LOCAL SOBOLEV CONSTANT ESTIMATE FOR INTEGRAL BAKRY-EMERY RICCI CURVATURE

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Abstract. We extend several geometrical results in [10] for Riemannian manifolds with integral curvature to complete smooth metric measure spaces with integral Bakry-Émery Ricci curvature.

1. Introduction

Sobolev inequalities not only encode rich analytical and geometrical information about the manifold, but also have wide applications in differential geometry. An useful method to estimate the Sobolev constant is to estimate the isoperimetric constant since they are equivalent [5,6,12]. A key issue for the isoperimetric constant study is the volume control, which is given by Ricci curvature lower bound After Petersen and Wei [16] generalized the classical Laplace and volume comparison to integral Ricci curvature lower bound, many results for pointwise Ricci lower bound have been extended to integral Ricci curvature lower bound, see e.g. [1,2,9,10,17–20,23, 24]. In particular, Dai, Wei and Zhang [10] obtained the local isoperimetric constant estimate for integral Ricci curvature. For smooth metric measure space $M^n_f := (M^n, g, e^{-f} d\,\mathrm{vol})$, a natural generalization of Ricci curvature is the Bakry-Émery Ricci curvature [3] defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}(f).$$

A great deal of efforts has been devoted to the study of smooth metric measure space with Bakry-Émery Ricci curvature bounded below, some of the earlier works are [4,14,15,21]. However, limited work has been done for the integral Bakry-Émery Ricci curvature.

Recently, Wu [22] extend the volume comparison in [16] to integral Bakry-Émery Ricci curvature case. In this paper we extend the local isoperimetric constant estimate in [10] to integral Bakry-Émery Ricci curvature and give applications.

To state the results, we fix some notations. Given $x \in M^n_f$, let $\rho_f(x)$ be the smallest eigenvalue of $\mathrm{Ric}_f : T_xM \to T_xM$, and

$$\mathrm{Ric}^H_{f-} = ((n-1)H - \rho_f(x))_+ = \max\{0, (n-1)H - \rho_f(x)\}.$$

Denote $B_x(R) \subset M$ the ball with radius R, center at x. Various weighted L^p norms of the function h on a smooth metric measure space M^n_f are

$$\|h\|_{p,f,B_x(R)} = \left(\int_{B_x(R)} |h|^p e^{-f} d\,\mathrm{vol}\right)^{\frac{1}{p}}$$

and

$$\|h\|_{p,f,a}(R) = \sup_{x\in M^n_f} \left[\int_0^R \int_{\mathbb{S}^{n-1}} |h|^p e^{-at} \mathcal{A}_f(t,\theta) d\theta dt\right]^{\frac{1}{p}}.$$

Key words and phrases. Integral Bakry-Émery Ricci curvature, Isoperimetric constant estimate, Sobolev constant.

Here $\mathcal{A}_f(t,\theta)$ is the volume element of weighted measure $e^{-f}d\operatorname{vol} = \mathcal{A}_f(t,\theta)d\theta\wedge dt$ and $d\theta$ is the volume element of unit sphere $\mathbb{S}^{n-1}$. Clearly, $\|\operatorname{Ric}_{f-}^H\|_{p,f,a}(R)\equiv 0$ iff $\operatorname{Ric}_f \geq (n-1)H$. For convenience, we will assume $H=0$. Then the following scale invariant curvature quantity is useful,

$$\bar{\kappa}(p,f,a,R) = R^2 \sup_{x\in M_f^n}\left[\fint_{B_x(R)} \rho_{f-}^p e^{-at}\mathcal{A}_f(t,\theta)d\theta\wedge dt\right]^{\frac{1}{p}},$$

where $\operatorname{vol}_f(B_x(R)) = \int_{B_x(R)} e^{-f}d\operatorname{vol}$, $\fint_{B_x(R)} = \frac{1}{\operatorname{vol}_f(B_x(R))}\int_{B_x(R)}$.

Our first result gives an estimation of the local normalized Dirichlet isoperimetric constant.

Theorem 1.1. Let M_f^n be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a\geq 0$. For $p>\frac{n}{2}$, there exists $\varepsilon = \varepsilon(n,p,a)>0$ such that if $\bar{\kappa}(p,f,a,1)\leq\varepsilon$, then for any $x\in M_f^n$, $\partial B_x(R)\neq\varnothing$, $R\leq 1$, we has the estimate

$$\operatorname{ID}_{n,f}^*\left(B_x(R)\right) \geq 10^{-2n}e^{-2a}R^{-1}, \tag{1.1}$$

where

$$\operatorname{ID}_{n,f}^*\left(B_x(R)\right) = \operatorname{vol}_f\left(B_x(R)\right)^{-\frac{1}{n}}\inf_{\Omega}\left\{\frac{\operatorname{vol}_f(\partial\Omega)}{\operatorname{vol}_f(\Omega)^{\frac{n-1}{n}}}\right\}.$$

Here the infimum runs over all subdomains $\Omega\subset B_x(R)$ with smooth boundary and $\partial\Omega\cap\partial B_x(R)=\varnothing$.

Remark 1.2. Clearly, the local normalized Dirichlet constant have explicit and accurate dependency of the growth of f, Theorem 1.1 will recover to [10, Theorem 1.1] when f is a constant. The smallness of $\bar{\kappa}(p,f,a,1)$ is necessary, see the counterexample in [10, Section 6] when f is constant and $\bar{\kappa}(p,f,a,1)$ is bounded. Also the result is not true when $p\leq\frac{n}{2}$, see details in [1].

It is well known that the classical Dirichlet isoperimetric and Sobolev constants are same, see e.g. [12]. In Section 3, we introduced these for smooth metric measure spaces, see Definitions 3.1, 3.2. Similar proof shows they are also same, see Theorem A.1. Hence we have

Proposition 1.3. With the same assumption as in Theorem 1.1, the Sobolev inequality

$$\fint_{B_x(R)}|\nabla h|e^{-f}d\operatorname{vol} \geq 10^{-2n}e^{-2a}R^{-1}\left(\fint_{B_x(R)} h^{\frac{n}{n-1}}e^{-f}d\operatorname{vol}\right)^{\frac{n-1}{n}} \tag{1.2}$$

holds for all $h\in C_0^\infty\left(B_x(R)\right)$.

Recall the f-Laplacian of M_f^n is

$$\Delta_f = \Delta - \nabla f\cdot\nabla.$$

Given the normalized form of integral in Proposition 1.3, we denote the normalized L^p norm for function h by

$$\|h\|_{p,f,B_x(R)}^* = \|h\|_{p,f,B_x(R)}\left(\operatorname{vol}_f(B_x(R))\right)^{-\frac{1}{p}},$$

and

$$\|h\|_{p,f,a}^*(R) = \sup_{x\in M_f^n}\left(\fint_{B_x(R)} he^{-at}\mathcal{A}_f(t,\theta)d\theta\wedge dt\right)^{\frac{1}{p}}$$

with the same $\mathcal{A}_f(t,\theta)d\theta\wedge dt$ as above. It easy to observe that

$$\|h\|_{p,f,B_x(R)}^* \leq e^{\frac{aR}{p}}\|h\|_{p,f,a}^*(R), \tag{1.3}$$

and the normalized L^∞ norm is independent of f satisfying $\|h\|^*_{\infty,f,B_x(R)} = \sup_{B_x(R)} h$. By employing above Sobolev inequality (1.2), we extend the maximum principle in [10, 17] to integral Bakry-Émery Ricci curvature situation.

Theorem 1.4. Let M^n_f be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{n}{2}$, there exists an $\varepsilon = \varepsilon(n,p,a) > 0$ and $C = C(n,p,a) > 1$ such that if $\bar{\kappa}(p,f,a,1) \leq \varepsilon$ and $R \leq 1$, then for any function $u : \Omega\,(\subset B_x(R)) \subset M^n_f \to \mathbb{R}$ with $\Delta_f u \geq h$, we have

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C \cdot R^2 \cdot \|h_-\|^*_{p,f,\Omega}.$$

Also we have the gradient estimate.

Theorem 1.5. Let M^n_f be a complete smooth metric measure space. Assume that $\partial_r f \geq -a$ along all minimal geodesics for some constant $a \geq 0$. For $p > \frac{n}{2}$, there exists an $\varepsilon = \varepsilon(n,p,a) > 0$ and $C(n,p,a) > 1$ such that if $\bar{\kappa}(p,f,a,1) \leq \varepsilon$ and $R \leq 1$ and u is a function on $B_x(R)$ satisfying

$$\Delta_f u = h,$$

then

$$\sup_{B(x,\frac{R}{2})} |\nabla u|^2 \leq C(n,p,a) R^{-2} \left[\left(\|h\|^*_{2p,f,B_x(R)} \right)^2 + \left(\|u\|^*_{2,f,B_x(R)} \right)^2 \right].$$

An outline of this paper is as follows. In §2, we review the Laplacian and volume comparison for integral Bakry-Émery Ricci curvature. In §3, we define local Dirichlet isoperimetric and Sobloev constants, as well as their normalized form in smooth metric measure space, moreover, we estimate the the normalized isoperimetric constant for integral Bakry-Émery Ricci curvature, see Theorem 1.1. In §4, as an applications, we establish the maximum principle (Theorem 1.4) and gradient estimate (Theorem 1.5) in a complete smooth metric measure space with integral Bakry-Émery Ricci curvature. In Appendix, we give the proof of equivalence between the two constants defined in §3.

Acknowledgments. Part of work was done while the first author was visiting UCSB. She would like to thank China Scholarship Council(CSC)(No. 201606140057) for the financial support and UCSB for the hospitality and providing great environment for research. She also would like to thank Professor Yu Zheng for his constant support and her office mate at UCSB, Jesús Núñez-Zimbrón, for his interest and helpful conversations. The second author is supported by National Science Foundation under Grant No. DMS-1506393.

2. Preliminary

In this section, we review Laplacian and volume comparison for smooth metric measure spaces. Let M^n_f be a complete smooth metric measure space and $r(y) = d(y,x)$ be a distance function from $x \in M^n_f$. Assume that f satisfies $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. By choosing the Euclidean space with a weighted function as the model space, that is $\mathbb{R}^n_a = \left(\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-h} d\,\mathrm{vol}\right)$ with $h(x) = -a|x|$ for $x \in \mathbb{R}^n$, then the f-Laplacian error term is

$$\psi(y) = \left(\Delta_f r - \frac{n-1}{r} - a \right)_+.$$

Wei-Wylie [21] has proved $\mathrm{Ric}_{f-} \geq 0$ yields $\Delta_f r \leq \frac{n-1}{r} + a$, that is $\| \mathrm{Ric}_{f-} \|_{p,f,a}(r) \equiv 0$ implies that $\psi \equiv 0$. In [22], this has been extended to integral Bakry-Émery Ricci curvature bound following the work of Petersen-Wei [16] for integral Ricci curvature.

Theorem 2.1. ([22, Theorem 1.1]) For any $p > \frac{n}{2}$, we have

$$\|\psi\|_{2p,f,a}(r) \le C(n,p)\left[\|\operatorname{Ric}_{f-}\|_{p,f,a}(r)\right]^{\frac{1}{2}}, \tag{2.1}$$

with $C(n,p) = \left(\frac{(n-1)(2p-1)}{2p-n}\right)^{\frac{1}{2}}$. Moreover, let $B_x(r_2)$ and $B_x(r_1)$ be geodesic balls centered at x with radius $r_2 \ge r_1 > 0$, we have

$$\left(\frac{\operatorname{vol}_f(B_x(r_2))}{V(n,a,r_2)}\right)^{\frac{1}{2p}} - \left(\frac{\operatorname{vol}_f(B_x(r_1))}{V(n,a,r_1)}\right)^{\frac{1}{2p}} \le C(n,p,a,r_2)\left(\|\operatorname{Ric}_{f-}\|_{p,f,a}(r_2)\right)^{\frac{1}{2}}, \tag{2.2}$$

where $V(n,a,t)$ denote the volume of geodesic ball $B_0(t)$ in the model space $\mathbb{R}^n_a$.

Remark 2.2. In fact, the proof of [22, Theorem 1.1] gives the following normalized form of Laplacian comparison

$$\|\psi\|^*_{2p,f,a}(r) \le C(n,p)\left(\|\operatorname{Ric}_f\|^*_{p,f,a}(r)\right)^{\frac{1}{2}} = C(n,p)r^{-1}\left(\kappa(p,f,a,r)\right)^{\frac{1}{2}}. \tag{2.3}$$

Remark 2.3. Here we choose the power $2p$ rather than $2p-1$, so the explicit expression of $C(n,p,a,r_2)$ similar to the one in [16, Lemma 2.1] rather than the one in [22]. If we denote the volume of $(n-1)$-dimensional unit ball in $\mathbb{R}^n$ and the weighted volume of geodesic sphere $\partial B_0(t)$ by ω_n and $A(n,a,t)$, respectively. Then $A(n,a,t) = \omega_n t^{n-1}e^{at}$ and

$$\begin{aligned} C(n,p,a,r_2) &= C(n,p)\int_0^{r_2} tA(n,a,t)\left(\frac{1}{V(n,a,t)}\right)^{1+\frac{1}{2p}} dt \\ &\le C(n,p)\int_0^{r_2} tA(n,a,t)\left(\int_0^t A(n,0,s)ds\right)^{-\left(1+\frac{1}{2p}\right)} dt \\ &= C(n,p)e^{ar_2}r_2^{1-\frac{n}{2p}}. \end{aligned}$$

Hence, (2.2) implies that

$$\begin{aligned} \left(\frac{\operatorname{vol}_f(B_x(r_1))}{\operatorname{vol}_f(B_x(r_2))}\right)^{\frac{1}{2p}} &\ge \left(\frac{V(n,a,r_1)}{V(n,a,r_2)}\right)^{\frac{1}{2p}}\left[1 - C(n,p)e^{ar_2}r_2^{1-\frac{n}{2p}}\left(V(n,a,r_2)\right)^{\frac{1}{2p}}\left(\|\operatorname{Ric}_{f-}\|^*_{p,f,a}(r_2)\right)^{\frac{1}{2}}\right] \\ &\ge e^{-\frac{ar_2}{2p}}\left(\frac{r_1}{r_2}\right)^{\frac{n}{2p}}\left[1 - C(n,p)e^{(1+\frac{1}{2p})ar_2}\bar{\kappa}^{\frac{1}{2}}(p,f,a,r_2)\right], \end{aligned} \tag{2.4}$$

where $C(n,p)$ is a constant depends on n and p. Hence, there exists a constant $\varepsilon_0 = \varepsilon_0(n,p,a,r_0) > 0$ such that if $\bar{\kappa}(p,f,a,r_0) \le \varepsilon_0$, then

$$\frac{\operatorname{vol}_f(x,r)}{\operatorname{vol}_f(x,r_0)} \ge \frac{e^{-ar_0}}{2}\left(\frac{r}{r_0}\right)^n, \forall r \le r_0. \tag{2.5}$$

For $r_0 \le 1$, from (2.4), it is easy to observe that there exists a $\varepsilon_0 = \varepsilon_0(n,p,a)$, independent of r, such that (2.5) holds for $\bar{\kappa}(p,f,a,r_0) \le \varepsilon_0$.

Remark 2.4. The scale invariant $\bar{\kappa}(p,f,a,r)$ has the curvature inequalities. For any $r_1 \le r_2$, and $\bar{\kappa}(p,f,a,r_2) \le \varepsilon_0$, on one hand,

$$\begin{aligned} \bar{\kappa}(p,f,a,r_1) &\le r_1^2\left(\frac{\operatorname{vol}_f(B_x(r_2))}{\operatorname{vol}_f(B_x(r_1))}\cdot\frac{1}{\operatorname{vol}_f(B_x(r_2))}\int_0^{r_2}\int_{S^{n-1}}\rho^p_{f-}e^{-at}\mathcal{A}_f(t,\theta)d\theta dt\right)^{\frac{1}{p}} \\ &\le 2^{\frac{1}{p}}e^{\frac{ar_2}{p}}\left(\frac{r_1}{r_2}\right)^{2-\frac{n}{p}}\bar{\kappa}(p,f,a,r_2). \end{aligned} \tag{2.6}$$

Hence, $\bar{\kappa}(p,f,a,r_1)\le\varepsilon_0$ holds for $r_1\le 2^{-\frac{1}{2p-n}}e^{-\frac{ar_2}{2p-n}}r_2$. On the other hand, if $\bar{\kappa}(p,f,a,r_1)\le\varepsilon_0$, using the same method as in [17, Section 2.3] and volume doubling property (2.5), we have

$$\begin{aligned}\bar{\kappa}(p,f,a,r_2)&=r_2^2\sup_{x\in M_f^n}\left(\frac{1}{\operatorname{vol}_f\left(B_x(r_2)\right)}\int_0^{r_2}\int_{S^{n-1}}\rho_{f-}^p e^{-at}\mathcal{A}_f(t,\theta)d\theta dt\right)^{\frac{1}{p}}\\&\le\left(\frac{r_2}{r_1}\right)^2 2^{\frac{n+1}{p}}e^{\frac{ar_1}{p}}\bar{\kappa}(p,f,a,r_1).\end{aligned}\tag{2.7}$$

Hence, it is sufficient to work for the case $\bar{\kappa}(p,f,a,1)$ is small and then scale the metric to obtain the curvature condition.

3. Local Dirichlet Isoperimetric constant estimate

In this section, we introduce the local isoperimetric and Sobolev constants in smooth metric measure spaces motivated by classical ones in [6,12]. Furthermore, we estimate the normalized form of the constants.

Definition 3.1. Let $B_x(r)$ be a geodesic ball with $\partial B_x(r)\neq\varnothing$ in a complete smooth metric measure space M_f^n. For $n\le\alpha\le\infty$, the Dirichlet α-isoperimetric constant of $B_x(r)$ is defined by

$$\mathrm{ID}_{\alpha,f}\left(B_x(r)\right)=\inf_{\Omega}\frac{\operatorname{vol}_f(\partial\Omega)}{\operatorname{vol}_f(\Omega)^{1-\frac{1}{\alpha}}}$$

where Ω is an open submanifold of $B_x(r)$ with $\partial\Omega\cap\partial B_x(r)=\varnothing$.

Clearly, $\mathrm{ID}_{n,f}\left(B_x(r)\right)$ is a scale invariant and $\mathrm{ID}_{\infty,f}\left(B_x(r)\right)$ is a weighted Cheeger constant.

Definition 3.2. The Dirichlet α-Sobolev constant of $B_x(r)\subset M_f^n$ is defined by

$$\mathrm{SD}_{\alpha,f}(B_x(r))=\inf_h\frac{\|\nabla h\|_{1,f,B_x(r)}}{\|h\|_{\frac{\alpha}{\alpha-1},f,B_x(r)}},$$

where the infimum is taken over all $h\in C_0^\infty\left(B_x(r)\right)$.

For convenience, we normalize the two kinds of constants above as following:

$$\begin{aligned}\mathrm{ID}^*_{\alpha,f}\left(B_x(r)\right)&=\mathrm{ID}_{\alpha,f}\left(B_x(r)\right)\operatorname{vol}_f\left(B_x(r)\right)^{-\frac{1}{\alpha}}.\\ \mathrm{SD}^*_{\alpha,f}\left(B_x(r)\right)&=\mathrm{SD}_{\alpha,f}\left(B_x(r)\right)\operatorname{vol}_f\left(B_x(r)\right)^{-\frac{1}{\alpha}}.\end{aligned}\tag{3.1}$$

To estimate $\mathrm{ID}^*_{\alpha,f}\left(B_x(R)\right)$, we need several lemmas. By suitable modification of Gromov's observation [7,11], we can show

Lemma 3.3. Let M_f^n be a complete smooth metric measure space. Assume that $\partial_r f\ge -a$ along all minimal geodesic segments for some constant $a\ge 0$. Let S be any hypersurface dividing M_f^n into two parts M_1, M_2. For any subsets $W_i\subset M_i$, there exists x_1 in one of W_i, say W_1, and a subset W in another one, W_2, such that the unique minimal geodesic joint x_1 and any $x_2\in W$ meet S at q with

$$d(x_1,q)\ge d(x_2,q),\tag{3.2}$$

and

$$\operatorname{vol}_f(W_2)\le 2\operatorname{vol}_f(W).\tag{3.3}$$

Analogous to [10, Lemma 4.2], we have the following volume estimate by using Laplacian comparison method in smooth metric measure space.

Lemma 3.4. Let M^n_f, S, W and x_1 be as in Lemma 3.3. Then for any $p > \frac{n}{2}$,

$$\operatorname{vol}_f(W) \le 2^{n-1}\left[e^{aD} D \operatorname{vol}_f(S') + e^{(a+\frac{a}{2p})D} C(n,p) \left(\bar{\kappa}(p,f,a,D)\right)^{\frac{1}{2}} \operatorname{vol}_f(B(x,D))\right], \tag{3.4}$$

where $D = \sup_{x\in W} d(x_1,x)$ and S' is the set of intersection points with S of geodesics γ_{xx_1} for all $x \in W$.

Proof. Let $\Gamma \subset S_{x_1}$ be the set of unit vectors v such that $\gamma_v = \gamma_{x_1x_2}$ for some $x_2 \in W$. Using the polar coordinate $(\theta,t) \in S_{x_1} \times \mathbb{R}^+$ and $e^{-f} d\operatorname{vol} = \mathcal{A}_f(\theta,t)d\theta \wedge dt$. Recall that [22, Theorem 3.1], we have

$$\frac{\partial}{\partial t}\frac{\mathcal{A}_f}{t^{n-1}e^{at}} = \left(\Delta_f t - \frac{n-1}{t} - a\right)\frac{\mathcal{A}_f}{t^{n-1}e^{at}} \le \psi \frac{\mathcal{A}_f}{t^{n-1}e^{at}} \tag{3.5}$$

with $\psi = \left(\Delta_f t - \frac{n-1}{t} - a\right)_+$. Integrating (3.5) from t to s gives

$$\begin{aligned}\mathcal{A}_f(s,\theta) &\le \left(\frac{s}{t}\right)^{n-1} e^{a(s-t)}\left(\mathcal{A}_f(t,\theta) + \int_t^s \psi \mathcal{A}_f(l,\theta)dl\right)\\ &\le 2^{n-1}e^{\frac{as}{2}}\left(\mathcal{A}_f(t,\theta) + \int_t^s \psi\mathcal{A}_f(l,\theta)dl\right)\end{aligned}$$

for any $\frac{s}{2} \le t \le s$. For any $\theta \in \Gamma$, let $s_1(\theta)$, $s_2(\theta)$ respectively the minimum and maximum radius such that $\exp_{x_1}(s_i\theta) \in W$, and $s(\theta)$ such that $\exp_{x_1}(s(\theta)\theta) \in S$. Then Lemma 3.3 implies that $2s(\theta) \ge s_2(\theta) \ge s_1(\theta) \ge s(\theta)$. Thus,

$$\begin{aligned}\operatorname{vol}_f(W) &\le \int_\Gamma \int_{s_1(\theta)}^{s_2(\theta)} \mathcal{A}_f(s,\theta)dsd\theta\\ &\le 2^{n-1}e^{as(\theta)}\int_\Gamma\int_{s_1(\theta)}^{s_2(\theta)}\left(\mathcal{A}_f(s(\theta),\theta) + \int_{s(\theta)}^s \psi\mathcal{A}_f(l,\theta)dl\right)dsd\theta\\ &\le 2^{n-1}e^{aD}D\left(\int_\Gamma \mathcal{A}_f(s(\theta),\theta)d\theta + \int_0^D\int_\Gamma \psi\mathcal{A}_f(s,\theta)dsd\theta\right).\end{aligned} \tag{3.6}$$

On the other hand,

$$\operatorname{vol}_f(S') = \int_\Gamma \frac{\mathcal{A}_f(s(\theta),\theta)}{\cos\alpha(\theta)}d\theta \ge \int_\Gamma \mathcal{A}_f\left(s(\theta),\theta\right)d\theta,$$

where $\alpha(\theta)$ is the angle between S and the radical geodesic $\exp_{x_1}(s\theta)$. Applying above result, Hölder inequality, (1.3), and Laplacian comparison (2.3) successively in (3.6), we obtain

$$\begin{aligned}\operatorname{vol}_f(W) &\le 2^{n-1}e^{aD}D\left[\operatorname{vol}_f(S') + \|\psi\|_{1,f,B_{x_1}(D)}\right]\\ &\le 2^{n-1}e^{aD}D\left[\operatorname{vol}_f(S') + \|\psi\|^*_{2p,f,B_{x_1}(D)}\operatorname{vol}_f(B_{x_1}(D))\right]\\ &\le 2^{n-1}e^{aD}D\left[\operatorname{vol}_f(S') + e^{\frac{aD}{2p}}\|\psi\|^*_{2p,f,a}(D)\operatorname{vol}_f(B_{x_1}(D))\right]\\ &\le 2^{n-1}e^{aD}D\left[\operatorname{vol}_f(S') + e^{\frac{aD}{2p}}C(n,p)D^{-1}\left(\bar{\kappa}(p,f,a,D)\right)^{\frac{1}{2}}\operatorname{vol}_f(B(x,D))\right]\end{aligned}$$

the required estimate. □

Lemma 3.4 enable us to obtain a local Cheeger's constant estimate.

Lemma 3.5. Let M^n_f, S, W and x_1 be the same as in Lemma 3.3. For $p > \frac{n}{2}$, there exists $\varepsilon = \varepsilon(p, n, a)$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$, then for a geodesic ball $B = B_x(r)$, $r \leq \frac{1}{2}$ which is divided equally by S, we have

$$\mathrm{vol}_f(B_x(r)) \leq 2^{n+3} r e^a \,\mathrm{vol}_f\left(S \cap B_x(2r)\right).$$

Proof. To begin with choosing $W_i = B_x(2r) \cap M_i$ with M_i as in Lemma 3.3, then

$$\mathrm{vol}_f\left(B \cap M_1\right) = \mathrm{vol}_f\left(B \cap M_2\right) \leq \min\left\{\mathrm{vol}_f(W_1), \mathrm{vol}_f(W_2)\right\} \leq 2\,\mathrm{vol}_f(W),$$

and $D \leq 2r$ and $S' \subset S \cap B_x(2r)$. Thus, by Lemma 3.4, we have

$$\begin{aligned} &\mathrm{vol}_f\left(B_x(r)\right) \leq 4\,\mathrm{vol}_f(W) \\ &\leq 2^{n+1}\left[2re^{2ar}\,\mathrm{vol}_f\left(S \cap B_x(2r)\right) + e^{2r(a+\frac{a}{2p})} C(n,p)\left(\bar{\kappa}(p,f,a,2r)\right)^{\frac{1}{2}} \mathrm{vol}_f(B_x(2r))\right]. \end{aligned} \tag{3.7}$$

Next we aim at canceling the curvature inequality. Since volume doubling property (2.5) implies that

$$\mathrm{vol}_f(B_x(2r)) \leq 2\frac{V(n,a,2r)}{V(n,a,r)} \cdot \mathrm{vol}_f(B_x(r)) \leq 2^{n+1} e^a \,\mathrm{vol}_f(B_x(r)) \tag{3.8}$$

holds for $\bar{\kappa}(p, f, a, 2r) \leq \varepsilon_0$. From the curvature inequality (2.6) and $r \leq \frac{1}{2}$, the curvature condition reduces to $\bar{\kappa}(p, f, a, 1) \leq 2^{-\frac{1}{p}} e^{-\frac{a}{p}} \varepsilon_0$, as well as,

$$e^{2r(a+\frac{a}{2p})} C(n,p)\left(\bar{\kappa}(p,f,a,2r)\right)^{\frac{1}{2}} \leq e^{a+\frac{a}{2p}} C(n,p)\left(2^{\frac{1}{p}} e^{\frac{a}{p}} \bar{\kappa}(p,f,a,1)\right)^{\frac{1}{2}}. \tag{3.9}$$

Inserting (3.8) and (3.9) into (3.7) gives

$$\mathrm{vol}_f\left(B_x(r)\right) \leq 2^{n+2} e^a r\,\mathrm{vol}_f\left(S \cap B_x(2r)\right) + 2^{2n+2+\frac{1}{2p}} e^{2a+\frac{a}{p}} C(n,p)\left(\bar{\kappa}(p,f,a,1)\right)^{\frac{1}{2}}. \tag{3.10}$$

Here we used $r \leq \frac{1}{2}$. Hence, we can get the required result by choosing

$$\varepsilon(n,p,a) = \min\left\{2^{-\frac{1}{p}} e^{-\frac{a}{p}} \varepsilon_0, \left(2 \cdot 2^{2n+2+\frac{1}{2p}} e^{2a+\frac{a}{p}} C(n,p)\right)^{-2}\right\}$$

and regrouping (3.10). □

Volume doubling property (2.5) indicate that the volume quotient of concentric geodesic ball lower bound by a function of the quotient of corresponding radius. Next theorem not only offer the other direction but also extend [10, Theorem 3.3] to the case of integral Bakry-Émery Ricci curvature. Actually, by fixing the upper bound $\frac{1}{2}$ and the bigger radius 1, we obtain the other radius.

Theorem 3.6. Let M^n_f be a complete smooth metric measure space with $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{n}{2}$, there exists $\varepsilon = \varepsilon(n, p, a) > 0$ and $r_0 = r_0(n, a) > 0$ such that if $\bar{\kappa}(p, f, a, 1) \leq \varepsilon$, then

$$\frac{\mathrm{vol}_f(B_x(r_0))}{\mathrm{vol}_f(B_x(1))} \leq \frac{1}{2}, \forall x \in M^n_f. \tag{3.11}$$

Our proof follows the idea in [10, Theorem 3.3], but using the approach directly runs into obstacle. The difficult was conquered by repeating the process of choosing the radius $k - 1$ times with k depends on a.

Proof. For any $x \in M^n_f$ and $i = 2, \cdots, k$, let $r_1 = 1$, choose points $x_i \in B_x(r_{i-1})$ with $r_i < \frac{1}{3} r_{i-1}$ and $d_i = d_i(x, x_i) = \frac{r_{i-1} - r_i}{2} > \frac{1}{3} r_{i-1}$, then

$$B_x(r_i) \subset B_{x_i}(d_i + r_i) \setminus B_{x_i}(d_i - r_i) \subset B_{x_i}(d_i + r_i) \subset B_x(r_{i-1}), \quad i = 2, 3, \cdots, k.$$

Moreover,

$$\frac{\mathrm{vol}_f\,(B_x(r_i))}{\mathrm{vol}_f\,(B_x(r_{i-1}))} \leq \frac{\mathrm{vol}_f\,(B_{x_i}(d_i + r_i) \setminus B_{x_i}(d_i - r_i))}{\mathrm{vol}_f\,(B_{x_i}(d_i + r_i))} \leq 1 - \frac{\mathrm{vol}_f(B_{x_i}(d_i - r_i))}{\mathrm{vol}_f(B_{x_i}(d_i + r_i))}.$$

By (2.4), we have

$$\begin{aligned} \frac{\mathrm{vol}_f(B_{x_i}(d_i - r_i))}{\mathrm{vol}_f(B_{x_i}(d_i + r_i))} &\geq \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a(d_i + r_i)} \left[1 - C(n,p) e^{\left(1 + \frac{1}{2p}\right)a} \bar{\kappa}^{\frac{1}{2}}(p, f, a, d_i + r_i)\right]^{2p} \\ &\geq \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a} \left[1 - C(n,p) e^{\left(1 + \frac{1}{2p}\right)a} \cdot (2e^a)^{\frac{1}{2p}}\, \bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)\right]^{2p} \\ &= \left(\frac{d_i - r_i}{d_i + r_i}\right)^n e^{-a} \left[1 - C(n,p) e^{\left(1 + \frac{1}{p}\right)a} 2^{\frac{1}{2p}} \bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)\right]^{2p}. \end{aligned}$$

Here we used curvature inequality (2.6) and $d_i + r_i \leq 1$ in the second inequality. Choose a $q = q(n)$ such that

$$\frac{1 - q}{1 + q} = \left(\frac{3}{4}\right)^{\frac{1}{n}},$$

then for any $r_i \leq \frac{1}{3} q r_{i-1}$, since $d_i > \frac{1}{3} r_{i-1}$, we have

$$\left(\frac{d_i - r_i}{d_i + r_i}\right)^n \geq \frac{3}{4}.$$

Choose $\varepsilon \leq \varepsilon_0$ such that

$$\left(1 - C(n,p) e^{\left(1 + \frac{3}{2p}\right)a} 2^{\frac{1}{p}} \bar{\kappa}^{\frac{1}{2}}(p, f, a, 1)\right)^{2p} \geq \frac{2}{3}. \tag{3.12}$$

Then

$$\frac{\mathrm{vol}_f(B_{x_i}(d_i - r_i))}{\mathrm{vol}_f(B_{x_i}(d_i + r_i))} \geq \frac{3}{4} \cdot e^{-a} \cdot \frac{2}{3} = \frac{e^{-a}}{2},$$

and

$$\frac{\mathrm{vol}_f(B_x(r_i))}{\mathrm{vol}_f(B_x(r_{i-1}))} \leq 1 - \frac{e^{-a}}{2}.$$

Hence,

$$\frac{\mathrm{vol}_f(B_x(r_k))}{\mathrm{vol}_f(B_x(r_1))} = \prod_{i=2}^{k} \frac{\mathrm{vol}_f(B_x(r_i))}{\mathrm{vol}_f(B_x(r_{i-1}))} = \left(1 - \frac{e^{-a}}{2}\right)^{k-1}$$

with $r_k = \left(\frac{q}{3}\right)^{k-1} r_1$. We choose the integer $k = k(a)$ such that

$$\left(1 - \frac{e^{-a}}{2}\right)^{k-1} \leq \frac{1}{2} < \left(1 - \frac{e^{-a}}{2}\right)^{k-2}.$$

Since $r_1 = 1$, so the proof is complete by choosing $r_0 = r_k$ and $\varepsilon \leq \varepsilon_0$ satisfying (3.12). □

We now turn to prove the Theorem 1.1.

Proof of Theorem 1.1. Our first step is to show the estimation (1.1) holds for some radius $r_0 = r_0(n, p, a)$ if $\bar{\kappa}(p, f, a, 1) \le \varepsilon_1$ for some small constant $\varepsilon_1 = \varepsilon_1(n, p, a) > 0$. By Theorem 3.6, we assume that $\varepsilon_1 = \varepsilon_1(n, p, a)$ is chosen such that

$$\frac{\mathrm{vol}_f(B_y(2r_0))}{\mathrm{vol}_f(B_y(\frac{1}{10}))} \le \frac{1}{2}, \ \forall y \in M_f^n$$

holds for some $r_0 = r_0(n, a)$. Given any $y_0 \in M_f^n$, let Ω be a smooth subdomain of $B_{y_0}(r_0)$. Assume that Ω is connected and its boundary $S = \partial\Omega$ divides M_f^n into Ω and Ω^c. For any $y \in \Omega$, let r_y be the smallest radius such that

$$\mathrm{vol}_f\left(B_y(r_y) \cap \Omega\right) = \mathrm{vol}_f\left(B_y(r_y) \cap \Omega^c\right) = \frac{1}{2}\mathrm{vol}_f\left(B_y(r_y)\right). \tag{3.13}$$

From $\Omega \subset B_y(2r_0)$ and $\mathrm{vol}_f(B_y(2r_0)) \le \frac{1}{2}\mathrm{vol}_f\left(B_y(\frac{1}{10})\right)$, it follows that $r_y \le \frac{1}{10}$. Since Ω has a covering

$$\Omega \subset \cup_{y\in\Omega} B_y(2r_y),$$

thanks to Vitali Covering Lemma (cf.[13, Section 1.3]), there exists a countable family of disjoint balls $\{B_{y_i}(2r_i)\}$ such that $\Omega \subset \cup_i B_{y_i}(10r_i)$. On one hand, choosing ε_1 such that $\bar{\kappa}(p, a, f, r) \le \varepsilon_0$ for all $r \le 1$, and using volume doubling property (2.5) leads to

$$\begin{aligned} \mathrm{vol}_f(\Omega) &\le \sum_i \mathrm{vol}_f\left(B_{y_i}(10r_i)\right) \le 2\cdot 10^n \cdot \sum_i e^{10r_i a}\,\mathrm{vol}_f\left(B_{y_i}(r_i)\right) \\ &\le 2\cdot 10^n\cdot e^a \sum_i \mathrm{vol}_f\left(B_{y_i}(r_i)\right). \end{aligned} \tag{3.14}$$

Moreover, choosing ε_1 as in Lemma 3.5 and using the disjoint of the balls $\{B_{x_i}(2r_i)\}$ gives

$$\mathrm{vol}_f(\partial\Omega) \ge \sum_i \mathrm{vol}_f\left(B_{y_i}(2r_i) \cap S\right) \ge 2^{-(n+3)} e^{-a} \sum_i \mathrm{vol}_f\left(B_{y_i}(r_i)\right) r_i^{-1}. \tag{3.15}$$

Combining (3.14) with (3.15) we obtain

$$\begin{aligned} \frac{\mathrm{vol}_f(\partial\Omega)}{(\mathrm{vol}_f(\Omega))^{\frac{n-1}{n}}} &\ge 10^{-(n-1)} 2^{-(n+4-\frac{1}{n})} e^{-2a+\frac{a}{n}} \frac{\sum_i \mathrm{vol}_f\left(B_{y_i}(r_i)\right) r_i^{-1}}{\left(\sum_i \mathrm{vol}_f(B_{y_i}(r_i))\right)^{\frac{n-1}{n}}} \\ &\ge 2^{-n-1}\cdot 10^{-n} e^{-2a+\frac{a}{n}} \frac{\sum_i \mathrm{vol}_f\left(B_{y_i}(r_i)\right) r_i^{-1}}{\sum_i \left(\mathrm{vol}_f(B_{y_i}(r_i))\right)^{\frac{n-1}{n}}} \\ &\ge 2^{-1}\cdot 10^{-2n} e^{-2a+\frac{a}{n}} \inf_i \frac{\mathrm{vol}_f\left(B_{y_i}(r_i)\right) r_i^{-1}}{\left(\mathrm{vol}_f(B_{y_i}(r_i))\right)^{\frac{n-1}{n}}} \\ &\ge 2^{-1}\cdot 10^{-2n} e^{-2a+\frac{a}{n}} \inf_i \left[r_i^{-1}\, \mathrm{vol}_f^{\frac{1}{n}}\left(B_{y_i}(r_i)\right)\right]. \end{aligned} \tag{3.16}$$

On the other hand, since $d(y_i, y_0) \le r_0$, then $B_{y_0}(r_0) \subset B_{y_i}(2r_0)$. Using the volume doubling property (2.5) with $r_i \le \frac{1}{10}$ and (3.13) yields

$$\mathrm{vol}_f\left(B_{y_i}(r_i)\right) \ge \frac{(10r_i)^n}{2} e^{-\frac{a}{10}}\,\mathrm{vol}_f\left(B_{y_i}(\tfrac{1}{10})\right) \ge (10r_i)^n e^{-\frac{a}{10}}\,\mathrm{vol}_f\left(B_{y_0}(r_0)\right).$$

Inserting above inequality into (3.16), we obtain

$$\frac{\mathrm{vol}_f(\partial\Omega)}{(\mathrm{vol}_f(\Omega))^{\frac{n-1}{n}}} \ge 5\cdot 10^{-2n} e^{-2a+\frac{a}{n}-\frac{a}{10n}}\,\mathrm{vol}_f^{\frac{1}{n}}(B_{y_0}(r_0)) \ge 10^{-2n} e^{-2a}\,\mathrm{vol}_f^{\frac{1}{n}}(B_{y_0}(r_0)).$$

Hence,

$$\operatorname{vol}_f\left(B(y_0,r_0)\right)^{-\frac{1}{n}}\inf_{\Omega}\left\{\frac{\operatorname{vol}_f(\partial\Omega)}{\operatorname{vol}_f(\Omega)^{\frac{n-1}{n}}}\right\}\geq 10^{-2n}e^{-2a}.$$

Our task now is to show (1.1) holds for any radius $R\leq 1$ and for $\bar{\kappa}(p,f,a,1)\leq\varepsilon_2$ with $\varepsilon_2=\varepsilon_2(n,p,a)>0$. Let $r_1=\frac{R}{r_0}\leq\frac{1}{r_0}$. After a scaling, its sufficient to check that $\bar{\kappa}(p,f,a,r_1)\leq\varepsilon_1$. Choose ε_2 satisfying $\varepsilon_2\leq\varepsilon_0$, then (2.5) holds for all $R\leq 1$. Now if $r_1\leq 1$, by (2.6),

$$\bar{\kappa}(p,f,a,r_1)\leq 2^{\frac{1}{p}}e^{\frac{a}{p}}\bar{\kappa}(p,f,a,1)\leq 2^{\frac{1}{p}}e^{\frac{a}{p}}\varepsilon_2.$$

If $1<r_1\leq\frac{1}{r_0}$, then by (2.7) we have

$$\bar{\kappa}(p,f,a,r_1)\leq 2^{\frac{n+1}{p}}e^{\frac{ar_1}{p}}\bar{\kappa}(p,f,a,1)\leq 2^{\frac{n+1}{p}}e^{\frac{a}{pr_0}}r_0^{-2}\varepsilon_2.$$

Using the two cases, let

$$\varepsilon_2=\min\{2^{-\frac{1}{p}}e^{-\frac{a}{p}}\varepsilon_1,2^{-\frac{n+1}{p}}e^{-\frac{a}{pr_0}}r_0^2\varepsilon_1,\varepsilon_0\}.$$

The proof is complete by setting $\varepsilon=\varepsilon_2$. □

Theorem A.1 in appendix implies that the normalized constants in (3.1) is equivalent, that is

$$\mathrm{ID}^*_{n,f}\left(B_x(R)\right)=\mathrm{SD}^*_{n,f}\left(B_x(R)\right).$$

Hence, Theorem 1.1 gives the following Sobolev inequality.

Corollary 3.7. If $\bar{\kappa}(p,f,a,1)\leq\varepsilon$ for the ε in Theorem 1.1, then for any $R\leq 1$,

$$\|\nabla h\|^*_{1,f,B_x(R)}\geq 10^{-2n}e^{-2a}R^{-1}\|h\|^*_{\frac{n}{n-1},f,B_x(R)},\quad\forall h\in C_0^\infty(B_x(R)).\tag{3.17}$$

and

$$\|\nabla h\|^*_{2,f,B_x(R)}\geq\frac{n-2}{2(n-1)}10^{-2n}e^{-2a}R^{-1}\|h\|^*_{\frac{2n}{n-2},f,B_x(R)},\quad\forall h\in C_0^\infty(B_x(R)).\tag{3.18}$$

Applying (3.17) to $h^{\frac{2(n-1)}{n-2}}$, together with Hölder inequality we can get (3.18).

The first eigenvalue of f-Laplacian is defined by

$$\lambda_1(B_x(R))=\inf_{h\in C_0^\infty}\frac{\int_{B_x(R)}|\nabla h|^2d\operatorname{vol}_f}{\int_{B_x(R)}h^2d\operatorname{vol}_f}.$$

As in Cheeger's inequality [6] we have

Corollary 3.8. With the same set up as Theorem 1.1. For $p>\frac{n}{2}$, there exists $\varepsilon=\varepsilon(n,p,a)>0$ such that $\bar{\kappa}(p,f,1)\leq\varepsilon$, then for any $R\leq 1$, the first eigenvalue of Dirichlet f-Laplacian has lower bound

$$\lambda_1(B_x(R))\geq C(n,p,a)R^{-2},\quad C(n,p,a)=\frac{(n-2)^2}{4(n-1)^2}10^{-4n}e^{-4a}.$$

Proof. Suppose $\Delta_f h = -\lambda h$ for some $\lambda > 0$, and normalized h such that $\fint_{B_x(R)} h^2 e^{-f} d\,\mathrm{vol} = 1$ and $h = 0$ on $\partial B_x(R)$. Then using (3.18) we have

$$\begin{aligned}\lambda = \left(\|\nabla h\|^*_{2,f,B_x(R)}\right)^2 &\geq \left(\frac{n-2}{2(n-1)}10^{-2n}e^{-2a}R^{-1}\|h\|^*_{\frac{2n}{n-2},f,B_x(R)}\right)^2 \\ &\geq \frac{(n-2)^2}{4(n-1)^2}10^{-4n}e^{-4a}R^{-2}\left(\|h\|^*_{2,f,B_x(R)}\right)^2 \\ &= \frac{(n-2)^2}{4(n-1)^2}10^{-4n}e^{-4a}R^{-2}.\end{aligned}$$

The proof is completed. □

4. APPLICATIONS

In this section, we prove the maximum principle and gradient estimate for integral Bakry-Émery Ricci curvature with the help of the normalized local Dirichlet Sobolev constant estimate.

Let $C_s(\Omega)$ be the normalized local Sobolev constant of $\Omega \subset B_x(R) \subset M^n_f$, that

$$\|h\|^*_{\frac{2n}{n-2},f,\Omega} \leq C_s(\Omega)\|\nabla h\|^*_{2,f,\Omega}, \quad \forall h \in C_0^\infty(\Omega). \tag{4.1}$$

Obviously, $C_s(\Omega)$ is the smallest constant such that (4.1) hold for all $h \in C_0^\infty(\Omega)$. Since $h \in C_0^\infty(B_x(R))$, then (3.18) gives

$$C_s(\Omega) \leq C_s(B_x(R)) \leq \frac{2(n-1)}{n-2}10^{2n}e^{2a}R. \tag{4.2}$$

Theorem 4.1. Let M^n_f be a smooth metric measure space and $\Omega \subset B_x(R) \subset M^n_f$ be a domain. For $p > \frac{n}{2}$ and any function u with $u|_{\partial\Omega} = 0$, we have

$$\|u\|^*_{\infty,f} \leq C_s^2(B_x(R))C(n,p)\|\Delta_f u\|_{p,f,B_x(R)}$$

where $C(n,p)$ is a constant depends on n and p.

The proof of this result is quite similar to the one used in [17, Theorem 3.1] with $s = \frac{n}{2}$ due to the Sobolev inequality and the self-adjoint of Δ_f, and so is omitted.

Corollary 4.2. With the same assumption as in Theorem 4.1, for $p > \frac{n}{2}$ and any function $u : \Omega \subset M^n_f \to \mathbb{R}$ with $\Delta_f u \geq -h$, where f is nonnegative on Ω, we have

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C(n,p) \cdot C_s^2\left(B_x(R)\right) \cdot \|h\|^*_{p,f,\Omega}.$$

Proof. Without lossing of generality, we can assume that $\sup_{x\in\partial\Omega} u(x) = 0$. Then the Dirichlet problem

$$\begin{cases}\Delta_f v = -\lambda v & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega.\end{cases} \tag{4.3}$$

Hence $u - v$ is f-subharmonic, and $u - v \leq 0$ on $\partial\Omega$. By the Maximum principle we get $u \leq v$ in $\overline{\Omega}$, that is $\sup u \leq \|v\|_{\infty,f}$. Using Theorem 4.1 we complete the proof. □

Combining Corollary 4.2 with (4.2) gives Theorem 1.4.

Following the idea in [10, Theorem 5.2], we use the standard and powerful Nash-Morser iteration and establish the gradient estimate. We give the detail of proof due to there are still many differences.

Theorem 4.3. Let M_f^n be a complete smooth metric measure space. Assume that M_f^n satisfies $\partial_r f \geq -a$ along all minimal geodesic segments for some constant $a \geq 0$. For $p > \frac{n}{2}$, u is a function on $B_x(R)$ with $u = 0$ on $\partial B_x(R)$ and satisfying

$$\Delta_f u = h,$$

then

$$\sup_{B(x,\frac{R}{2})} |\nabla u|^2 \leq C(n,p) R^{-2} \frac{\mathrm{vol}_f\left(B_x(R)\right)}{\mathrm{vol}_f\left(B_x(\frac{3}{4}R)\right)} \left[\left(\|h\|^*_{2p,f,B_x(R)}\right)^2 + \left(\|u\|^*_{2,f,B_x(R)}\right)^2 \right]$$
$$\times \left[R^{-2} C_s^2\left(B_x(R)\right) \left(1 + a + R^{-2} C_s^2\left(B_x(R)\right) e^{\frac{a}{p}} \bar{\kappa}(p,f,R)\right) + \left(R^{-2} C_s^2(B_x(R)) e^{\frac{a}{p}} \bar{\kappa}(p,f,R)\right)^{\frac{2p}{2p-n}} \right]^{\frac{n}{2}}.$$

Proof. By scaling we assume $R = 1$. We omit the volume form $e^{-f} d\,\mathrm{vol}$ for convenience. From the Bochner formula, we have

$$\Delta_f |\nabla u|^2 = 2|\operatorname{Hess} u|^2 + 2\langle \nabla u, \nabla \Delta_f u\rangle + 2\operatorname{Ric}_f(\nabla u, \nabla u) \geq 2\langle \nabla u, \nabla h\rangle - 2|\operatorname{Ric}_{f-}|v. \tag{4.4}$$

Let $v = |\nabla u|^2 + \|h\|^*_{p,f}$ and $v = |\nabla u|^2$ if h is constant. For any $\eta \in C_0^\infty\left(B_x(1)\right)$, $l > 1$, we have

(4.5)
$$\begin{aligned}
\int |\nabla(\eta v^l)|^2 &= -\int \left(\eta v^l\right)\left[v^l \Delta_f \eta + 2\langle \nabla \eta, \nabla v^l\rangle + \eta \Delta_f v^l\right] \\
&= \int v^{2l}\left(-\eta\Delta_f\eta\right) - 2\int v^l \langle \nabla\eta, \eta\nabla v^l\rangle - l\int \eta^2 v^{2l-1}\Delta_f v - l(l-1)\int \eta^2 v^{2l-2}|\nabla v|^2 \\
&\leq \int v^{2l}\left(-\eta\Delta_f\eta\right) + \frac{l}{l-1}\int v^{2l}|\nabla\eta|^2 + \frac{l-1}{l}\int \eta^2|\nabla v^l|^2 \\
&\quad - l\int \eta^2 v^{2l-1}\left(2\langle\nabla u, \nabla h\rangle - 2|\operatorname{Ric}_{f-}|v\right) - \frac{l-1}{l}\int \eta^2 |\nabla v^l|^2 \\
&= \int v^{2l}\left(-\eta\Delta_f\eta\right) + \frac{l}{l-1}\int v^{2l}|\nabla\eta|^2 - 2l\int \eta^2 v^{2l-1}\langle\nabla u, \nabla h\rangle + 2l\int \eta^2 v^{2l}|\operatorname{Ric}_{f-}|.
\end{aligned}$$

Here we used Young's inequality $2xy \leq \varepsilon x^2 + \frac{1}{\varepsilon}y^2$ with $\varepsilon = \frac{l}{l-1}$ and (4.4) in the inequality. Integrating by parts and using $|\nabla u| \leq v^{\frac{1}{2}}$ gives

(4.6)
$$\begin{aligned}
&-2l\int \eta^2 v^{2l-1}\langle \nabla u, \nabla h\rangle \\
&= 2l\int \eta^2 v^{2l-1}h^2 + 4l\int \eta h v^{2l-1}\langle\nabla\eta, \nabla u\rangle + 2l(2l-1)\int \eta^2 h v^{2l-2}\langle\nabla u, \nabla v\rangle \\
&\leq 2l\int \eta^2 v^{2l-1}h^2 + 2\int\left(2\sqrt{l}\eta v^{l-\frac{1}{2}}h\right)\left(\sqrt{l}v^l|\nabla\eta|\right) + 2\int\left((4l-2)\eta v^{l-\frac{1}{2}}h\right)\left(\frac{1}{2}\eta|\nabla v^l|\right) \\
&= \left[6l + (4l-2)^2\right]\int \eta^2 v^{2l-1}h^2 + l\int v^{2l}|\nabla\eta|^2 + \frac{1}{4}\int\left|\nabla(\eta v^l) - v^l\nabla\eta\right|^2 \\
&\leq (16l^2 - 10l + 4)\int \eta^2 v^{2l-1}h^2 + \left(l + \frac{1}{2}\right)\int v^{2l}|\nabla\eta|^2 + \frac{1}{2}|\nabla(\eta v^l)|^2.
\end{aligned}$$

Here we used the L^2 Hölder inequality and

$$\eta^2|\nabla v^l|^2 = \left|\nabla(\eta v^l) - v^l\nabla\eta\right|^2 \leq 2|\nabla(\eta v^l)|^2 + 2v^{2l}|\nabla\eta|^2.$$

Inserting (4.6) into (4.5) and regrouping gives

$$\begin{aligned}\int |\nabla(\eta v^l)|^2 \leq& \int v^{2l}\left(-2\eta\Delta_f\eta\right) + \left(\frac{2l}{l-1}+2l+1\right)\int v^{2l}|\nabla\eta|^2 \\ &+ 4(8l^2-5l+2)\int \eta^2 v^{2l-1}h^2 + 4l\int \eta^2 v^{2l}|\operatorname{Ric}_{f-}|.\end{aligned} \tag{4.7}$$

In order to control $\Delta_f\eta$, choose a cut-off function $\phi\in C_0^\infty\left(B_x(1)\right)$ such that $0\leq\phi\leq 1$. For $0<r<1$, $\phi(t)\equiv 1$ for $t\in[0,r]$, $\phi(t)\equiv 0$ for $t\geq 1$, and $\phi'\leq 0$. Then define $\eta(y)=\phi(r(y))$ and $r(y)=d(x,y)$ be a distance function from x. Thus $|\nabla\eta|=|\phi'|$, and

$$\begin{aligned}\Delta_f\eta &= \phi''+\phi'\Delta_f r \\ &\geq \phi''+\phi'\left(\psi+\frac{n-1}{r}+a\right) \\ &\geq -|\phi''|-|\phi'|\psi-\frac{n-1}{r}|\phi'|-a|\phi'|,\end{aligned}$$

where $\psi=\left(\Delta_f r-\frac{n-1}{r}-a\right)_+$. Hence, for $l\geq\frac{n}{n-2}$, (4.7) becomes

$$\int|\nabla(\eta v^l)|^2\leq C(n)l^2\int\left[\left(|\phi''|+|\phi'|r^{-1}+a\right)\eta v^{2l}+|\phi'|\psi\eta v^{2l}+|\phi'|^2v^{2l}+\eta^2h^2v^{2l-1}+\eta^2v^{2l}|\operatorname{Ric}_{f-}|\right].$$

Notice that this formula remains valid for $l=1$. Indeed,

$$\left|\nabla\left(\eta v^{\frac12}\right)\right|^2=\left|v^{\frac12}\nabla\eta+\eta\frac{|\nabla u|}{v^{\frac12}}\nabla|\nabla u|\right|^2\leq 2v|\nabla\eta|^2+2\eta^2|\operatorname{Hess}u|^2,$$

and

$$\begin{aligned}\int\eta^2|\operatorname{Hess}u|^2 &= -\int\nabla_i u\left(2\eta\nabla_i\eta\nabla_i\nabla_j u+\eta^2\nabla_i\Delta_f u+\eta^2(\operatorname{Ric}_f)_{ij}\nabla_j u\right) \\ &\leq\frac12\int\eta^2|\operatorname{Hess}u|^2+3\int|\nabla\eta|^2v+2\int\eta^2h^2+\int\eta^2|\operatorname{Ric}_{f-}|v.\end{aligned}$$

Next we use C_s denote $C_s\left(B_x(1)\right)$ for simplicity. Let $\beta=\frac{n}{n-2}$, applying Sobolev inequality (4.1), then for $l\geq\frac{n}{n-2}$ and $l=1$,

$$\begin{aligned}\left(⨍_{B_x(1)}\left(\eta^2v^{2l}\right)^\beta\right)^{\frac1\beta} &\leq C_s^2⨍_{B_x(1)}\left|\nabla\left(\eta v^l\right)\right|^2 \\ &\leq C_s^2C(n)l^2⨍_{B_x(1)}\left(|\varphi''|+|\varphi'|r^{-1}+a\right)\eta v^{2l} \\ &\quad+C_s^2C(n)l^2⨍_{B_x(1)}\left(|\varphi'|\psi\eta v^{2l}+|\varphi'|^2v^{2l}+\eta^2h^2v^{2l-1}+\eta^2v^{2l}|\operatorname{Ric}_{f-}|\right).\end{aligned} \tag{4.8}$$

The integration involving Bakry-Émery Ricci curvature can be estimated as follows,

$$\begin{aligned}
\fint_{B_x(1)} \eta^2 v^{2l} |\operatorname{Ric}_{f-}| &\le \|\operatorname{Ric}_{f-}\|^*_{p,f,B_x(1)} \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\le e^{\frac{a}{p}} \|\operatorname{Ric}_{f-}\|^*_{p,a,f}(1) \left(\fint_{B_x(1)} \eta^2 v^{2l}\right)^{\frac{p-1}{p}q} \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta}\right)^{\frac{p-1}{p}(1-q)} \\
&\le e^{\frac{a}{p}} \bar{\kappa}(p,f,1) \left[\varepsilon \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta}\right)^{\frac{1}{\beta}} + \varepsilon^{-\frac{n}{2p-n}} \left(\fint_{B_x(1)} \eta^2 v^{2l}\right)\right],
\end{aligned}$$

where $q = q(n,p) = \frac{2p-n}{2(p-1)} > 0$ determined by $q + (1-q)\beta = \frac{p}{p-1}$, we also used Young's inequality

$$xy \le \varepsilon x^b + \varepsilon^{-\frac{b^*}{b}} y^{b^*}, \ \forall x, y \ge 0, \ b > 1, \ \frac{1}{b^*} + \frac{1}{b} = 1,$$

where

$$b = \frac{p}{(1-q)(p-1)\beta}, \quad b^* = \frac{p}{(p-1)q}.$$

By choosing $\varepsilon = \left(4C_s^2 C(n) l^2 e^{\frac{a}{p}} \bar{\kappa}(p,f,1)\right)^{-1}$, we obtain

$$\begin{aligned}
&C_s^2 C(n) l^2 \fint_{B_x(1)} \eta^2 v^{2l} |\operatorname{Ric}_{f-}| \\
&\le \frac{1}{4} \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta}\right)^{\frac{1}{\beta}} + C(n,p) \left(C_s^2 l^2 e^{\frac{a}{p}} \bar{\kappa}(p,f,1)\right)^{\frac{2p}{2p-n}} \fint_{B_x(1)} \eta^2 v^{2l}.
\end{aligned} \tag{4.9}$$

For the term $\fint_{B_x(1)} \eta^2 h^2 v^{2l-1}$, since $v \ge \|h^2\|^*_{p,f,B_x(1)}$, we have

$$\fint_{B_x(1)} \eta^2 h^2 v^{2l-1} \le \frac{1}{\|h^2\|^*_{p,f,B_x(1)}} \fint_{B_x(1)} \eta^2 h^2 v^{2l} \le \left(\fint_{B_x(1)} \left(\eta^2 h^2 v^{2l}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Now the same argument as above with $\varepsilon = (4C_s^2 C(n) l^2)^{-1}$ gives

$$C_s^2 C(n) l^2 \fint_{B_x(1)} \eta^2 h^2 v^{2l-1} \le \frac{1}{4} \left(\fint_{B_x(1)} (\eta^2 v^{2l})^{\beta}\right)^{\frac{1}{\beta}} + C(n,p) \left(C_s^2 l^2\right)^{\frac{2p}{2p-n}} \fint_{B_x(1)} \eta^2 v^{2l}. \tag{4.10}$$

For the term with ψ, applying Hölder inequality, (1.3) and the Laplacian comparison (2.3) gives

$$\begin{aligned}
C_s^2 C(n) l^2 \fint_{B_x(1)} |\phi'| \psi \eta v^{2l} &\le C_s^2 C(n) l^2 \|\psi\|^*_{2p,f,B_x(1)} \cdot \|\eta \phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)} \\
&\le C_s^2 C(n) l^2 e^{\frac{a}{2p}} \|\psi\|^*_{2p,a,f}(1) \cdot \|\eta \phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)} \\
&\le C_s^2 C(n) l^2 e^{\frac{a}{2p}} C(n,p) \left(\bar{\kappa}(p,f,1)\right)^{\frac{1}{2}} \cdot \|\eta \phi' v^{2l}\|^*_{\frac{2p}{2p-1},f,B_x(1)}.
\end{aligned} \tag{4.11}$$

Note that for $\alpha = \frac{p(n-2)}{n(2p-1)} = \frac{1}{\beta}\frac{p}{2p-1} < 1$, we have

$$
\begin{aligned}
\|\eta\phi' v^{2l}\|^*_{\frac{2p}{2p-1}} &= \left[\fint_{B_x(1)} \left(\eta^2 v^{2l}\right)^{\alpha\beta} \left(|\phi'|^2 v^{2l}\right)^{\frac{p}{2p-1}}\right]^{\frac{2p-1}{2p}} \\
&\le \left[\left(\fint_{B_x(1)} \left(\eta^2 v^{2l}\right)^{\beta}\right)^{\alpha} \left(\fint_{B_x(1)} \left(|\phi'|^2 v^{2l}\right)^{\frac{np}{np+2p-n}}\right)^{\frac{np+2p-n}{n(2p-1)}}\right]^{\frac{2p-1}{2p}} \\
&\le \left[\left(\fint_{B_x(1)} \left(\eta^2 v^{2l}\right)^{\beta}\right)^{\alpha} \left(\fint_{B_x(1)} |\phi'|^2 v^{2l}\right)^{\frac{p}{2p-1}}\right]^{\frac{2p-1}{2p}} \\
&\le \varepsilon \left(\fint_{B_x(1)} \left(\eta^2 v^{2l}\right)^{\beta}\right)^{\frac{1}{\beta}} + \frac{1}{4\varepsilon}\fint_{B_x(1)} |\phi'|^2 v^{2l},
\end{aligned}
\tag{4.12}
$$

where we used Hölder inequlity due to $p > \frac{n}{2}$ implies that $\frac{np}{np+2p-n} < 1$ in the second inequality. By setting $\varepsilon = \left(4C_s^2 C(n) l^2 e^{\frac{a}{2p}} C(n,p) \left(\bar{\kappa}(p,f,1)\right)^{1/2}\right)^{-1}$ and inserting (4.12) into (4.11) we obtain

$$
\begin{aligned}
C_s^2 C(n) l^2 \fint_{B_x(1)} \psi\eta|\phi'| v^{2l} \le{}& \frac{1}{4}\left(\fint_{B_x(1)} \left(\eta^2 v^{2l}\right)^{\beta}\right)^{\frac{1}{\beta}} \\
&+ \left(C_s^4 C^2(n) l^4 e^{\frac{a}{p}} C^2(n,p)\bar{\kappa}(p,f,1)\right)\fint_{B_x(1)} |\phi'|^2 v^{2l}.
\end{aligned}
\tag{4.13}
$$

Inserting (4.9), (4.10), (4.13) into (4.8) gives

$$
\begin{aligned}
\left(\fint_{B_x(1)} \left(\eta^2 v^{2l}\right)^{\beta}\right)^{\frac{1}{\beta}} \le{}& 4C_s^2 C(n) l^2 \fint_{B_x(1)} \left(|\phi''| + \frac{|\phi'|}{r} + a\right)\eta v^{2l} \\
&+ 4C_s^2 C(n) l^2 \left(1 + C_s^2 l^2 e^{\frac{a}{p}} C^2(n,p)\bar{\kappa}(p,f,1)\right)\fint_{B_x(1)} |\phi'|^2 v^{2l} \\
&+ C(n,p)\left(C_s^2 l^2\right)^{\frac{2p}{2p-n}}\left[1 + \left(e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right)^{\frac{2p}{2p-n}}\right]\left(\fint_{B_x(1)} \eta^2 v^{2l}\right).
\end{aligned}
\tag{4.14}
$$

Define $l_i = \frac{\beta^i}{2}$, $i \ge 0$, and $r_i = \frac{3}{4} - \sum_{j=0}^{i} 2^{-j-1}$. Choose cut-off function $\eta_i = \phi_i(r) \in C_0^\infty\left(B_x(r_i)\right)$ such that

$$
\eta_i \equiv 1 \quad \text{on } B\left(x, r_{i+1}\right); \quad |\phi_i'| \le 2^{i+1}, \quad |\phi_i''| \le 2^{2i+2}.
$$

Then (4.14) becomes

$$
\begin{aligned}
&\|v\|^*_{\beta^{i+1},f,B_x(r_{i+1})} \\
&\le C(n,p)\left\{C_s^2\left(\beta^{4i}\right)\left(1 + a + C_s^2 e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right) + \left(C_s^2\beta^{2i}\right)^{\frac{2p}{2p-n}}\left[1 + \left(e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right)^{\frac{2p}{2p-n}}\right]\right\}\|v\|^*_{\beta^i,f,B_x(r_i)} \\
&\le C(n,p)\beta^{si}\left\{C_s^2\left(1 + a + C_s^2 e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right) + C_s^{\frac{4p}{2p-n}}\left[1 + \left(e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right)^{\frac{2p}{2p-n}}\right]\right\}\|v\|^*_{\beta^i,f,B_x(r_i)}
\end{aligned}
$$

where $s = \max\{4, \frac{4p}{2p-n}\}$. Then substituting η_i into the estimate and running the iteration from $i = 0$ gives

$$
\|v\|^*_{\infty,B_x(\frac{1}{2})} \le C(n,p) A^{\frac{n}{2}} \|v\|^*_{1,f,B_x(\frac{3}{4})}.
\tag{4.15}
$$

$$A = C_s^2\left(1 + a + C_s^2 e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right) + C_s^{\frac{4p}{2p-n}}\left[1 + \left(e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right)^{\frac{2p}{2p-n}}\right].$$

Finally, we estimate the term $\|v\|^*_{1,f,B_x(\frac{3}{4})}$. For $\eta \in C_0^\infty(B_x(1))$ with $\eta \equiv 1$ in $B_x(\frac{3}{4})$ and $|\nabla\eta| \le 5$, then

$$\|v\|^*_{1,f,B_x(\frac{3}{4})} \le \frac{\mathrm{vol}_f\left(B_x(1)\right)}{\mathrm{vol}_f\left(B_x(\frac{3}{4})\right)}\|v\|^*_{1,f,B_x(1)} = \frac{\mathrm{vol}_f\left(B_x(1)\right)}{\mathrm{vol}_f\left(B_x(\frac{3}{4})\right)}\fint_{B_x(1)}\eta^2\left(|\nabla u|^2 + \|h\|^*_{p,f,B_x(1)}\right). \tag{4.16}$$

Using the integrating by parts and Young's inequality, we have

$$\begin{aligned}\fint_{B_x(1)}\eta^2|\nabla u|^2 &= -2\fint_{B_x(1)}\eta\langle\nabla\eta,\nabla u\rangle u - \fint_{B_x(1)}\eta^2 hu\\ &\le \frac{1}{2}\fint_{B_x(1)}\eta^2|\nabla u|^2 + 2\fint_{B_x(1)}|\nabla\eta|^2u^2 + \frac{1}{2}\fint_{B_x(1)}\left(\eta^2h^2 + \eta^2u^2\right).\end{aligned}$$

Regrouping above inequality and inserting it into (4.16) gives

$$\begin{aligned}\|v\|^*_{1,f,B_x(\frac{3}{4})} &= \frac{\mathrm{vol}_f\left(B_x(1)\right)}{\mathrm{vol}_f\left(B_x(\frac{3}{4})\right)}\left[4\fint_{B_x(1)}|\nabla\eta|^2u^2 + \fint_{B_x(1)}\eta^2u^2 + \fint_{B_x(1)}\eta^2h^2 + \|h^2\|^*_{p,f,B_x(1)}\right]\\ &\le \frac{\mathrm{vol}_f\left(B_x(1)\right)}{\mathrm{vol}_f\left(B_x(\frac{3}{4})\right)}\left[101\left(\|u\|^*_{2,f,B_x(1)}\right)^2 + 2\left(\|h\|^*_{2p,f,B_x(1)}\right)^2\right].\end{aligned} \tag{4.17}$$

Combing (4.15) with (4.17) yields

$$\begin{aligned}\sup_{B(x,\frac{1}{2})}|\nabla u|^2 &\le C(n,p)\frac{\mathrm{vol}_f\left(B_x(1)\right)}{\mathrm{vol}_f\left(B_x(\frac{3}{4})\right)}\left[\left(\|h\|^*_{2p,f,B_x(1)}\right)^2 + \left(\|u\|^*_{2,f,B_x(1)}\right)^2\right]\\ &\quad\times\left\{C_s^2\left(1 + a + C_s^2e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right) + C_s^{\frac{4p}{2p-n}}\left[1 + \left(e^{\frac{a}{p}}\bar{\kappa}(p,f,1)\right)^{\frac{2p}{2p-n}}\right]\right\}^{\frac{n}{2}}.\end{aligned}$$

Thus we can get the desired result by scaling. □

Combining Theorem 4.3, local Sobolev constant estimate (4.2), with the volume doubling property (3.18) gives Theorem 1.5.

Appendix A. Equivalence of Isoperimetric and Sobolev Constants for Weighted Measure

In this appendix we show the equivalence between the local isoperimetric and Sobolev constants defined in Section 3 by adapting the proof of [12, Theorem 9.5]. A special case of Theorem A.1 can be found in [8, Proposition 1.1].

Theorem A.1. For all $n \le \alpha \le \infty$, we have

$$\mathrm{ID}_{\alpha,f}(B_x(R)) = \mathrm{SD}_{\alpha,f}(B_x(R)). \tag{A.1}$$

Proof. Let σ_f be $(n-1)$ dimensional Hausdorff measure. We omit the weighted measure $e^{-f}d\,\mathrm{vol}$ for convenience. For $\Omega \subset B_x(R)$ with $\Omega \cap \partial B_x(R) = \varnothing$, let $\Omega_\varepsilon = \{y \in \Omega : d(y,\partial\Omega) \ge \varepsilon\}$. Constructing a function by

$$h_\varepsilon(y) = \begin{cases}0, & B_x(R)\setminus\Omega\\ \frac{1}{\varepsilon}d(y,\partial B_x(R)), & \Omega\setminus\Omega_\varepsilon\\ 1, & \Omega_\varepsilon.\end{cases}$$

Since the distance function d is Lipschitz, then h_ε is Lipschitz with $h_\varepsilon|_{\partial\Omega} = 0$. Applying Sobolev inequality to h_ε gives

$$\int_{B_x(R)} |\nabla h_\varepsilon| \geq \mathrm{SD}_{\alpha,f}(B_x(R)) \left(\int_{B_x(R)} |h_\varepsilon|^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \geq \mathrm{SD}_{\alpha,f}(B_x(R)) \operatorname{vol}_f(\Omega_\varepsilon)^{\frac{\alpha-1}{\alpha}}. \tag{A.2}$$

As well as, Coarea formula implies that

$$\lim_{\varepsilon\to 0} \int_{B_x(R)} |\nabla h_\varepsilon| e^{-f} d\operatorname{vol} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \sigma_f(\partial\Omega_t)dt = \sigma_f(\partial\Omega). \tag{A.3}$$

Combining (A.2) with (A.3), together with Definition 3.1, we have $\mathrm{ID}_{\alpha,f}(B_x(R)) \geq \mathrm{SD}_{\alpha,f}(B_x(R))$.

To see that $\mathrm{ID}_{\alpha,f}(B_x(R)) \leq \mathrm{SD}_{\alpha,f}(B_x(R))$, its suffices to show that

$$\int_{B_x(R)} |\nabla h| \geq \mathrm{ID}_{\alpha,f}(B_x(R)) \left(\int_{B_x(R)} h^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}} \tag{A.4}$$

hold for $h|_{\partial B_x(R)} = 0$. Without loss of generality, we may assume $h \geq 0$. Let $B_t := \{y \in B_x(R) | h(y) > t\}$ to be the sublevel set of h. By co-area formula,

$$\int_{B_x(R)} |\nabla h| = \int_0^\infty \sigma_f(\partial B_t)dt \geq \mathrm{ID}_{\alpha,f}(B_x(R)) \int_0^\infty \operatorname{vol}_f^{\frac{\alpha-1}{\alpha}}(B_t)\, dt. \tag{A.5}$$

Let

$$F(s) = \left(\int_0^s \operatorname{vol}_f^{\frac{\alpha-1}{\alpha}}(B_t)\, dt \right)^{\frac{\alpha}{\alpha-1}} - \frac{\alpha}{\alpha-1} \int_0^s t^{\frac{1}{\alpha-1}} \operatorname{vol}_f(B_t) dt.$$

Obviously, $F(0) = 0$ and

$$F'(s) = \frac{\alpha}{\alpha-1} \left(\int_0^s \operatorname{vol}_f^{\frac{\alpha-1}{\alpha}}(B_t)\, dt \right)^{\frac{1}{\alpha-1}} \operatorname{vol}_f^{\frac{\alpha-1}{\alpha}}(B_s) - \frac{\alpha}{\alpha-1} s^{\frac{1}{\alpha-1}} \operatorname{vol}_f(B_s).$$

Since $B_s \subset B_t$ for $t \leq s$, then $\int_0^s \operatorname{vol}_f^{\frac{\alpha-1}{\alpha}}(B_t)\, dt \geq s \operatorname{vol}_f^{\frac{\alpha-1}{\alpha}}(B_s)$ yields $F'(s) \geq 0$, hence $F(s) \geq 0$. Applying this inequality to (A.5) yields

$$\int_{B_x(R)} |\nabla h| \geq \mathrm{ID}_{\alpha,f}(B_x(R)) \left(\frac{\alpha}{\alpha-1} \int_0^\infty t^{\frac{1}{\alpha-1}} \operatorname{vol}_f(B_t) dt \right)^{\frac{\alpha-1}{\alpha}}. \tag{A.6}$$

Integrating by parts and using the co-area formula,

$$\begin{aligned} \frac{\alpha}{\alpha-1} \int_0^\infty t^{\frac{1}{\alpha-1}} \operatorname{vol}_f(B_t) dt &= \int_0^\infty \left(\frac{d(t^{\frac{\alpha}{\alpha-1}})}{dt} \int_t^\infty \int_{\partial B_s} \frac{d\sigma_f(\partial B_s)}{|\nabla h|} ds \right) dt \\ &= \int_0^\infty t^{\frac{\alpha}{\alpha-1}} \int_{\partial B_t} \frac{d\sigma_f(\partial B_t)}{|\nabla h|} dt = \int_{B_x(R)} h^{\frac{\alpha}{\alpha-1}}. \end{aligned}$$

Inserting above equality into (A.6) gives (A.4). □

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