)\frac{c}{R}G^{\frac12}\\ & -\frac{c}{R}\left(\frac{(n-1)(2n-1)}{R}+4(n-1)^2H+\left(\frac{3n}{2}-2\right)\right)-2(n-1)\left(|K'|+\tfrac{|K|}{u}\right)-2(n-1)^2H^2.\end{aligned}$$

Solving this quadratic inequality gives

$$(G(q_0))^{\frac12} \le (n-1)H + c(n,H)R^{-1} + \left[(n-1)\sup_{B(q,2R)}\left(|K'|+\tfrac{|K|}{u}\right)\right]^{1/2}. \tag{2.14}$$

Therefore

$$\sup_{B(q,R)}|\nabla h| = \sup_{B(q,R)}G^{1/2} \le \sup_{B(q,2R)}G^{1/2} \le (n-1)H+c(n,H)R^{-1}+\left[(n-1)\sup_{B(q,2R)}\left(|K'|+\tfrac{|K|}{u}\right)\right]^{1/2}.$$

If q_0 is in the cut locus of q, use the upper barrier $r_{q_0,\epsilon}(x)$ for $r(x)$ and let $\epsilon\to 0$ gives the same estimate. □

2.4. **Integral estimate of Hessian.** Idea: Find function f such that $\Delta f\equiv 1$ and $⨍_{B(p,R)}(\text{Hess} f - \frac1n g) \le \Psi(\delta|n,R)$.

In $\mathbb{M}^n_{-\delta}$, the Riemmannian metric in polar coordinate is given by

$$g = dr^2 + \sinh^2_{-\delta}(r)ds^2$$

and the Laplacian is

$$\Delta_{-\delta} = \frac{\partial^2}{\partial r^2} + (n-1)\coth_\delta(r)\frac{\partial}{\partial r} + \frac{1}{\sinh^2_{-\delta}(r)}\bar\Delta_{\mathbb{S}^{n-1}}.$$

Let $\bar f$ be the function on $\mathbb{M}^n_{-\delta}$ such that $\Delta_{-\delta}\bar f\equiv 1$, $\bar f(0)=0$, $\bar f'\ge 0$, $\bar f'(0)=0$. Namely

$$\bar f'' + (n-1)\coth_{-\delta}(r)\bar f' = 1$$

This is first order linear equation for $\bar f'$. Solving gives $\bar f' = \frac{\text{vol}B(\bar p,r)}{\text{vol}\partial B(\bar p,r)}$. In polar coordinate the volume element is

$$A_{-\delta}(r,\theta) = \sinh^{n-1}_{-\delta}(r),$$

and $\text{vol}\partial B(\bar p,r) = \int_{\mathbb{S}^{n-1}} A_{-\delta}(r,\theta)$.

In $\mathbb{R}^n$, $\bar f = \frac{r^2}{2n}$ and $(\bar f')^2 = \frac{2\bar f}{n}$. In general

$$(\bar f')^2 - \frac{2\bar f}{n} = \Psi(\delta|n,R). \tag{2.15}$$

Consider $f: B(p,R)\to\mathbb{R}$ such that $\Delta f\equiv 1$, $f|_{\partial B} = \bar f(R)$. By maximal principle $f\le \bar f(R)$ on $B(p,R)$.

Now $\Delta(\bar f - f) = \Delta\bar f - \Delta f \le \Delta_{-\delta}\bar f - \Delta f = 1-1 = 0$. Hence

$$0 \le \bar f - f \le C_1(n,\delta,R).$$

Lemma 2.1. 1) $\fint_{B(p,R)} |\nabla f - \nabla \bar{f}|^2 \leq \Psi(\delta|n, R))$.
2) $|f - \bar{f}| \leq \Psi(\delta|n, R))$ on $B(p, \frac{R}{2})$.
3) $\fint_{B(p,\frac{R}{2})} |\operatorname{Hess} f - \frac{1}{n} g|^2 \leq \Psi(\delta|n, R))$.

Sketch of Proof. : 1)

$$\fint_{B(p,R)} |\nabla f - \nabla \bar{f}|^2 = -\fint (\Delta f - \Delta \bar{f})(f - \bar{f}) \leq C(n, \delta, R) \left| \fint (\Delta f - \Delta \bar{f}) \right|$$

$$\begin{aligned} 0 \leq \int_{B(p,R)} (\Delta f - \Delta \bar{f}) &= \operatorname{vol} B(p, R) - \int_{\partial B(p,R)} \langle \nabla \bar{f}, \ \vec{n} \rangle \\ &= \operatorname{vol} B(p, R) - \frac{\operatorname{vol} B(\bar{p}, r)}{\operatorname{vol} \partial B(\bar{p}, r)} \operatorname{vol} \partial B(p, R) \\ &\leq \operatorname{vol} B(p, R) - (1 - \delta) \operatorname{vol} B(p, R) = \delta \cdot \operatorname{vol} B(p, R). \end{aligned}$$

Hence we get $\fint_{B(p,R)} |\nabla f - \nabla \bar{f}|^2 \leq \Psi(\delta|n, R))$.

For 2), Poincare inequality gives

$$\fint_{B(p,R)} |f - \bar{f}|^2 \leq \Psi(\delta|n, R)). \tag{2.16}$$

Now $|\nabla \bar{f}| = |\bar{f}'| \leq C_2(n, \delta, R)$, $f \geq \bar{f} - C_1(n, \delta, R)$, apply Cheng-Yau's gradient estimate (Theorem 2.5) to $f + 2C_1$ gives $|\nabla f| \leq C(n, \delta, R)$ on $B(p, \frac{R}{2})$. (2.16) combines with these gradient bounds gives 2).

Recall $|A|^2 = \operatorname{tr}(A \cdot A^T)$.

$$|\operatorname{Hess} f - \frac{1}{n} g|^2 = |\operatorname{Hess} f|^2 - 2\frac{\Delta f}{n} + \frac{1}{n} = |\operatorname{Hess} f|^2 - \frac{\Delta f}{n}. \tag{2.17}$$

From Bochner formula

$$\frac{1}{2} \Delta |\nabla f|^2 = |\operatorname{Hess} f|^2 + \langle \nabla f, \nabla \Delta f \rangle + \operatorname{Ric}(\nabla f, \nabla f) = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f).$$

Let φ be a cut-off function of $B(p, R)$ such that $\varphi \equiv 1$ on $B(p, \frac{R}{2})$ and $\Delta \varphi + |\nabla \varphi| \leq C(n, \delta, R)$.

$$\begin{aligned} \int \varphi |\operatorname{Hess} f|^2 &= \frac{1}{2} \int \Delta \left(|\nabla f|^2 - \tfrac{2f}{n} + \tfrac{2f}{n} \right) \varphi - \int \varphi \operatorname{Ric}(\nabla f, \nabla f) \\ &\leq \frac{1}{2} \int_{B(x,R)} \Delta \varphi (|\nabla f|^2 - \tfrac{2f}{n}) + \int \varphi \tfrac{\Delta f}{n} + \delta \int_B |\nabla f|^2 \end{aligned}$$

Since $|\nabla f| \leq C(n, \delta, R)$, and

$$\fint_{B(x,R)} \left| |\nabla f|^2 - \tfrac{2f}{n} \right| \leq \fint_{B(x,R)} \left| |\nabla f|^2 - |\nabla \bar{f}|^2 \right| + \left| |\nabla \bar{f}|^2 - \tfrac{2\bar{f}}{n} \right| + \left| \tfrac{2\bar{f}}{n} - \tfrac{2f}{n} \right| \leq \Psi(\delta|n, R)),$$

we have

$$\fint \varphi \left(|\operatorname{Hess} f|^2 - \tfrac{\Delta f}{n} \right) \leq \Psi(\delta|n, R)).$$

This combines with (2.17) gives 3). □

The integral estimate of the Hessian combines with Cheeger-Colding's segment inequality gives Theorem 2.2.

3. Lecture 3: Busemann function of manifolds with nonnegative Ricci curvature

3.1. Busemann function. Let M^n be a complete noncompact manifold. For any $p \in M$, there exists a ray $\gamma(t) : [0, +\infty) \to M$, i.e. $\mathrm{d}(\gamma(t), \gamma(s)) = |t - s|$. The Busemann function is a renormalized distance function from infinity, which plays an important in the study of noncompact manifolds.

Definition 3.1. The Busemann function associated to a ray γ is a function $b_\gamma : M \to \mathbb{R}$ defined by

$$b_\gamma(x) = \lim_{t\to\infty} \Big(t - \mathrm{d}(x, \gamma(t))\Big).$$

Note that the sequence is monotone and bounded so the limit exists. Namely by triangle inequality

$$|t - \mathrm{d}(x, \gamma(t))| = |\,\mathrm{d}(p, \gamma(t)) - \mathrm{d}(x, \gamma(t))| \le \mathrm{d}(p, x).$$

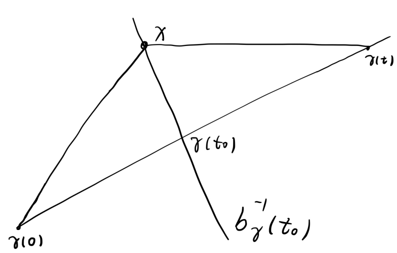
Also if $s < t$, then

$$\begin{aligned}\Big(s - \mathrm{d}(x, \gamma(s))\Big) - \Big(t - \mathrm{d}(x, \gamma(t))\Big) &= s - t - \mathrm{d}(x, \gamma(s)) + \mathrm{d}(x, \gamma(t)) \\ &= -\,\mathrm{d}(\gamma(t), \gamma(s)) - \mathrm{d}(x, \gamma(s)) + \mathrm{d}(x, \gamma(t)) \\ &\le 0.\end{aligned}$$

Hence $t - \mathrm{d}(x, \gamma(t))$ is nondecreasing in t.

Remark 3.1.
- b_γ is Lipschitz with Lipschitz constant 1.
- Along γ, $b_\gamma(\gamma(t)) = t$ is linear in t.

Example 3.1. Let $M = \mathbb{R}^n$ with the usual Euclidean metric. Then all rays are of the form $\gamma(t) = \gamma(0) + t\gamma'(t)$, and $b_\gamma(x) = \langle x - \gamma(0), \gamma'(0)\rangle$.



Proof. Write

$$x - \gamma(0) = a\gamma'(0) + v,$$

where $a = \langle x - \gamma(0), \gamma'(0)\rangle$ and $v \perp \gamma'(0)$. Then

$$\mathrm{d}(x, \gamma(t)) = \|x - \gamma(0) - t\gamma'(0)\| = \sqrt{(a - t)^2 + \|v\|^2}.$$

Thus we have

$$\lim_{t\to\infty} (t - \mathrm{d}(x, \gamma(t))) = \lim_{t\to\infty} \frac{t^2 - (a - t)^2 - v^2}{t + \sqrt{(a - t)^2 + \|v\|^2}} = a.$$

□

Theorem 3.1 (Cheeger-Gromoll, 71', 72')**.**

- If the sectional curvature $K_M \geq 0$, then $\operatorname{Hess} b_\gamma \geq 0$.
- If the Ricci curvature $\operatorname{Ric}_M \geq 0$, then $\Delta b_\gamma \geq 0$.
  (both in barrier sense)

Remark 3.2. (1) The first result plays an important role in the proof of Soul's theorem, while the second one leads to the splitting theorem.

(2) By Laplacian comparison we have $\Delta r \leq \Delta_{\mathbb{R}^n} \bar{r} = \frac{n-1}{r}$. So intuitively, $\Delta\Big(t - \mathrm{d}(x, \gamma(t))\Big) \geq -\frac{n-1}{\mathrm{d}(x,\gamma(t))} \to 0$ as $t \to \infty$.

Definition 3.2 (Busemann function of a point)**.** $b_p(x) := \sup_\gamma b_\gamma(x)$, where the supremum is taken among all rays γ starting from p.

When M^n is polar with pole at p, then $b_p(x) = \mathrm{d}(p, x)$. Still $b_p(x)$ is convex when $K_M \geq 0$, and subharmonic when $\operatorname{Ric}_M \geq 0$ in barrier sense.

The convex of $b_p(x)$ implies $b_p(x)$ is proper. In fact, it imples $b_p^{-1}(-\infty, a] = \cap_\gamma b_\gamma^{-1}(-\infty, a]$ is compact. Here is a quick proof.

Proof. If for the sake of contradiction it is not true, then there exists an $a > 0$ and a sequence $\{x_i\} \subset M$ such that $b_p(x_i) \leq a$ and $\mathrm{d}(p, x_i) \to \infty$. Now we connect p with x_i by minimal geodesics γ_i. Then a subsequence of γ_i converges to a ray γ. Since b_p is convex,

$$b_p(p) = 0, b_p(x_i) \leq a \implies b_p(\gamma_i) \leq a.$$

Hence $b_\gamma(\gamma(t)) \leq a$ for all t. But $b_\gamma(\gamma(t)) = t$, this is a contradiction. □

Question 3.1 (Open problem since 70's)**.** Is b_p proper when $\operatorname{Ric}_M \geq 0$?

It has been shown that the answer is yes in many special cases:

- When M is polar with pole at p, we have $b_p(x) = \mathrm{d}(p, x)$, proper.
- If $\lim_{r\to\infty} \sup \frac{\operatorname{diam}\left(\partial B(p,r)\right)}{r} = \epsilon < 1$, then $\lim_{x\to\infty} \inf \frac{b_p(x)}{\mathrm{d}(p,x)} \geq 1 - \epsilon > 0$, which implies b_p is proper.
- **Shen, 1996**: When M^n has Euclidean volume growth, b_p is proper.
- **Sormani 1998**: When M^n has linear volume growth (i.e. $Cr \leq \operatorname{vol} B(x, r) \leq C' r$), b_p is proper.

3.2. **Nabonnand's example of manifolds with positive Ricci curvature.** [15]

We first recall Nabonnand's example [15], which is the first example of a manifold with positive Ricci curvature with infinite fundamental group π_1.

Let $M = \mathbb{R}^k \times \mathbb{S}^1$ equipped with the double warped metric

$$g = \mathrm{d}r^2 + f^2(r)\, \mathrm{d}s_{k-1}^2 + h^2(r)\, \mathrm{d}s_1^2$$

with $f(0) = 0, f'(0) = 1, f''(0) = 0, h(0) > 0, h'(0) = 0$. Denote $H = \frac{\partial}{\partial r}$, u a unit vector tangent to $\mathbb{S}^{k-1}$, and v a unit vector tangent to $\mathbb{S}^1$. Then one can compute

$$\mathrm{Ric}(H,H) = -(k-1)\frac{f''}{f} - \frac{h''}{h} \tag{3.1}$$

$$\mathrm{Ric}(u,u) = -\frac{f''}{f} - \frac{k-2}{f^2}\left(1-(f')^2\right) - \frac{f'h'}{fh} \tag{3.2}$$

$$\mathrm{Ric}(v,v) = -\frac{h''}{h} - (k-1)\frac{f'h'}{fh}. \tag{3.3}$$

When $0 < f' < 1, f'' < 0$, $h' < 0$, and $k \geq 2$ then it is easy to see that $\mathrm{Ric}(u,u) > 0$.

Choose $h = f'$, then $\mathrm{Ric}(v,v) = \mathrm{Ric}(H,H)$. Let f be the solution of the ODE:

$$\begin{cases} f' = (1-\varphi(f))^{\frac{1}{2}} \\ f(0) = 0, \end{cases}$$

where $\varphi(x) = \frac{\sqrt{3}}{\pi}\int_0^x \frac{\arctan u^3}{u^2}\,\mathrm{d}u$. Here we choose an explicit φ, there are many other choices of φ for the construction. As $\int_0^\infty \frac{\arctan u^3}{u^2}\,\mathrm{d}u = \frac{\pi}{\sqrt{3}}$, we have $0 < \varphi(x) < 1$ for $x \in (0,\infty)$. Note that $h \to 0$ and $h \sim r^{-1/2}$, $f \sim r^{1/2}$ as $r \to +\infty$.

Then one computes that $\mathrm{Ric}(H,H) > 0$ when $k \geq 3$.

Same construction works for $\mathbb{R}^k \times M^q$, where M has nonnegative Ricci curvature by modifying with $h = (f')^{1/q}$ [3].

In [26] the author constructed a metric with positive Ricci curvature on $\mathbb{R}^k \times N$, where N is a nilmanifold for k big. This is the first example of manifolds with positive Ricci curvature with nilpotent fundamental group. See [1, 2] for more constructions along the line.

3.3. **Example of manifolds with positive Ricci curvature and non-proper Busemann function.** [18]

Theorem 3.2 (Pan-Wei 2022)**.** Given any integer $n \geq 4$, there is an open n-manifold with positive Ricci curvature and a non-proper Busemann function.

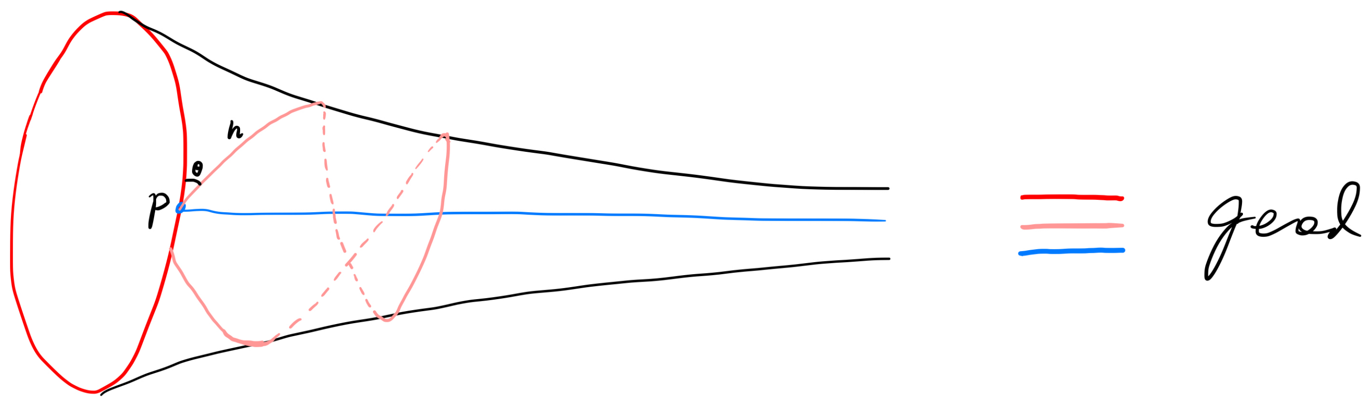
Proof. First we study the geodesics in Nabonnand's example.

Given any fixed point p with $r = 0$ in $\mathbb{R}^k \times S^1$, there are three types of geodesics:

(i) moving purely in $\mathbb{R}^k$,

(ii) moving purely in the S^1 direction,

(iii) a mixture of both.



In case (i), the geodesic is a ray and in case (ii), the geodesic is a closed circle. In case (iii), by Clairaut's relation $(\cos\theta(r))h(r) = \text{const} = \cos\theta(0)$. Since h goes to zero as r increases and $\cos\theta$ is bounded, the geodesic stays in a bounded region and cross $\{r = 0\}$ transversely infinite many times.

On the universal cover $\tilde{M}$ of M, a geodesic $\tilde{\gamma}$ starting at $\tilde{p}$ is a ray precisely when the projection $\pi(\tilde{\gamma})$ is case (i). This is the case as in (ii) and (iii), the geodesics are bounded in M, they can not lift to rays. If yes, then the group action $\mathbb{Z}$ would give a line, contradicting $Ric > 0$.

Let a be a generator of $\pi_1(M, p) = \mathbb{Z}$. We claim $b_{\tilde{p}}(a^l(\tilde{p})) = 0$ for all $l \in \mathbb{Z}$. The orbit of $\tilde{p}$ is noncompact as $\mathbb{Z}$ is an infinite group. Hence $b_{\tilde{p}}$ is not proper.

To show the claim, since there is only one ray, we have $b_{\tilde{p}} = b_{\tilde{\gamma}}$. Now

$$b_{\tilde{\gamma}}(a^l\tilde{p}) = \lim_{t\to\infty}(t - d(\tilde{\gamma}(t), a^l\tilde{p})).$$

Let $\tilde{\alpha}$ be a minimal geodesic connecting $\tilde{\gamma}(t)$ and $a^l\tilde{p}$, then

$$d(\tilde{\gamma}(t),\ a^l\tilde{p}) = \text{the shortest representative in the homotopy class} \le t + l\cdot 2\pi h(t).$$

Covering map is distance non-increasing, hence

$$d(\tilde{\gamma}(t),\ a^l\tilde{p}) \ge d(p, \gamma(t)) = t.$$

Therefore

$$-2\pi l\cdot h(t) \le t - d(\tilde{\gamma}(t), a^l\tilde{p}) \le 0.$$

Since $h(t) \to 0$ as $t \to \infty$, we have $b_{\tilde{\gamma}}(a^l\tilde{p}) = 0$ and proved the claim. □

Question 3.2. What about $n = 3$?

Bruè-Naber-Semola [4] gave examples of manifolds M^n with nonnegative Ricci curvature and π_1 not finitely generated for $n \ge 7$.

Question 3.3. Is the properness of the Busemann function related to the finitely generation of the π_1?

4. Lecture 4: Hausdorff dimension of Ricci limit spaces

4.1. **Hausdorff dimension.** Hausdorff measure is an outer measure on subsets of a general metric space (X, d).

Definition 4.1 (Hausdorff Measure)**.** Let $0 \le \alpha < \infty$ and $0 < \delta < \infty$. Let $A \subset X$. Define an outer measure

$$\mathcal{H}^\alpha_\delta(A) := \omega_\alpha \inf \left\{ \sum_{i\in\mathbb{N}} \left(\tfrac{\operatorname{diam} C_i}{2}\right)^\alpha \,\middle|\, A \subset \bigcup_{i=1}^\infty C_i, \text{ with } \operatorname{diam} C_i < \delta \text{ for every } i \in \mathbb{N} \right\},$$

the infimum is taken over all countable covers of A.

The α-dimensional *Hausdorff measure* of $A \subset X$ is the outer measure

$$\mathcal{H}^\alpha(A) := \lim_{\delta\to 0} \mathcal{H}^\alpha_\delta(A).$$

Note that $\mathcal{H}^s_\delta(A)$ is monotone nonincreasing in δ, so the limit is well defined. The normalization constant ω_α is chosen so that if $(X, d) = (\mathbb{R}^n, |\cdot|)$ then $\mathcal{H}^n(A) = \mathcal{L}^n(A)$, where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure. On a Riemannian manifold (M^n, g), we have a natural measure induced by the volume form, and the n-dimensional Hausdorff measure on (M^n, g) also coincides with the measure induced by the volume form.

Definition 4.2. The quantity

$$\dim_{\mathcal{H}}(A) := \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) = 0\} = \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) \ne \infty\}$$

is called the *Hausdorff dimension* of A.

Observe that there is an obvious restriction of these definitions to open covers by **balls** (rather than arbitrary sets) or sets with diameter exactly δ. This will, in general give a larger measure, as there are fewer covers to choose from, hence potentially increasing the infimum. This gives rise to quantities known as *spherical Hausdroff measure/dimension* $\mathcal{S}^\alpha$ or *Minkowski measure/dimension*.

Lemma 4.1. If $\mathcal{H}^\alpha$ on a metric space X satisfies the doubling property, then for any compact subset $A \subset X$, $\mathcal{H}^\alpha(A) = \mathcal{S}^\alpha(A)$.

Hint: Use Vitali covering

In what follows we give some examples of spaces which have a lower topological dimension than their Hausdorff dimension.

See this article about various dimensions.

Example 4.1. Let $(\mathbb{R}, d_\alpha)$ be defined by

$$d_\alpha(t_1, t_2) = |t_1 - t_2|^{1/\alpha}.$$

One can see that $\dim_{\mathcal{H}}(\mathbb{R}, d_\alpha) = \alpha$.

Indeed, taking a cover of any arbitrarily large compact interval $I = [-R, R]$, observe that for any $\delta > 0$ and any cover of I by C_i sets of diameter δ, this corresponds in $\mathbb{R}$ to a collection $\{\tilde{C}_i\}$ of diameter $\sim (\operatorname{diam} C_i)^\alpha$. Therefore we have that

$$\sum_{i\in\mathbb{N}} (\operatorname{diam} C_i/2)^\alpha = \sum_{i\in\mathbb{N}} ((\operatorname{diam} \tilde{C}_i)^{1/\alpha}/2)^\alpha = \sum_{i\in\mathbb{N}} (\operatorname{diam} \tilde{C}_i/2)^1.$$

As this is true for every such δ, we draw the desired conclusion.

As a sort of generalization of this, we introduce the Nilpotent Lie Group, the Heisenberg group.

Definition 4.3. The *Heisenberg Group* $\mathbb{H}$ is the set of upper-triangluar 3×3 matrices with 1's populating the main diagonal

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

This is obviously diffeomorphic to $\mathbb{R}^3$, and we equip it with a left-invariant vector field

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$$
$$Z = \frac{\partial}{\partial z}$$

One easily checks that the only nontrivial commutator identity is $[X, Y] = Z$.

This means in particular we have a left-invariant metric on $\mathbb{H}$. However, there is a different metric, known as *sub-Riemannian* or *Carnot-Carathéodory metric* which amounts to "forgetting" that we can move along the z-axis, or equivalently that lines in the z-direction have infinite length.

More concretely, we define the CC metric on $\mathbb{H}$ as follows:

Definition 4.4. Define $d = d_{CC}$ by

$$d_{CC}(p, q) = \inf \left\{ \int_0^1 \|\gamma'(t)\| dt \,\middle|\, \gamma(0) = p, \gamma(1) = q, \gamma'(t) \in \operatorname{span}(X, Y) \right\}$$

where γ is a piecewise smooth curve connecting p and q.

Definition 4.5. We call $\operatorname{span}(X, Y)$ the *horizontal* directions.

Definition 4.6. Given $\lambda \in \mathbb{R}$, define the dilation operation

$$\delta_\lambda : \mathbb{H} \to \mathbb{H}$$

by

$$\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z).$$

One checks immediately that $(\delta_\lambda)_* : \mathfrak{h} \to \mathfrak{h}$ satisfies the commutator relations, where $\mathfrak{h} = Lie(\mathbb{H})$.

Proposition 4.1. Dilation by $\lambda \in \mathbb{R}$ satisfies the equation

$$d(\delta_\lambda(p), \delta_\lambda(q)) = |\lambda| d(p, q),$$

which is to say that distances scale appropriately under this dilation.

Proof. One easily computes that $(\delta_\lambda)_* X = \lambda X$ and $(\delta_\lambda)_* Y = \lambda Y$.

Therefore, observe that all derivatives $\gamma'(t)$ lie in the horizontal direction, and are scaled linearly. Take for granted that the distance reached by a curve. Hence, if γ from p to q is a minimal length curve, define a new curve $\gamma_\lambda(t) := \delta_\lambda(\gamma(t))$. Observe that obviously $\gamma_\lambda(0) = \delta_\lambda(p)$ and $\gamma_\lambda(1) = \delta_\lambda(q)$.

By the fact that γ is length-minimizing, so is γ_λ, and it remains to compute the distance in question.

Indeed, if $\gamma'(t) = a(t)X + b(t)Y$, then $\gamma'_\lambda(t) = \lambda a(t)X + \lambda b(t)Y = \|\lambda\gamma'(t)\|$ by the above observations, and therefore

$$\begin{aligned} d(\delta_\lambda(p), \delta_\lambda(q)) &= \int_0^1 \|\gamma'_\lambda(t)\| dt \\ &= \int_0^1 \|\lambda\gamma'(t)\| dt \\ &= |\lambda| \int_0^1 \|\gamma'(t)\| dt \\ &= |\lambda| d(p, q) \end{aligned}$$

as sought. □

Proposition 4.2. With Lebesgue measure we see that $\mathrm{vol}(B(0,\varepsilon)) = \mathrm{vol}(B(0,1))\varepsilon^4$ because moving in the z-direction yields ε^2 volume, and therefore $\dim_{\mathcal{H}}(\mathbb{H}) = 4$.

If $(M_i^n, p_i, \mu_i) \xrightarrow[mGH]{} (Y, p, \mu)$ and each M_i has $\mathrm{Ric} \geq -(n-1)H$, then under the μ-renormalzied limit measure we have $r_1 \geq r_2$ implying

$$\frac{\mu(B(y,r_1))}{\mu(B(y,r_2))} \geq \frac{v(n,H,r_1)}{v(n,H,r_2)} \tag{4.1}$$

Proposition 4.3. Any such space Y has $\dim_{\mathcal{H}}(Y) \leq n$.

Proof. We use Minkowski measure by balls instead. If A is compact, we can cover A by $Cov(A,\varepsilon)$ many ε-balls. Let $p_0 = p_i$ be the point which realizes the minimum $\min\{\mathrm{vol}(B(p_i,\varepsilon)\}$ over an ε-cover of A. Observe that

$$\begin{aligned} Cov(A,\varepsilon) &\leq \frac{\mathrm{vol}(A)}{\mathrm{vol}(B(p_i,\varepsilon))} \\ &\leq \frac{\mathrm{vol}(B(p_i, \mathrm{diam}\, A + \varepsilon))}{\mathrm{vol}(B(p_i,\varepsilon/2))} \\ \text{(by 4.1)} \quad &\leq \frac{v(n,H,\varepsilon + \mathrm{diam}(A))}{v(n,H,\varepsilon/2)} \\ &\sim \varepsilon^{-n} \end{aligned}$$

Therefore, if $s = \dim_H(A) > n$, we have

$$Cov(A,\varepsilon)\varepsilon^s \leq \varepsilon^{-n}\varepsilon^s \xrightarrow[\varepsilon\to 0]{} 0$$

□

Let (M_i^n, g_i, p_i) be a sequence of Riemannian manifolds with $\mathrm{Ric}_{M_i} \geq (n-1)H$ and $M_i^n \to Y$. Recall that Y is called a non-collapsed limit if

$$\mathrm{vol}(B(p_i,1)) \geq v > 0$$

and collapsed if

$$\mathrm{vol}(B(p_i,1)) \to 0$$

Let μ be a renormalized measure on Y. We have the following result

Proposition 4.4. If Y is a non-collapsed limit, then

$$\mu = c\mathcal{H}^n$$

for some constant c and $\dim_{\mathcal{H}}(Y) = n$.

If Y is a collapsed limit, then

$$\dim_{\mathcal{H}}(Y) \leq n - 1$$

In particular, the Hausdorff dimension of Y cannot be between n and $n-1$.

4.2. **Rectifiable dimension of Ricci limit spaces.**

Definition 4.7 (tangent cone at a point)**.** Let (X, d) be a metric space. Let $p \in X$. We take the pGH limit of $(X, p, \lambda_n d)$ where $\lambda_n \to \infty$. If such a limit exists, then it is called a tangent cone of X at p.

Remark 4.1. Note that the limit may depend on the choice of sequence λ_n, i.e. tangent cones may not be unique. If it is unique, we denote the tangent cone at p as $C_p(X)$. The intuition is that we are zooming in the space at p.

Example 4.2.
(1) For a Riemannian manifold, the tangent cone at a point is just the tangent space and is isometric to $\mathbb{R}^n$.
(2) For a folded paper with length metric, let p be a point at the boundary. The space is actully isometric to $\mathbb{R}^n$ (imagine an ant on the surface and it cannot tell the space from the plane), so is the tangent cone at p.
(3) We glue two copies of 2-disks togther along the boundary. Topologically the space is homeomorphic to a 2-sphere. If p is a point at the boundary, then due to same reason as the previous example, the tangent cone at p is isometric to $\mathbb{R}^n$.
(4) Consider a cone and its vertex p. The tangent cone at p is just the cone itself, and is homeomorphic to $\mathbb{R}^n$ but not isometric to it.
(5) Let X be a cube. If p is a point at an edge, then $C_p(X)$ isometric to $\mathbb{R}^2$ just as the folded paper. If q is at one of the vertices, then the tangent cone is a cone with crossection being an equilateral triangle.

Definition 4.8 (asymptotic cone)**.** Let (X, d) be a metric space. Let $p \in X$. We take the pGH limit of $(X, p, \lambda_n d)$ where $\lambda_n \to 0$. If such a limit exists, then it is called a asymptotic cone of X.

Remark 4.2. The asymptotic cone does not depend on $p \in X$, but may depend on λ_n.The intuition is that we are zooming out the space, looking from somewhere far away. Asymptotic cone is especially useful when study spaces with $\text{Ric} \geq 0$.

Definition 4.9 (Regular and singular points)**.** A point $p \in X$ is called a regular point if the tangent cone at p exists, and is unique and isometric to $\mathbb{R}^k$ for some integer k.

Remark 4.3. Different regular points in X may have different k. For example, consider a suitable CW complex. $\mathcal{R} = \{p \in X | p \textit{ is regular}\}$ is the collection of all regular points.

Example 4.3. Consider the tangent bundle $T\mathbb{S}^2$ with metric $g = dr^2 + \varphi(r)^2[\psi(r)^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2]$. As $r \to \infty$, we let $\varphi(r) \to r, \psi(r) \to 1$. Hence at infinity it looks like $\mathbb{R}^4, g = dr^2 + d\mathbb{S}^3$. The unit sphere bundle of $T\mathbb{S}^2$ is $\mathbb{RP}^3$. The asymptotic cone is a cone over $\mathbb{RP}^3$, hence not even topologically a manifold. (Manifolds are all cones over spheres.)

Denote the k-regular set,

$$\mathcal{R}_k = \{y \in Y \mid C_y(Y) \text{ is } \mathbb{R}^k\}$$

We also have the following general result for Ricci limit space Y.

Theorem 4.1 (Colding-Naber)**.** There exists a unique integer k, $0 \le k \le n$ such that $\mathcal{R}_k$ has full measure, i.e.,

$$\mu(Y \setminus \mathcal{R}_k) = 0$$

This k is called the rectified dimension or essential dimension. In general, k is not equal to the Hausdorff dimension of Y. However, in the non-collapsed case, we have

$$k = \dim_{\mathcal{H}}(Y)$$

4.3. **Examples of Ricci limit space with Hausdorff dimension different from the rectifiable dimension.** [17] [12]

Consider $M = \mathbb{R}^k \times S^1, g = \mathrm{d}r^2 + f^2(r)\,\mathrm{d}s^2_{k-1} + h^2(r)\,\mathrm{d}s_1^2$ again as in Subsection 3.2, but with warping functions as in [26]. Namely let $f(r) = r(1+r^2)^{-\frac{1}{4}} \sim \sqrt{r},\ h(r) = (1+r^2)^{-\alpha} \sim r^{-2\alpha}$.

From the curvature formulas (3.1)-(3.3) one can check that $\mathrm{Ric} > 0$ if $k \ge \max\{4\alpha + 3, 16\alpha^2 + 8\alpha + 1\}$.

Let $\tilde{M} \cong \mathbb{R}^k \times \mathbb{R}^1$ be its universal cover. We denote the asymptotic cone, singular set, regular set of $\tilde{M}$ by $Y, \mathcal{S}, \mathcal{R}_2$ respectively.

Theorem 4.2 (Pan-Wei, GAFA, 2022)**.**

(1) $Y = [0, +\infty) \times \mathbb{R},\ \mathcal{S} = \{0\} \times \mathbb{R},\ \mathcal{R}_2 = (0, +\infty) \times \mathbb{R}$
(2) $\dim_{\mathcal{H}} \mathcal{S} = 1 + 2\alpha,\ \dim_{\mathcal{H}} \mathcal{R}_2 = 2$

Theorem 4.3 ([12])**.** *Y is equipped with incomplete Riemannian metric $g_Y = \mathrm{d}r^2 + r^{-4\alpha}\,\mathrm{d}v^2$ on $(0, +\infty) \times \mathbb{R}$*

Remark 4.4. Y with this metric is called Grushin-2α half plane, a subRiemannian and RCD space at the same time. But the whole plane is not an RCD space as the singular set cuts the plane into two disjoint parts and the regular set of RCD space is connected,

Theorem 4.3 has some immediate interesting consequences.

For $\lambda > 0$, consider $F_\lambda(r, v) = (\lambda r, \lambda^{1+2\alpha} v)$. Then we have $F_* g_Y = \lambda^2 g_Y$, therefore

$$\mathrm{d}\Big(F_\lambda(y_1), F_\lambda(y_2)\Big) = \lambda\,\mathrm{d}(y_1, y_2). \tag{4.2}$$

Apply above with $\lambda = v^{\frac{1}{1+2\alpha}}$. Then

$$\mathrm{d}\Big((0, v), (0, 0)\Big) = \mathrm{d}\Big(F_\lambda(0, 1), F_\lambda(0, 0)\Big) = v^{\frac{1}{1+2\alpha}}\,\mathrm{d}\Big((0, 1), (0, 0)\Big).$$

This implies $\dim_{\mathcal{H}} \mathcal{S} = 1 + 2\alpha$.

Proof of Theorem 4.3. We have $\tilde{M} = \mathbb{R}^k \times \mathbb{R}^1$, with double warped metric

$$g = dr^2 + r(1+r^2)^{\frac{1}{4}} dS^2_{k-1} + (1+r^2)^{-\alpha} dv^2.$$

Given any $\lambda > 1$, let $s = \lambda^{-1} r$, $w = \lambda^{-2\alpha} v$. Then we get

$$\begin{aligned}\lambda^{-2} g_{\tilde{M}} =& \lambda^{-2}\left[dr^2 + r^2(1+r^2)^{-\frac{1}{2}} dS_{k-1}^2 + (1+r^2)^{-2\alpha} dv^2\right]\\ =& ds^2 + \frac{s^2}{1+\lambda^2 s^2} dS_{k-1}^2 + (1+\lambda^2 s^2)^{-2\alpha}\lambda^{4\alpha} dw^2\end{aligned}$$

As we take the limit $\lambda \to \infty$, this metric approaches $ds^2 + s^{-4\alpha} dw^2$. □

5. Lecture 5: Topology of Ricci limit/RCD spaces

There is a lot of work on geometric structures of Ricci limit spaces. (Cheeger-Colding, Naber, Jiang)

For a non-collapsing Ricci limit space X, (i.e. $(M_i, p_i) \to X$, $Ric_{M_i} \geq k$, non-collapsing condition: $volB(p_i, 1) > v > 0$) Then:

(1) The Riemann measure $dvol_{M_i}$ converges to the n-dimensional Hausdorff measure of X.
(2) Regular points have full measure and form a (topological) manifold.
(3) All tangent cones of all points are metric cones
(4) $dim_{\mathcal{H}}(\mathcal{S}) \leq n-2$
(5) $dim_{\mathcal{H}}(\mathcal{S}) \leq n-4$ if also $Ric_{M_i} \leq \bar{k}$

For a collapsing Ricci limit space X, there exists $k \in \mathbb{N} \cap [0, n-1]$ such that $\mathcal{R}_k = \{p \in X | C_p(X) \cong \mathbb{R}^k\}$ has full measure with respect to the limit of the renormalised measure. Also in this case, $dim_{\mathcal{H}}(X)$ may not be an integer.

5.1. Relative δ-covers. [22]

The universal cover is often defined as the simply connected cover. Here we do not assume it is simply connected, instead as the cover of all covers.

Definition 5.1. [23, Page 82] We say $\widetilde{X}$ is a universal cover of a path-connected space X if $\widetilde{X}$ is a cover of X such that for any other cover $\bar{X}$ of X, there is a commutative triangle formed by a covering map $f : \widetilde{X} \to \bar{X}$ and the two covering projections as below:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & \bar{X} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Let $\mathcal{U}$ be any open covering of X. For any $x \in X$, by [23, Page 81], there is a covering space $\widetilde{X}_{\mathcal{U}}$ of X with covering group $\pi_1(X, \mathcal{U}, p)$, where $\pi_1(X, \mathcal{U}, x)$ is a normal subgroup of $\pi_1(X, p)$ generated by homotopy classes of closed paths having a representative of the form $\alpha^{-1} \circ \beta \circ \alpha$, where β is a closed path lying in some element of $\mathcal{U}$ and α is a path from x to $\beta(0)$.

Now we recall the notion of δ-covers introduced in [21] which plays an important role in studying the existence of the universal cover.

Definition 5.2. Given $\delta > 0$, the δ-cover, denoted $\widetilde{X}^\delta$, of a length space X is defined to be $\widetilde{X}_{\mathcal{U}_\delta}$, where $\mathcal{U}_\delta$ is the open covering of X consisting of all balls of radius δ.

Intuitively, a δ-cover is the result of unwrapping all but the loops generated by small loops in X. Clearly $\widetilde{X}^{\delta_1}$ covers $\widetilde{X}^{\delta_2}$ when $\delta_1 \leq \delta_2$.

Definition 5.3 (Relative δ-cover)**.** Suppose X is a length space, $x \in X$ and $0 < r < R$. Let

$$\pi^\delta : \widetilde{B}_R(x)^\delta \to B_R(x)$$

be the δ-cover of the open ball $B_R(x)$. A connected component of

$$(\pi^\delta)^{-1}(B(x,r)),$$

where $B(x,r)$ is a closed ball, is called a relative δ-cover of $B(x,r)$ and is denoted $\widetilde{B}(x,r,R)^\delta$.

5.2. **Universal cover of Ricci limit/RCD space exists.** [14, 21, 22]

In [22, Lemma 2.4, Theorem 2.5] it is shown that if the relative δ-cover stabilizes, then universal cover exists. This is the key tool for showing the existence of the universal cover.

Theorem 5.1. Let (X, d) be a length space and assume that there is $x \in X$ with the following property: for all $r > 0$, there exists $R \geq r$, such that $\tilde{B}(x,r,R)^\delta$ stabilizes for all δ sufficiently small. Then (X, d) admits a universal cover $\tilde{X}$. More precisely $\tilde{X}$ is obtained as covering space $\tilde{X}_{\mathcal{U}}$ associated to a suitable open cover $\mathcal{U}$ of X satisfying the following property: for every $x \in X$ there exists $U_x \in \mathcal{U}$ such that U_x is lifted homeomorphically by any covering space of (X, d).

Theorem 5.2 ([21, 22])**.** If X is the Gromov-Hausdorff limit of a sequence of complete Riemannian manifolds M_i^n with Ricci curvature $\geq K$, then X has a universal cover.

Theorem 5.3 ([14])**.** Any $\mathsf{RCD}^*(K,N)$ space $(X, \mathsf{d}, \mathfrak{m})$ admits a universal cover $(\tilde{X}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}})$, which is itself $\mathsf{RCD}^*(K,N)$, where $K \in \mathbb{R}$, $N \in (1, +\infty)$.

By Theorem 5.1 it is enough to show the relative covers stabilize. The following result plays an important role in [24] showing that RCD spaces are semi-locally simply connected, which follows from [14, Theorem 4.5].

Theorem 5.4. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(K,N)$ space for some $K \in \mathbb{R}, N \in (1,\infty)$. For all $R > 0$ and $x \in X$, there exists $\delta_{x,R}$ depending on X, x, R such that

$$\tilde{B}(x, \tfrac{R}{10}, R)^{\delta_{x,R}} = \tilde{B}(x, \tfrac{R}{10}, R)^{\delta} \qquad \forall \delta < \delta_{x,R}. \tag{5.1}$$

The proof is divided into two steps.

Step I: Show the stability of relative δ covers at regular points.

Intuitively, if not there are shorter and shorter based closed geodesic loops shrinking toward x, find corresponding closed curve in the tangent cone $\mathbb{R}^k$ which are "alomost closed based geodeisc loops", this is a contradiction.

For rigorous proof, one needs quantitative estimates. The midpoint m of the geodesic loop is a cut point of x. In particular, for $y \in X$ with $d(x,y) > D$, where $D = d(x,m)$, we have

$$d(x,y) < D + d(y,m).$$

Using Abresch-Gromoll inequality on δ-covers one gets Sormani's uniform cut lemma [20] — quantitative version.

Lemma 5.1. For all $D \leq \frac{1}{2}$, y with $d(x,y) \geq D + S(K,N)D$, we have

$$d(x,y) \leq D + d(y,m) - S(K,N)D,$$

where $S(K,N) > 0$ is a small constant.

Now pass this distance estimate to the tangent cone at x, which is $\mathbb{R}^k$, but this is not true on $\mathbb{R}^k$.

Step II: With one regular point, use Bishop-Gromov relative volume comparison and a packing argument to show the stability of relative δ covers everywhere.

5.3. **Ricci limit/RCD spaces are semi-locally simply connected.** [16, 24]

Recall that the universal cover $\tilde{X}$ is simply connected iff X is semilocally simply connected, which means that there exists a neighbourhood such that every loop is contractible in X.

Definition 5.4 (1-contractibility radius)**.**

$$\rho(t,x) = \inf\{\infty, \rho \geq t | \text{ any loop in } B_t(x) \text{ is contractible in } B_\rho(x)\}.$$

X is semi-locally simply connected if for any $x \in X$, there is $T > 0$ such that $\rho(T,x) < \infty$.

In [16], for noncollapsing Ricci limit space we show it is essentially locally simply connected.

Theorem 5.5. Any non-collapsing Ricci limit space is semi-locally simply connected. Therefore the universal cover is simply connected. In fact

$$\lim_{t\to 0} \frac{\rho(t,x)}{t} = 1.$$

In the paper we illustrate several ways of constructing homotopy. One way is to construct a homotopy by defining it on finer and finer skeletons of closed unit disk, see [17, Lemma 4.1].

Theorem 5.6 ([24])**.** For a locally compact metric space, if any local relative δ-cover is stable, then it is semi-locally simply connected.

Combining this with Theorem 5.4 it shows that the universal cover of $\mathsf{RCD}(k,N), N < \infty$ is simply connected.

To prove this the key lemma is to use stability of the local relative δ-cover to show any loop in a small neighborhood of an $\mathsf{RCD}(K,N)$ space is homotopic to some loops in very small balls by a controlled homotopy image.

Lemma 5.2 (Key Lemma)**.** For any $x \in (X, \mathsf{d}, \mathfrak{m})$, an $\mathsf{RCD}(K,N)$, any $l < 1/2$, and small $\delta > 0$, there exists $\rho < l$ and $k \in \mathbb{N}$ so that any loop $\gamma \subset B_\rho(x)$ is homotopic to the union of some loops γ_i $(1 \leq i \leq k)$ in δ-balls and the homotopy image is in $B_{4l}(x)$.

Apply above lemma iteratively one can construct the needed homotopy. Namely first shrink γ to loops in δ_1-balls, the second step is to shrink each new loop to smaller loops in δ_2-balls, etc. Since the homotopy to shrink each loop is contained in a l_i-ball in the i-th step, this process converges to a homotopy map which contracts γ while the image is contained in a ball with radius $\sum_{i=1}^{\infty} l_i < R$ as in the construction of [16, Lemma 4.1].

Question 5.1. $\lim_{t\to 0} \frac{\rho(t,x)}{t} = 1$?

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Department of Mathematics, University of California, Santa Barbara, CA 93106